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Deriving effective models for multiscale systems via evolutionary Γ-convergence

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Abstract

We discuss possible extensions of the recently established theory of evolutionary Γ-convergence for gradient systems to nonlinear dynamical systems obtained by perturbation of a gradient systems. Thus, it is possible to derive effective equations for pattern forming systems with multiple scales. Our applications include homogenization of reaction-diffusion systems, the justification of amplitude equations for Turing instabilities, and the limit from pure diffusion to reaction-diffusion. This is achieved by generalizing the Γ-limit approaches based on the energy-dissipation principle or the evolutionary variational estimate.

1 Introduction

The theory of evolutionary Γ-convergence was developed for families of gradient systems $(\mathbf{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)_{\varepsilon\in[0,1]}$, which define the family of gradient flows

$$\mathrm{D}_{\dot u}\mathcal{R}_\varepsilon(u^\varepsilon, \dot u^\varepsilon) = -\mathrm{D}\mathcal{E}_\varepsilon(u^\varepsilon), \qquad u^\varepsilon(0) = u^0_\varepsilon.$$

The aim of the theory is to provide as general conditions as possible for the convergence of the energy functionals $\mathcal{E}_\varepsilon \rightsquigarrow \mathcal{E}_0$ and of the dissipation potentials $\mathcal{R}_\varepsilon \rightsquigarrow \mathcal{R}_0$ for $\varepsilon \to 0$, that still guarantee that the solutions $u^\varepsilon : [0,T] \to \mathbf{X}$ converge to a solution $u^0 : [0,T] \to \mathbf{X}$ of the limiting gradient flow as $\varepsilon \to 0$. We refer to the surveys [SaS04, Ste08, Ser11, Bra13, Mie15b]. We emphasize here that there are numerous much older works relating to the case that $\mathbf{X}$ is a Hilbert space $\mathbf{H}$ and $\mathcal{R}_\varepsilon(u,\dot u) = \frac12\|\dot u\|^2_{\mathbf{H}}$ is independent of ε such that only equation has the form $\dot u = -\mathrm{D}\mathcal{E}_\varepsilon(u)$ where A_ε is a maximal monotone operator, see [Bré73, Att84].

Here we are interested in perturbed gradient systems, where we allow the energy functional $\mathcal{E}_\varepsilon$ to depend on the time $t \in [0,T]$ and the equation to contain a non-gradient term h_ε. We use the quadruple $(\mathbf{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon, h_\varepsilon)$ to denote the perturbed gradient system, which then defines an evolutionary equation

$$\mathrm{D}_{\dot u}\mathcal{R}_\varepsilon(u^\varepsilon, \dot u^\varepsilon) = -\mathrm{D}\mathcal{E}_\varepsilon(u^\varepsilon) + h_\varepsilon(t, u^\varepsilon), \qquad u^\varepsilon(0) = u^0_\varepsilon. \tag{1.1}$$

Here we understand that h_ε is a lower order perturbation of the gradient system obtained for $h_\varepsilon \equiv 0$. Thus, the hope is that it is possible to generalize the strong results on evolutionary convergence of gradient systems (see [Ser11, Mie15b]) to the perturbed case without adding too much technicalities.

Hence, there are two major motivations for considering perturbed gradient systems. On the one hand, there may be cases where a given system has a particular gradient structure $(\widehat{\mathbf{X}}, \widehat{\mathcal{E}}_\varepsilon, \widehat{\mathcal{R}}_\varepsilon)$, but it may be easier to treat it as a perturbed gradient system $(\mathbf{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$. We highlight this by looking at the reaction-diffusion system

$$\dot u = \mathrm{div}\,\big(a_\varepsilon(x)\nabla u\big) + \frac{c_\varepsilon(x)(1{-}uv)}{d_\varepsilon(x){+}u{+}v}, \quad \dot v = \mathrm{div}\,\big(b_\varepsilon(x)\nabla v\big) + \frac{c_\varepsilon(x)(1{-}uv)}{d_\varepsilon(x){+}u{+}v},$$

where $u, v > 0$ are densities and $a_\varepsilon, b_\varepsilon, c_\varepsilon$ and d_ε are positive ε-periodic coefficients. It was shown in [Mie11] that this system is has a gradient system with

$$\widehat{\mathcal{E}}(u,v) = \int_\Omega \lambda_{\mathrm{B}}(u) + \lambda_{\mathrm{B}}(v)\,\mathrm{d}x \quad \text{with } \lambda_{\mathrm{B}}(u) := u\log u - u + 1,$$
$$\widehat{\mathcal{R}}^*_\varepsilon(u,v,\mu,\nu) = \int_\Omega \Big(\frac{a_\varepsilon}{2}|\nabla\xi|^2 + \frac{b_\varepsilon}{2}|\nabla\nu|^2 + C_\varepsilon(x,u,v)(\xi+\nu)^2\Big)\,\mathrm{d}x,$$

where $c_\varepsilon(x,u,v) = \frac{c_\varepsilon(x)uv}{(d_\varepsilon(x)+u+v)(\log(uv)-1)} > 0$, and $\mathcal{R}^*_\varepsilon$ is the Legendre dual potential of $\mathcal{R}_\varepsilon$, see (4.1). However, doing a multiscale analysis for the limit $\varepsilon \to 0$ is very difficult because of the dependence of $\widehat{\mathcal{R}}^*_\varepsilon$ on u and v.

For a perturbed gradient structure we may choose the classical L^2 gradient structure for the leading terms and treat the reactions as perturbations, i.e.

$$\mathcal{E}_\varepsilon(u,v) = \int_\Omega \Big(\frac{a_\varepsilon}{2}|\nabla u|^2 + \frac{b_\varepsilon}{2}|\nabla v|^2\Big)\,\mathrm{d}x, \quad \mathcal{R}_\varepsilon(\dot u,\dot v) = \frac{1}{2}\|\dot u\|^2_{\mathrm{L}^2} + \frac{1}{2}\|\dot v\|^2_{\mathrm{L}^2},$$

and the perturbation $h_\varepsilon(x,u,v) = \frac{c_\varepsilon(x)(1-uv)}{d_\varepsilon(x)+u+v}\big(1,1\big)^\top$. For such a system the limit $\varepsilon \to 0$ can be taken much more easily, see Section 5.1 and [MRT14, Rei15].

On the other hand, the treatment of perturbed gradient systems is important, since the dynamics of pure gradient systems is completely different from perturbed ones. In gradient systems, typical solutions converge to local minimizer of the energy for $t \to \infty$. In a perturbed s gradient system, much more complicated dynamics can happen, like Hopf bifurcations or chaos, see e.g. [FiP90].

Section 2 provides a priori estimates for the perturbed gradient system $(\mathbf{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon, h_\varepsilon)$. In additions to the standard conditions on gradient systems the new assumption is an estimate of the form $\mathcal{R}^*_\varepsilon(u, \frac{1}{c}h_\varepsilon(t,u)) \le C\mathcal{E}(t,u)$ (cf. (2.2)). Based only on these simply estimates we provide two abstract results on evolutionary Γ-convergence.

The first result on evolutionary Γ-convergence for perturbed gradient systems is given in Theorem 3.2 and relies on the rather strong assumption of λ-convexity. For this we assume that $\mathbf{X}$ is a Hilbert space $\mathbf{H}$, that the dissipation potentials have the quadratic form $\mathcal{R}_\varepsilon(u,\dot u) = \frac{1}{2}\langle \mathbb{G}_\varepsilon \dot u, \dot u\rangle$, and that there exists a $\lambda \in \mathbb{R}$ such that $u \mapsto \mathcal{E}_\varepsilon(u) - \lambda\mathcal{R}_\varepsilon(u)$ is convex for all $\varepsilon \in [0,1]$. Otherwise the assumptions are rather weak, since the simple Γ-convergence of $\mathcal{E}_\varepsilon(t,\cdot) \xrightarrow{\Gamma} \mathcal{E}_0(t,\cdot)$ and pointwise convergence of $\mathcal{R}_\varepsilon$ are essentially sufficient.

The second result on evolutionary Γ-convergence for perturbed gradient systems relies on De Giorgi's energy-dissipation principle. It is much more flexible, since no λ-convexity is needed and $\mathcal{R}_\varepsilon$ can be much more general. The major new quantity in this approach is the dissipation functional

$$\mathfrak{D}_\varepsilon(u(\cdot)) := \int_0^T \Big(\mathcal{R}_\varepsilon(u(t),\dot u(t)) + \mathcal{R}^*_\varepsilon\big(u(t), h_\varepsilon(t,u(t)) - \mathrm{D}_u\mathcal{E}_\varepsilon(t,u(t))\big)\Big)\,\mathrm{d}t.$$

As a major assumption of the abstract result in Theorem 4.3 one needs the liminf estimate $\liminf_{\varepsilon\to0}\mathfrak{D}_\varepsilon(u^\varepsilon(\cdot)) \ge \mathfrak{D}(u(\cdot))$ if $u^\varepsilon \rightharpoonup u$ in $\mathrm{W}^{1,p}([0,T];\mathbf{X})$.

In Section 5 we discuss a few possible applications of the general results. We first consider the classical question of homogenization of reaction-diffusion systems as a didactical example. There we treat the diffusion part as a gradient part associated with the convex and quadratic Dirichlet energy. Because of the semilinear structure, all the nonlinear reaction terms can be treated as non-gradient perturbations. We are able to apply the λ-convex theory and refer to [LiR15] for a comparison of the strengths and weaknesses of the two different approaches discussed on the basis of the homogenization of a Cahn–Hilliard-type problem.

In Section 5.2 we reconsider the theory developed in [Mie15a] for pure gradient systems. There it was shown that the Ginzburg–Landau equations can be understood as the evolutionary Γ-limit of the suitable scaled Swift–Hohenberg equation. We discuss the usage of perturbed gradient systems to analyze a coupled system of Swift–Hohenberg equations introduced in [SA*14].

Finally we speculate concerning the usage of evolutionary Γ-convergence to derive a nonlinear reaction-diffusion system from a single Fokker–Planck-type master equation of diffusion in physical space as well as along a chemical reaction path. This follows the spirit of [PSV10, PSV12, AM*12, LM*15], where chemical reaction is understood as a limit of diffusion.

2 Energy control and a priori estimates

As was announced earlier, we will consider the non-gradient term h_ε as a lower-order perturbation of the gradient system. Before specifying this, we fix the major properties of the energy functionals $\mathcal{E}_\varepsilon$. We assume that the reflexive and separable Banach space $\mathbf{Z}$ is compactly embedded into $\mathbf{X}$ and

$$\operatorname{dom}\mathcal{E}_\varepsilon := \{\,(t,u)\mid \mathcal{E}_\varepsilon(t,u)<\infty\,\} = [0,T]\times\mathbf{D}_\varepsilon, \quad \mathbf{D}_\varepsilon := \operatorname{dom}\mathcal{E}_\varepsilon(0,\cdot), \tag{2.1a}$$

$$\exists\, c_0,\alpha>0 \ \ \forall\,(\varepsilon,t,u)\in[0,1]\times[0,T]\times\mathbf{X}: \quad \mathcal{E}_\varepsilon(t,u)\geq c_0\|u\|_{\mathbf{Z}}^\alpha, \tag{2.1b}$$

$$\exists\,\Lambda_{\mathrm{na}}\geq 0 \ \ \forall\,(\varepsilon,t,u)\in[0,1]\times[0,T]\times\mathbf{D}_\varepsilon: \quad |\partial_t\mathcal{E}_\varepsilon(t,u)|\leq\Lambda_{\mathrm{na}}\mathcal{E}_\varepsilon(t,u), \tag{2.1c}$$

where $\|u\|_{\mathbf{Z}}=\infty$ for $u\in\mathbf{X}\setminus\mathbf{Z}$. Note that the energies are only defined up to a constant, so we can choose $C=0$ in the usual condition of coercivity $\mathcal{E}_\varepsilon(t,u)\geq c_0\|u\|_{\mathbf{Z}}^\alpha - C$.

In this section we consider general dissipation potentials $\mathcal{R}_\varepsilon:\mathbf{X}\times\mathbf{X}\to[0,\infty]$, which means that $\mathcal{R}_\varepsilon(u,\cdot):\mathbf{X}\to[0,\infty]$ is a lower semicontinuous and convex functional satisfying additionally $\mathcal{R}_\varepsilon(u,0)=0$. The first condition on the perturbation $h_\varepsilon:[0,T]\times\mathbf{X}\to\mathbf{X}^*$ is the following bound:

$$\begin{aligned}&\exists\,\Lambda_{\mathrm{ng}}\geq 0,\ c\in\,]0,1[\ \forall\,(\varepsilon,t,u)\in[0,1]\times[0,T]\times\mathbf{X}:\\ &\qquad\qquad \mathcal{R}_\varepsilon^*(u,\tfrac{1}{c}h_\varepsilon(t,u))\leq\frac{\Lambda_{\mathrm{ng}}}{c}\mathcal{E}(t,u).\end{aligned} \tag{2.2}$$

Based on these assumptions we first derive a control of the energy $\mathcal{E}_\varepsilon$ for fixed w and along solutions $u:[0,T]\to\mathbf{X}$ of the perturbed gradient flow

$$\mathrm{D}_{\dot u}\mathcal{R}_\varepsilon(u,\dot u(t)) = -\mathrm{D}_u\mathcal{E}_\varepsilon(t,u(t)) + h_\varepsilon(t,u(t)) \text{ for a.a. } t\in[0,T]. \tag{2.3}$$

Note that all the estimates are uniform in $\varepsilon \in [0,1]$.

Proposition 2.1 *If* (2.1c) *holds, then for all* $(\varepsilon,s,t,w) \in [0,1]\times[0,T]^2\times\mathbf{D}_\varepsilon$*:*

$$\mathrm{e}^{-\Lambda_{na}|t-s|}\mathcal{E}_\varepsilon(s,w) \le \mathcal{E}_\varepsilon(t,w) \le \mathrm{e}^{\Lambda_{na}|t-s|}\mathcal{E}_\varepsilon(s,w). \tag{2.4}$$

Assuming additionally (2.2) *and setting* $\Lambda := \Lambda_{na}+\Lambda_{ng}$*, every solution* $u:[0,T]\to\mathbf{X}$ *of* (2.3) *satisfies, for* $0\le s<t\le T$*, the estimate*

$$\mathcal{E}_\varepsilon(t,u(t)) + \int_s^t (1-c)\mathcal{R}_\varepsilon(u(r),\dot u(r))\,\mathrm{d}r \le \mathrm{e}^{\Lambda(t-s)}\mathcal{E}_\varepsilon(s,u(s)). \tag{2.5}$$

Proof. Equation (2.4) follows by a simple Gronwall estimate based on (2.1c).

For the second result we apply $\langle\cdot,\dot u\rangle$ to (2.3) and use $\langle \mathrm{D}_{\dot u}\mathcal{R}_\varepsilon(u,\dot u),\dot u\rangle \ge \mathcal{R}_\varepsilon(u,\dot u)$ and the chain rule for $\mathcal{E}_\varepsilon$ to obtain the energy estimate

$$\mathcal{E}_\varepsilon(t,u(t)) + \int_s^t \mathcal{R}_\varepsilon(u,\dot u)\,\mathrm{d}r \le \mathcal{E}_\varepsilon(s,u(s)) + \int_s^t \partial_r\mathcal{E}_\varepsilon(r,u(r)) + \langle h_\varepsilon(r,u(r)),\dot u(r)\rangle\,\mathrm{d}r.$$

Estimating $\langle h_\varepsilon,\dot u(r)\rangle \le c\mathcal{R}^*_\varepsilon(u,\frac{1}{c}h_\varepsilon) + c\mathcal{R}_\varepsilon(u,\dot u) \le \Lambda_{\mathrm{ng}}\mathcal{E}_\varepsilon(t,u) + c\mathcal{R}_\varepsilon(u,\dot u)$ we find the purely energetic a priori estimate

$$\mathcal{E}_\varepsilon(t,u(t)) + \int_s^t (1-c)\mathcal{R}_\varepsilon(u,\dot u)\,\mathrm{d}r \le \mathcal{E}_\varepsilon(s,u(s)) + \int_s^t \Lambda\mathcal{E}_\varepsilon(r,u(r))\,\mathrm{d}r. \tag{2.6}$$

Neglecting $\mathcal{R}_\varepsilon$, Gronwall's estimate gives $\mathcal{E}_\varepsilon(t,u(t)) \le \mathrm{e}^{\Lambda(t-s)}\mathcal{E}_\varepsilon(s,u(s))$ for all $t\in[s,T]$. Next we replace t by r in the latter relation and insert it into the right-hand side of (2.6), which provides the assertion (2.5). ■

The main point of this proposition is that we are able to derive uniform a priori estimates as follows:

Corollary 2.2 (Uniform a priori estimates) *Assume that the dissipation potentials are equicoercive:*

$$\exists\, c_R>0,\ p>1\ \ \forall\,(\varepsilon,u,v)\in[0,1]\times\mathbf{X}^2:\quad \mathcal{R}_\varepsilon(u,v)\ge c_R\|v\|_X^p, \tag{2.7}$$

and that the initial energies satisfy $\mathcal{E}_\varepsilon(0,u^0_\varepsilon)\le C_E<\infty$*. Then, there exists* $C_*<\infty$ *such that the solutions* $u_\varepsilon:[0,T]\to\mathbf{X}$ *of* (2.3) *satisfy*

$$\|u_\varepsilon(\cdot)\|_{\mathrm{L}^\infty([0,T];\mathbf{Z})} + \|u_\varepsilon(\cdot)\|_{\mathrm{W}^{1,p}([0,T];\mathbf{X})} \le C. \tag{2.8}$$

Proof. We use (2.5) for $s=0$ and $t\le T$, where the right-hand side is estimated by $\mathrm{e}^{\Lambda T}C_E<\infty$. Now, the coercivity (2.1b) of $\mathcal{E}_\varepsilon$ gives the bound in $\mathrm{L}^\infty([0,T];\mathbf{Z})$. Then, the coercivity (2.7) of $\mathcal{R}_\varepsilon$ gives the bound in $\mathrm{W}^{1,p}([0,T];\mathbf{X})$. ■

3 Perturbed evolutionary variational estimate

In this section we consider a simple Hilbert-space setting, i.e. the dynamic space $\mathbf{X}$ is a Hilbert space $\mathbf{H}$ with norm $\|\cdot\|$, and the dissipation potentials $\mathcal{R}_\varepsilon$ are one-half of the square of Hilbert-space norms. Nevertheless, we do not work with one Hilbert space but with a family of norms:

$$\begin{aligned}&\exists\, C>0\ \ \forall\,\varepsilon\in[0,1]\ \ \exists\,\mathbb{G}_\varepsilon=\mathbb{G}_\varepsilon^*\in \mathrm{Lin}(\mathbf{H},\mathbf{H}):\\ &\qquad \mathcal{R}_\varepsilon(v)=\frac12\langle \mathbb{G}_\varepsilon v,v\rangle \text{ and } \frac{1}{2C}\|v\|^2\leq \mathcal{R}_\varepsilon(v)\leq \frac{C}{2}\|v\|^2.\end{aligned}\tag{3.1}$$

For the energies $\mathcal{E}_\varepsilon:[0,T]\times\mathbf{H}\to\mathbb{R}_\infty$ we assume that they are uniformly λ-convex:

$$\begin{aligned}&\exists\,\lambda_*\in\mathbb{R}\ \ \forall\,(\varepsilon,t)\in[0,1]\times[0,T]:\ \ \mathcal{E}_\varepsilon(t,\cdot)+\lambda_*\mathcal{R}_\varepsilon(\cdot)\text{ is convex on }\mathbf{H};\\ &\qquad \mathcal{E}_\varepsilon(t,u_\theta)\leq(1{-}\theta)\mathcal{E}_\varepsilon(t,u_0)+\theta\mathcal{E}_\varepsilon(t,u_1)+\lambda_*\theta(1{-}\theta)\mathcal{R}_\varepsilon(u_1{-}u_0),\end{aligned}\tag{3.2}$$

where $u_\theta:=(1{-}\theta)u_0+\theta u_1$. For sufficiently smooth $\mathcal{E}_\varepsilon$ condition (3.2) simply means

$$\mathcal{E}_\varepsilon(t,w)\geq\mathcal{E}_\varepsilon(t,u)+\langle \mathrm{D}\mathcal{E}_\varepsilon(t,u),w{-}u\rangle+\lambda_*\mathcal{R}_\varepsilon(w{-}u).\tag{3.3}$$

For the non-gradient term $h_\varepsilon:[0,T]\times\mathbf{H}\to\mathbf{H}^*$ we assume that it is controlled by the gradient parts as in (2.2).

Here we do not address the question of existence and uniqueness of solutions, which we assume to hold. (For this one may additionally impose a global Lipschitz continuity of h_ε.) Our concern is the convergence of the solutions $u_\varepsilon:[0,T]\to\mathbf{H}$ for the perturbed gradient system $(\mathbf{H},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon,h_\varepsilon)$, i.e. u_ε satisfies (2.3).

Our next result provides a reformulation of this equation in terms of a *perturbed evolutionary variational estimate* (PEVE), which is a direct generalization of the metric theory in [AGS05,DaS14], where $\Lambda_{\mathrm{na}}=\Lambda_{\mathrm{ng}}=0$. Since it is a statement for fixed $\varepsilon\in[0,1]$, we can drop the index ε here.

Proposition 3.1 *Assume that the assumptions* (3.1)*,* (2.1)*,* (3.2)*, and* (2.2) *hold and set* $\Lambda:=\Lambda_{na}+\Lambda_{ng}$*. Then, a function* $u\in \mathrm{H}^1([0,T];\mathbf{H})\cap \mathrm{L}^\infty(0,T;\mathbf{Z})$ *solves* (2.3) *if and only if (PEVE) holds:*

$$\begin{aligned}&\forall\,0\leq s<t\ \ \forall\,w\in\mathbf{H}:\\ &\mathrm{e}^{\lambda_*(t-s)}\mathcal{R}(u(t){-}w)-\mathcal{R}(u(s){-}w)+A_*^+(t{-}s)\mathcal{E}(t,u(t))\\ &\qquad\leq A_*^-(t{-}s)\mathcal{E}(t,w)-\int_s^t \mathrm{e}^{\lambda_*(r-s)}\langle h(r,u(r)),w{-}u(r)\rangle\,\mathrm{d}r,\end{aligned}\tag{PEVE}$$

where $A_*^\pm(r)=\big(\mathrm{e}^{\lambda_* r}-\mathrm{e}^{\mp\Lambda r}\big)/(\lambda_*\pm\Lambda)$ *(giving* $A_*^\pm(0)=0$ *and* $(A_*^\pm)'(0)=1$*).*

Proof. We first show that (2.3) implies (PEVE). For this, we choose arbitrary w and apply $\langle\cdot,u(t){-}w\rangle$ to (2.3) to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}(u(t){-}w)&\overset{(3.1)}{=}\langle \mathrm{D}\mathcal{R}(\dot u),u{-}w\rangle\overset{(2.3)}{=}\langle \mathrm{D}\mathcal{E}(t,u)-h(t,u),w{-}u\rangle\\ &\overset{(3.3)}{\leq}\mathcal{E}(t,w)-\mathcal{E}(t,u)-\lambda_*\mathcal{R}(w{-}u)-\langle h(t,u),w{-}u\rangle.\end{aligned}$$

Moving $-\lambda_*\mathcal{R}(w{-}u)$ to the left-hand side and multiplying by $\mathrm{e}^{\lambda_*(t-s)}$ we can integrate over $t \in [s, t_1]$. Renaming t and t_1 into r and t, respectively, we find

$$\begin{aligned}&\mathrm{e}^{\lambda_*(t-s)}\mathcal{R}(u(t){-}w) - \mathcal{R}(u(s){-}w)\\&\qquad\leq \int_s^t \mathrm{e}^{\lambda_*(r-s)}\Big(\mathcal{E}(r,w) - \mathcal{E}(r,u(r)) - \langle h(r,u(r)), w{-}u(r)\rangle\Big)\,\mathrm{d}r.\end{aligned}$$

From (2.4) we obtain $\mathcal{E}(r,w) \leq \mathrm{e}^{\Lambda(t-r)}\mathcal{E}(t,w)$, and (2.5) implies $\mathcal{E}(r,u(r)) \geq \mathrm{e}^{-\Lambda(t-r)}\mathcal{E}(t,u(t))$. Inserting this into the last estimate and doing the integration in $r \in [s,t]$ explicitly for the first two terms leads to the desired result (PEVE).

We now show that (PEVE) implies (2.3). For this we divide both sides by $t-s>0$ and then take the limit $s \nearrow t$. Using $A_*^{\pm}(r)/r \to 1$ for $r \searrow 0$ we obtain the differential form again, namely

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}(u{-}w) &= \langle \mathbb{G}\dot u, u{-}w\rangle = \langle \mathrm{D}_u\mathcal{E}(t,u), w{-}u\rangle\\&\leq \mathcal{E}(t,w) - \mathcal{E}(t,u) - \lambda_*\mathcal{R}(u{-}w) + \langle h(t,u), u{-}w\rangle.\end{aligned}$$

Keeping t fixed and inserting the test function $w = u(t) - \delta v$ with $\delta > 0$, we divide by δ first and then pass to the limit to obtain $\langle \dot u, v\rangle \leq \langle -\mathrm{D}\mathcal{E}(t,u) + h(t,u), v\rangle$. Since v is arbitrary, we also have the opposite sign (replace v by $-v$), and (2.3) is established. ■

The above characterization of solutions of the perturbed gradient system $(\mathbf{H}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon, h_\varepsilon)$, which give rise to the evolution equation (2.3), allows us to formulate a result concerning evolutionary Γ-convergence. For this we use the notion of (strong) Γ-convergence of the energies, continuous convergence of the dissipation potentials, and strong convergence of the perturbations:

$$\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}_0, \quad \text{i.e.} \begin{cases} w_\varepsilon \to w \text{ in } \mathbf{H} \implies \liminf\limits_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t,w_\varepsilon) \geq \mathcal{E}_0(t,w_0), \\ \forall\, \widehat w_0\ \exists\, \widehat w_\varepsilon \to \widehat w_0 \text{ in } \mathbf{H}: \quad \mathcal{E}_\varepsilon(t,w_\varepsilon) \to \mathcal{E}_0(t,\widehat w_0); \end{cases} \tag{3.4a}$$

$$\mathcal{R}_\varepsilon \xrightarrow{\mathsf{C}} \mathcal{R}_0, \quad \text{i.e. } w_\varepsilon \rightharpoonup w_0 \text{ in } \mathbf{Z} \implies \mathcal{R}_\varepsilon(w_\varepsilon) \to \mathcal{R}_0(w_0); \tag{3.4b}$$

$$w_\varepsilon \rightharpoonup w_0 \text{ in } \mathbf{Z} \implies \quad h_\varepsilon(t,w_\varepsilon) \rightharpoonup h_0(t,w_0) \text{ in } \mathbf{H}^*. \tag{3.4c}$$

Concerning the static Γ-convergence in (3.4a) we refer to the standard textbooks [Dal93, Bra02, Bra13]. In these statements the weak convergence in $\mathbf{Z}$ can be replaced by the more general and maybe more flexible statement of convergence within sublevels of $\mathcal{E}_\varepsilon$, namely $w_\varepsilon \to w_0$ in $\mathbf{H}$ and $\mathcal{E}_\varepsilon(t,w_\varepsilon) \leq C$. Clearly, the equicoercivity (2.1b) implies weak convergence in $\mathbf{Z}$.

The following result relies on PEVE and the a priori estimate provided in Corollary 2.2. The latter shows that the desired accumulating points exist, since the unit ball in $\mathrm{W}^{1,p}([0,T];\mathbf{Z})$ is weakly compact, i.e. converging subsequences as assumed in the following result always exist.

Theorem 3.2 (Evolutionary Γ-convergence via PEVE) *Let the assumptions of Proposition 3.1 and* (3.4) *hold. If for a family of solutions* $u^\varepsilon : [0,T] \to \mathbf{H}$ *of* (2.3) *a subsequence* $(u^{\varepsilon_k})_{k\in\mathbb{N}}$ *satisfies*

$$\varepsilon_k \to 0 \quad \textit{and} \quad u^{\varepsilon_k} \rightharpoonup u \quad \textit{in } \mathrm{H}^1([0,T];\mathbf{H}),$$

then u *is a solution of the limiting perturbed gradient system* $(\mathbf{H}, \mathcal{E}_0, \mathcal{R}_0, h_0)$*, i.e.* u *solves* (2.3) *for* $\varepsilon = 0$*.*

Proof. By the a priori estimate in Corollary 2.2 we can assume $u^{\varepsilon_k} \rightharpoonup u$ in $\mathrm{H}^1([0,T];\mathbf{H})$ and

$$\forall\, t \in [0,T]: \quad u^{\varepsilon_k}(t) \rightharpoonup u(t) \text{ in } \mathbf{Z} \text{ and } u^{\varepsilon_k}(t) \to u(t) \text{ in } \mathbf{H}.$$

We now exploit that the perturbed evolutionary variational estimate (PEVE) holds with λ_* and Λ independently of ε. For $0 \le s < t \le T$ and $w \in \mathbf{H}$ we have

$$\begin{aligned} &\mathrm{e}^{\lambda_*(t-s)}\mathcal{R}_\varepsilon(u^{\varepsilon_k}(t){-}w) - \mathcal{R}_\varepsilon(u^{\varepsilon_k}(s){-}w) + A_*^+(t{-}s)\mathcal{E}_\varepsilon(t,u^{\varepsilon_k}(t)) \\ &\qquad \le A_*^-(t{-}s)\mathcal{E}_\varepsilon(t,w) - \int_s^t \mathrm{e}^{\lambda_*(r-s)}\langle h_\varepsilon(r,u^{\varepsilon_k}(t)), w{-}u^{\varepsilon_k}(r)\rangle\,\mathrm{d}r. \end{aligned} \tag{3.5}$$

Fixing s and t we may now choose a suitable test function $w = w^{\varepsilon_k}$, namely such that $w^{\varepsilon_k} \to w^0$ and $\mathcal{E}(t,w^{\varepsilon_k}) \to \mathcal{E}(t,w^0)$ (cf. (3.4a)). Note that the equicoercivity implies $w^{\varepsilon_k} \rightharpoonup w^0$ in $\mathbf{Z}$.

Hence, we can pass to the limit inferior for $\varepsilon_k \to 0$ in (3.5). Indeed, on the left-hand side the first two terms converge to $\mathrm{e}^{\lambda_*(t-s)}\mathcal{R}_0(u(t){-}w^0) - \mathcal{R}_0(u(s){-}w^0)$ because of (3.4b), whereas the third term has a liminf bounded from below by $A_*^+(t{-}s)\mathcal{E}_0(t,u(t))$, where we use $A_*^+(t{-}s) > 0$. On the right-hand side the first term converges to $A_*^-(t{-}s)\mathcal{E}_0(t,w^0)$ by the choice of w^{ε_k}, whereas the second term converges to $\int_s^t \mathrm{e}^{\lambda_*(r-s)}\langle h_0(r,u(r)), w^0{-}u(r)\rangle\,\mathrm{d}r$ by strong convergence of $w^{\varepsilon_k}{-}u^{\varepsilon_k}(r)$ and weak convergence of $h_{\varepsilon_k}(r,u^{\varepsilon_k}(r))$. Thus, since w^0 is arbitrary, (PEVE) is established for u, and by Proposition (3.1) we know that u is a solution of (2.3) for $\varepsilon = 0$. ■

4 De Giorgi's energy-dissipation principle

To prepare for De Giorgi's reformulation of gradient flows in terms, we recall the following fact from convex analysis. For a convex function $\Psi : \mathbf{X} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ the Legendre–Fenchel dual $\Psi^* : \mathbf{X}^* \to \mathbb{R}_\infty$ is defined via

$$\Psi^*(\xi) := \sup\{\, \langle \xi, v\rangle - \Psi(v) \mid v \in \mathbf{X} \,\} \tag{4.1}$$

and the convex subdifferential via

$$\partial\Psi(v) = \{\, \xi \in \mathbf{X}^* \mid \Psi(w) \ge \Psi(v) + \langle \xi, w{-}v\rangle \text{ for all } w \in \mathbf{X} \,\}. \tag{4.2}$$

Proposition 4.1 *Let* $\mathbf{X}$ *be a reflexive Banach space and* $\Psi : \mathbf{X} \to \mathbb{R}_\infty$ *be proper, convex, and lower semi-continuous. Then, the following holds:*

(A) Young-Fenchel estimate: $\forall\, (v,\xi) \in \mathbf{X}{\times}\mathbf{X}^* : \quad \Psi(v) + \Psi^*(\xi) \geq \langle \xi, v\rangle$.

(B) Fenchel equivalence ([Fen49, EkT76]): for all $(v,\xi) \in \mathbf{X}{\times}\mathbf{X}^*$ *we have*

$$\text{(i) } \xi \in \partial\Psi(v) \iff \text{(ii) } v \in \partial\Psi^*(\xi) \iff \text{(iii) } \Psi(v) + \Psi^*(\xi) = \langle \xi, v\rangle.$$

We emphasize that the relation (i) is a relation in dual space $\mathbf{X}^*$, (ii) is a relation in $\mathbf{X}$, and (iii) is a relation in $\mathbb{R}$. Using (A), it is immediate that (iii) can be replaced by the estimate (iii)′ $\Psi(v) + \Psi^*(\xi) \leq \langle \xi, v\rangle$.

We can apply these equivalences with $\Psi(\cdot) = \mathcal{R}_\varepsilon(u, \cdot)$ to the formulation of the gradient flow associated with our perturbed gradient system $(\mathbf{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon, h_\varepsilon)$ and obtain three equivalent formulations:

$$\begin{aligned}
&\text{force balance } \; \mathrm{D}_{\dot u}\mathcal{R}_\varepsilon(u, \dot u) = -\mathrm{D}_u\mathcal{E}_\varepsilon(t, u) + h_\varepsilon(t, u);\\
&\text{rate equation } \; \dot u = \mathrm{D}_\xi\mathcal{R}^*_\varepsilon\big(u, -\mathrm{D}_u\mathcal{E}_\varepsilon(t, u){+}h_\varepsilon(t, u)\big);\\
&\text{power balance } \mathcal{R}_\varepsilon(u, \dot u){+}\mathcal{R}^*_\varepsilon(u, h_\varepsilon(t, u)\text{-}\mathrm{D}_u\mathcal{E}_\varepsilon(t, u)){=}\langle h_\varepsilon(t, u)\text{-}\mathrm{D}_u\mathcal{E}_\varepsilon(t, u), \dot u\rangle.
\end{aligned}$$

The main point is that a time-integrated version of the third formulation can be used to characterize solutions of perturbed gradient systems. For this we need an *abstract chain rule* for $\mathcal{E}_\varepsilon$. We say that $(\mathbf{X}, \mathcal{E})$ satisfies the chain rule if for all $p \geq 1$ the following holds. If $u \in \mathrm{W}^{1,p}([0, T]; \mathbf{X})$, $\mathcal{E}(\cdot, u(\cdot)) \in \mathrm{L}^1([0, T]$, and $\mathrm{D}_u\mathcal{E}(\cdot, u(\cdot)) \in \mathrm{L}^{p*}([0, T]; \mathbf{X}^*)$, then $t \mapsto \mathcal{E}(t, u(t))$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) = \langle \xi(t), \dot u(t)\rangle + \partial_t\mathcal{E}(t, u(t)) \text{ a.e. in } [0, T]. \tag{4.3}$$

We refer to [RoS06, MRS13] for general treatments and derivations of such abstract chain rules. Using this chain rule, we can integrate the power balance in time and replace $\langle \mathrm{D}_u\mathcal{E}_\varepsilon(t, u), \dot u\rangle$ by the difference of the initial and final energies plus an integral over $\partial_t\mathcal{E}_\varepsilon$. De Giorgi's energy-dissipation principle (EDP) states that this integrated version of the power estimate (iii)′ is equivalent to the force balance (2.3) for a.a. $t \in [0, T]$. Again we can drop the parameter $\varepsilon > 0$.

Theorem 4.2 (De Giorgi's EDP) *Assume that* $(\mathbf{X}, \mathcal{E})$ *satisfies the chain rule* (4.3) *and that there exists* $C, p > 1$ *such that* $(1{+}\|\dot u\|^p)/C \leq \mathcal{R}(u, \dot u) \leq C(1{+}\|\dot u\|^p)$. *Then a function* $u \in \mathrm{W}^{1,p}([0, T]; \mathbf{X})$ *is a solution of the perturbed gradient system* $(\mathbf{X}, \mathcal{E}, \mathcal{R}, h)$ *if and only if it satisfies the Upper Energy-Dissipation Estimate*

$$\mathcal{E}(T, u(T)) + \mathfrak{D}(u(\cdot)) \leq \mathcal{E}(0, u(0)) + \int_0^T \partial_t\mathcal{E}(t, u(t)) + \langle h(t, u(t)), \dot u(t)\rangle \,\mathrm{d}t, \tag{UEDE}$$

where De Giorgi's dissipation functional $\mathfrak{D}$ *is given by*

$$\mathfrak{D}(u(\cdot)) := \int_0^T \mathcal{R}(u(t), \dot u(t)){+}\mathcal{R}^*\big(u(t), h(t, u(t)){-}\mathrm{D}_u\mathcal{E}(t, u(t))\big)\,\mathrm{d}t. \tag{4.4}$$

This result is a simple generalization of [Mie15b, Thm. 3.3], where the proof for the case $h \equiv 0$ is given. We remark that the EDP relates the final energy $\mathcal{E}(T, u(T))$ plus the dissipated energy $\int_0^T \mathcal{R}+\mathcal{R}^* \,\mathrm{d}t$ to the initial energy $\mathcal{E}(0, u(0))$ plus the external work $\int_0^T \partial_t \mathcal{E}(t, u(t)) \,\mathrm{d}t$ and the work due to the non-gradient terms $\int_0^T \langle h(t, u(t)), \dot{u}(t)\rangle \,\mathrm{d}t$. It is sufficient to establish the UEDE, then by the chain rule one obtains an equality in UEDE giving the power balance.

The EDP is ideal for proving evolutionary Γ-convergence. In fact, it is the basis of the famous Sandier–Serfaty approach, see [SaS04, Ser11]. For this we look at the ε-dependent UEDE:

$$\mathcal{E}_\varepsilon(T, u^\varepsilon(T)) + \mathfrak{D}_\varepsilon(u^\varepsilon(\cdot)) \leq \mathcal{E}_\varepsilon(0, u^0_\varepsilon) + \int_0^T \partial_t \mathcal{E}_\varepsilon(t, u^\varepsilon(t)) + \langle h_\varepsilon(t, u^\varepsilon(t)), \dot{u}^\varepsilon(t)\rangle \,\mathrm{d}t. \quad (4.5)$$

The main importance of the EDP is that it involves the UEDE, which states that the final and the dissipated energies only need to have a good upper bound. Hence, in passing to the Γ-limit it will be sufficient to have good liminf estimates for these terms, while the right-hand side can be controlled by the well-preparedness of the initial conditions and proper assumptions on the power of the external forces $\partial_t \mathcal{E}_\varepsilon(t, u)$ and the power $\langle h_\varepsilon(t, u), \dot{u}\rangle$. The following result gives sufficient conditions for evolutionary Γ-convergence, in fact for "pE-convergence" in the sense of [Mie15b].

Theorem 4.3 (Evolutionary Γ-convergence via EDP) *Assume that the perturbed gradient systems* $(\mathbf{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon, h_\varepsilon)$ *satisfy* (2.1), (2.2), (2.7) *and that*

$$\mathcal{E}_\varepsilon(t,\cdot) \xrightarrow{\Gamma} \mathcal{E}_0(t,\cdot) \quad \textit{and} \quad \mathcal{E}_\varepsilon(0, u^0_\varepsilon) \to \mathcal{E}_0(0, u^0_0); \quad (4.6a)$$

$$(\mathbf{X}, \mathcal{E}_0) \textit{ satisfies the chain rule}; \quad (4.6b)$$

$$w_\varepsilon \overset{\mathbf{Z}}{\rightharpoonup} w_0 \implies \big(\partial_t \mathcal{E}_\varepsilon(t, w_\varepsilon) \to \partial_t \mathcal{E}_0(t,, w_0) \;\&\; h_\varepsilon(t, w_\varepsilon) \overset{\mathbf{Z}^*}{\to} h_0(t, w_0)\big); \quad (4.6c)$$

$$\widehat{w}_\varepsilon(\cdot) \rightharpoonup \widehat{w}_0(\cdot) \textit{ in } \mathrm{W}^{1,p}([0,T]; \mathbf{X}) \implies \mathfrak{D}_0(\widehat{w}_0) \leq \liminf_{\varepsilon\to 0} \mathfrak{D}_\varepsilon(\widehat{w}^\varepsilon). \quad (4.6d)$$

If $u^\varepsilon : [0,T] \to \mathbf{X}$ *is a family of solutions for* (2.3) *with* $u^\varepsilon(0) = u^0_\varepsilon$ *and*

$$\varepsilon_k \to 0 \quad \textit{and} \quad u^{\varepsilon_k} \rightharpoonup u \textit{ in } \mathrm{W}^{1,p}([0,T]; \mathbf{X}) \quad \textit{as } k \to \infty,$$

then u *is a solution for the perturbed system* $(\mathbf{X}, \mathcal{E}_0, \mathcal{R}_0, h_0)$ *with* $u(0) = u^0_0$.

The crucial and most difficult condition here is the liminf estimate for De Giorgi's dissipation potential, where $\mathcal{D}_0$ again must have the form (4.4). The liminf estimate is then sufficient, since the duality of $\mathcal{R}_0$ and $\mathcal{R}^*_0$ and the chain rule (4.6b) imply equality again.

Proof. Because of the assumptions we can use the a priori estimates of Corollary 2.2 and may assume the additional convergences

$$\forall\, t \in [0,T]: \quad u^{\varepsilon_k}(t) \rightharpoonup u(t) \text{ in } \mathbf{Z} \text{ and } u^{\varepsilon_k}(t) \to u(t) \text{ in } \mathbf{X}.$$

Using the EDP in Theorem 4.2 we know that u^ε satisfies the UEDE (4.5). Using the assumptions (4.6a) and (4.6b) and the a priori estimates, we easily see that the right-hand side in (4.5) converges to $\mathcal{E}_0(0,u_0^0) + \int_0^T \partial_t\mathcal{E}_0(t,u(t)) + \langle h_0(t,u(t)),\dot u(t)\rangle\,\mathrm{d}t$.

On the left-hand side we have $\mathcal{E}_0(T,u(T)) \leq \liminf_{\varepsilon_k\to0}\mathcal{E}_{\varepsilon_k}(T,u^{\varepsilon_k}(T))$ and $\mathfrak{D}_0(u(\cdot)) \leq \liminf_{\varepsilon_k\to0}\mathfrak{D}_{\varepsilon_k}(u^{\varepsilon_k}(\cdot))$. Thus, the UEDE for u with $\varepsilon=0$ is established, and the EDP in Theorem 4.2 implies that u solves (2.3) for $\varepsilon=0$. ■

Based on the philosophy of this result, the notion of "EDP-convergence" was introduced in [LM*15] by asking $\mathfrak{D}_\varepsilon \xrightarrow{\Gamma} \mathfrak{D}_0$ in $\mathrm{W}^{1,p}([0,T];\mathbf{X})$. This convergence is in fact much more than what is needed for evolutionary Γ-convergence. In principle, in (4.6d) it is sufficient to obtain the desired liminf estimate only along solutions. In contrast, EDP-convergence asks for a Γ-convergence along arbitrary functions. This is physically justified by fluctuation theory, which gives the proper justification of gradient structures, see e.g. [OnM53].

Remark 4.4 *A similar theory may be derived for perturbed gradient systems in the form*

$$\dot u = \mathrm{D}_\xi\mathcal{R}_\varepsilon^*\big(u,-\mathrm{D}_u\mathcal{E}_\varepsilon(t,u)\big) + g_\varepsilon(t,u).$$

The corresponding energy-dissipation principle takes the form

$$\mathcal{E}_\varepsilon(T,u(T)) + \widehat{\mathfrak{D}}_\varepsilon(u) \leq \mathcal{E}_\varepsilon(0,u(0)) + \int_0^T \big(\partial_t\mathcal{E}_\varepsilon(t,u) + \langle \mathrm{D}_u\mathcal{E}_\varepsilon(t,u), g_\varepsilon(t,u)\rangle\big)\,\mathrm{d}t,$$

$$\text{where } \widehat{\mathfrak{D}}_\varepsilon(u) = \int_0^T \big(\mathcal{R}_\varepsilon(u,\dot u - g_\varepsilon(t,u)) + \mathcal{R}_\varepsilon^*(u,-\mathrm{D}_u\mathcal{E}_\varepsilon(u))\big)\,\mathrm{d}t.$$

*We refer to [DPZ13, Bud14, DL*15] for the usage of this variational principle, where the term $\langle \mathrm{D}_u\mathcal{E}_\varepsilon(t,u), g_\varepsilon(t,u)\rangle$ even disappears because of a Hamiltonian structure of g_ε.*

5 Applications of evolutionary Γ-convergence

We provide a few possible applications of the two theories developed above.

5.1 Homogenization of reaction-diffusion system

We only discuss a few simple results, where we emphasize that scalar reaction-diffusion equations can easily be treated as unperturbed gradient systems. However, for general systems no gradient structure exists. We consider a vector $u=(u_1,...,u_I)\in\mathbb{R}^I$ of concentrations depending on $(t,x)\in[0,T]\times\Omega$, where Ω is a bounded smooth domain in $\mathbb{R}^d$, which we may consider as a periodically structured solid, surface or interface. The reaction-diffusion system reads

$$M_\varepsilon(x)\dot u = \mathrm{div}\,\big(A_\varepsilon(x)\nabla u\big) - F_\varepsilon(x,u) \text{ in } \Omega, \quad A_\varepsilon(x)\nabla u\cdot\nu = 0 \text{ on } \partial\Omega. \tag{5.1}$$

Here M_ε, A_ε, and F_ε depend periodically on x in the form

$$M_\varepsilon(x) = \mathbb{M}(\frac{1}{\varepsilon}x), \quad A_\varepsilon(x) = \mathbb{A}(\frac{1}{\varepsilon}x), \quad F_\varepsilon(x,u) = \mathbb{F}(\frac{1}{\varepsilon}x, u),$$

where the functions $\mathbb{M}$, $\mathbb{A}$, and $\mathbb{F}$ are 1-periodic in the variable $y = \frac{1}{\varepsilon}x \in \mathbb{R}^d$, viz. $\mathbb{M}(y+k) = \mathbb{M}(y)$ for all $y \in \mathbb{R}^d$ and all $k \in \mathbb{Z}^d$.

We can apply the theory of perturbed gradient systems by using the spaces $\mathbf{X} = \mathrm{L}^2(\Omega;\mathbb{R}^I)$ and $\mathbf{Z} = \mathrm{H}^1(\Omega;\mathbb{R}^I)$ and the functionals

$$\mathcal{E}_\varepsilon(u) = \int_\Omega \frac{1}{2}\nabla u \cdot A_\varepsilon(x)\nabla u + \frac{1}{2}|u|^2 \,\mathrm{d}x \quad \text{and} \quad \mathcal{R}_\varepsilon(\dot u) = \int_\Omega \frac{1}{2}\dot u \cdot M_\varepsilon(x)\dot u \,\mathrm{d}x.$$

For the perturbation h_ε we choose $h_\varepsilon(t,x,u) = u - F_\varepsilon(x,u)$.

In addition to the 1-periodicity, the main assumptions on the functions $\mathbb{M}$, $\mathbb{A}$, and $\mathbb{F}$ are the following. There exists $C, c_0 > 0$ such that

$$\begin{aligned}
&\mathbb{M} = \mathbb{M}^\top \in \mathrm{L}^\infty(\mathbb{R}^d;\mathbb{R}^{I\times I}), \quad \xi\cdot\mathbb{M}(y)\xi \geq c_0|\xi|^2,\\
&\mathbb{A} = \mathbb{A}^\top \in \mathrm{L}^\infty(\mathbb{R}^d;\mathbb{R}^{(I\times d)\times(I\times d)}), \quad \Xi : \mathbb{A}(y)\Xi \geq c_0|\Xi|^2,\\
&\mathbb{F}(\cdot,u) \in \mathrm{L}^\infty(\mathbb{R}^d;\mathbb{R}^I), \quad |\mathbb{F}(y,u)-\mathbb{F}(y,\widetilde u)| \leq C|u-\widetilde u|,
\end{aligned}$$

for all $u, \widetilde u \in \mathbb{R}^I$, $y, \xi \in \mathbb{R}^d$, and $\Xi \in \mathbb{R}^{I\times d}$.

First we observe that the general assumptions (2.1) hold with $\mathbf{D}_\varepsilon = \mathbf{Z} = \mathrm{H}^1(\Omega;\mathbb{R}^I)$, $\alpha = 2$, and $\Lambda_{\mathrm{na}} = 0$. Moreover, (2.2) holds since

$$\mathcal{R}^*_\varepsilon(u, h_\varepsilon) \leq C_1\|h_\varepsilon\|^2_{\mathrm{L}^2} \leq C(1+\|u\|^2_{\mathrm{L}^2}) \leq \Lambda_{\mathrm{ng}}\mathcal{E}_\varepsilon(u).$$

We now show that the theory developed in Section 3 for the perturbed evolutionary variational estimate holds. By the definition of $\mathcal{R}_\varepsilon$ it is quadratic on the Hilbert space $\mathbf{H} = \mathrm{L}^2(\Omega;\mathbb{R}^I)$, i.e. (3.1) holds. Moreover, $\mathcal{E}_\varepsilon$ is convex, so (3.2) holds with $\lambda_* = 0$.

To apply Theorem 3.2 we need to establish convergence for $\mathcal{E}_\varepsilon$, $\mathcal{R}_\varepsilon$, and h_ε. Strong Γ-convergence of $\mathcal{E}_\varepsilon$ in $\mathbf{H}$ (or similarly weak Γ-convergence in $\mathbf{Z}$) holds with

$$\mathcal{E}_0(u) = \int_\Omega \Big(\frac{1}{2}\nabla u : A_{\mathrm{eff}}\nabla u + \frac{1}{2}|u|^2\Big)\,\mathrm{d}x,$$

where the effective tensor follows from linear homogenization, see e.g. [Dal93, Bra06]. Since weak convergence in $\mathbf{Z} = \mathrm{H}^1(\Omega;\mathbb{R}^I)$ implies strong convergence in $\mathbf{H} = \mathrm{L}^2(\Omega;\mathbb{R}^I)$, it is easy to show that $w_\varepsilon \rightharpoonup w_0$ in $\mathbf{Z}$ implies

$$\begin{aligned}
&\mathcal{R}_\varepsilon(w_\varepsilon) \to \mathcal{R}_0(w_0) = \int_\Omega \frac{1}{2} w_0 \cdot M_{\mathrm{eff}} w_0 \,\mathrm{d}x \quad \text{with } M_{\mathrm{eff}} = \int_{[0,1]^d} \mathbb{M}(y)\,\mathrm{d}y,\\
&h_\varepsilon(w_\varepsilon) \to h_0(w_0) = w_0 - F_{\mathrm{eff}}(w_0) \text{ in } \mathbf{H}, \quad \text{where } F_{\mathrm{eff}}(w) = \int_{[0,1]^d} \mathbb{F}(y,w)\,\mathrm{d}y.
\end{aligned}$$

We refer to [MRT14] for the last convergence. Thus, assumption (3.4) is established, Theorem 3.2 is applicable, and the limiting perturbed gradient system $(\mathbf{H}, \mathcal{E}_0, \mathcal{R}_0, h_0)$ is identified by using $A_{\rm eff}$, $M_{\rm eff}$, and $F_{\rm eff}$ in its definition. In particular, the limiting perturbed gradient flow is given by the effective reaction-diffusion system

$$M_{\rm eff}\dot u = \mathrm{div}\,\big(A_{\rm eff}\nabla u\big) - F_{\rm eff}(u).$$

Of course, the above homogenization problem only serves as a didactical example, since the result is well known. However, the theory allows for significant generalizations. We first mention the homogenization of the Cahn-Hilliard equation in [LiR15], where also a comparison between the two abstract approaches (PEVE versus EDP) is done. In [MRT14, Rei15] the case of ε-dependent diffusion constants is two-scale convergence and proving strong convergence via a suitable Gronwall estimates.

5.2 Justification of amplitude equations

An application of the theory developed in Section 3 to the justification of amplitude equations is given in [Mie15a] for the case of pure gradient systems. The suitably rescaled fourth-order Swift–Hohenberg equation with periodic boundary condition on the circle $\mathbb{S}$ reads

$$\dot w = -\frac{1}{\varepsilon^2}\big(1{+}\varepsilon^2\partial_x^2\big)^2 w + \mu w + \beta\varepsilon w_x - w^3 \quad \text{on } \mathbb{S} := \mathbb{R}/_{2\pi\mathbb{Z}} \tag{5.2}$$

and is a gradient system for $\beta = 0$ on the Hilbert space $\mathrm{L}^2(\mathbb{S})$ for the energy functional $\mathcal{F}^{\rm SH}_\varepsilon(w) = \int_{\mathbb{S}} \frac{1}{2\varepsilon^2}(w{+}\varepsilon^2 w_{xx})^2 - \frac{\mu}{2}w^2 + \frac14 w^4 \,\mathrm{d}x$ and the dissipation potential $\mathcal{R}^{\rm SH}(\dot w) = \frac12\|\dot w\|^2_{\mathrm{L}^2}$. Here we show that the case $\beta \neq 0$ can be treated as a perturbed gradient system.

Because of the special form of the linear operator all typical solutions of (5.2) will spatially oscillate on the scale ε and are approximately of the form $w(t,x) \approx \mathrm{Re}\,\big(A(t,x)\mathrm{e}^{\mathrm{i}x/\varepsilon}\big)$. Using a suitable bijection $\mathbb{M}_\varepsilon$ between $\mathrm{L}^2(\mathbb{S})$ and a proper subspace of $\mathbf{H} := \mathrm{L}^2(\mathbb{S};\mathbb{C})$, which satisfies $w = \mathrm{Re}\,\big((\mathbb{M}_\varepsilon w)\mathrm{e}^{\mathrm{i}x/\varepsilon}\big)$, one can define the amplitudes $A^\varepsilon = \mathbb{M}_\varepsilon w^\varepsilon \in \mathbf{H}$ and finds perturbed gradient systems $(\mathbf{H}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon, h_\varepsilon)$ with $\mathcal{E}_\varepsilon(\mathbb{M}_\varepsilon w) = \mathcal{F}^{\rm SH}_\varepsilon(w)$, $\mathcal{R}_\varepsilon(\mathbb{M}_\varepsilon \dot w) = \mathcal{R}^{\rm SH}(\dot w)$, and the non-gradient part $h_\varepsilon(A) = \beta\big(\mathrm{i}A + \varepsilon\partial_x A\big)/2$.

Using the theory developed in [Mie15a] (cf. Thm. 2.3 there with $\gamma = 0$) one can show that Theorem 3.2 applies with $\mathbf{Z} = \mathrm{H}^1(\mathbb{S};\mathbb{C})$, and we find evolutionary Γ-convergence to the perturbed gradient system $(\mathbf{H}, \mathcal{E}^{\rm GL}, \mathcal{R}^{\rm GL}, h_0)$ with

$$\mathcal{E}^{\rm GL}(A) = \int_{\mathbb{S}} \Big(|A'|^2 - \frac{\mu}{4}|A|^2 + \frac{3}{32}|A|^4\Big)\,\mathrm{d}x \quad \text{and } \mathcal{R}^{\rm GL}(\dot A) = \frac14\|\dot A\|^2_{\mathrm{L}^2}$$

and $h_0(A) = \mathrm{i}\beta A/2$, which leads to the limiting perturbed gradient flow given by the Ginzburg-Landau equation

$$\dot A = 4A_{xx} + (\mu{+}\mathrm{i}\beta)A - \tfrac34\,|A|^2 A.$$

This result is not too surprising, since the perturbation introduced by $\beta \neq 0$ can be compensated by a rotation of the form $w(t,x) = \widetilde w(t, x{-}\varepsilon\beta t)$, which then transforms into a phase shift $A(t,x) = \widetilde A(t,x)\mathrm{e}^{\mathrm{i}\beta t}$ via $\mathbb{M}_\varepsilon$.

The theory for perturbed gradient systems can be used in much more general situations. We may consider a system of two Swift–Hohenberg equations with different critical wave lengths that are coupled in a non-gradient manner:

$$\dot u = -\frac{1}{\varepsilon^2}\big(1+\varepsilon^2\partial_x^2\big)^2 u + \mu_1 u + (\eta+\beta) w - u^3,$$
$$\dot w = -\frac{1}{\varepsilon^2}\big(1+\mu^2\varepsilon^2\partial_x^2\big)^2 w + \mu_2 w + (\eta-\beta) u - w^3.$$

We refer to [SA*14] for this model in the case $\mu_1 = \mu_2$ and $\eta = 0$. Here u has the critical wave length $2\pi\varepsilon$ while that of w is $2\pi\mu\varepsilon$. The coupling between the two system occurs through a gradient term η or a non-gradient term β.

Thus, we can define the associated perturbed gradient system via

$$\mathbf{H} = \mathrm{L}^2(\mathbb{S})^2, \quad \mathcal{R}^{\mathrm{cSH}}(\dot u, \dot w) = \frac{1}{2}\|\dot u\|^2_{\mathrm{L}^2} + \frac{1}{2}\|\dot w\|^2_{\mathrm{L}^2}, \quad h(u,w) = \beta\begin{pmatrix} w \\ -u \end{pmatrix},$$
$$\mathcal{E}^{\mathrm{cSH}}_\varepsilon(u,w) = \int_{\mathbb{S}}\Big(\frac{(u+\varepsilon^2 u_{xx})^2 + (w+\mu^2\varepsilon^2 w_{xx})^2}{2\varepsilon^2} + E(u,w)\Big)\,\mathrm{d}x$$

with $E(u,w) = -(\mu_1 u^2 + \mu_2 w^2)/2 - \eta uw + (u^4 + w^4)/4$. It is clear that the theory developed in Section 3 is principally applicable and that the induced limiting system for $\varepsilon \to 0$ will again be a perturbed gradient system given in terms of two possibly coupled Ginzburg–Landau equations. However, the critical bifurcations do no longer occur at $\mu_j = 0$. So, one needs to do a careful linear bifurcation analysis first. This and the justification of the arising amplitude equations will be the content of subsequent work.

5.3 From diffusion to reaction

In a series of papers it was shown that simple reactions can be understood as evolutionary Γ-limits of diffusion systems, if the occurrence of a reaction is measured moving along a reaction path. In particular, for an interchange reaction $A \rightleftharpoons B$ one should consider A and B are minima, which are separated by a saddle point. We refer to [PSV10,PSV12,AM*12] for a series of papers along this spirit.

In [LM*15] a systematic approach based on the energy-dissipation principle was developed allowing for a simultaneous treatment of diffusion in a physical domain $\Omega \in \mathbb{R}^d$ with points $x \in \Omega$ and the diffusion along the chemical reaction variable $y \in [0,1] =: \Upsilon$. Denoting by $u(t,x,y)$ the concentration of particles one can write the master equation based on a gradient system, where the energy functional is the relative entropy with respect to the equilibrium state w_ε, namely

$$\mathcal{E}_\varepsilon(u) = \int_{\Omega\times\Upsilon} \lambda_{\mathrm{B}}\big(u(x,y)/w_\varepsilon(y)\big) w_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}x \quad \text{with } \lambda_{\mathrm{B}}(z) := z\log z - z + 1,$$

where the equilibrium state is $w_\varepsilon(y) = \mathrm{e}^{-V(y)/\varepsilon}/Z_\varepsilon$ with $Z_\varepsilon = \int_\Upsilon \mathrm{e}^{-V(y)/\varepsilon}\,\mathrm{d}y$. Here $y = 0$ corresponds to the pure state A, while $y = 1$ corresponds to the pure state B. We assume

$V(0) = V(1) = 0$ and $0 < V(y) < 1 = V(1/2)$ for $y \in \Upsilon \setminus \{0, 1/2, 1\}$. The full state space $\mathbf{X}$ is the set $\mathrm{M}(\Omega{\times}\Upsilon)$ of all non-negative Radon measures on $\Omega{\times}\Upsilon$.

Since in general the mass per particle can change during reactions we define a function $m : \Upsilon \to \mathbb{R}_{>0}$ such that the total mass $\int_{\Omega\times\Upsilon} m(y)u(t,x,y)\,\mathrm{d}y\,\mathrm{d}x$ is conserved. E.g. for the reaction $3\,O_2 \rightleftharpoons 2\,O_3$ one may set $m(0) = 2$, $m(1/2) = 1$, and $m(1) = 3$, where we assume that $y = 1/2$ corresponds to O_1. Using the function m we can define a dissipation potential $\mathcal{R}_\varepsilon$ via its Legendre dual

$$\mathcal{R}^*_\varepsilon(u,\xi) = \int_{\Omega\times\Upsilon} \frac{1}{2}\Big(\mu(y)|\nabla_x\xi|^2 + \tau_\varepsilon\big[\frac{\partial_y\xi}{m(y)}\big]^2\Big)\,u\,\mathrm{d}y\,\mathrm{d}x,$$

where μ is a possibly y-dependent spatial mobility and $\tau_\varepsilon \gg 1$ is the chemical mobility. The latter has to be scaled in a suitable manner to allow the particles to overcome the potential barrier of size $1/\varepsilon$ at $y = 1/2$.

Using that $\mathrm{D}\mathcal{E}_\varepsilon(u) = \log(u/w_\varepsilon)$, the master equation (Kolmogorov's forward equation) for u is given via $\dot u = \mathrm{D}_\xi\mathcal{R}^*_\varepsilon(u, -\mathrm{D}\mathcal{E}_\varepsilon(u))$ and takes the explicit form

$$\dot u = \mu(y)\Delta_x u + \frac{\tau_\varepsilon}{m(y)}\partial_y\Big(u\,\partial_y\Big[\frac{\log u + V(y)/\varepsilon}{m(y)}\Big]\Big).$$

Generalizing the results in [LM*15], where only the case $m \equiv 1$ was treated, it should be possible to show that the gradient systems $(\mathrm{M}(\Omega{\times}\Upsilon), \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ have the evolutionary Γ-limit $(\mathrm{M}(\Omega{\times}\Upsilon), \mathcal{E}_0, \mathcal{R}_0)$, where the limit energy $\mathcal{E}_0$ is only finite if all the particles are in pure states $y = 0$ or $y = 1$, i.e.

$$\mathcal{E}_0(u) = \int_\Omega \Big(\lambda_{\mathrm{B}}(c_0/c_0^*)c_0^* + \lambda_{\mathrm{B}}(c_1/c_1^*)c_1^*\Big)\,\mathrm{d}x \quad \text{if } u = c_0\delta_{y=0} + c_1\delta_{y=1}$$

and $+\infty$ else. This means that we now have two concentrations c_0 and c_1 depending only on time t and the physical position $x \in \Omega$.

Fixing $m(1/2) = 1$ the limiting dissipation potential $\mathcal{R}^*_0$ takes the form

$$\mathcal{R}^*_0(c_0, c_1; \eta_0, \eta_1) = \int_\Omega \Big(\sum_{j=0}^{1} \tfrac{\mu_j c_j}{2}|\nabla_x\eta_j|^2 + k\big(c_0^{m_1}c_1^{m_0}\big)^{1/2}\mathfrak{C}^*(m_1\eta_0 - m_0\eta_1)\Big)\mathrm{d}x$$

where $\mathfrak{C}^*(\eta) = 4(\cosh(\eta/2){-}1)$, $\mu_j = \mu(j)$, and $m_j = m(j)$. Thus, we expect evolutionary convergence to the nonlinear reaction-diffusion system

$$\dot c_0 = \mu_0\Delta_x c_0 + m_1 k\big((c_0/c_0^*)^{m_1} - (c_1/c_1^*)^{m_0}\big),$$
$$\dot c_1 = \mu_1\Delta_x c_1 - m_0 k\big((c_0/c_0^*)^{m_1} - (c_1/c_1^*)^{m_0}\big).$$

It is interesting to note that $\mathcal{R}^*_0$ is no longer quadratic in the chemical potentials η_j, but contains exponential terms through $\mathfrak{C}^*$. This seems to correspond nicely to the de Donder–Marcelin kinetics as described in [Fei72, Def. 3.3], [GK*00, Eqn. (11)], or [Grm10, Eqn. (69)], and generalizes the usual quadratic fluctuation theory, cf. [OnM53]. The importance of the function $\mathfrak{C}^*$ for fluctuations in reactions and jump processes was first highlighted in [MPR14] based on large-deviation principles. Further discussions are found in [LM*15, MP*15].

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