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Obstacle mean-field game problem

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Abstract. In this paper, we introduce and study a first-order mean-field game obstacle problem. We examine the case of local dependence on the measure under assumptions that include both the logarithmic case and power-like nonlinearities. Since the obstacle operator is not differentiable, the equations for first-order mean field game problems have to be discussed carefully. Hence, we begin by considering a penalized problem. We prove this problem admits a unique solution satisfying uniform bounds. These bounds serve to pass to the limit in the penalized problem and to characterize the limiting equations. Finally, we prove uniqueness of solutions.

1. Introduction

The mean-field game framework [HMC06, HCM07, LL06a, LL06b, LL07a, LL07b] is a class of methods that model the behavior of large populations of rational agents under a non-cooperative dynamic behavior. This research area has applications ranging from economics to engineering, as discussed in the recent surveys [LLG10, Car11, GS14], the additional references therein, and the lectures by P. L. Lions in Collége de France [Lio11].

In this paper, we investigate first-order mean-field game obstacle problems in the stationary periodic setting. To our knowledge, in the context of mean-field games, these problems were not studied previously. Before describing the problem, we start by recalling the original stationary mean-field game problem from [LL06a], as well as the obstacle problem for Hamilton-Jacobi (H-J) equations [Lio82].

Let $\mathbb{T}^N$ be the N-dimensional torus identified when convenient with $[0,1]^N$. Consider a continuous function, $H : \mathbb{R}^N \times \mathbb{T}^N \to \mathbb{R}$, the Hamiltonian, and a continuous increasing function, $g : \mathbb{R}_0^+ \to \mathbb{R}$. In [LL06a], the authors consider the stationary mean-field game system

$$\begin{cases} H(Du, x) = g(\theta) + \overline{H} \\ \operatorname{div}(D_p H \theta) = 0, \end{cases} \tag{1.1}$$

where the unknowns are a function $u : \mathbb{T}^N \to \mathbb{R}$, a probability measure identified with its density $\theta : \mathbb{T}^N \to \mathbb{R}$ and a constant $\overline{H}$. The second equation is the adjoint of the linearization of the first equation in the variable u. This system (1.1) has the canonical structure of a mean-field game problem: a nonlinear elliptic or parabolic nonlinear partial differential equation (PDE) coupled with a PDE given by the adjoint of its linearization.

The existence of weak solutions for (1.1) was considered in [LL06a]. In [Eva09] mean-field games are not mentioned explicitly, however, the results there yield the existence of smooth solutions of (1.1) for $g(\theta) = \ln\theta$. The second order case, was also studied in [GSM14] (see also [GISMY10]), [GPSM12], and [GPV14]). Stationary mean-field games with congestion were considered in [GM]. The time-dependent problem was addressed for parabolic mean-field games in [LL06b], [CLLP12], [Por13], [GPSM14], [GPSM13], [GPa], [GPb], and in [Car13a], and [Car13b] for first-order mean-field games.

The first-order obstacle problem arises in optimal stopping (see [Lio82], [BP87], [BP88], [BCD97] and the references therein). In the periodic setting, a model problem is the following: let $\psi : \mathbb{T}^N \to \mathbb{R}$, and $H : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ be continuous functions. The obstacle problem is defined by

$$\max\{H(Du, x), u - \psi(x)\} = 0, \tag{1.2}$$

where $u : \mathbb{T}^N \to \mathbb{R}$ is a bounded continuous function.

The linearization of the obstacle operator is not well-defined since the left-hand side of (1.2) may fail to be differentiable. Thus, it is not clear what should be the corresponding mean-field model. One of the contributions of this paper is the characterization of the appropriate analog to (1.1) for obstacle problems. This is achieved by applying the penalization method. This is a standard technique employed in many related problems, e.g. [Lio82]. In the classical obstacle problem, to do so, one considers a family of smooth functions, $\beta_\epsilon : \mathbb{R} \to \mathbb{R}_0^+$, which vanish identically in $\mathbb{R}_0^-$ and satisfy $\beta_\epsilon(z) = \frac{z-\epsilon}{\epsilon}$ for $z > \epsilon$. Then, obstacle problem is approximated by the equation

$$H(Du_\epsilon, x) + \beta_\epsilon(u_\epsilon - \psi) = \epsilon \Delta u_\epsilon. \tag{1.3}$$

This equation admits viscosity solutions that satisfy uniform Lipschitz bounds. By sending $\epsilon \to 0$, one obtains a solution to (1.2).

Thanks to [Lio82], for every $\epsilon > 0$ there exists a smooth solution u^ε to (1.3). It is also well known that, up to subsequences, u^ϵ converges uniformly to a viscosity solution u of (1.2). The rate of convergence of this approximation was investigated using the nonlinear adjoint method in [CGT].

We then are led naturally to the approximate mean-field obstacle problem

$$\begin{cases} H(Du_\epsilon, x) + \beta_\epsilon(u_\epsilon - \psi) = g(\theta_\epsilon) \\ -\operatorname{div}(D_pH(Du_\epsilon, x)\theta_\epsilon) + \beta_\epsilon'(u_\epsilon - \psi)\theta_\epsilon = \gamma(x). \end{cases} \tag{1.4}$$

The additional term γ in the right-hand side of (1.4) arises for the following reason: the mean-field obstacle problem models a population of agents trying to move optimally up to a certain stopping time at which they switch to the obstacle (the term $\beta_\epsilon'\theta_\epsilon$ is the flow of agents switching to the obstacle). Without a source term introducing new agents in the system, we could fall into the pathological situation $\theta_\epsilon \equiv 0$. As it will be clear from the discussion, the approximate problem (1.4) admits smooth solutions even without additional elliptic regularization terms. This remarkable property is also true for certain first-order mean-field games, see, for instance, [Eva03]. The function u_ϵ in (1.4) is the value function for an optimal stopping problem. This problem may not admit a continuous solution, [BP87], [BP88]. Owing to the structure of (1.4), we were able to prove regularity estimates that hold uniformly in ϵ. However, in other related important situations, this may not be the case. It would be extremely interesting to consider a discontinuous viscosity solution approach for such problems.

As we will show in Section 4, by passing to the limit in (1.4), we obtain the mean-field obstacle problem

$$\begin{cases} H(Du, x) = g(\theta) & \text{in} \quad \mathbb{T}^N, \\ -\operatorname{div}(D_pH(Du, x)\theta) \leq \gamma(x) & \text{in} \quad \mathbb{T}^N, \\ -\operatorname{div}(D_pH(Du, x)\theta) = \gamma(x) & \text{in} \quad \{u < \psi\}. \\ u \leq \psi. \end{cases} \tag{1.5}$$

This paper is structured as follows: after discussing the main hypothesis in Section 2, we prove, in Section 3, various estimates for (1.4) that are uniform in ϵ. Namely, we obtain:

Theorem 1.1. *Under the assumptions of Section 2, let* $(u_\epsilon, \theta_\epsilon)$ *be the solution to* (1.4). *Then, there exists a constant* C *independent of* ϵ *such that*

$$\|u_\epsilon\|_{W^{2,2}(\mathbb{T}^N)} \leq C, \tag{1.6}$$

$$\|\theta_\epsilon\|_\infty \leq C, \tag{1.7}$$

$$\|\theta_\epsilon\|_{W^{1,2}(\mathbb{T}^N)} \leq C, \tag{1.8}$$

and

$$\|Du_\epsilon\|_\infty \leq C. \tag{1.9}$$

Applying these estimates, we consider the limit $\epsilon \to 0$ in Section 4. There we get the following result:

Theorem 1.2. *Under the assumptions of Section 2, let* $(u_\epsilon, \theta_\epsilon)$ *be the solution to* (1.4). *Then there exists* $u \in W^{1,\infty}(\mathbb{T}^d) \cap W^{2,2}(\mathbb{T}^d)$, *and* $\theta \in L^\infty(\mathbb{T}^d) \cap W^{1,2}(\mathbb{T}^d)$ *such that, through some subsequence,*

$$u_\epsilon \to u \quad \text{in } L^\infty(\mathbb{T}^N),$$

$$Du_\epsilon \to Du, \quad \theta_\epsilon \to \theta \quad \text{in } L^2(\mathbb{T}^N),$$

$$D^2 u_\epsilon \rightharpoonup D^2 u \quad \text{in } L^2(\mathbb{T}^N),$$

as $\epsilon \to 0$. *Furthermore,* (u, θ) *solves* (1.5).

Finally, in Section 5 we establish the uniqueness of solution of the limit problem. More precisely, our main result is:

Theorem 1.3. *Under the assumptions of Section 2, there exists a unique solution* (u, θ) $u \in W^{1,\infty}(\mathbb{T}^N) \cap W^{2,2}(\mathbb{T}^N)$ *and* $\theta \in L^\infty(\mathbb{T}^N) \cap W^{1,2}(\mathbb{T}^N)$ *of the mean-field obstacle problem* (1.5).

2. Assumptions

In this section, we describe our main assumptions. First, to ease the presentation, we assume the obstacle to vanish, that is, $\psi \equiv 0$. This entails no loss of generality as we can always redefine the Hamiltonian and the solution so that the new obstacle vanishes. In addition, we will take the source term $\gamma(x) = 1$. However, our results can be easily adapted to deal with a non-vanishing smooth source γ.

On the Hamiltonian H and the function g we assume:

(i) $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is smooth and positive;
(ii) For each $p \in \mathbb{R}^N$, $x \to H(p, x)$ is periodic;
(iii) There exists a constant $\lambda > 0$ such that

$$H_{p_i p_j}(p, x)\xi_i \xi_j \geq \lambda |\xi|^2 \tag{2.1}$$

for all p, x, $\xi \in \mathbb{R}^N$;

(iv) There exists $C > 0$ such that

$$
\begin{aligned}
|D^2_{pp}H| &\leq C \\
|D^2_{xp}H| &\leq C(1+|p|) \\
|D^2_{xx}H| &\leq C(1+|p|^2)
\end{aligned}
\tag{2.2}
$$

and

$$H(p,x) - D_pH(p,x)p \leq C \tag{2.3}$$

for all p, $x \in \mathbb{R}^N$.

(vi) $g : \mathbb{R}^+ \to \mathbb{R}$ is smooth and such that
  (a) $g' > 0$,
  (b) $g^{-1}(0) > 0$,
  (c) $\theta \to \theta g(\theta)$ is convex,
  (d) there exist $C, \widetilde{C} > 0$ and $\alpha \in [0, \alpha_0)$ with α_0 the solution of

$$2\alpha_0 = (\alpha_0 + 1)\beta(\beta - 1), \qquad \beta = \sqrt{\frac{2^*}{2}}, \tag{2.4}$$

if $N > 2$, and $\alpha_0 = \infty$ if $N \leq 2$, such that

$$C\theta^{\alpha-1} \leq g'(\theta) \leq \widetilde{C}\theta^{\alpha-1} + \widetilde{C}, \tag{2.5}$$

  (f) for any $C_0 > 0$ there exists $C_1 > 0$ such that

$$C_0\theta \leq \frac{1}{2}g(\theta)\theta + C_1, \tag{2.6}$$

for any $\theta \geq 0$.

We choose a penalization term $\beta_\epsilon : \mathbb{R} \to \mathbb{R}$, smooth, with $0 \leq \beta'_\epsilon \leq \frac{1}{\epsilon}$, $\beta''_\epsilon \geq 0$ and such that

$$\beta_\epsilon(s) = 0 \quad \text{for } s \leq 0, \quad \beta_\epsilon(s) = \frac{s-\epsilon}{\epsilon} \quad \text{for } s > 2\epsilon \tag{2.7}$$

$$|\beta_\epsilon(s) - s\beta'_\epsilon(s)| \leq C \quad \text{for } s \in \mathbb{R}. \tag{2.8}$$

Remark 2.1. *The typical examples we have in mind for g are*

$$g(\theta) = \log(\theta),$$

and

$$g(\theta) = \theta^\alpha + \theta_0,$$

for some $\theta_0 > 0$, $\alpha \in (0, \alpha_0)$ with α_0 as in Assumption 2.4.

Remark 2.2. *The assumptions on the Hamiltonian imply that*

$$\frac{\gamma}{2}|p|^2 - C \leq H(p,x) \leq C|p|^2 + C \tag{2.9}$$

and

$$
\begin{aligned}
|D_pH(p,x)| &\leq C(1+|p|) \\
|D_xH(p,x)| &\leq C(1+|p|^2)
\end{aligned}
\tag{2.10}
$$

for all p, $x \in \mathbb{R}^N$.

3. A-priori estimates

In this section, we will establish various a-priori estimates for smooth solutions of the approximate mean-field obstacle problem. Because these estimates will be uniform in ϵ, we can pass to an appropriate limit as $\epsilon \to 0$, as explained in the next section.

In what follows, we denote by (u, θ) a classical solution of (1.4), and we will omit the subscript ϵ for convenience.

Lemma 3.1. *Under the assumptions of Section 2, there exist constants C, $\theta_0 > 0$ independent of ϵ such that for any solution (u, θ) of (1.4),*

$$\theta \geq \theta_0 \quad \text{in } \mathbb{T}^N, \tag{3.1}$$

$$\int_{\mathbb{T}^N} \theta dx \leq C, \tag{3.2}$$

$$\left| \int_{\mathbb{T}^N} \theta g(\theta) dx \right| \leq C, \tag{3.3}$$

$$\int_{\mathbb{T}^N} |u| dx \leq C, \tag{3.4}$$

and

$$\int_{\mathbb{T}^N} |Du|^2 \theta dx \leq C. \tag{3.5}$$

Proof. The lower bound on θ is a consequence of the fact that g^{-1} is increasing with $g^{-1}(0) > 0$, and H and β_ϵ are non-negative:

$$\theta = g^{-1}(H(Du, x) + \beta_\epsilon(u)) \geq g^{-1}(0) =: \theta_0 > 0.$$

Next, multiplying the first equation of (1.4) by θ, the second equation by u, integrating and subtracting, we get

$$\begin{aligned} \int_{\mathbb{T}^N} g(\theta)\theta dx &= \int_{\mathbb{T}^N} (H(Du, x) + \beta_\epsilon(u))\theta dx \\ &= \int_{\mathbb{T}^N} (H(Du, x) - D_p H(Du, x) Du)\theta dx \\ &+ \int_{\mathbb{T}^N} (\beta_\epsilon(u) - \beta_\epsilon'(u)u)\theta dx + \int_{\mathbb{T}^N} u dx. \end{aligned}$$

Then, using (2.3) and (2.8), we can find a constant $C_0 > 0$ such that

$$\int_{\mathbb{T}^N} g(\theta)\theta dx \leq C_0 \int_{\mathbb{T}^N} \theta dx + \int_{\mathbb{T}^N} u dx \leq C_0 \int_{\mathbb{T}^N} \theta dx + \int_{\mathbb{T}^N} u^+ dx. \tag{3.6}$$

Since, g satisfies (2.6), we deduce that

$$\frac{1}{2} \int_{\mathbb{T}^N} g(\theta)\theta dx \leq \int_{\mathbb{T}^N} u^+ dx + C_1. \tag{3.7}$$

Since $H \geq 0$, $\beta_\epsilon(u) \leq g(\theta)$. In particular, (3.7) implies

$$\int_{\mathbb{T}^N} \beta_\epsilon(u) dx \leq \int_{\mathbb{T}^N} g(\theta) dx \leq \frac{1}{\theta_0} \int_{\mathbb{T}^N} g(\theta)\theta dx \leq C \int_{\mathbb{T}^N} u^+ dx + C.$$

At the same time, by (2.8),

$$\int_{\mathbb{T}^N} \beta_\epsilon(u)dx \geq \int_{\{u>2\epsilon\}} \beta_\epsilon'(u)udx - C = \frac{1}{\epsilon}\int_{\{u>2\epsilon\}} udx - C.$$

Hence

$$\frac{1}{\epsilon}\int_{\mathbb{T}^N} u^+dx \leq C\int_{\mathbb{T}^N} u^+dx + C,$$

from which, for ϵ small enough, we get

$$\int_{\mathbb{T}^N} u^+dx \leq C\epsilon. \tag{3.8}$$

We infer, in particular, that $\int_{\mathbb{T}^N} g(\theta)\theta dx \leq C$ from which (3.2) follows. On the other hand, the convexity of $\theta g(\theta)$ implies

$$\int_{\mathbb{T}^N} \theta g(\theta)dx \geq \left(\int_{\mathbb{T}^N} \theta dx\right) g\left(\int_{\mathbb{T}^N} \theta dx\right) \geq -C$$

and (3.3) is then proven.

Estimate (3.4) can be proven observing that (3.6) combined with (3.2), (3.3), and estimate (3.8) yields

$$\left|\int_{\mathbb{T}^N} udx\right| \leq C.$$

This estimate, combined with (3.8) implies,

$$\int_{\mathbb{T}^N} u^-dx \leq C$$

from which then (3.4) follows.

Finally, using the first equation of (1.4) and (2.9) we get

$$\int_{\mathbb{T}^N} |Du|^2\theta dx \leq C\int_{\mathbb{T}^N} g(\theta)\theta dx + C\int_{\mathbb{T}^N} \theta dx$$

and then (3.5) is a consequence of (3.2) and (3.3). □

Lemma 3.2. *Under the assumptions of Section 2, there exists a constant $C > 0$ independent of ϵ such that for any solution (u, θ) of (1.4)*

$$\|u\|_{W^{2,2}(\mathbb{T}^N)} \leq C, \tag{3.9}$$

$$\int_{\mathbb{T}^N} g'(\theta)|D\theta|^2dx \leq C \tag{3.10}$$

and

$$\|\theta^{\frac{\alpha+1}{2}}\|_{W^{1,2}(\mathbb{T}^N)} \leq C. \tag{3.11}$$

Proof. Using (2.9), (3.1) and (3.3) , we get

$$\begin{aligned}\int_{\mathbb{T}^N} |Du|^2dx &\leq C\int_{\mathbb{T}^N} H(Du,x)dx + C \leq C\int_{\mathbb{T}^N} g(\theta)dx + C\\ &\leq C\int_{\mathbb{T}^N} \theta g(\theta)dx + C \leq C.\end{aligned}$$

The previous bound on $\int_{\mathbb{T}^N} |Du|^2 dx$, estimate (3.4), and the Poincaré inequality imply

$$\|u\|_{L^2(\mathbb{T}^N)} \leq C.$$

Next, differentiating twice with respect to x_i the first equation in (1.4), and then summing on i (we use Einstein's convention, that is, summing over repeated indices) we get

$$\begin{aligned} D_pH \cdot D(\Delta u) + \Delta_x H + 2H_{x_ip_j}u_{x_jx_i} + H_{p_jp_l}u_{x_jx_i}u_{x_lx_i} \\ + \beta_\epsilon'(u)\Delta u + \beta_\epsilon''(u)|Du|^2 = \Delta(g(\theta)). \end{aligned}$$

Multiplying the previous equation by θ and using that

$$\begin{aligned} \int_{\mathbb{T}^N} (D_pH \cdot D(\Delta u) + \beta_\epsilon'(u)\Delta u)\theta dx &= \int_{\mathbb{T}^N} (-\mathrm{div}(D_pH\theta) + \beta_\epsilon'(u)\theta)\Delta u dx \\ = \int_{\mathbb{T}^N} \Delta u dx = 0, \end{aligned}$$

we obtain

$$\begin{aligned} \int_{\mathbb{T}^N} (\Delta_x H + 2H_{x_ip_j}u_{x_jx_i} + H_{p_jp_l}u_{x_jx_i}u_{x_lx_i})\theta dx \\ = -\int_{\mathbb{T}^N} \beta_\epsilon''(u)|Du|^2\theta dx + \int_{\mathbb{T}^N} \Delta(g(\theta))\theta dx. \end{aligned}$$

The uniformly convexity of H, properties (2.2), the convexity of β_ϵ and (3.5) then imply

$$C\int_{\mathbb{T}^N} |D^2u|^2\theta dx + \int_{\mathbb{T}^N} g'(\theta)|D\theta|^2 dx \leq C\int_{\mathbb{T}^N} |Du|^2\theta dx + C \leq C$$

which gives, in particular, (3.10). Moreover, from the previous inequality and (3.1), we infer that

$$\int_{\mathbb{T}^N} |D^2u|^2 dx \leq C.$$

This concludes the proof of (3.9). Finally, from (3.10) and (2.5) we infer that

$$\int_{\mathbb{T}^N} \theta^{\alpha-1}|D\theta|^2 dx \leq C\int_{\mathbb{T}^N} g'(\theta)|D\theta|^2 dx \leq C,$$

that is $|D\theta^{\frac{\alpha+1}{2}}| \in L^2(\mathbb{T}^N)$. Since, in addition $\theta \in L^1(\mathbb{T}^N)$, the previous estimate and the Poincaré inequality imply that $\theta^{\frac{\alpha+1}{2}} \in L^2(\mathbb{T}^N)$ and so (3.11) holds. □

We end this section with the proof of Theorem 1.1.

Proof of Theorem 1.1. Estimate (1.6) follows from lemma 3.2. Therefore, we proceed to prove the remaining bounds. First, we remark that from assumption (2.5), g satisfies

$$g(\theta) \leq C\theta^\alpha + C, \quad \text{when } \alpha > 0, \tag{3.12}$$

and

$$g(\theta) \leq C\log(\theta) + C, \quad \text{when } \alpha = 0. \tag{3.13}$$

Next, we show that $\beta_\epsilon'(u)$ is bounded uniformly in ϵ. The function $s \to \beta_\epsilon'(s)$ is increasing. Hence $\beta_\epsilon'(u)$ attains its maximum where u has the maximum. Let x_0 be a maximum point of u, then $Du(x_0) = 0$ and $D^2u(x_0) \leq 0$ and from (1.4), at $x = x_0$ we have

$$\begin{aligned}1 &= -\mathrm{div}(D_pH(Du,x)\theta) + \beta_\epsilon'(u)\theta \\ &= -H_{p_ip_j}u_{x_ix_j}\theta - H_{p_ix_i}\theta - \frac{1}{g'(\theta)}(u_{x_ix_j}H_{p_i}H_{p_j} + H_{x_i}H_{p_i} + \beta_\epsilon'(u)u_{x_i}H_{p_i}) + \beta_\epsilon'(u)\theta \\ &\geq -H_{p_ix_i}\theta - \frac{H_{x_i}H_{p_i}}{g'(\theta)} + \beta_\epsilon'(u)\theta.\end{aligned}$$

Then

$$\max \beta_\epsilon'(u) = \beta_\epsilon'(u(x_0)) \leq H_{p_ix_i}(0,x_0) + \frac{H_{x_i}(0,x_0)H_{p_i}(0,x_0)}{g'(\theta(x_0))\theta(x_0)} + \frac{1}{\theta(x_0)}.$$

Using the properties of the Hamiltonian, (2.5) and (3.1), we conclude that

$$\max \beta_\epsilon'(u) \leq C. \tag{3.14}$$

Next, we claim that, for some constant C independent on p,

$$\int_{\mathbb{T}^N} \theta^{p-1}|D\theta|^2 dx \leq C \int_{\mathbb{T}^N} \theta^{p+1}(1 + |Du|^4)dx. \tag{3.15}$$

In order to prove (3.15), we use the technique from [Eva03] (see the proof of Theorem 5.1) and multiply equation the second equation in (1.4) by $\mathrm{div}(\theta^p D_pH(Du,x))$, for $p > 0$, and integrate by parts:

$$\begin{aligned}\int_{\mathbb{T}^N} \beta_\epsilon'\theta\mathrm{div}(\theta^p D_pH)dx &= \int_{\mathbb{T}^N} (\theta H_{p_i})_{x_i}(\theta^p H_{p_j})_{x_j} dx \\ &= \int_{\mathbb{T}^N} (\theta H_{p_i})_{x_j}(\theta^p H_{p_j})_{x_i} dx \\ &= \int_{\mathbb{T}^N} (\theta(H_{p_i})_{x_j} + \theta_{x_j}H_{p_i})(\theta^p(H_{p_j})_{x_i} + p\theta^{p-1}\theta_{x_i}H_{p_j})dx \\ &= \int_{\mathbb{T}^N} \theta^{p+1}(H_{p_i})_{x_j}(H_{p_j})_{x_i} + p\theta^{p-1}H_{p_i}\theta_{x_i}H_{p_j}\theta_{x_j} \\ &\quad + (p+1)\theta^p\theta_{x_i}H_{p_j}(H_{p_i})_{x_j}dx \\ &=: \int_{\mathbb{T}^N} I_1 + I_2 + I_3 dx.\end{aligned} \tag{3.16}$$

Using assumptions (2.2) on H, we get

$$\begin{aligned}I_1 &= \theta^{p+1}(H_{p_ip_k}u_{x_kx_j} + H_{p_ix_j})(H_{p_jp_l}u_{x_lx_i} + H_{p_jx_i}) \\ &\geq \theta^{p+1}[\gamma^2|D^2u|^2 - C(1+|Du|)|D^2u| - C(1+|Du|^2)] \\ &\geq \theta^{p+1}\tilde{\gamma}^2|D^2u|^2 - C\theta^{p+1}(1+|Du|^2),\end{aligned}$$

for some $\tilde{\gamma} > 0$. Clearly

$$I_2 = p\theta^{p-1}|D_pH \cdot D\theta|^2.$$

Let us estimate I_3 from below. From the first equation of (1.4), we gather that

$$H_{p_j}u_{x_jx_l} = g'(\theta)\theta_{x_l} - H_{x_l} - \beta_\epsilon' u_{x_l}. \tag{3.17}$$

Assumption (2.5) and the lower bound on θ (3.1), imply the existence of a positive constant C_0 such that

$$g'(\theta)\theta \geq C_0 > 0. \tag{3.18}$$

Then, using the properties of the Hamiltonian, (3.14) , (3.17) and (3.18), we get

$$
\begin{aligned}
I_3 &= (p+1)\theta^p \theta_{x_i} H_{p_j}(H_{p_i p_l} u_{x_l x_j} + H_{p_i x_j}) \\
&= (p+1)g'(\theta)\theta^p H_{p_i p_l}\theta_{x_i}\theta_{x_l} + (p+1)\theta^p\theta_{x_i}(H_{p_j}H_{p_i x_j} - H_{p_i p_l}H_{x_l}) - (p+1)\theta^p \beta_\epsilon' H_{p_i p_l}\theta_{x_i} u_{x_l} \\
&\geq (p+1)\gamma C_0 \theta^{p-1}|D\theta|^2 - C(p+1)\theta^p|D\theta|(1+|Du|^2) - C(p+1)\theta^p|D\theta||Du| \\
&\geq C(p+1)\theta^{p-1}|D\theta|^2 - C(p+1)\theta^{p+1}(1+|Du|^4).
\end{aligned}
$$

Next, let us bound from above the left-hand side of (3.16). We have

$$
\begin{aligned}
\beta_\epsilon'\theta \mathrm{div}(\theta^p D_p H) &= \beta_\epsilon'\theta(p\theta^{p-1}D_pH\cdot D\theta + \theta^p H_{p_i p_j}u_{x_j x_i} + \theta^p H_{p_i x_i}) \\
&\leq p\theta^{p-1}|D_pH\cdot D\theta|^2 + \tilde{\gamma}^2\theta^{p+1}|D^2u|^2 + Cp\theta^{p+1}(1+|Du|),
\end{aligned}
$$

where, again, we used the properties of the Hamiltonian and (3.14).

From the preceding estimates, we conclude that

$$
\begin{aligned}
C(p+1)\int_{\mathbb{T}^N}\theta^{p-1}|D\theta|^2dx - C(p+1)\int_{\mathbb{T}^N}\theta^{p+1}(1+|Du|^4)dx + p\int_{\mathbb{T}^N}\theta^{p-1}|D_pH\cdot D\theta|^2dx& \\
+\tilde{\gamma}^2\int_{\mathbb{T}^N}\theta^{p+1}|D^2u|^2 - C\int_{\mathbb{T}^N}\theta^{p+1}(1+|Du|^2)dx& \\
\leq \int_{\mathbb{T}^N} I_1 + I_2 + I_3 dx& \\
= \int_{\mathbb{T}^N}\beta_\epsilon'\theta\mathrm{div}(\theta^p D_pH)dx& \\
\leq p\int_{\mathbb{T}^N}\theta^{p-1}|D_pH\cdot D\theta|^2dx + \tilde{\gamma}^2\int_{\mathbb{T}^N}\theta^{p+1}|D^2u|^2dx& \\
+ Cp\int_{\mathbb{T}^N}\theta^{p+1}(1+|Du|)dx.&
\end{aligned}
$$

The previous inequalities imply (3.15).

By Lemma 3.2, if $N>2$, we have $\theta \in L^{\frac{2^*(1+\alpha)}{2}}$, and for $N=2$, $\theta\in L^p$, for all p. If $N=1$, the (1.7) holds trivially by Morrey's theorem.

Assume $N>2$, then Sobolev's inequality provides the bound

$$
\begin{aligned}
\left(\int_{\mathbb{T}^N}\theta^{\frac{p+1}{2}2^*}dx\right)^{\frac{2}{2^*}} &\leq C\int_{\mathbb{T}^N}\theta^{p+1}dx + C\int_{\mathbb{T}^N}|D(\theta^{\frac{p+1}{2}})|^2dx \\
&= C\int_{\mathbb{T}^N}\theta^{p+1}dx + C(p+1)^2\int_{\mathbb{T}^N}\theta^{p-1}|D\theta|^2dx.
\end{aligned}
\tag{3.19}
$$

Let $\beta := \sqrt{\frac{2^*}{2}} = \sqrt{\frac{N}{N-2}} > 1$, then assumption (2.4) can be rewritten in the following way

$$
2\alpha \leq (\alpha+1)\beta^2\frac{\beta-1}{\beta}
$$

and, for $\alpha>0$, it implies, together with (2.9) and (3.12) that

$$
|Du|^4 \leq C(g(\theta))^2 + C \leq C\theta^{2\alpha} + C \leq C(1+\theta^{(\alpha+1)\beta^2\frac{\beta-1}{\beta}}).
$$

The same inequality holds when $\alpha=0$, using (3.13):

$$
|Du|^4 \leq C(g(\theta))^2 + C \leq C(\log(\theta))^2 + C \leq C(1+\theta^{(\alpha+1)\beta^2\frac{\beta-1}{\beta}}).
$$

Therefore, from Hölder inequality we get

$$
\begin{aligned}
\int_{\mathbb{T}^N} \theta^{p+1}(1+|Du|^4)dx &\leq C\int_{\mathbb{T}^N} \theta^{p+1}(1+\theta^{(\alpha+1)\beta^2\frac{\beta-1}{\beta}})dx \\
&\leq C\int_{\mathbb{T}^N} \theta^{p+1}dx + C\left(\int_{\mathbb{T}^N} \theta^{(p+1)\beta}dx\right)^{\frac{1}{\beta}}\left(\int_{\mathbb{T}^N} \theta^{(\alpha+1)\beta^2}dx\right)^{\frac{\beta-1}{\beta}} \\
&\leq C\int_{\mathbb{T}^N} \theta^{p+1}dx + C\left(\int_{\mathbb{T}^N} \theta^{(p+1)\beta}dx\right)^{\frac{1}{\beta}} \\
&\leq C\left(\int_{\mathbb{T}^N} \theta^{(p+1)\beta}dx\right)^{\frac{1}{\beta}}.
\end{aligned}
$$

The last inequality, (3.15) and (3.19) give the estimate

$$
\left(\int_{\mathbb{T}^N} \theta^{(p+1)\beta^2}dx\right)^{\frac{1}{\beta^2}} \leq Cp^2\left(\int_{\mathbb{T}^N} \theta^{(p+1)\beta}dx\right)^{\frac{1}{\beta}}.
$$

Arguing as in [Eva03], we get (1.7) and hence (1.9) for $N > 2$.

When $N \leq 2$, the reasoning is similar, because $\theta \in L^p$ for any p, and so (1.7) holds too.

Finally, (1.8) is a consequence of (3.11) and the estimate (1.7) just proven. □

4. Convergence

In this section, we present the proof of Theorem 1.2 using the previous estimates.

Proof of Theorem 1.2. Let $(u_\epsilon, \theta_\epsilon)$ be a solution of (1.4). The estimates obtained in the previous section, namely in Theorem 1.1, imply the existence of functions $u \in W^{2,2}(\mathbb{T}^N) \cap W^{1,\infty}(\mathbb{T}^N)$ and $\theta \in W^{1,2}(\mathbb{T}^N) \cap L^\infty(\mathbb{T}^N)$ such that, up to subsequence, as $\epsilon \to 0$

$$
\begin{gathered}
u_\epsilon \to u \quad \text{in } L^\infty(\mathbb{T}^N), \\
Du_\epsilon \to Du, \quad \theta_\epsilon \to \theta \quad \text{in } L^2(\mathbb{T}^N), \\
D^2u_\epsilon \rightharpoonup D^2u \quad \text{in } L^2(\mathbb{T}^N).
\end{gathered}
$$

Furthermore, the sequence u_ϵ converges uniformly to a non-positive function u.

For $s > 0$, since β'_ϵ is increasing ($\beta''_\epsilon > 0$) and $\beta_\epsilon(0) = 0$ we have

$$
\beta_\epsilon(s) = \beta_\epsilon(0) + \beta'_\epsilon(\xi_s)s \leq \max_{t\in[0,s]} \beta'_\epsilon(t)s = \beta'_\epsilon(s)s.
$$

Therefore, for any $s \in \mathbb{R}$

$$
\beta_\epsilon(s) \leq \beta'_\epsilon(s)s^+.
$$

This implies, using (3.14)

$$
0 \leq \beta_\epsilon(u_\epsilon) \leq \beta'_\epsilon(u_\epsilon)(u_\epsilon)^+ \leq C(u_\epsilon)^+.
$$

We conclude that $\beta_\epsilon(u_\epsilon) \to 0$ uniformly as $\epsilon \to 0$. Hence, the limit (u, θ) solves

$$
\begin{gathered}
H(Du, x) = g(\theta) \quad \text{in} \quad \mathbb{T}^N, \\
-\mathrm{div}(D_pH(Du, x)\theta) \leq 1 \quad \text{in} \quad \mathbb{T}^N, \\
-\mathrm{div}(D_pH(Du, x)\theta) = 1 \quad \text{in} \quad \{u < 0\}. \\
u \leq 0.
\end{gathered}
$$

□

5. Uniqueness

We end the paper with the proof of uniqueness of solutions to (1.5). This will be based upon a modified monotonicity argument inspired by the original technique by Lasry and Lions, see [LL06a, LL06b, LL07a].

Proof of Theorem 1.3. Let (u_1, θ_1) and (u_2, θ_2) be distinct solutions of (1.5). Set

$$A := \{u_1 - u_2 > 0\}.$$

A is an open set. Moreover, $A \subset \{u_2 < 0\}$ since $u_1 - u_2 = u_1 \leq 0$ in $\{u_2 = 0\}$, therefore,

$$-\mathrm{div}(D_pH(Du_2, x)\theta_2) = \gamma(x) \quad \text{in } A.$$

From the first equality in (1.5), we have

$$\int_A [H(Du_1, x) - H(Du_2, x)](\theta_1 - \theta_2)dx = \int_A (g(\theta_1) - g(\theta_2))(\theta_1 - \theta_2)dx.$$

Using the second inequality for u_1 and the third equality for u_2 in (1.5), multiplying by $u_1 - u_2 > 0$ in A and integrating by parts, we obtain

$$\begin{aligned} 0 &\leq \int_A \mathrm{div}(D_pH(Du_1, x)\theta_1 - D_pH(Du_2, x)\theta_2)(u_1 - u_2)dx \\ &= -\int_A (D_pH(Du_1, x)\theta_1 - D_pH(Du_2, x)\theta_2)D(u_1 - u_2)dx. \end{aligned}$$

Note that there is no boundary data since $u_1 - u_2 = 0$ on ∂A. Adding the two inequalities and using the convexity of H, we get

$$\begin{aligned} 0 \leq \int_A (g(\theta_1) - g(\theta_2))(\theta_1 - \theta_2) &\leq \int_A [H(Du_1, x) - H(Du_2, x)](\theta_1 - \theta_2)dx \\ &\quad - \int_A (D_pH(Du_1, x)\theta_1 - D_pH(Du_2, x)\theta_2)D(u_1 - u_2)dx \\ &= -\int_A [H(Du_2, x) - H(Du_1, x) - D_pH(Du_1, x)D(u_2 - u_1)]\theta_1 dx \\ &\quad - \int_A [H(Du_1, x) - H(Du_2, x) - D_pH(Du_2, x)D(u_1 - u_2)]\theta_2 dx \\ &\leq -C\int_A |D(u_1 - u_2)|^2 dx. \end{aligned}$$

Thus we infer that $|A| = 0$, i.e., $u_1 \leq u_2$ almost everywhere. □

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