Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Asymptotics and stability of a periodic solution to a singularly perturbed parabolic problem in case of a double root of the degenerate equation

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submitted: June 29, 2015

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No. 2141

Berlin 2015

2010 *Mathematics Subject Classification.* 35B25, 35B10, 35K20, 35K57.

Key words and phrases. Singularly perturbed reaction-diffusion equation, double root of the degenerate equation, initial boundary value problem, asymptotic expansion, asymptotically stable periodic solution, region of attraction.

This work was partially supported by RFBR-DFG grant N14-01-91333, by the DFG grant RE1336/1-1 and by the program of cooperation of the Moscow State University and the Humboldt University of Berlin.

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Abstract

For a singularly perturbed parabolic problem with Dirichlet conditions we prove the existence of a solution periodic in time and with boundary layers at both ends of the space interval in the case that the degenerate equation has a double root. We construct the corresponding asymptotic expansion in the small parameter. It turns out that the algorithm of the construction of the boundary layer functions and the behavior of the solution in the boundary layers essentially differ from that ones in case of a simple root. We also investigate the stability of this solution and the corresponding region of attraction.

1 Formulation of the problem

Consider the singularly perturbed parabolic equation

$$\varepsilon^2 \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} \right) = f(u, x, t, \varepsilon), \tag{1.1}$$

$$(x, t) \in D = (0 < x < 1) \times (-\infty < t < \infty),$$

where $\varepsilon > 0$ is a small parameter and f is T-periodic in t, under the boundary conditions

$$u(0, t, \varepsilon) = u^0(t), \quad u(1, t, \varepsilon) = u^1(t) \text{ for } -\infty < t < \infty, \tag{1.2}$$

where u^0 and u^1 are T-periodic functions. We are interested in the existence of a solution $u(x, t, \varepsilon)$ of (1.1), (1.2) which is T-periodic in t

$$u(x, t + T, \varepsilon) = u(x, t, \varepsilon) \quad \text{for } (x, t) \in \bar{D}. \tag{1.3}$$

Such equations are often used as mathematical models of reaction-diffusion processes in chemical kinetics, astrophysics, biology and in another applications. Some traditional fields of applications can be found in the book [1]. It is well-known (see, e.g. [2] and [3]) that if the functions f, u^0 and u^1 are sufficiently smooth and if the degenerate equation

$$f(u, x, t, 0) = 0 \tag{1.4}$$

has a smooth root

$$u = \varphi(x, t) \text{ for } (x, t) \in \bar{D} \tag{1.5}$$

which is T-periodic in t and stable, that is, the condition

$$f_u(\varphi(x, t), x, t, 0) > 0 \text{ for } (x, t) \in \bar{D} \tag{1.6}$$

is fulfilled, and if the boundary functions u^0 and u^1 are located in the region of attraction of the root $\varphi(x,t)$, then for sufficiently small ε the boundary value problem (1.1) – (1.3) has a solution $u(x,t,\varepsilon)$ with the asymptotic representation

$$u(x,t,\varepsilon) = \bar{u}(x,t,\varepsilon) + \Pi(\xi,t,\varepsilon) + \tilde{\Pi}(\tilde{\xi},t,\varepsilon) \text{ for } (x,t) \in \bar{D}, \tag{1.7}$$

where

$$\bar{u}(x,t,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \bar{u}_i(x,t) \tag{1.8}$$

is the regular part of the asymptotic representation whose main term is the root $\varphi(x,t)$ of the degenerate equation, that is, $\bar{u}_0(x,t) = \varphi(x,t)$,

$$\Pi\left(\xi,t,\varepsilon\right) = \sum_{i=0}^{\infty} \varepsilon^i \Pi_i(\xi,t), \quad \text{Ãl} \quad \tilde{\Pi}\left(\tilde{\xi},t,\varepsilon\right) = \sum_{i=0}^{\infty} \varepsilon^i \tilde{\Pi}_i(\tilde{\xi},t) \tag{1.9}$$

are the boundary layer parts of the asymptotic representation whose coefficients (the boundary layer functions Π_i and $\tilde{\Pi}_i$) depend on t and on the boundary layer variables $\xi = \frac{x}{\varepsilon}$ and $\tilde{\xi} = \frac{1-x}{\varepsilon}$, respectively and exponentially decay to zero as the corresponding boundary variable tends to ∞, that is, the following estimates hold true

$$\begin{aligned} |\Pi_i(\xi,t)| &\le c\exp(-\kappa\xi) \text{ for } \xi \ge 0, \quad -\infty < t < \infty, \\ |\tilde{\Pi}_i(\tilde{\xi},t)| &\le c\exp(-\kappa\tilde{\xi}) \text{ for } \tilde{\xi} \ge 0, \quad -\infty < t < \infty, \end{aligned} \tag{1.10}$$

where the positive constants c and κ do not depend on ε.

We note that according to the condition (1.6) the root (1.5) of the degenerate equation (1.4) is a simple root. It is known, that this assumption can be violated in some interesting applications. Particularly, the case of intersecting roots of the degenerate equation has been considered by the authors in the papers [4] and [5]. In what follows we study the problem (1.1) – (1.3) under the condition that the function f has the form

$$f(u,x,t,\varepsilon) \equiv h(x,t)\left(u - \varphi(x,t)\right)^2 - \varepsilon f_1(u,x,t,\varepsilon). \tag{1.11}$$

In that case the root $u = \varphi(x,t)$ of the degenerate equation is a double root.

We will show that under some conditions (e.g. under the conditions (A1)–(A3) below) problem (1.1) – (1.3) in case of a double root has a solution with the asymptotics (1.7) but the series (1.8) and (1.9) change their qualitative behavior (cf. (2.2) – (2.4)), and the boundary layers can be devided into three zones in which the solution of our problem exhibits different behavior, and also the algorithm for the construction of the boundary layer functions differs from that one in case of a simple root.

We note that in the more general case when the function h in (1.11) also depends on u, that is, $h = h(u,x,t)$, where $h(\varphi(x,t),x,t) \neq 0$ for $(x,t) \in \bar{D}$, all features of the asymptotics associated with the existence of a double root $\varphi(x,t)$ of the degenerate equation remain, but their implementations are more sophisticated.

We mention that the analogous problem of the existence of a T-periodic solution with boundary layers of the equation (1.1), where the function f has the structure (1.11) but satisfying Neumann boundary conditions

$$\frac{\partial u}{\partial x}(0,t,\varepsilon) = 0, \quad \frac{\partial u}{\partial x}(1,t,\varepsilon) = 0 \quad \text{for } -\infty < t < \infty$$

has been considered in [6]. In that case, different to our situation, the asymptotics of the form (1.7) is constructed by using the standard algorithm, and the boundary layers show a one-zone character with exponentially decaying boundary layer functions as in case of a simple root of the degenerate equation. The difference to the case of a simple root is manifested only in another scale of the boundary layer variables ξ and $\tilde{\xi}$, namely

$$\xi = \frac{x}{\varepsilon^{\frac{3}{4}}},\ \tilde{\xi} = \frac{1-x}{\varepsilon^{\frac{3}{4}}}.$$

Note also, that a specific case of the double root has been considered in our paper [7], the phenomenon of multi-zones behaviour in the boundary layer has been studied for the initial value problem with a double root in [8].

The structure of our paper is as follows. In section 2 we construct a formal asymptotic expansion of a solution to the problem (1.1) – (1.3) in case that the function f has the form (1.11), in section 3 we prove the existence of a solution with the constructed asymptotics. Finally, in section 4 we study the stability of this solution and its region of attraction.

2 Construction of the asymptotics of a solution

2.1 Assumptions and form of the asymptotics

Concerning the functions f, u^0, u^1 we assume

$(A1)$. *The function f has the form (1.11), the functions h, φ, f_1, u^0 and u^1 are sufficiently smooth and T-periodic in t, additionally we assume*

$$h(x,t) > 0 \quad \text{for} \quad (x,t) \in \bar{D}. \tag{2.1}$$

We note that the required smoothness is determined by the order of the constructed asymptotics. In case of arbitrarily high order asymptotics we suppose that the functions are infinitely often differentiable.

$(A2)$.

$$\bar{f}_1(x,t) := f_1(\varphi(x,t), x, t, 0) > 0 \ \text{ for } \ (x,t) \in \bar{D}.$$

As can be seen in the following, this assumption plays a fundamental role in the construction of the boundary layer asymptotics as well as in the proof of the existence of a solution with the constructed asymptotics. This condition implies, that in the considered case of a double root of the degenerate equation (different from the case of a simple root) the terms of order $O(\varepsilon)$ on the right hand side of equation (1.1) play a crucial role.

$(A3)$.

$$u^0(t) > \varphi(0,t), \quad u^1(t) > \varphi(1,t) \quad \text{for} \quad -\infty < t < \infty.$$

This condition is essential for the construction of the boundary layer functions.

Under the conditions $(A1)-(A3)$ we construct an asymptotic expansion of the solution to the problem (1.1)–(1.3) in the form (1.7), but an essential difference to the case of a simple root consists in the fact that the regular part $\bar{u}(x,t,\varepsilon)$ of the asymptotics will be now a power series in $\sqrt{\varepsilon}$ (and not in ε as in the case of a simple root), and the boundary layer series are series in $\sqrt[4]{\varepsilon}$, where their coefficients, that is, the functions Π_i and $\tilde{\Pi}_i$ depend not only on ξ, t and $\tilde{\xi}, t$, respectively but also on ε, that is, we have

$$\bar{u}(x,t,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{\frac{i}{2}} \bar{u}_i(x,t), \tag{2.2}$$

$$\Pi(\xi,t,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{\frac{i}{4}} \Pi_i(\xi,t,\varepsilon), \tag{2.3}$$

$$\tilde{\Pi}(\tilde{\xi},t,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{\frac{i}{4}} \tilde{\Pi}_i(\tilde{\xi},t,\varepsilon). \tag{2.4}$$

The boundary variables ξ and $\tilde{\xi}$ have the same scaling as in the case of a simple root of the degenerate equation

$$\xi = \frac{x}{\varepsilon}, \quad \tilde{\xi} = \frac{1-x}{\varepsilon}.$$

2.2 Regular part of the asymptotics

By the standard approach, we have for the series (2.2) the relation

$$\varepsilon^2 \left(\frac{\partial^2 \bar{u}}{\partial x^2} - \frac{\partial \bar{u}}{\partial t} \right) = f(\bar{u}, x, t, \varepsilon)$$

where we compare the coefficients belonging to the same powers of $\sqrt{\varepsilon}$ of the expansions on both sides. By this way we get equations for the determination of the coefficients $\bar{u}_i(x,t)$ of the series (2.2). For $\bar{u}_0(x,t)$ we obtain the equation

$$h(x,t) \left(\bar{u}_0 - \varphi(x,t) \right)^2 = 0,$$

which implies $\bar{u}_0(x,t) = \varphi(x,t)$.

For $\bar{u}_1(x,t)$ we get the quadratic equation

$$h(x,t)\bar{u}_1^2 - \bar{f}_1(x,t) = 0.$$

Taking into account (2.1) and condition (A2), this equation has two roots. As $\bar{u}_1(x,t)$ we take the positive root

$$\bar{u}_1(x,t) = \left[h^{-1}(x,t) \bar{f}_1(x,t) \right]^{\frac{1}{2}} > 0. \tag{2.5}$$

This choice will turn out to be suitable for the study of the equations for the boundary layer functions and for the proof of the existence of a solution with the constructed asymptotics.

For the determination of the coefficients $\bar{u}_i(x,t), \;\; i=2,3,\dots$, of the series (2.2) we get the linear algebraic equations

$$[2h(x,t)\bar{u}_1(x,t)]\,\bar{u}_i = F_i(x,t),$$

where the function $F_i(x,t)$ depends on the functions $\bar{u}_j(x,t)$ with $j<i$. Since $h(x,t)\bar{u}_1(x,t)$ is different from zero for all $(x,t)\in\bar{D}$, these equations uniquely define the functions $\bar{u}_i(x,t)$. Obviously, all functions $\bar{u}_i(x,t)$ are T-periodic functions in t.

2.3 Boundary layer part of the asymptotics

The determination of the boundary layer functions $\Pi_i(\xi,t)$ is based on the relation

$$\begin{aligned}\frac{\partial^2\Pi}{\partial\xi^2}-\varepsilon^2\frac{\partial\Pi}{\partial t} = \Pi f := f(\bar{u}(\varepsilon\xi,t,\varepsilon)+\Pi(\xi,t,\varepsilon),\varepsilon\xi,t,\varepsilon)\\ -f(\bar{u}(\varepsilon\xi,t,\varepsilon),\varepsilon\xi,t,\varepsilon) = h(\varepsilon\xi,t)\big[\,(\bar{u}(\varepsilon\xi,t,\varepsilon)+\Pi(\xi,t,\varepsilon)-\varphi(\varepsilon\xi,t))^2\\ -(\bar{u}(\varepsilon\xi,t,\varepsilon)-\varphi(\varepsilon\xi,t))^2\,\big]-\varepsilon\Pi f_1 \;\text{ for }\; \xi>0,\; -\infty<t<\infty\end{aligned} \tag{2.6}$$

with the boundary conditions

$$\Pi(0,t,\varepsilon)=-\bar{u}(0,t,\varepsilon), \quad \Pi(\infty,t,\varepsilon)=0 \;\text{ for }\; -\infty<t<\infty. \tag{2.7}$$

Equation (2.6) and the boundary conditions (2.7) follow from the standard approach for the boundary functions (see [2]), but the method to derive the equations for the functions $\Pi_i(\xi,t,\varepsilon)$ of the series (2.3) from the relation (2.6) after substituting (2.3) into (2.6) will be different from the standard approach since that approach cannot be applied in the case of a multiple root (see [9]).

Now we describe the algorithm for deriving the equations which determine the functions $\Pi_i(\xi,t,\varepsilon)$.

To determine $\Pi_0(\xi,t,\varepsilon)$ we use the differential equation

$$\frac{\partial^2\Pi_0}{\partial\xi^2} = h(0,t)\left[\Pi_0^2+2\sqrt{\varepsilon}\bar{u}_1(0,t)\Pi_0\right] \;\text{ for }\; \xi>0. \tag{2.8}$$

If we would apply the standard scheme to derive the equation for Π_0, then the term multiplied by $\sqrt{\varepsilon}$ would not appear in the differential equation (2.8). As a consequence, Π_0 would tend to zero of order $O(\frac{1}{\xi^2})$ as $\xi\to\infty$ which contradicts the true behavior of the solution of problem (1.1) – (1.3) in the boundary layer.

From (2.7) we get the following boundary conditions for $\Pi_0(\xi,t,\varepsilon)$

$$\Pi_0(0,t,\varepsilon)=u^0(t)-\varphi(0,t), \quad \Pi_0(\infty,t,\varepsilon)=0 \;\text{ for }\; -\infty<t<\infty. \tag{2.9}$$

The boundary value problem (2.8), (2.9) can be reduced to the following initial value problem

$$\frac{\partial\Pi_0}{\partial\xi} = -\left[2h(0,t)\left(\frac{1}{3}\Pi_0+\sqrt{\varepsilon}\bar{u}_1(0,t)\right)\right]^{\frac{1}{2}}\Pi_0 \;\text{ for }\; \xi>0, \tag{2.10}$$

$$\Pi_0(0,t,\varepsilon)=u^0(t)-\varphi(0,t)=:\Pi^0(t). \tag{2.11}$$

Problem (2.10), (2.11) can be solved explicitly. Using the relations $\Pi^0(t) > 0$ which follows from assumption (A3) and $\bar{u}_1(0,t) > 0$ (see (2.5)), we can conclude that $\Pi_0(\xi,t,\varepsilon)$ tends monotonically to zero as $\xi \to \infty$.

We write the solution in the following form

$$\Pi_0(\xi,t,\varepsilon) = \frac{12\sqrt{\varepsilon}\bar{u}_1(0,t)\left[1+O(\varepsilon^{\frac{1}{4}})\right]\exp\left(-\varepsilon^{\frac{1}{4}}k_0(t)\xi\right)}{\left\{1-\left[1-(12\bar{u}_1(0,t)(\Pi^0(t))^{-1})^{\frac{1}{2}}\,\varepsilon^{\frac{1}{4}}+O(\sqrt{\varepsilon})\right]\exp\left(-\varepsilon^{\frac{1}{4}}k_0(t)\xi\right)\right\}^2}, \tag{2.12}$$

where $k_0(t) := [2h(0,t)\bar{u}_1(0,t)]^{\frac{1}{2}} > 0$. The terms $O(\varepsilon^{\frac{1}{4}})$ and $O(\sqrt{\varepsilon})$ have the required decay order as $\varepsilon \to 0$ uniformly on the half line $\xi \geq 0$.

We note that in case $\Pi^0(t) < 0$ the boundary value problem (2.8), (2.9) has no solution. The case $\Pi^0(t) \equiv 0$ which implies $\Pi_0(\xi,t,\varepsilon) \equiv 0$ requires a special investigation.

A straightforward analysis of the expression (2.12) yields that the decay of the function $\Pi_0(\xi,t,\varepsilon)$ for increasing ξ exhibits a different behavior in different intervals of the variable ξ. We can distinguish three zones.

As the first zone we consider the interval $0 \leq \xi \leq \varepsilon^{-\gamma}$ (that is, $0 \leq x \leq \varepsilon^{1-\gamma}$), where γ is any number of the interval $[0, \frac{1}{4})$. In this zone we have $\Pi_0(\xi,t,\varepsilon) = O\left(\frac{1}{1+\xi^2}\right)$, that means, the function $\Pi_0(\xi,t,\varepsilon)$ exhibits a polynomial decay.

As the second zone (transition zone) we take the interval $\varepsilon^{-\gamma} \leq \xi \leq \varepsilon^{-\frac{1}{4}}$ (that is, $\varepsilon^{1-\gamma} \leq x \leq \varepsilon^{\frac{3}{4}}$). Here, we can observe a change of the character of the decay and of the scaling of the boundary layer variable.

Finally, as the third zone we take the interval $\xi \geq \varepsilon^{-\frac{1}{4}}$ (that is, $x \geq \varepsilon^{\frac{3}{4}}$). Here, the function $\Pi_0(\xi,t,\varepsilon)$ satisfies the estimate

$$\Pi_0(\xi,t,\varepsilon) = O(\sqrt{\varepsilon})\exp(-k_0(t)\zeta), \quad \text{where} \quad \zeta = \varepsilon^{\frac{1}{4}}\xi = \frac{x}{\varepsilon^{\frac{3}{4}}},$$

that means, the boundary layer variable ζ for this zone has another scaling as the former variable ξ, and the function $\Pi_0(\zeta,t,\varepsilon)$ decays exponentially in the third zone as $\zeta \to \infty$. We emphasize once more that the positivity of the function $\bar{u}_1(0,t)$ plays a fundamental role for the described behavior of the function $\Pi_0(\xi,t,\varepsilon)$.

From (2.12) we get for $\Pi_0(\xi,t,\varepsilon)$ the estimate

$$|\Pi_0(\xi,t,\varepsilon)| \leq c\,\Pi_\kappa(\xi,\varepsilon) \text{ for } \xi \geq 0, \quad -\infty < t < +\infty, \tag{2.13}$$

where

$$\Pi_\kappa(\xi,\varepsilon) := \frac{\sqrt{\varepsilon}\exp\left(-\varepsilon^{\frac{1}{4}}\kappa\xi\right)}{\left[1-(1-\varepsilon^{\frac{1}{4}})\exp\left(-\varepsilon^{\frac{1}{4}}\kappa\xi\right)\right]^2}, \tag{2.14}$$

and κ satisfies $0 < \kappa < \min_{0\leq t\leq T} k_0(t)$. The function $\Pi_\kappa(\xi,\varepsilon)$ has the same three-zones behavior as the function $\Pi_0(\xi,t,\varepsilon)$. It plays the role of an estimator for the functions $\Pi_i(\xi,t,\varepsilon)$

of the series (2.3) as an analogon to the function $\exp(-\kappa\xi)$ which plays the same role for the boundary layer functions in case of a simple root of the degenerate equation (see (1.10)).

The derivative $\frac{\partial \Pi_0}{\partial t}(\xi,t,\varepsilon)$ satisfies the same estimate as the function $\Pi_0(\xi,t,\varepsilon)$

$$\left|\frac{\partial \Pi_0}{\partial t}(\xi,t,\varepsilon)\right| \le c\,\Pi_\kappa(\xi,\varepsilon) \ \text{ for } \ \xi \ge 0, \quad -\infty < t < +\infty.$$

This can be verified by differentiating the expression (2.12) with respect to t or by studying the boundary value problem for $\frac{\partial \Pi_0}{\partial t}$, which we get from the boundary value problem (2.8), (2.9) by differentiating with respect to t.

The boundary value problems for determining the functions $\Pi_i(\xi,t,\varepsilon)$, $i = 1,2,\ldots$, of the series (2.3) read as follows

$$\frac{\partial^2 \Pi_i}{\partial \xi^2} = \alpha(\xi,t,\varepsilon)\Pi_i + \pi_i(\xi,t,\varepsilon) \ \text{ for } \ \xi > 0, \ -\infty < t < \infty \tag{2.15}$$

$$\Pi_i(0,t,\varepsilon) = \begin{cases} -\bar{u}_{\frac{i}{2}}(0,t) & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd,} \end{cases} \qquad \Pi_i(\infty,t,\varepsilon) = 0,$$

where

$$\alpha(\xi,t,\varepsilon) := 2h(0,t)\left[\Pi_0(\xi,t,\varepsilon) + \sqrt{\varepsilon}\bar{u}_1(0,t)\right],$$

and the functions $\pi_i(\xi,t,\varepsilon)$ are recursively defined by means of the functions $\Pi_j(\xi,t,\varepsilon)$ with $j < i$ by an approach which differs from the standard one. In order to describe this procedure, we first rewrite the right hand side of equation (2.6) in the following form, where we take into account the relation $x = \varepsilon^{\frac{3}{4}}\zeta$:

$$\Pi f = h(\varepsilon^{\frac{3}{4}}\zeta,t)\Big[\Big(\bar{u}(\varepsilon^{\frac{3}{4}}\zeta,t,\varepsilon) + \Pi(\xi,t,\varepsilon) - \varphi(\varepsilon^{\frac{3}{4}}\zeta,t)\Big)^2$$

$$- \Big(\bar{u}(\varepsilon^{\frac{3}{4}}\zeta,t,\varepsilon) - \varphi(\varepsilon^{\frac{3}{4}}\zeta,t)\Big)^2\Big] - \varepsilon\Pi f_1.$$

Now we expand the right hand side of this relation into a power series of $\varepsilon^{\frac{1}{4}}$ and denote by $\beta_i(\zeta,\Pi_0,\ldots,\Pi_{i-1})$ the coefficient belonging to $\varepsilon^{\frac{i}{4}}$. In this coefficient we do not include the summand $2h(0,t)\Pi_0(\xi,t,\varepsilon)\Pi_i$ which is included into the expression $\alpha(\xi,t,\varepsilon)\Pi_i$ in equation (2.15).

If the modulus of the summand $\beta_{ij}(\zeta,\Pi_0,\ldots,\Pi_{i-1})$ belonging to $\beta_i(\zeta,\Pi_0,\ldots,\Pi_{i-1})$ can be estimated by a product containing at least two factors $|\Pi_k(\xi,t,\varepsilon)|$ with $k < i$, that is,

$$|\beta_{ij}| \le c|\Pi_k(\xi,t,\varepsilon)|\,|\Pi_l(\xi,t,\varepsilon)|, \quad k < i, \quad l < i,$$

then this summand after replacing ζ by $\varepsilon^{\frac{1}{4}}\xi$ belongs to the expression $\pi_i(\xi,t,\varepsilon)$; if the estimate of $|\beta_{ij}|$ contains only one factor $|\Pi_k(\xi,t,\varepsilon)|$ with $k < i$, then this summand after replacing ζ by $\varepsilon^{\frac{1}{4}}\xi$ and multiplying it by $\sqrt{\varepsilon}$ belongs to the expression $\pi_{i-2}(\xi,t,\varepsilon)$.

Moreover, for $i \ge 6$ we include the summand $\sqrt{\varepsilon}\frac{\partial \Pi_{i-6}}{\partial t}(\xi,t,\varepsilon)$ appearing as part of the term $-\varepsilon^2\frac{\partial \Pi}{\partial t}$ on the left hand side of (2.6) into $\pi_i(\xi,t,\varepsilon)$.

We note that it holds $\pi_1(\xi, t, \varepsilon) \equiv 0$. Thus we have

$$\Pi_1(\xi, t, \varepsilon) \equiv 0$$

and we can represent the expression $\pi_2(\xi, t, \varepsilon)$ in the form

$$\pi_2(\xi, t, \varepsilon) = \sqrt{\varepsilon}\left(2h(0,t)\bar{u}_2(0,t)\Pi_0(\xi, t, \varepsilon) - \Pi_0 f_1\right),$$

where

$$\Pi_0 f_1 = f_1\left(\varphi(0,t) + \Pi_0(\xi, t, \varepsilon), 0, t, 0\right) - f_1\left(\varphi(0,t), 0, t, 0\right).$$

The described procedure to determine the functions $\pi_i(\xi, t, \varepsilon)$ permits to derive the following estimates

$$|\pi_i(\xi, t, \varepsilon)| \leq c\left[\Pi_\kappa^2(\xi, \varepsilon) + \sqrt{\varepsilon}\Pi_\kappa(\xi, \varepsilon)\right], \tag{2.16}$$

where the functions $\Pi_\kappa(\xi, \varepsilon)$ are defined in (2.14). The constant c and the index κ depend on the index i.

The inequality (2.16) implies the following estimate of the type (2.13) (see [9])

$$|\Pi_i(\xi, t, \varepsilon)| \leq c\Pi_\kappa(\xi, \varepsilon) \;\text{ for }\; \xi \geq 0, \quad -\infty < t < +\infty, \quad i = 2, 3, \ldots, \tag{2.17}$$

where c and κ depend on i. From this estimate we get that all $\Pi_i(\xi, t, \varepsilon)$ feature the same three-zones behavior as $\Pi_0(\xi, t, \varepsilon)$. The proof of the inequality (2.17) is based on the use of the explicit representation of $\Pi_i(\xi, t, \varepsilon)$

$$\Pi_i(\xi, t, \varepsilon) = \Phi(\xi, t, \varepsilon)\Phi^{-1}(0, t, \varepsilon)\Pi_i(0, t, \varepsilon)$$

$$+\Phi(\xi, t, \varepsilon)\int_0^\xi \Phi^{-2}(s, t, \varepsilon)\int_\infty^s \Phi(\sigma, t, \varepsilon)\pi_i(\sigma, t, \varepsilon)d\sigma ds,$$

where

$$\Phi(\xi, t, \varepsilon) = \frac{\partial \Pi_0}{\partial \xi}(\xi, t, \varepsilon).$$

It follows the same line as in [9].

The derivatives $\frac{\partial \Pi_i}{\partial t}$ also satisfy an estimate of type (2.17). We note that all functions $\Pi_i(\xi, t, \varepsilon)$ and their derivatives are T-periodic in t.

The coefficients of the series (2.4), that is, the boundary layer functions $\tilde{\Pi}_i(\tilde{\xi}, t, \varepsilon)$ are defined analogously to the functions $\Pi_i(\xi, t, \varepsilon)$ and satisfy estimates also of type (2.17).

By this way, we have constructed the formal asymptotics of the solution of the boundary value problem (1.1) – (1.3) in the form (1.7), and the corresponding summands appearing in (1.7) are determined in form of the series (2.2), (2.3) and (2.4).

3 Proof of the existence of a solution with the constructed asymptotics

We denote by $U_n(x,t,\varepsilon)$ the partial sum of the constructed expansion (1.7) consisting of the three series (2.2), (2.3) and (2.4)

$$U_n(x,t,\varepsilon) = \sum_{i=0}^{n} \varepsilon^{\frac{i}{2}} \bar{u}_i(x,t) + \sum_{i=0}^{2n+1} \varepsilon^{\frac{i}{4}} \left[\Pi_i \left(\frac{x}{\varepsilon}, t, \varepsilon \right) + \tilde{\Pi}_i \left(\frac{1-x}{\varepsilon}, t, \varepsilon \right) \right]. \tag{3.1}$$

It is obvious that $U_n(x,t,\varepsilon)$ is a T-periodic function in t.

Theorem 3.1 *Under the assumptions (A1) – (A3) there is a sufficiently small $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ problem (1.1) – (1.3) has a solution $u_T(x,t,\varepsilon)$ for which the function $U_n(x,t,\varepsilon)$ is an asymptotic approximation of order $O\left(\varepsilon^{\frac{n+1}{2}}\right)$ uniformly in $\bar{D}$ for any $n = 0, 1, 2, \ldots$, that is,*

$$u_T(x,t,\varepsilon) = U_n(x,t,\varepsilon) + O\left(\varepsilon^{\frac{n+1}{2}}\right) \quad \textit{for} \ \ (x,t) \in \bar{D}. \tag{3.2}$$

Proof. The proof is based on the asymptotic method of differential inequalities. The essence of this approach is to construct upper and lower solutions to the problem (1.1) – (1.3) by means of the partial sum $U_n(x,t,\varepsilon)$ defined in (3.1) which yield the formal asymptotics of the solution of problem (1.1) – (1.3).

From the construction of the function $U_n(x,t,\varepsilon)$ it follows that it satisfies the asymptotic relations

$$L_\varepsilon U_n := \varepsilon^2 \left(\frac{\partial^2 U_n}{\partial x^2} - \frac{\partial U_n}{\partial t} \right) - f(U_n, x, t, \varepsilon) = O\left(\varepsilon^{\frac{n+1}{2}}\right) \quad \text{for} \ \ (x,t) \in D, \tag{3.3}$$

$$U_n(0,t,\varepsilon) = u^0(t) + o(\varepsilon^N), \ \ U_n(1,t,\varepsilon) = u^1(t) + o(\varepsilon^N)$$
$$\text{for any natural number } N \ \text{ and } \ -\infty < t < +\infty. \tag{3.4}$$

These relations imply that $U_n(x,t,\varepsilon)$ yields the formal asymptotics of the solution of (1.1) – (1.3).

Now we recall the definition of the lower and upper solutions for the problem (1.1) – (1.3).

Definition 3.1 *The functions $\underline{U}(x,t,\varepsilon)$ and $\bar{U}(x,t,\varepsilon)$ are called lower and upper solutions of (1.1) – (1.3) in D if they satisfy the conditions*

(1). $L_\varepsilon \underline{U} := \varepsilon^2 \left(\frac{\partial^2 \underline{U}}{\partial x^2} - \frac{\partial \underline{U}}{\partial t} \right) - f(\underline{U}, x, t, \varepsilon) \geq 0 \geq L_\varepsilon \bar{U}$ *for* $(x,t) \in D$;

(2). $\underline{U}(0,t,\varepsilon) \leq u^0(t) \leq \bar{U}(0,t,\varepsilon), \quad \underline{U}(1,t,\varepsilon) \leq u^1(t) \leq \bar{U}(1,t,\varepsilon)$
for $-\infty < t < +\infty$;

(3). $\underline{U}(x,t,\varepsilon)$ *and* $\bar{U}(x,t,\varepsilon)$ *are* T*-periodic functions in* t.

The lower and upper solutions are called ordered if it holds

$$\underline{U}(x,t,\varepsilon) \le \bar{U}(x,t,\varepsilon) \ \text{ for } \ (x,t) \in \bar{D}. \tag{3.5}$$

It is well known (see, e.g. [1]) that the existence of ordered lower and upper solutions to the problem (1.1) – (1.3) implies the existence of a solution $u(x,t,\varepsilon)$ of (1.1) – (1.3) satisfying the inequalities

$$\underline{U}(x,t,\varepsilon) \le u(x,t,\varepsilon) \le \bar{U}(x,t,\varepsilon) \ \text{ for } \ (x,t) \in \bar{D}. \tag{3.6}$$

We will show that for $n \ge 2$ and for sufficiently small ε the function

$$\underline{U}(x,t,\varepsilon) = U_n(x,t,\varepsilon) - M\varepsilon^{\frac{n}{2}} \tag{3.7}$$

is a lower solution of (1.1)–(1.3), where $U_n(x,t,\varepsilon)$ is defined in (3.1) and $M > 0$ is a sufficiently large number not depending on ε.

As we already noticed, the function $\underline{U}(x,t,\varepsilon)$ is T-periodic in t, that is, condition (3) in Definition 3.1 is fulfilled.

Next, using the first relation in (3.4) we get

$$\underline{U}(0,t,\varepsilon) = u^0(t) + o(\varepsilon^N) - M\varepsilon^{\frac{n}{2}}.$$

Hence, we have $\underline{U}(0,t,\varepsilon) < u^0(t)$ for sufficiently small ε and for $-\infty < t < \infty$, that is, the first inequality of condition (2) in Definition 3.1 is satisfied.

By the same way, we can prove that the second inequality of condition (2) in Definition 3.1 is fulfilled, that is, we have $\underline{U}(1,t,\varepsilon) \le u^1(t)$ for sufficiently small ε and for $-\infty < t < \infty$.

Now we will verify condition (1) in Definition 3.1. Using the relations (3.3) and (3.7) we obtain

$$\begin{aligned} L_\varepsilon \underline{U} &= L_\varepsilon U_n - \left[f(U_n - M\varepsilon^{\frac{n}{2}}, x, t, \varepsilon) - f(U_n, x, t, \varepsilon)\right] \\ &= O\left(\varepsilon^{\frac{n+1}{2}}\right) - \left[f_u(U_n, x, t, \varepsilon)\left(-M\varepsilon^{\frac{n}{2}}\right) + \frac{1}{2} f^*_{uu}\left(-M\varepsilon^{\frac{n}{2}}\right)^2\right], \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} &f_u(U_n, x, t, \varepsilon) = 2h(x,t)(U_n(x,t,\varepsilon) - \varphi(x,t)) + \varepsilon f_{1u}(U_n, x, t, \varepsilon), \\ &f^*_{uu} = 2h(x,t) + \varepsilon f^*_{1uu} = 2h(x,t) + O(\varepsilon) \end{aligned} \tag{3.9}$$

(the mark $*$ means that the derivative is taken at some intermediate point).

Now we show that for sufficiently small ε the inequality

$$f_u(U_n, x, t, \varepsilon) > c_0\sqrt{\varepsilon} \ \text{ for } \ (x,t) \in \bar{D} \tag{3.10}$$

holds, where c_0 is some positive number not depending on ε. From (3.9) we get that for $-\infty < t < \infty$, $0 \le x \le \frac{1}{2}$ the relation

$$f_u(U_n, x, t, \varepsilon) = 2h(x,t)\left[\sqrt{\varepsilon}\bar{u}_1(x,t) + \Pi_0(\xi, t, \varepsilon) + \sqrt{\varepsilon}\Pi_2(\xi, t, \varepsilon)\right] + O\left(\varepsilon^{\frac{3}{4}}\right) \quad (3.11)$$

is valid since in that interval we have $\tilde{\Pi}_i(\tilde{\xi}, t, \varepsilon) = o(\varepsilon^N)$ for any natural number N. For $0 \le \xi \le \xi_0$, where ξ_0 is any fixed number not depending on ε, we get for $\Pi_0(\xi, t, \varepsilon)$ from (2.12) the estimate

$$\Pi_0(\xi, t, \varepsilon) \ge \frac{c}{(1+\xi_0)^2}.$$

Therefore, for sufficiently small ε it holds

$$\Pi_0(\xi, t, \varepsilon) + \sqrt{\varepsilon}\Pi_2(\xi, t, \varepsilon) > 0 \quad \text{for} \quad 0 \le \xi \le \xi_0.$$

Because of $\bar{u}_1(x,t) > 0$ (see (2.5)), we obtain from (3.11)

$$\begin{aligned} &f_u(U_n, x, t, \varepsilon) \ge 2h(x,t)\sqrt{\varepsilon}\bar{u}_1(x,t) + O(\varepsilon^{\frac{3}{4}}) > 2c_0\sqrt{\varepsilon} \\ &\text{for } \ 0 \le x \le \xi_0\varepsilon, \quad -\infty < t < +\infty, \end{aligned} \quad (3.12)$$

where

$$0 < c_0 < \min_{(x,t) \in \bar{D}} [h(x,t)\bar{u}_1(x,t)].$$

Now we choose ξ_0 sufficiently large such that the inequality holds

$$\frac{1}{2} \min_{(x,t) \in \bar{D}} \bar{u}_1(x,t) + \Pi_2(\xi, t, \varepsilon) \ge 0 \quad \text{for} \quad \xi \ge \xi_0, \quad -\infty < t < +\infty.$$

The existence of such ξ_0 follows from the property $\Pi_2(\xi, t, \varepsilon) \to 0$ as $\xi \to \infty$. Taking into account $\Pi_0(\xi, t, \varepsilon) > 0$ we get from (3.11)

$$\begin{aligned} &f_u(U_n, x, t, \varepsilon) \ge 2h(x,t)\sqrt{\varepsilon}\frac{1}{2}\bar{u}_1(x,t) + O\left(\varepsilon^{\frac{3}{4}}\right) > c_0\sqrt{\varepsilon} \\ &\text{for } \ \xi_0\varepsilon \le x \le \frac{1}{2}, \quad -\infty < t < +\infty. \end{aligned} \quad (3.13)$$

The inequalities (3.12) and (3.13) imply the validity of (3.10) for $0 \le x \le \frac{1}{2}$, $-\infty < t < \infty$.

Analogously we prove that for sufficiently small ε the inequality (3.10) is valid for $\frac{1}{2} \le x \le 1$, $-\infty < t < \infty$.

Using the inequality (3.10) and the fact that for $n \ge 2$ the last summand $\frac{1}{2}f^*_{uu}M^2\varepsilon^n$ on the right hand side of (3.8) for any fixed number M has a higher order of smallness than $O\left(\varepsilon^{\frac{n+1}{2}}\right)$ as $\varepsilon \to 0$, then we get from (3.8)

$$L_\varepsilon \underline{U} > O\left(\varepsilon^{\frac{n+1}{2}}\right) + c_0 M \varepsilon^{\frac{n+1}{2}}.$$

Consequently, for sufficiently small ε and sufficiently large M the inequality

$$L_\varepsilon \underline{U} \geq 0 \text{ for } (x,t) \in D$$

is valid, that is, the function $\underline{U}(x,t,\varepsilon)$ satisfies condition (1) of Definition 3.1 for a lower solution. By this way, the function $\underline{U}(x,t,\varepsilon)$ defined in (3.7) is for sufficiently small ε and sufficiently large M a lower solution of the boundary value problem (1.1) – (1.3).

Analogously it can be proved that the function

$$\bar{U}(x,t,\varepsilon) = U_n(x,t,\varepsilon) + M\varepsilon^{\frac{n}{2}} \tag{3.14}$$

is for sufficiently small ε and sufficiently large M an upper solution of (1.1) – (1.3).

The lower and upper solutions defined in (3.7) and (3.14) are obviously ordered lower and upper solutions. Hence, we can conclude that there exists a solution of problem (1.1) – (1.3) (we denote it by $u_T(x,t,\varepsilon)$) satisfying the inequalities in (3.6). Thus, we have

$$u_T(x,t,\varepsilon) = U_n(x,t,\varepsilon) + O\left(\varepsilon^{\frac{n}{2}}\right) \text{ for } (x,t) \in \bar{D}. \tag{3.15}$$

Taking into account

$$U_n(x,t,\varepsilon) = U_{n-1}(x,t,\varepsilon) + O\left(\varepsilon^{\frac{n}{2}}\right)$$

we obtain from (3.15) (for $n \geq 2$)

$$u_T(x,t,\varepsilon) = U_{n-1}(x,t,\varepsilon) + O\left(\varepsilon^{\frac{n}{2}}\right) \text{ for } (x,t) \in \bar{D}.$$

Replacing in this relation n by $n+1$ we obtain for $n \geq 1$

$$u_T(x,t,\varepsilon) = U_n(x,t,\varepsilon) + O\left(\varepsilon^{\frac{n+1}{2}}\right) \text{ for } (x,t) \in \bar{D}.$$

Putting $n = 1$ we get

$$u_T(x,t,\varepsilon) = U_1(x,t,\varepsilon) + O(\varepsilon).$$

Taking into account $U_1(x,t,\varepsilon) = U_0(x,t,\varepsilon) + O(\sqrt{\varepsilon})$ we obtain

$$u_T(x,t,\varepsilon) = U_0(x,t,\varepsilon) + O(\sqrt{\varepsilon}),$$

that is, (3.2) is valid also for $n = 0$.

By this way, the proof of Theorem 3.1 has been completed. □

4 Stability and region of attraction of the solution $u_T(x,t,\varepsilon)$.

Theorem 4.1 *Assume the conditions (A1) – (A3) to be satisfied. Then for sufficiently small ε the solution $u_T(x,t,\varepsilon)$ of (1.1) – (1.3) is asymptotically stable in the sense of Lyapunov.*

This theorem claims that any solution $u(x,t,\varepsilon)$ of the equation (1.1) satisfying the boundary conditions (1.2) and the initial condition

$$u(x,t_0,\varepsilon) = u_0(x,\varepsilon) \text{ for } 0 \leq x \leq 1,$$

where t_0 is an arbitrary time moment and where $u_0(x,\varepsilon)$ is sufficiently near to $u_T(x,t_0,\varepsilon)$, that is,

$$\max_{0 \le x \le 1} |u_0(x,\varepsilon) - u_T(x,t_0,\varepsilon)|$$

is sufficiently small, remains near $u_T(x,t,\varepsilon)$ for $t > t_0$, additionally it holds

$$\lim_{t\to+\infty} [u(x,t,\varepsilon) - u_T(x,t,\varepsilon)] = 0 \ \text{ for } \ 0 \le x \le 1. \tag{4.1}$$

Proof. We consider the derivative of the function $f(u,x,t,\varepsilon)$ with respect to u along the solution $u_T(x,t,\varepsilon)$ of problem (1.1) – (1.3)

$$f_u(u_T(x,t,\varepsilon),x,t,\varepsilon) = 2h(x,t)\,(u_T(x,t,\varepsilon) - \varphi(x,t)) + \varepsilon f_{1u}(u_T(x,t,\varepsilon),x,t,\varepsilon).$$

Using the asymptotics of the solution $u_T(x,t,\varepsilon)$ (see (3.2)) we get

$$f_u(u_T(x,t,\varepsilon),x,t,\varepsilon) = f_u(U_n(x,t,\varepsilon),x,t,\varepsilon) + O\left(\varepsilon^{\frac{n+1}{2}}\right).$$

For $n \ge 1$, sufficiently small ε and taking into account the relation (3.10) it follows from this relation

$$f_u(u_T(x,t,\varepsilon),x,t,\varepsilon) > c_0\sqrt{\varepsilon} \ \text{ for } \ (x,t) \in \bar{D}. \tag{4.2}$$

The inequality (4.2) implies the asymptotic stability of the solution $u_T(x,t,\varepsilon)$ for sufficiently small ε (see [1]). This completes the proof of Theorem 4.1. □

Now we study the region of attraction of the solution $u_T(x,t,\varepsilon)$. We define to any initial time t_0 the region of attraction of the asymptotically stable solution $u_T(x,t,\varepsilon)$ of (1.1) – (1.3) as the set $\mathcal{A}$ of functions $u_0(x,\varepsilon)$ such that to each $u_0 \in \mathcal{A}$ there exists a solution $u(x,t,\varepsilon)$ of equation (1.1) with the following properties

(i). $u(x,t,\varepsilon)$ satisfies the boundary conditions (1.2).

(ii). $u(x,t,\varepsilon)$ satisfies the initial condition

$$u(x,t_0,\varepsilon) = u_0(x,\varepsilon) \ \text{ for } \ 0 \le x \le 1, \tag{4.3}$$

and the compatibility condition

$$u_0(0,\varepsilon) = u^0(t_0), \quad u_0(1,\varepsilon) = u^1(t_0)$$

and exists for $t > t_0$.

(iii). $u(x,t,\varepsilon)$ obeys the limit relation (4.1).

The following theorem provides a sufficient condition for a function $u_0(x,\varepsilon)$ to belong to the set $\mathcal{A}$.

Theorem 4.2 *Suppose the assumptions (A1) – (A3) to be valid. Let $u_0(x,\varepsilon)$ be any smooth function satisfying the condition*

$$u_0(x,\varepsilon) \geq u_T(x,t_0,\varepsilon) \ \textit{ for } \ 0 \leq x \leq 1. \tag{4.4}$$

Then, for sufficiently small ε, problem (1.1), (1.2), (4.3) has a solution $u(x,t,\varepsilon)$ which exists for all $t > t_0$, and this solution obeys the limit relation (4.1).

Proof. Again we use the method of differential inequalities.

Definition 4.1 *The functions $\underline{U}(x,t,\varepsilon)$ and $\bar{U}(x,t,\varepsilon)$ are called lower and upper solutions of the problem (1.1), (1.2), (4.3), if they satisfy the following conditions:*

(1). $L_\varepsilon \underline{U} \geq 0 \geq L_\varepsilon \bar{U}$ for $0 < x < 1, t > t_0$;

(2). $\underline{U}(0,t,\varepsilon) \leq u^0(t) \leq \bar{U}(0,t,\varepsilon)$ and $\underline{U}(1,t,\varepsilon) \leq u^1(t) \leq \bar{U}(1,t,\varepsilon)$ for $t \geq t_0$;

(3) $\underline{U}(x,t_0,\varepsilon) \leq u_0(x,\varepsilon) \leq \bar{U}(x,t_0,\varepsilon)$ for $0 \leq x \leq 1$.

We construct an upper solution in the form

$$\bar{U}(x,t,\varepsilon) = u_T(x,t,\varepsilon) + AE(t,\varepsilon), \tag{4.5}$$

where

$$E(t,\varepsilon) = \exp\left[-\frac{p(t-t_0)}{\varepsilon^{\frac{3}{2}}}\right],$$

A and p are positive numbers not depending on ε chosen below.

It is obvious that for any $A > 0$ and $p > 0$ the function $\bar{U}(x,t,\varepsilon)$ fulfills the inequalities of condition (2) in Definition 4.1, and for sufficiently large A the inequality of condition (3) is satisfied too.

Now we show that for sufficiently small p the inequality for $\bar{U}(x,t,\varepsilon)$ of condition (1) in Definition 4.1 is fulfilled. We have

$$\begin{aligned} L_\varepsilon \bar{U} &= \varepsilon^2\left(\frac{\partial^2 \bar{U}}{\partial x^2} - \frac{\partial \bar{U}}{\partial t}\right) - f(\bar{U},x,t,\varepsilon) = \varepsilon^2\left(\frac{\partial^2 u_T}{\partial x^2} - \frac{\partial u_T}{\partial t}\right) \\ &+ \sqrt{\varepsilon} pAE(t,\varepsilon) - h(x,t)\big[\,(u_T - \varphi(x,t))^2 + 2(u_T - \varphi(x,t))AE(t,\varepsilon) \\ &+ A^2E^2(t,\varepsilon)\big] + \varepsilon f_1(u_T,x,t,\varepsilon) + \varepsilon f^*_{1u} AE(t,\varepsilon) \\ &= \left\{\varepsilon^2\left(\frac{\partial^2 u_T}{\partial x^2} - \frac{\partial u_T}{\partial t}\right) - h(x,t)(u_T - \varphi(x,t))^2 + \varepsilon f_1(u_T,x,t,\varepsilon)\right\} \\ &+ \left[\sqrt{\varepsilon}p - 2h(x,t)(u_T - \varphi(x,t)) - h(x,t)AE(t,\varepsilon) + \varepsilon f^*_{1u}\right] AE(t,\varepsilon). \end{aligned} \tag{4.6}$$

The expression in the curly brackets vanishes since $u_T(x,t,\varepsilon)$ is a solution of equation (1.1). The second summand in the squared brackets can be estimated as follows

$$-2h(x,t)(u_T - \varphi(x,t)) < -c_0\sqrt{\varepsilon} \ \text{ for } \ (x,t) \in \bar{D}, \tag{4.7}$$

which can be verified in the same way as it was done for (3.10).

Therefore, we get from (4.6) the inequality

$$L_\varepsilon \bar{U} \le \sqrt{\varepsilon}\left[p - c_0 + O(\sqrt{\varepsilon})\right] AE(t,\varepsilon).$$

If we choose p less than c_0 then we obtain for sufficiently small ε the inequality

$$L_\varepsilon \bar{U} < 0 \ \text{ for } \ 0 < x < 1, t > t_0,$$

that is, $\bar{U}(x,t,\varepsilon)$ satisfies the inequality of condition (1) in Definition 4.1.

Hence, the function $\bar{U}(x,t,\varepsilon)$ defined in (4.5) is for sufficiently large A, sufficiently small p and sufficiently small ε an upper solution of problem (1.1), (1.2), (4.3).

The function

$$\underline{U}(x,t,\varepsilon) = u_T(x,t,\varepsilon) - \varepsilon E(t,\varepsilon) \tag{4.8}$$

is a lower solution of the same problem. Indeed, the inequalities for $\underline{U}(x,t,\varepsilon)$ of conditions (2) and (3) in Definition 4.1 are fulfilled obviously.

Now we show that $\underline{U}(x,t,\varepsilon)$ satisfies the inequalities of condition (1). Analogously to (4.6), we obtain the relation

$$L_\varepsilon \underline{U} = \varepsilon\left[-\sqrt{\varepsilon}p + 2h(x,t)(u_T - \varphi(x,t)) - \varepsilon h(x,t)E(t,\varepsilon) - \varepsilon f^*_{1u}\right] E(t,\varepsilon).$$

Using (4.7) we get from this relation the inequality

$$L_\varepsilon \underline{U} \ge \varepsilon^{\frac{3}{2}}\left[-p + c_0 + O(\sqrt{\varepsilon})\right].$$

Choosing $p < c_0$ we have for sufficiently small ε

$$L_\varepsilon \underline{U} > 0 \ \text{ for } \ 0 < x < 1 \ \text{ and } t > t_0.$$

Therefore, the function $\underline{U}(x,t,\varepsilon)$ defined in (4.8) is for sufficiently small ε a lower solution of problem (1.1), (1.2), (4.3).

The existence of a lower and of an upper solution of problem (1.1), (1.2), (4.3) implies the existence of a solution $u(x,t,\varepsilon)$ of this problem satisfying the inequalities [10]

$$\underline{U}(x,t,\varepsilon) \le u(x,t,\varepsilon) \le \bar{U}(x,t,\varepsilon) \ \text{ for } \ 0 \le x \le 1 \ \text{ and } \ t \ge t_0.$$

Using the expressions (4.5) and (4.8) for the functions $\bar{U}(x,t,\varepsilon)$ and $\underline{U}(x,t,\varepsilon)$, we get from these inequalities

$$-\varepsilon E(t,\varepsilon) \le u(x,t,\varepsilon) - u_T(x,t,\varepsilon) \le AE(t,\varepsilon) \ \text{ for } \ 0 \le x \le 1 \ \text{ and } \ t \ge t_0.$$

From $E(t,\varepsilon) \to 0$ as $t \to +\infty$ we get

$$\lim_{t\to\infty}\left[u(x,t,\varepsilon) - u_T(x,t,\varepsilon)\right] = 0 \ \text{ for } \ 0 \le x \le 1.$$

Consequently, the solution $u(x,t,\varepsilon)$ of problem (1.1), (1.2), (4.3) satisfies the limit relation (4.1). This completes the proof of Theorem 4.2. □

5 References

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