Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Amplitude equations for collective spatio-temporal dynamics in arrays of coupled systems

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submitted: January 26, 2015

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No. 2070
Berlin 2015

Key words and phrases. coupled oscillators, amplitude equations, Ginzburg-Landau equation, spatio-temporal chaos.

S. Yanchuk acknowledges the support by the European Research Council under ERC-2010-AdG 267802 Analysis of Multiscale Systems Driven by Functionals. M. Wolfrum is supported by the German Research Foundation (DFG) in the framework of the Collaborative Research Center SFB 910. T. Kapitaniak has been supported by the Polish National Science Centre, MAESTRO Programme - Project No 2013/08/A/ST8/00/780. P. Perlikowski acknowledges the Scholarships for Young Scientists from Minister of Science and High Education of Poland .

Edited by
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Abstract

We study the coupling induced destabilization in an array of identical oscillators coupled in a ring structure where the number of oscillators in the ring is large. The coupling structure includes different types of interactions with several next neighbors. We derive an amplitude equation of Ginzburg-Landau type, which describes the destabilization of a uniform stationary state and close-by solutions in the limit of a large number of nodes. Studying numerically an example of unidirectionally coupled Duffing oscillators, we observe a coupling induced transition to collective spatio-temporal chaos, which can be understood using the derived amplitude equations.

The understanding of the complex dynamical behavior of networks of coupled nonlinear units can contribute to the explanation of various collective phenomena that can be observed in coupled systems in biology, economy, or physics [30, 17, 26]. Discrete media, e.g. in neural systems, can exhibit complex coupling structures including feed-forward loops, large coupling ranges, and interaction mechanisms of different kind. In such systems, the influence of the coupling can transform simple equilibrium dynamics of a single unit into complicated spatio-temporal structures of the network dynamics. For continuous media, amplitude equations of Ginzburg-Landau type have been a powerful tool for a universal description of diffusion induced spatio-temporal dynamics and pattern formation in a spatially homogeneous system. In this paper, we apply this technique to a large class of spatially discrete systems, where a one dimensional array of identical dynamical units is coupled in a homogeneous but possibly complicated structure including different types of interactions to several next neighbors. Under the assumption that the number of units is large, we can derive an amplitude equations of Ginzburg-Landau type not only for discrete versions of well understood examples of diffusion induced pattern formation in continuous media, but also for e.g. unidirectional or anti-diffusive interactions. We show that in all these cases, the collective dynamics close to the destabilized homogeneous state can be described by the amplitude equations, and illustrate this by an example of a chain of unidirectionally coupled Duffing oscillators, showing a coupling induced transition from equilibrium to spatio-temporal defect dynamics.

1 Introduction

Dynamics on coupled networks have been the subject of extensive research in the last decade. Coupled systems can display a huge variety of dynamical phenomena, starting from synchronization phenomena in various types of inhomogeneous or irregular networks, up to complex collective behavior, such as for example various forms of phase transitions, traveling waves [8, 25, 21, 14], phase-locked patterns, amplitude death states [5], or so called chimera states that display a regular pattern of coherent and incoherent motion [1, 20, 28, 32]. Of particular interest are situations, where complex spatio-temporal structures can emerge in regular arrays of identical units induced only by the coupling interaction. In many cases, the resulting phenomena differ substantially from corresponding situations in continuous media [18] and depend strongly on the underlying network topology.

Our specific interest is in the emergence of spatio-temporal structures in a homogeneous array of identical units that have a stable uniform equilibrium at which the coupling vanishes. In continuous media, the Turing instability gives a classical paradigm of a coupling induced instability. There is of course a natural direct counterpart in the discrete setting, but it turns out that in addition there appear also some genuinely new phenomena. In Refs [36, 23, 35] it has been shown that in a ring of unidirectionally coupled oscillators, i.e. in a purely convective setting, the Eckhaus scenario can be observed that is characterized by a coexistence of multiple periodic patterns, whose number is proportional to the domain size [6]. In Ref. [24] it has been shown that Duffing oscillators coupled in the same way, exhibit a complex transition to spatio-temporal chaos. In this paper we develop a general theoretical framework for such phenomena in large arrays, using the amplitude equation approach, see e.g. Refs [19, 15, 13, 27, 34, 37]. We derive an amplitude equation of Ginzburg-Landau type that governs the local dynamics close to the destabilizing uniform steady state. It provides a universal destabilization scenario that is already well known in the context of reaction-diffusion systems [6, 31, 4, 3]. However, as we show, it can be applied to a much larger class of coupling mechanisms, including the case of unidirectional and anti-diffusive interaction and allowing for a mixture of such interactions in the coupling to several next neighbors. As a specific feature, the convective part will appear in the amplitude equation as a rotation of the coordinates in an intermediate time scale that is faster than the diffusive processes described by the Ginzburg-Landau equation. Having deduced the amplitude equation and the corresponding scaling laws in terms of the number of oscillators, which is assumed to tend to infinity, we use this theory for the explanation of spatio-temporal chaos in a ring of unidirectionally coupled Duffing oscillators.

2 Model equation, spectral conditions, and notations

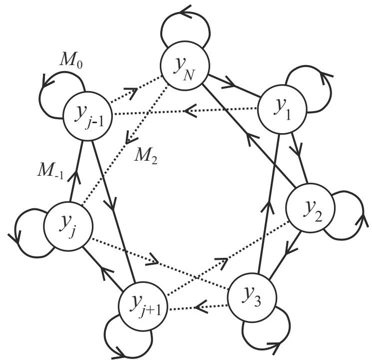

Figure 1: An example of a ring of N coupled oscillators. Apart from the self-coupling M_0, each oscillator y_j is also coupled with y_{j+2} (M_2) as well as y_{j-1} (M_{-1}). See Eq. (1) for the equation of motion.

We are interested in a system of N identical coupled oscillators that has a uniform equilibrium, where the coupling vanishes. The coupling network is organized as an array with possibly different types of interactions between several next neighbors, closed to a ring by periodic boundary conditions. Such systems can be written in general form as

$$\dot{y}_j = \sum_{m=-R}^{R} M_m(p) y_{j+m} + h(y_{j-R}, y_{j-R+1}, \dots y_{j+R}; p), \tag{1}$$

where $y_j \in \mathbb{R}^n,\ j = 1, \dots, N$ describes the state of the j-th oscillator and $R \leq N/2$ the coupling range. Closing the system with periodic boundary conditions, all indexes have to be considered modulo

N. The linear part of the dynamics is given by the $n \times n$ matrices $M_m(p),\ m = 1, \dots, N$, depending on the bifurcation parameter p and accounting for the coupling to the m-th neighbor; in particular $M_0(p)$ describes the linear part of the local dynamics (self-coupling). The nonlinear part h, again including a local dependence and a dependence on the m-th neighbor, should vanish at the origin $h(0, \dots, 0; p) = 0$ and have also zero derivatives there. Note that this system is symmetric (equivariant) with respect to index shift. Figure 1 illustrates an example with self coupling and coupling to the neighbor on the left and to the second neighbor on the right. The specific form of (1) also implies that the coupling vanishes at the equilibrium $y_1 = \cdots = y_N = 0$, which is true e.g. when the coupling is a function of the difference $y_j - y_m$ for any two coupled oscillators j, m.

Due to the spatial homogeneity, the modes of the linearization at the zero equilibrium of (1) are plane waves of the form $y_{j+1} = e^{2\pi ik/N} y_j$ with discrete wave numbers $k = 1, \dots, N$. Hence, the characteristic equation can be factorized as

$$\chi(p, \lambda, e^{2\pi ik/N}) = \det \left[\lambda \mathrm{Id} - \sum_m e^{2\pi imk/N} M_m(p) \right] = 0,$$

where Id denotes the identity matrix in $\mathbb{R}^n$ and the index $k = 1, \dots, N$ accounts for the N-th roots of unity that appear as the eigenvalues of the circular coupling structure [22]. Following the approach in Refs [36, 23], we replace for large N the discrete numbers $2\pi k/N$ by a continuous parameter φ, and obtain the *asymptotic continuous spectrum*

$$\Lambda_p = \left\{ \lambda \in \mathbb{C} :\ \chi(p, \lambda, e^{i\varphi}) = \det \left[\lambda \mathrm{Id} - \sum_m e^{im\varphi} M_m(p) \right] = 0,\ \varphi \in [0, 2\pi) \right\}, \tag{2}$$

containing all eigenvalues and, for large N, being densely covered by the eigenvalues. Since the expression (2) is periodic in φ, the asymptotic continuous spectrum Λ_p has generically the form of one or several closed curves $\lambda_p(\varphi)$ in the complex plane, parametrized by φ.

At the bifurcation value $p = 0$, we assume that the asymptotic continuous spectrum touches the imaginary axis at some point $i\omega_0$ (see Fig. 2), i.e. the following conditions are fulfilled

$$\lambda(\varphi_0) = i\omega_0, \quad \frac{\partial \lambda}{\partial \varphi}(\varphi_0) = i\kappa_1, \quad \kappa_1 \in \mathbb{R}, \tag{3}$$

$$\mathrm{Re}\lambda(\varphi) < \mathrm{Re}\lambda(\varphi_0) \ \text{ for all } \varphi \neq \varphi_0. \tag{4}$$

The first condition from (3) means that the point $i\omega_0$ belongs to the asymptotic continuous spectrum, while the second condition from (3) guarantees that the real part $\Re(\lambda(\varphi))$ is tangent to the zero axis at $\varphi = \varphi_0$ (see Fig. 2). This tangency describes the condition for the destabilization (or bifurcation) of the zero solution with respect to a plane wave solution with temporal frequency ω_0 and wave number $k_0 \approx \frac{\varphi_0 N}{2\pi}$. Indeed, if the spectrum Λ_p is contained in the left half of the complex plane with $\mathrm{Re}\lambda < 0$, then the uniform equilibrium $y_1 = \cdots = y_N = 0$ is asymptotically stable. As soon as Λ_p crosses the imaginary axis, it becomes unstable for sufficiently large N. The conditions (3)–(4) are, in fact, conditions on the matrices M_m of the linearization.

Before we present our main result, we now introduce some useful notations and formulate a technical lemma, which follows from the bifurcation conditions (3). With v_0 and v_1 we denote the eigenvector and the adjoint eigenvector to the critical eigenvalue $\lambda_0(\varphi_0) = i\omega_0$, which we assumed in (3) to exist for $p = 0$. Moreover, it will be convenient to denote by

$$L_0 = \sum_m e^{im\varphi_0} M_m(0),$$

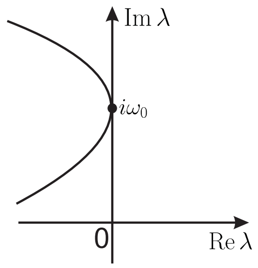

Figure 2: Asymptotic continuous spectrum Λ_p at the destabilization; schematically.

$$L_1 = \sum_m m e^{im\varphi_0} M_m(0),$$
$$L_2 = \sum_m m^2 e^{im\varphi_0} M_m(0)$$

the "moments" of the coupling matrices M_m. In this notation, the equations for the eigenvectors v_0 and v_1 read as

$$[i\omega_0 \mathrm{Id} - L_0]\, v_0 = 0, \tag{5}$$

$$[-i\omega_0 \mathrm{Id} - L_0^*]\, v_1 = 0. \tag{6}$$

These vectors can be normalized as

$$|v_0|^2 = 1, \quad \langle v_0, v_1 \rangle = 1. \tag{7}$$

Finally, we expand the matrices $M_m(p)$ with respect to the parameter p and write them as $M_m(0) + pK_m + \mathcal{O}\left(p^2\right)$. In this way, we can define

$$L_K = \sum_m e^{im\varphi_0} K_m.$$

Lemma 1. *Assume that φ_0 is a regular point of the asymptotic continuous spectrum (2), such that $\lambda_0(\varphi)$ exists and is locally differentiable in a small neighborhood of φ_0. Further, let the bifurcation condition (3) hold. Then*

$$\langle L_1 v_0, v_1 \rangle = \kappa_1. \tag{8}$$

The proof of this Lemma will be given together with the proof of the main result in Section 5.

3 Main Result: Reduction to Ginzburg-Landau Equation

In this section, we present an amplitude equation that describes for system (1) the dynamics close to the destabilization threshold in the limit of large N. We perform a limiting procedure, where for a fixed coupling range R the total number of oscillators N tends to infinity. In this way, the coupling becomes local in the limit $N \to \infty$, and coupling terms will be approximated by derivatives of the amplitude. Consequently, our results will be valid for large arrays, where the coupling range is small compared to the total size. It will be shown that the amplitude equation has the form of a complex Ginzburg-Landau equation with periodic boundary conditions. With Proposition 2 we present here the main assertions, while the technical details of the derivation and the proof is deferred to Section 5.

Proposition 2. *Assume that the bifurcation conditions (3)–(4) hold and φ_0 is a regular point of the asymptotic continuous spectrum Λ_0. Additionally, let the points $\pm 3i\omega_0$ not belong to the asymptotic continuous spectrum with $\varphi = \varphi_0$ (non-resonance condition). Let the nonlinearity h be of third order. Introduce the small parameter*

$$\varepsilon = \frac{1}{N}$$

and apply the multiple scale ansatz

$$y_j(t) = \varepsilon e^{i\omega_0 t + i\varphi_0 j} v_0 A(T_1, x_1, T_2) + \varepsilon^3 e^{3i(\omega_0 t + \varphi_0 j)} v_2 A^3(T_1, x_1, T_2) + c.c., \tag{9}$$

with the amplitude $A \in \mathbb{C}$ depending on the rescaled coordinates $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t$, and $x_1 = \varepsilon j$ ($c.c.$ denotes complex conjugated terms, $v_2 \in \mathbb{C}^n$) to the following system

$$\dot{y}_j = \sum_{m=-R}^{R} \left(M_m(0) + \varepsilon^2 r K_m + \mathcal{O}\left(\varepsilon^4\right) \right) y_{j+m} + h(y_{j-R}, \ldots y_{j+R}; p), \tag{10}$$

with the rescaled parameter $p = \varepsilon^2 r$ and $j = 1, \ldots, N$ with periodic boundary conditions. Then, the solvability conditions up to the order ε^3 imply the following partial differential equation of Ginzburg-Landau type

$$\partial_{T_2} u = r\kappa_2 u + \frac{\kappa_3}{2} \partial_\xi^2 u + \zeta u \left| u \right|^2 \tag{11}$$

with periodic boundary conditions

$$u(\xi, T_2) = u(\xi + 1, T_2),$$

where $u(\xi, T_2)$ with $\xi \in [0, 1]$ is related to the amplitude A by

$$A(T_1, x_1, T_2) = u(\kappa_1 T_1 + x_1, T_2). \tag{12}$$

The coefficient κ_1 in (12) is given by (3) or (8), and

$$\kappa_2 = \left\langle L_K v_0, v_1 \right\rangle, \quad \kappa_3 = \left\langle L_2 v_0, v_1 \right\rangle. \tag{13}$$

Finally, the coefficient ζ of the cubic nonlinearity and the vector $v \in \mathbb{C}^n$ have to be determined by the nonlinearity h according to expressions (18) and (19), see Appendix.

According to this proposition, small solutions of a coupled system of the form (10) close to criticality, i.e. $p = O(\varepsilon^2)$ can be approximated in the form (9), where the amplitude A is related to a solution of the Ginzburg-Landau equation (11) via (12). Note that the relation (12) introduces a rotating frame on the timescale $T_1 = \varepsilon t$ with rotation velocity κ_1. κ_1 is called also the group velocity of the basic pattern [16]. The time evolution of the Ginzburg-Landau equation enters only on the slowest time scale $T_2 = \varepsilon^2 t$.

Remark 3. For the case of a symmetric coupling $M_m = M_{-m}$, $K_m = K_{-m}$, the rotation group velocity vanishes $\kappa_1 = 0$. This follows from Eq. (3) and $i\kappa_1 = \frac{\partial\lambda(\varphi_0)}{\partial\varphi_0} = -\frac{\partial\chi/\partial\varphi}{\partial\chi/\partial\lambda}$. Indeed, since the characteristic polynomial χ is real-valued for a symmetric coupling, its derivatives are also real-valued, and hence $\kappa_1 = 0$. Note that such a symmetric coupling induces an additional reflection symmetry $m \to -m$.

If, additionally to the symmetric coupling, the matrices M_m and K_m are symmetric, i.e. $M_m = M_m^T$ and $K_m = K_m^T$, then the coefficients κ_2 and κ_3 of the amplitude equation (11) are real. Indeed, in such a case, the characteristic matrix L_0 is symmetric and real implying that $\omega_0 = 0$ and $v_1 = v_0 = v_0^*$. Moreover

$$L_2 = \sum_{m>0} m^2 \left(e^{im\varphi_0} + e^{-im\varphi_0}\right) M_m(0) = 2 \sum_{m>0} m^2 \cos(m\varphi_0) M_m(0)$$

is real, similarly also L_K. This implies that κ_2 and κ_3 are real as well. Further symmetry properties determine whether the nonlinear term is real or complex. Finally note that, for the case $\varphi_0 = 0$, one obtains spatially homogeneous Hopf bifurcation.

4 Example: spatio-temporal dynamics in a ring of unidirectionally coupled Duffing oscillators

In this section, we illustrate the obtained result by considering a ring of unidirectionally coupled Duffing oscillators. A similar system has been studied theoretically and experimentally in Ref. [24], where the bifurcation mechanisms, leading subsequently to the destabilization of the homogeneous steady state, to the appearance of multiple stable periodic solutions, and to chaotic behavior, have been studied. We show that our approach allows to reveal the spatio-temporal features in the dynamics of this system, which can be traced back to the amplitude equation, derived in the previous section.

The autonomous Duffing oscillator is described by the following second order ordinary differential equation

$$\ddot{y} + d\dot{y} + ay + y^3 = 0, \tag{14}$$

where d and a are positive constants. System (14) is a single-well Duffing oscillator, which has a single equilibrium point at $y = \dot{y} = 0$. Due to the presence of damping $(d > 0)$ this equilibrium is an attractor for all initial conditions. We consider now a ring of N such oscillators with a linear unidirectional coupling to the next neighbor. Introducing new coordinates $x = y$, $z = \dot{y}$ and the coupling into Eq. (14), the equations of motion have the form

$$\begin{aligned} \dot{x}_j &= z_j, \\ \dot{z}_j &= -dz_j - ax_j - x_j^3 + k\left(x_{j+1} - x_j\right), \end{aligned} \tag{15}$$

where k is the coupling coefficient and indices are considered modulo N. As follows from the results in Ref. [23], for at least three coupled oscillators and increasing coupling strength k, one can observe rich dynamics starting from periodic oscillations to hyperchaos.

In the case of large N, we can use for system (15) the characteristic equation (2) for the asymptotic spectrum given by

$$\det\left[\lambda \mathrm{Id} - M_0 - e^{i\varphi} M_1\right] = 0, \tag{16}$$

where $M_0 = \begin{pmatrix} 0 & 1 \\ -a-k & -d \end{pmatrix}$ and $M_1 = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}$. Equation (16) can be solved explicitly with respect to λ:

$$\lambda_{1,2}(\varphi) = -\frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 - a - k\left(1 - e^{i\varphi}\right)}. \tag{17}$$

In the following we fix the $a = 0.1$ and $d = 0.3$. For these values a simple numerical analysis shows that the spectral curves (17) are stable (Re$\lambda < 0$) for all values of the coupling satisfying $k < k_0 \approx 0.1399$.

At $k = k_0$, the spectrum is tangent to the imaginary axis for $\varphi_0 \approx 1.245$ (cf. Fig. 2) leading to a purely imaginary eigenvalue $\lambda = i\omega_0$with $\omega_0 \approx 0.441$. For $k > k_0$, the zero steady state is unstable.

In a neighborhood of the destabilization at $k = k_0$, the dynamics of system (15) can be approximated by the normal form (11), and the correspondence between the solutions of the oscillator chain (15) and the normal form is given by Eq. (9). The leading terms in this approximation are

$$\begin{pmatrix} x_j(t) \\ z_j(t) \end{pmatrix} \approx \varepsilon e^{i\omega_0 t + i\varphi_0 j} v_0 A(\varepsilon t, \varepsilon j, \varepsilon^2 t) + c.c.$$

showing that the dynamics are given to leading order by a small amplitude wave with the temporal frequency ω_0, corresponding time period $P_t = 2\pi/\omega_0 \approx 14.25$, and the spatial wave number φ_0 corresponding to a wavelength of $2\pi/\varphi_0 \approx 5$. The amplitude of this wave is given by a solution of equation (11) inducing in this way modulations on a slow timescale and a large scale in space. Using formulas (8), (13), and (18), the coefficients of the normal form (11) for the chosen parameter values are $\kappa_1 = \kappa_3 = 0.15$, $\kappa_2 = 0.73 - 1.02i$, and $\zeta = -0.29 - 0.85i$. According to the numerical results on the one-dimensional Ginzburg-Landau equation on large domain [29, 2, 7], the obtained parameter values correspond to the spatio-temporal intermittency region, where, despite of the fact that there exist stable periodic patterns, a chaotic attractor is typically reached from a random initial condition, which is characterized by turbulent bursts interrupted by regions of periodic patterns. Moreover, the modulus of u is touching zero leading to the phase defects.

The chaotic behavior observed in the chain of oscillators (15), possesses all the above mentioned qualitative features. Indeed, as it has been shown in Ref. [24], the coexistence of the stable periodic patterns, as well as a chaotic intermittent patterns. In the following, we present some additional numerical results for the corresponding oscillator system. In particular, we will show the existence of the phase defects, which have not been mentioned in Ref. [24], but which are predicted by the corresponding Ginzburg-Landau system.

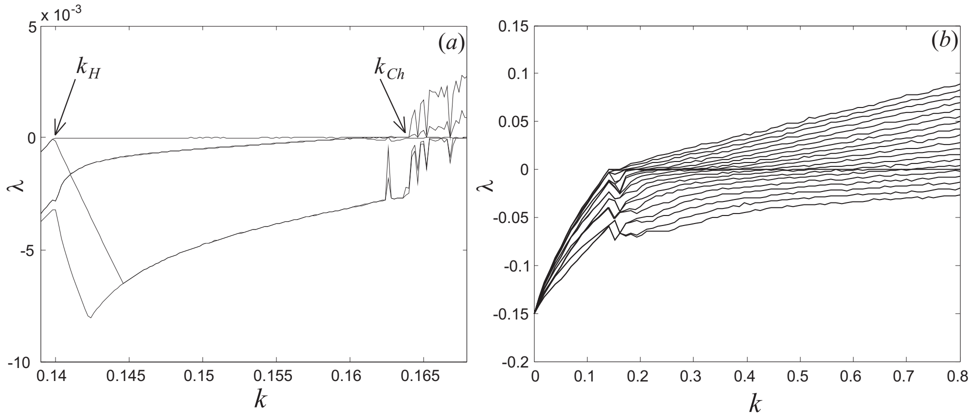

Figure 3: Lyapunov exponents for $N = 30$ unidirectionally coupled Duffing oscillators (a) five largest Lyapunov exponents, k_H and k_{Ch} indicate the Hopf bifurcation and the transition to chaos, $k \in (0.139, 0.17)$, (b) twenty largest Lyapunov exponents for $k \in (0.0, 0.8)$. At $k = 0.8$, there are fourteen positive Lyapunov exponents.

As already observed in Ref. [23], there is a coupling induced destabilization of the zero equilibrium. Increasing the coupling strength k, one can observe a transition to periodic oscillations and shortly after that to high dimensional chaos. In Fig. 3 we present the Lyapunov spectrum for $N = 30$ coupled oscillators and increasing coupling parameter k. Panel (a) shows a Hopf bifurcation close to $k = k_0$and

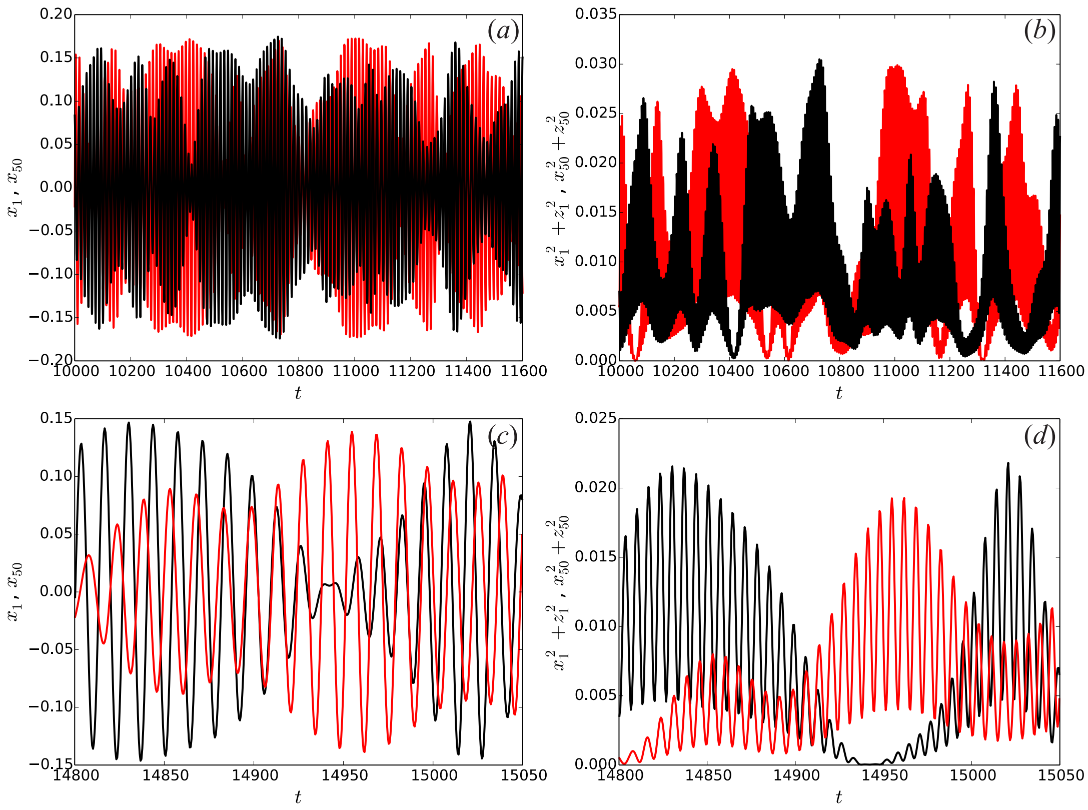

Figure 4: Numerical solutions for the ring of coupled Duffing oscillators (15) for $a = 0.1$, $d = 0.3$, $k = 0.15$, and $N = 200$. Slow modulations of $x_1(t)$ and $x_{50}(t)$ in (a) and of $x_1^2(t) + z_1^2(t)$ and $x_{50}^2(t) + z_{50}^2(t)$ in (c); corresponding fast oscillations in (b) and (d). Phase defect close to $t \approx 14950$, with a drop of the amplitude and a phase jump for $j = 1$.

a transition to chaos at $k_{Ch} = 0.164$ where two Lyapunov exponents become positive almost simultaneously. With further increasing coupling coefficient k (Fig. 3(b)) one can observe that more and more Lyapunov exponents become positive, indicating the presence of high dimensional spatio-temporal chaos [10, 12, 11]. For larger system size, the transition to chaos occurs at coupling values k_{Ch} closer to k_0 such that for $N = 200$ we are in the regime of spatio-temporal chaos already for $k = k_0 + 0.01$.

Figure 4 shows the corresponding time traces for two arbitrarily chosen oscillators in the chaotic regime. The panels (a) and (c) show the time traces $x_j(t)$ on a slow (a) and fast (c) time scale. Panels (b) and (d) show the corresponding amplitudes $x^2 + z^2$ of the oscillators. One can clearly observe fast oscillations with the frequency ω_0 and their slow modulation. The regime of collective macroscopic spatio-temporal chaos is characterized by the fact that the modulations of the time traces are chaotic on a large time scale and for two sufficiently distant oscillators they are not correlated. An interesting feature is the phase jump at $t \approx 14950$, where the amplitude $x_1^2 + z_1^2 \approx 0$ Fig. 4. This indicates so called *defect chaos*, a typical regime in the complex Ginzburg-Landau equation beyond the Benjamin-Feir instability, which is characterized by localized phase defects appearing irregularly in space and time, see e.g. Refs [4, 9]. A representation of the amplitude dynamics can be seen on the space-time plot in Fig. 5, where the numerical time traces $x_j(t)$ and $x_j^2(t) + z_j^2(t)$ are plotted for all oscillators $j = 1 \ldots 200$ (horizontal axis). The figure shows nicely the slow modulations of the amplitude, where the locations of the defects can be seen as the points where the amplitude of oscillations tends to zero. Moreover, one can see the convective motion given in the amplitude equation ansatz 12 by the rotating frame on the timescale

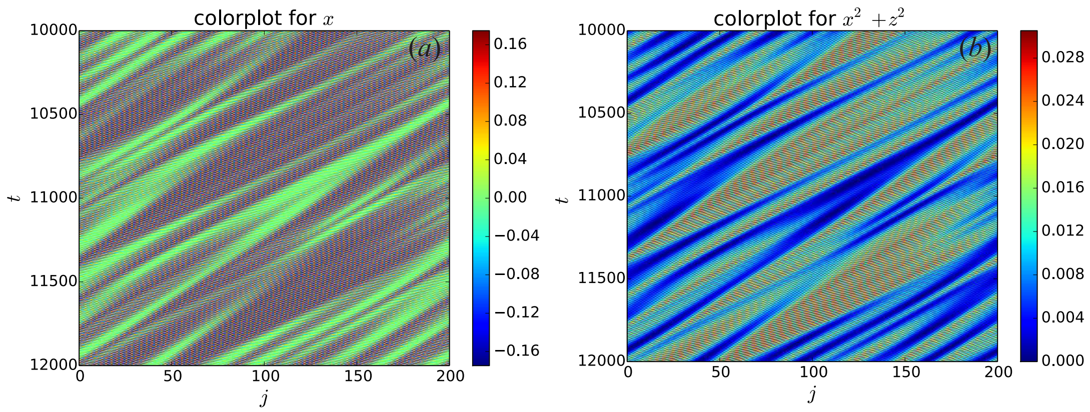

Figure 5: Spatio-temporal plots of the solutions $x_j(t)$ (a) and $x_j^2(t)+z_j^2(t)$ (b). Magnitude of the solution is shown in color. Phase defects at the spots where the amplitude approaches zero.

$T_1 = \varepsilon t$. Note that also the fast oscillations with the frequencies ω_0 (temporal) and φ_0 (spatial) can be also seen as a fine structure in the plot.

5 Proofs

Proof of Lemma 1. We have to show that κ_1, defined by (3), can be calculated as given in (8). To this end, we differentiate the eigenvalue equation (cf. (5))

$$\left[\lambda_0\left(\varphi\right)\mathrm{Id} - \sum_m e^{im\varphi}M_m(0)\right]v\left(\varphi\right) = 0,$$

with respect to φ:

$$\left[\mathrm{Id}\frac{\partial}{\partial\varphi}\lambda_0(\varphi) - i\sum_m me^{im\varphi}M_m(0)\right]v(\varphi) + \left[\lambda_0(\varphi)\mathrm{Id} - \sum_m e^{im\varphi}M_m(0)\right]\frac{\partial}{\partial\varphi}v(\varphi) = 0.$$

Evaluating the obtained expression at $\varphi = \varphi_0$, $\lambda(\varphi_0) = i\omega_0$, and $v(\varphi_0) = v_0$, we obtain

$$\left[\mathrm{Id}i\kappa_1 - iL_1\right]v_0 + \left[i\omega_0\mathrm{Id} - L_0\right]\frac{\partial v}{\partial\varphi}(\varphi_0) = 0.$$

The projection onto v_1 gives

$$i\kappa_1\left\langle v_0, v_1\right\rangle - i\left\langle L_1v_0, v_1\right\rangle + \left\langle\left[i\omega_0\mathrm{Id} - L_0\right]\frac{\partial v}{\partial\varphi}(\varphi_0), v_1\right\rangle = 0,$$

which implies

$$i\kappa_1 - i\left\langle L_1v_0, v_1\right\rangle + \left\langle\frac{\partial v}{\partial\varphi}(\varphi_0), \left[-i\omega_0\mathrm{Id} - L_0^*\right]v_1\right\rangle = 0.$$

Taking into account (6), we obtain the relation (8), which proves the Lemma.

Proof of Proposition 2. Substituting the multiple scale ansatz (9) into (10), we obtain

$$\varepsilon\frac{d}{dt}\left(e^{i\omega_0t+i\varphi_0j}v_0A + \varepsilon^2e^{3i(\omega_0t+\varphi_0j)}v_2A^3 + c.c.\right)$$

$$= \varepsilon \sum_m \left(M_m(0) + \varepsilon^2 r K_m\right) e^{i\omega_0 t + i\varphi_0 j} e^{im\varphi_0} v_0 A(T_1, x_1 + m\varepsilon, T_2)$$

$$+\varepsilon^3 \sum_m \left(M_m(0) + \varepsilon^2 r K_m\right) e^{3i(\omega_0 t + \varphi_0 j)} e^{3im\varphi_0} v_2 A^3(T_1, x_1 + m\varepsilon, T_2) + c.c.$$

$$+h(y_{j-R}, \ldots, y_{j+R}; \varepsilon^2 r)$$

Dividing the obtained equation by ε, expanding necessary arguments of A, we obtain up to the terms of the order ε^2 (the complex conjugated terms are omitted here for brevity)

$$e^{i\omega_0 t + i\varphi_0 j} v_0 \left(i\omega_0 A + \varepsilon \partial_{T_1} A + \varepsilon^2 \partial_{T_2} A\right) + 3i\omega_0 \varepsilon^2 e^{3i(\omega_0 t + \varphi_0 j)} v_2 A^3$$

$$= e^{i\omega_0 t + i\varphi_0 j} \sum_m \left(M_m(0) + \varepsilon^2 r K_m\right) e^{im\varphi_0} v_0 \left(A + m\varepsilon \partial_{x_1} A + \frac{1}{2} m^2 \varepsilon^2 \partial_{x_1}^2 A\right)$$

$$+\varepsilon^2 \sum_m M_m(0) e^{3i(\omega_0 t + \varphi_0 j)} e^{3im\varphi_0} v_2 A^3$$

$$+\varepsilon^2 e^{i\omega_0 t + i\varphi_0 j} A \left|A\right|^2 h_1\left(v_0\right) + \varepsilon^2 e^{3i(\omega_0 t + \varphi_0 j)} A^3 h_2\left(v_0\right)$$

where $h_1\left(v_0\right)$ and $h_2\left(v_0\right)$ are determined by the leading terms in the expansion of the nonlinearity. Note that due to our assumption, h is of third order and the leading order terms are given by expanding $h(y_{j-R}, \ldots, y_{j+R}; 0)$ in the homogeneous state $y_m = \alpha v_0 + c.c, \quad m = j-R, \ldots, j+R$ with respect to $\alpha \in \mathbb{C}$. The solvability condition requires that different harmonics as well as different orders of ε up to ε^2 are equal. We start with the first harmonic. By equating the terms containing $e^{i\omega_0 t + i\varphi_0 j}$ we obtain the following equation

$$v_0 \left(i\omega_0 A + \varepsilon \partial_{T_1} A + \varepsilon^2 \partial_{T_2} A\right) =$$

$$= \sum_m \left(M_m(0) + \varepsilon^2 r K_m\right) e^{im\varphi_0} v_0 \left(A + m\varepsilon \partial_{x_1} A + \frac{1}{2} m^2 \varepsilon^2 \partial_{x_1}^2 A\right) + \varepsilon^2 A \left|A\right|^2 h_1\left(v_0\right).$$

Since it should be satisfied for all ε, we first consider the ε^0 equation

$$i\omega_0 A v_0 = A L_0 v_0,$$

which holds according to the spectral condition (5). The ε^1 terms result into

$$v_0 \partial_{T_1} A - L_1 v_0 \partial_{x_1} A = 0.$$

Multiplication with v_1^T and using (8) from Lemma 1, we obtain

$$\partial_{T_1} A - \kappa_1 \partial_{x_1} A = 0.$$

This will be accounted for by introducing the new amplitude u by

$$u(\xi, T_2) = u(\kappa_1 T_1 + x_1, T_2) = A(T_1, x_1, T_2)$$

in a correspondingly rotating coordinate $\xi = \kappa_1 T_1 + x_1$. Finally, the ε^2 terms result into

$$v_0 \partial_{T_2} A = r A L_K v_0 + \frac{1}{2} \partial_{x_1}^2 A L_2 v_0 + A \left|A\right|^2 h_1\left(v_0\right)$$

Note that the dependence on T_1 does not show up in this equation. Hence, after multiplication with v_1^T, we can write it in terms of u as

$$\partial_{T_2} u = r\kappa_2 u + \frac{\kappa_3}{2}\partial_\xi^2 u + \zeta u \left|u\right|^2 ,$$

where

$$\zeta = \left\langle h_1\left(v_0\right), v_1\right\rangle . \tag{18}$$

Finally, it is simple to check that the solvability of the terms for the third harmonic leads to the expression

$$v_2 = \left[3i\omega_0 - \sum_m M_m(0)e^{3im\varphi_0}\right]^{-1} h_2\left(v_0\right) \tag{19}$$

Here, the existence of a nonzero solution v_2 is guaranteed by the non-resonance condition. Indeed, since the points $\pm 3i\omega_0$ do not belong to the asymptotic continuous spectrum with $\varphi = \varphi_0$, the matrix $\left[3i\omega_0 - \sum_m M_m(0)e^{3im\varphi_0}\right]$ is non-singular. Finally, note that the set of equations should be complemented by periodic boundary conditions in ξ, taking into account the ring structure of system (10). The proposition is proved.

6 Conclusions

In this paper, we studied coupling induced instabilities in large arrays of coupled systems. It turned out that also in such systems the Ginzburg-Landau equation can be used as a normal form for spatio-temporal dynamics close to the destabilization of a homogeneous equilibrium. This approach is classical for reaction-diffusion systems [6, 31] in continuous media and recently has been also applied to systems with large delay [33]. For discrete systems, a similar behavior should be obviously expected in systems resulting from a spatial discretization of a diffusive coupling in continuous media. However, our results show that in coupled oscillator systems applicability of the Ginzburg-Landau equation goes far beyond these obvious generalizations and can also be used to describe the dynamics in much more general systems e.g. with unidirectional, i.e. purely convective coupling or even with anti-diffusive interaction. In this sense, the class of coupled oscillators that we treated in this paper differs substantially from discrete analogs of the classical results for continuous media.

We illustrated our theoretical results by a detailed study of a ring of unidirectionally coupled Duffing oscillators in the regime of spatio-temporal chaos. Using the insight obtained from the amplitude equation, one can clearly describe the collective motion as a slow modulation of a rapidly oscillating plane wave as a basic solution. According to the amplitude equation, the spatio-temporal chaos observed in the slow dynamics of the oscillator system for suitably chosen parameters, could be identified as defect chaos.

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