Weierstraß-Institut
für Angewandte Analysis und Stochastik
Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Extremes of the supercritical Gaussian Free Field

Alberto Chiarini[1], Alessandra Cipriani[2], Rajat Subhra Hazra[3]

submitted: May 4, 2015

[1] Technische Universität Berlin
MA 766
Straße des 17. Juni 136
10623 Berlin
Germany
E-Mail: chiarini@math.tu-berlin.de

[2] Weierstrass Institute
Mohrenstraße 39
10117 Berlin
Germany
E-Mail: Alessandra.Cipriani@wias-berlin.de

[3] Theoretical Statistics and Mathematics Unit
Indian Statistical Institute
203 B.T. Road
Kolkata, 700108
India
E-Mail: rajatmaths@gmail.com

No. 2105
Berlin 2015

2010 *Mathematics Subject Classification.* 60K35, 60G15, 60G60, 60G70.

Key words and phrases. Discrete Gaussian free field, extreme value theory, Gumbel distribution, Stein-Chen method.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

ABSTRACT. We show that the rescaled maximum of the discrete Gaussian Free Field (DGFF) in dimension larger or equal to 3 is in the maximal domain of attraction of the Gumbel distribution. The result holds both for the infinite-volume field as well as the field with zero boundary conditions. We show that these results follow from an interesting application of the Stein-Chen method from Arratia et al. (1989).

1. INTRODUCTION

In this article we consider the problem of determining the scaling limit of the maximum of the discrete Gaussian free field (DGFF) on $\mathbb{Z}^d$, $d \geq 3$. Recently the maximum of the DGFF in the critical dimension $d = 2$ was resolved in Bramson et al. (2013). In this case, due to the presence of the logarithmic behavior of covariances, the problem is connected to extremes of various other models, for example the Branching Brownian motion and the Branching random walk. In $d \geq 3$, the presence of long-range dependence decaying polynomially changes the setting but the behavior of maxima is still hard to determine (Chatterjee, 2014, Section 9.6). This dependence also becomes a hurdle in various properties of level set percolation of the DGFF which were exhibited in a series of interesting works (Drewitz and Rodriguez (2013), Rodriguez and Sznitman (2013), Sznitman (2012)). The behavior of local extremes in the critical dimension has also been unfolded recently in the papers Biskup and Louidor (2013, 2014).

We consider the lattice $\mathbb{Z}^d$, $d \geq 3$ and take the infinite-volume Gaussian free field $(\varphi_\alpha)_{\alpha \in \mathbb{Z}^d}$ with law P on $\mathbb{R}^{\mathbb{Z}^d}$. The covariance structure of the field is given by the Green's function g of the standard random walk, namely $\mathrm{E}\left[\varphi_\alpha \varphi_\beta\right] = g(\alpha - \beta)$, for $\alpha, \beta \in \mathbb{Z}^d$. For more details of the model we refer to Section 2. It is well- known (see for instance Lawler (1991)) that for $\alpha \neq \beta$, $g(\alpha - \beta)$ behaves likes $\|\alpha - \beta\|^{2-d}$ and hence for $\|\alpha - \beta\| \to +\infty$, the covariance goes to zero. However this is not enough to conclude that the scaling is the same of an independent ensemble. To give an example where this is not the case, when V_N is the box of volume N, $\sum_{\alpha \in V_N} \varphi_\alpha$ is of order $N^{1/2+1/d}$, unlike the i. i. d. setting (see for example Funaki (2005, Section 3.4)).

The expected maxima over a box of volume N behaves like $\sqrt{2g(0)\log N}$. An independent proof of this fact is provided in Proposition 4 below; this confirms the idea that the extremes of the field resemble that of independent $\mathcal{N}(0, g(0))$ random variables. In this article we show that the similarity is even deeper, since the fluctuations of the maximum after recentering and scaling converge to a Gumbel distribution. Note that in $d = 2$ the limit is also Gumbel, but with a random shift (see Bramson et al. (2013, Theorem 2.5), Biskup and Louidor (2013)). The main results of this article is the following.

Theorem 1. *Let A be a subset of $\mathbb{Z}^d$ with $|A| = N$[a]. We set centering and scaling as follows:*

$$b_N = \sqrt{g(0)}\left[\sqrt{2\log N} - \frac{\log\log N + \log(4\pi)}{2\sqrt{2\log N}}\right] \quad \text{and} \quad a_N = g(0)(b_N)^{-1} \tag{1}$$

so that for all $z \in \mathbb{R}$

$$\lim_{N \to +\infty} \mathrm{P}\left(\frac{\max_{\alpha \in A}\varphi_\alpha - b_N}{a_N} < z\right) = \exp(-\mathrm{e}^{-z})$$

and the convergence is uniform in z.

Note that scaling and centering are exactly the same as in the i. i. d. $\mathcal{N}(0, g(0))$ case, see for example Hall (1982). As in $d = 2$, the argument depends on a comparison lemma. We show that in fact the proof is an interesting application of a Stein-Chen approximation result. Not only does the result depend on the correlation decay, but also crucially on the Markov property of the Gaussian free field. We use Theorem 1

[a] $|A|$ denotes the cardinality of A.

of the paper by Arratia et al. (1989) which approximates an appropriate dependent Binomial process with a Poisson process, and gives some calculable error terms. In general showing that the error terms go to zero is a non-trivial task. In the DGFF case, thanks to estimates on the Green's function and the Markov property, the error terms are negligible.

The techniques used for the infinite-volume DGFF allows us to draw conclusions also for the field with boundary conditions. For $n > 0$ let $N := n^d$; we consider the discrete hypercube $V_N := [0, n-1]^d \cap \mathbb{Z}^d$. We define therein a mean zero Gaussian field $(\psi_\alpha)_{\alpha \in \mathbb{Z}^d}$ whose covariance matrix $(g_N(\alpha, \beta))_{\alpha, \beta \in V_N}$ is the Green's function of the discrete Laplacian with Dirichlet boundary conditions outside V_N (again for a more precise definition see Section 2). The convergence result is the following:

Theorem 2. *Let V_N be as above and $(\psi_\alpha)_{\alpha \in \mathbb{Z}^d}$ be a DGFF with zero boundary conditions outside V_N with law $\widetilde{\mathrm{P}}_{V_N}$. Let the centering and scaling be as in (1). Then for all $z \in \mathbb{R}$*

$$\lim_{N \to +\infty} \widetilde{\mathrm{P}}_{V_N} \left(\frac{\max_{\alpha \in V_N} \psi_\alpha - b_N}{a_N} < z \right) = \exp(-\mathrm{e}^{-z}).$$

The core of the proof is an application of Slepian's Lemma and a re-run of the proof of Theorem 1.

The structure of the article is as follows. In Section 2 we recall the main facts on the DGFF that will be used in Section 3 to prove the main theorem.

2. Preliminaries on the DGFF

Let $d \geq 3$ and denote with $\|\cdot\|$ the ℓ_∞-norm on the lattice. Let $\psi = (\psi_\alpha)_{\alpha \in \mathbb{Z}^d}$ be a discrete Gaussian Free Field with zero boundary conditions outside $\Lambda \subset \mathbb{Z}^d$. On the space $\Omega := \mathbb{R}^{\mathbb{Z}^d}$ endowed with its product topology, its law $\widetilde{\mathrm{P}}_\Lambda$ can be explicitly written as

$$\widetilde{\mathrm{P}}_\Lambda(\mathrm{d}\,\psi) = \frac{1}{Z_\Lambda} \exp\left(-\frac{1}{2d} \sum_{\alpha, \beta \in \mathbb{Z}^d:\, \|\alpha - \beta\| = 1} (\psi_\alpha - \psi_\beta)^2 \right) \prod_{\alpha \in \Lambda} \mathrm{d}\,\psi_\alpha \prod_{\alpha \in \mathbb{Z}^d \setminus \Lambda} \delta_0(\mathrm{d}\,\psi_\alpha).$$

In other words $\psi_\alpha = 0$ $\widetilde{\mathrm{P}}_\Lambda$-a. s. if $\alpha \in \mathbb{Z}^d \setminus \Lambda$, and $(\psi_\alpha)_{\alpha \in \Lambda}$ is a multivariate Gaussian random variable with mean zero and covariance $(g_\Lambda(\alpha, \beta))_{\alpha, \beta \in \mathbb{Z}^d}$, where g_Λ is the Green's function of the discrete Laplacian problem with Dirichlet boundary conditions outside Λ. For a thorough review on the model the reader can refer for example to Sznitman (2012). It is known (Georgii, 1988, Chapter 13) that the finite-volume measure ψ admits an infinite-volume limit as $\Lambda \uparrow \mathbb{Z}^d$ in the weak topology of probability measures. This field will be denoted as $\varphi = (\varphi_\alpha)_{\alpha \in \mathbb{Z}^d}$. It is a centered Gaussian field with covariance matrix $g(\alpha, \beta)$ for $\alpha, \beta \in \mathbb{Z}^d$. With a slight abuse of notation, we write $g(\alpha - \beta)$ for $g(0, \alpha - \beta)$ and also $g_\Lambda(\alpha) = g_\Lambda(\alpha, \alpha)$. It will be convenient for us to view g through its random walk representation: if $\mathbb{P}_\alpha$ denotes the law of a simple random walk S started at $\alpha \in \mathbb{Z}^d$, then

$$g(\alpha, \beta) = \mathbb{E}_\alpha \left[\sum_{n \geq 0} \mathbb{1}_{\{S_n = \beta\}} \right].$$

In particular this gives $g(0) < +\infty$ for $d \geq 3$.

A key fact for the Gaussian Free Field is its spatial Markov property, which will be used in the paper. The proof of the following Lemma can be found in Rodriguez and Sznitman (2013, Lemma 1.2).

Lemma 3 (Markov property of the Gaussian Free Field)**.** *Let* $\emptyset \neq K \Subset \mathbb{Z}^{d}$[b], $U := \mathbb{Z}^d \setminus K$ *and define* $(\widetilde{\varphi}_\alpha)_{\alpha \in \mathbb{Z}^d}$ *by*

$$\varphi_\alpha = \widetilde{\varphi}_\alpha + \mu_\alpha, \quad \alpha \in \mathbb{Z}^d$$

where μ_α *is the* $\sigma(\varphi_\beta,\, \beta \in K)$*-measurable map defined as*

$$\mu_\alpha = \sum_{\beta \in K} \mathbb{P}_\alpha \left(H_K < +\infty,\, S_{H_K} = \beta \right) \varphi_\beta, \quad \alpha \in \mathbb{Z}^d. \tag{2}$$

Here $H_K := \inf\left\{ n \geq 0 :\ S_n \in K \right\}$. *Then, under* P, $(\widetilde{\varphi}_\alpha)_{\alpha \in \mathbb{Z}^d}$ *is independent of* $\sigma(\varphi_\beta,\, \beta \in K)$ *and distributed as* $(\psi_\alpha)_{\alpha \in \mathbb{Z}^d}$ *under* $\widetilde{\mathsf{P}}_U$.

As an immediate consequence of the Lemma (see Rodriguez and Sznitman (2013, Remark 1.3))

$$\mathsf{P}\left((\varphi_\alpha)_{\alpha \in \mathbb{Z}^d} \in \cdot \,|\sigma(\varphi_\beta,\, \beta \in K) \right) = \widetilde{\mathsf{P}}_U \left((\psi_\alpha + \mu_\alpha)_{\alpha \in \mathbb{Z}^d} \in \cdot \right) \quad \mathsf{P} - a.s.$$

where μ_α is given in (2), $\widetilde{\mathsf{P}}_U$ does not act on $(\mu_\alpha)_{\alpha \in \mathbb{Z}^d}$ and $(\psi_\alpha)_{\alpha \in \mathbb{Z}^d}$ has the law $\widetilde{\mathsf{P}}_U$.

2.0.1. *Law of large numbers of the recentered maximum.* Although this can be obtained directly by Theorem 1, we think it is interesting to insert an independent proof of the behavior of the maximum of the DGFF.

Proposition 4 (LLN for the maximum)**.** *Let* $V_N := [0,\, n-1]^d \cap \mathbb{Z}^d$, $n := N^d > 0$. *The following limit holds:*

$$\lim_{N \to +\infty} \frac{\mathsf{E}\left[\max_{\alpha \in V_N} \varphi_\alpha \right]}{\sqrt{2 \log N}} = g(0).$$

Proof. Observe first that under the assumptions of the theorem $g(0) \geq 1$ (Lawler, 1991, Exercise 1.5.7). The upper bound follows from Talagrand (2003, Prop. 1. 1. 3) with $\tau := g(0)$ and $M := N$. As for the lower bound, we will use Sudakov-Fernique inequality (Adler and Taylor, 2007, Theorem 2. 2. 3). We first need a lower bound for $d(\alpha,\, \beta) := \sqrt{\mathsf{E}\left[(\varphi_\alpha - \varphi_\beta)^2 \right]}$: we will apply here the bound

$$g(\alpha) \leq \left(\frac{c\sqrt{d}}{\|\alpha\|} \right)^{d-2}, \quad \|\alpha\| \geq d \tag{3}$$

whose proof is provided in Sznitman (2011). The key to obtain the result is to use a diluted version of the DGFF as follows. Consider $V_N^{(k)} := V_N \cap k\mathbb{Z}^d$, where $k := \lfloor \log n \rfloor \in \{1,\, 2,\, \ldots\}$ is a constant. Without loss of generality we can assume also that n is large enough so that

$$g(0) - \left(\frac{c\sqrt{d}}{\lfloor \log n \rfloor} \right)^{d-2} > 0 \iff n \geq \left\lceil \exp\left(\frac{c\sqrt{d}}{g(0)^{\frac{1}{d-2}}} \right) \right\rceil \tag{4}$$

Note the fact that

$$\mathsf{E}\left[\max_{\alpha \in V_N} \varphi(\alpha) \right] \geq \mathsf{E}\left[\max_{\alpha \in V_N^{(k)}} \varphi_\alpha \right]. \tag{5}$$

[b] $A \Subset B$ means that A is a finite subset of B.

Now for $\alpha,\,\beta \in T := V_N^{(k)}$ and $k > d$

$$d(\alpha,\,\beta) = \sqrt{2g(0) - 2g(\alpha-\beta)} \overset{(3)}{\geq} \sqrt{2}\sqrt{g(0) - \left(\frac{c\sqrt{d}}{\|\alpha-\beta\|}\right)^{d-2}}$$

$$\geq \sqrt{2}\sqrt{g(0) - \left(\frac{c\sqrt{d}}{\lfloor\log n\rfloor}\right)^{d-2}} =: \nu(n,\,d) \overset{(4)}{>} 0.$$

Notice also that $\lim_{N\to+\infty}\nu(n,\,d) = \sqrt{2g(0)}$. Hence by (5) and an application of Sudakov-Fernique inequality

$$\frac{\mathsf{E}\left[\max_{\alpha\in V_N}\varphi_\alpha\right]}{\sqrt{\log N}} \geq \nu(n,\,d)\sqrt{\frac{\log|T|}{\log N}}.$$

We obtain $\log|T| = d\log\left\lfloor\frac{n}{k}\right\rfloor(1+\mathrm{o}(1)) = d\log\left\lfloor\frac{n}{\lfloor\log n\rfloor}\right\rfloor(1+\mathrm{o}(1))$[c]. It follows that $\frac{\log|T|}{\log N} = 1+\mathrm{o}(1)$ and

$$\lim_{N\to+\infty}\frac{\mathsf{E}\left[\max_{\alpha\in V_N}\varphi_\alpha\right]}{\sqrt{\log N}} \geq \sqrt{2g(0)}.$$

□

3. Proof of the main result

The proof of the main result is an application of the Stein-Chen method. To keep the article self contained we recall the result from Arratia et al. (1989).

3.1. **Poisson approximation for extremes of random variables.** The main tool we will use relies on a two-moment condition to determine the convergence of the number of exceedances for a sequence of random variables. Let $(X_\alpha)_{\alpha\in\mathcal{A}}$ be a sequence of (possibly dependent) Bernoulli random variables of parameter p_α. Let $W := \sum_{\alpha\in\mathcal{A}} X_\alpha$ and $\lambda := \mathsf{E}[W]$. Now for each α we define a subset $B_\alpha \subseteq \mathcal{A}$ which we consider a "neighborhood" of dependence for the variable X_α, such that X_α is nearly independent from X_β if $\beta \in A\setminus B_\alpha$. Set

$$b_1 := \sum_{\alpha\in A}\sum_{\beta\in B_\alpha} p_\alpha p_\beta,$$

$$b_2 := \sum_{\alpha\in A}\sum_{\alpha\neq\beta\in B_\alpha}\mathsf{E}\left[X_\alpha X_\beta\right],$$

$$b_3 := \sum_{\alpha\in A}\mathsf{E}\left[\left|\mathsf{E}\left[X_\alpha - p_\alpha \mid \mathcal{H}_1\right]\right|\right]$$

where

$$\mathcal{H}_1 := \sigma\left(X_\beta :\ \beta\in A\setminus B_\alpha\right).$$

Theorem 5 (Theorem 1, Arratia et al. (1989)). *Let Z be a Poisson random variable with $\mathsf{E}[Z] = \lambda$ and let d_{TV} denote the total variation distance between probability measures. Then*

$$d_{TV}(\mathcal{L}(W),\,\mathcal{L}(Z)) \leq 2(b_1+b_2+b_3)$$

and

$$\left|P(W=0) - \mathrm{e}^{-\lambda}\right| < \min\left\{1,\,\lambda^{-1}\right\}(b_1+b_2+b_3).$$

[c] $f(N) = \mathrm{o}(1)$ means $\lim_{N\to+\infty} f(N) = 0$.

Let $A \Subset \mathbb{Z}^d$ with $N := |A|$, $u_N(z) := a_N z + b_N$, and define for all $\alpha \in A$

$$X_\alpha = \mathbb{1}_{\{\varphi_\alpha > u_N(z)\}} \sim Be(p).$$

A standard tool to determine the asymptotic of p is Mills ratio:

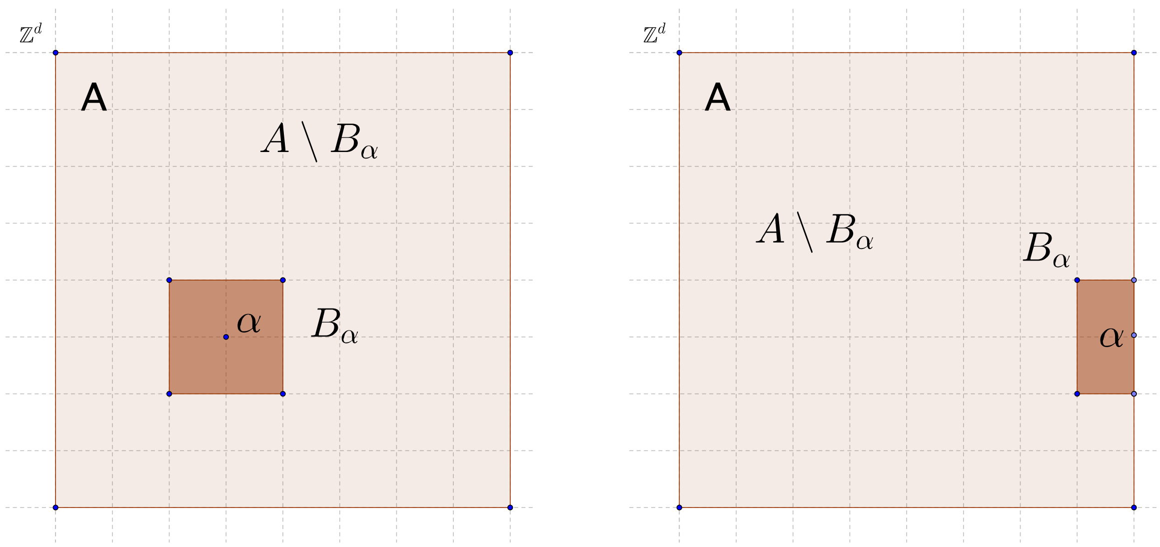
$$\left(1 - \frac{1}{t^2}\right) \frac{\mathrm{e}^{-t^2/2}}{\sqrt{2\pi} t} \leq \mathrm{P}\left(\mathcal{N}(0, 1) > t\right) \leq \frac{\mathrm{e}^{-t^2/2}}{\sqrt{2\pi} t}, \quad t > 0. \tag{6}$$

This yields $p \sim N^{-1} \exp(-z)$[d]. We furthermore introduce $W := \sum_{\alpha=1}^{N} X_\alpha$ and see that $\mathrm{E}\left[W\right] \sim \mathrm{e}^{-z}$. Of course W is closely related to the maximum since $\{\max_{\alpha \in A} \varphi_\alpha \leq u_N(z)\} = \{W = 0\}$. We will now fix $z \in \mathbb{R}$ and $\lambda := \mathrm{e}^{-z}$. We are now ready to prove our main result.

Proof. Our main idea is to apply Theorem 5. The proof will first show that the limit is Gumbel, and in the second part we will prove uniform convergence. To this scope we define, for a fixed but small $\epsilon > 0$,

$$B_\alpha := B\left(\alpha, (\log N)^{2+2\epsilon}\right) \cap A$$

where $B(\alpha, L)$ denotes the ball of center α of radius L in the ℓ_∞-distance. We draw below examples of such neighborhoods when $\alpha \in \partial A := \left\{\gamma \in A : \exists \beta \in \mathbb{Z}^d \setminus A, \, \|\beta - \gamma\| = 1\right\}$ and $\alpha \in \mathrm{int}(A) = A \setminus \partial A$.



(A) B_α when $\alpha \in \mathrm{int}(A)$.

(B) B_α when $\alpha \in \partial A$.

FIGURE 1. Examples of B_α

[d] $f \sim g$ means that $\lim_{N \to +\infty} f(N)/g(N) = 1$.

Convergence. The method is based on the estimate of three terms (cf. Subsec. 3.1).

i. $b_1 = \sum_{\alpha\in A}\sum_{\beta\in B_\alpha} p^2$. Using Mills ratio we have

$$\begin{aligned} b_1 &\le cN(\log N)^{d(2+2\epsilon)}\left(\frac{\sqrt{g(0)}\,\mathrm{e}^{-\frac{1}{2g(0)}u_N(z)^2}}{\sqrt{2\pi}u_N(z)}\right)^2 \\ &= N^{-1}(\log N)^{d(2+2\epsilon)}\,\mathrm{e}^{-2z+\mathrm{o}(1)} = \mathrm{o}(1). \end{aligned} \tag{7}$$

ii. $b_2 = \sum_{\alpha\in A}\sum_{\alpha\neq\beta\in B_\alpha} \mathsf{E}\left[X_\alpha X_\beta\right]$. First we need to estimate the joint probability

$$\mathsf{P}\left(\varphi_\alpha > u_N(z), \varphi_\beta > u_N(z)\right).$$

Denote the covariance matrix

$$\Sigma_2 = \begin{bmatrix} g(0) & g(\alpha-\beta) \\ g(\alpha-\beta) & g(0) \end{bmatrix}$$

Note that, for $w\in\mathbb{R}^2$, one has

$$w^t\Sigma_2^{-1}w = \frac{1}{g(0)^2-g(\alpha-\beta)^2}\left(g(0)\left(w_1^2+w_2^2\right)-2g(\alpha-\beta)w_1w_2\right).$$

Using $\mathbf{1} := (1,1)^t$ we denote by

$$\Delta_i := u_N(z)\left(\mathbf{1}^t\Sigma_2^{-1}\right)_i = \frac{u_N(z)(g(0)-g(\alpha-\beta))}{g(0)^2-g(\alpha-\beta)^2} = \frac{u_N(z)}{g(0)+g(\alpha-\beta)},\quad i=1,2.$$

Exploiting an easy upper bound on bi-variate Gaussian tails (see Savage (1962)) we have

$$\begin{aligned} &\mathsf{P}(\varphi_\alpha > u_N(z), \varphi_\beta > u_N(z)) \le \frac{1}{2\pi}\frac{1}{|\det\Sigma_2|^{1/2}\Delta_1\Delta_2}\exp\left(-\frac{u_N(z)^2}{2}\mathbf{1}^t\Sigma_2^{-1}\mathbf{1}\right) \\ &= \frac{1}{2\pi}\frac{(g(0)+g(\alpha-\beta))^2}{(g(0)^2-g(\alpha-\beta)^2)^{1/2}u_N(z)^2}\exp\left(-\frac{u_N(z)^2}{2}\frac{2(g(0)-g(\alpha-\beta))}{g(0)^2-g(\alpha-\beta)^2}\right) \\ &\le \frac{1}{4\pi\log N}\frac{\left(1+\frac{g(\alpha-\beta)}{g(0)}\right)^{3/2}}{\left(1-\frac{g(\alpha-\beta)}{g(0)}\right)^{1/2}}N^{-\frac{2g(0)}{g(0)+g(\alpha-\beta)}}\left(4\pi\log N\right)^{\frac{g(0)}{g(0)+g(\alpha-\beta)}}\,\mathrm{e}^{-\frac{2g(0)z}{g(0)+g(\alpha-\beta)}+\mathrm{o}(1)} \\ &\le \frac{\left(1+\frac{g(\alpha-\beta)}{g(0)}\right)^{3/2}}{\left(1-\frac{g(\alpha-\beta)}{g(0)}\right)^{1/2}}N^{-\frac{2g(0)}{g(0)+g(\alpha-\beta)}}\,\mathrm{e}^{-\frac{2g(0)z}{g(0)+g(\alpha-\beta)}+\mathrm{o}(1)} \end{aligned}$$

where in the second-to-last inequality we used $u_N(z)^2 = b_N^2 + 2g(0)z + g(0)^2z^2/b_N^2$ and the bound of b_N^2 (Hall, 1982, Equation 3)

$$g(0)(2\log N - \log\log N - \log 4\pi) \le b_N^2 \le 2g(0)\log N.$$

Also note that for $x\neq 0$, $g(\|x\|)/g(0) \le g(e_1)/g(0) = 1-\kappa$ where $\kappa := \mathbb{P}_0\left(\widetilde{H}_0 = +\infty\right) \in (0,1)$ and $\widetilde{H}_0 = \inf\{n\ge 1:\ S_n = 0\}$. Hence we have that

$$\frac{g(0)}{g(0)+g(\alpha-\beta)} \ge \frac{1}{2-\kappa} \qquad\text{and}\qquad \frac{g(\alpha-\beta)}{g(0)+g(x-y)} \le 1-\kappa.$$

We obtain thus

$$\mathsf{P}\left(\varphi_\alpha > u_N(z), \varphi_\beta > u_N(z)\right) \le \frac{(2-\kappa)^{3/2}}{\kappa^{1/2}}N^{-\frac{2}{(2-\kappa)}}\max\left(\mathrm{e}^{-2z}\,\mathbb{1}_{\{z\le 0\}}, \mathrm{e}^{-2z/(2-\kappa)}\,\mathbb{1}_{\{z>0\}}\right).$$

We get finally for some constants $c,\, c' > 0$ depending only on d and κ

$$\begin{aligned} b_2 &\le cN(\log N)^{d(2+2\epsilon)} \frac{(2-\kappa)^{3/2}}{\kappa^{1/2}} N^{-\frac{2}{(2-\kappa)}} \max\left(\mathrm{e}^{-2z}\,\mathbb{1}_{\{z\le 0\}}, \mathrm{e}^{-2z/(2-\kappa)}\,\mathbb{1}_{\{z>0\}}\right) \\ &\le c' N^{-\frac{\kappa}{(2-\kappa)}} (\log N)^{d(2+2\epsilon)} \max\left(\mathrm{e}^{-2z}\,\mathbb{1}_{\{z\le 0\}}, \mathrm{e}^{-2z/(2-\kappa)}\,\mathbb{1}_{\{z>0\}}\right). \end{aligned} \tag{8}$$

Since $\kappa/(2-\kappa) > 0$ we have that $b_2 = o(1)$.
iii. $b_3 = \sum_{\alpha\in A} \mathsf{E}\left[\left|\mathsf{E}\left[X_\alpha - p_\alpha \mid \mathcal{H}_1\right]\right|\right]$. It will be convenient to introduce also another σ-algebra which strictly contains $\mathcal{H}_1 = \sigma\left(X_\beta :\, \beta \in A\setminus B_\alpha\right)$, that is

$$\mathcal{H}_2 = \sigma\left(\varphi_\beta :\, \beta \in A\setminus B_\alpha\right).$$

Using the tower property of the conditional expectation and Jensen's inequality

$$\begin{aligned} \mathsf{E}\left[\left|\mathsf{E}\left[X_\alpha - p \mid \mathcal{H}_1\right]\right|\right] &= \mathsf{E}\left[\left|\mathsf{E}\left[\mathsf{E}\left[X_\alpha - p \mid \mathcal{H}_2\right] \mid \mathcal{H}_1\right]\right|\right] \\ &\le \mathsf{E}\left[\mathsf{E}\left[\left|\mathsf{E}\left[X_\alpha - p \mid \mathcal{H}_2\right]\right| \mid \mathcal{H}_1\right]\right] = \mathsf{E}\left[\left|\mathsf{E}\left[X_\alpha - p \mid \mathcal{H}_2\right]\right|\right]. \end{aligned}$$

At this point we recognize, thanks to Corollary 3, that

$$\mathsf{E}\left[X_\alpha \mid \mathcal{H}_2\right] = \widetilde{\mathsf{P}}_{\mathbb{Z}^d\setminus(A\setminus B_\alpha)}(\psi_\alpha + \mu_\alpha > u_N(z)) \quad \mathsf{P}-a.s.$$

where $(\psi_\alpha)_{\alpha\in\mathbb{Z}^d}$ is a Gaussian Free Field with zero boundary conditions outside $A\setminus B_\alpha$. In addition, observe that $g_{U_\alpha}(\alpha) \le g(0)$ (Lawler, 1991, Section 1.5). We will write more compactly $U_\alpha := \mathbb{Z}^d\setminus(A\setminus B_\alpha)$.

We will make use of the fact that μ_α is a centered Gaussian, and apply the same estimates of Popov and Ráth (2013): first we make use of the strong Markov property. Denoting by $S_\cdot \circ \theta_m = S_{m+\cdot}$ the time shift by m of the random walk, we observe that for $\beta \in A\setminus B_\alpha$

$$\begin{aligned} g(\alpha,\beta) &= \mathbb{E}_\alpha\left[\sum_{n\ge 0}\mathbb{1}_{\{S_n=\beta\}}\right] = \mathbb{E}_\alpha\left[\left(\sum_{n\ge 0}\mathbb{1}_{\{S_n=\beta\}}\right)\circ\theta_{H_{A\setminus B_\alpha}}\right] \\ &= \mathbb{E}_\alpha\left[\mathbb{E}_{S_{H_{A\setminus B_\alpha}}}\left[\sum_{n\ge 0}\mathbb{1}_{\{S_n=\beta\}}\right]\right] = \mathbb{E}_\alpha\left[g\left(S_{H_{A\setminus B_\alpha}},\beta\right),\, H_{A\setminus B_\alpha} < +\infty\right] \\ &= \sum_{\gamma\in A\setminus B_\alpha}\mathbb{P}_\alpha\left(H_{A\setminus B_\alpha} < +\infty,\, S_{H_{A\setminus B_\alpha}} = \gamma\right) g(\gamma,\beta). \end{aligned} \tag{9}$$

We can plug this in to obtain

$$\begin{aligned} &\mathrm{Var}\left[\mu_\alpha\right] \\ &= \sum_{\beta,\gamma\in A\setminus B_\alpha}\mathbb{P}_\alpha\left(H_{A\setminus B_\alpha} < +\infty,\, S_{H_{A\setminus B_\alpha}} = \beta\right)\mathbb{P}_\alpha\left(H_{A\setminus B_\alpha} < +\infty,\, S_{H_{A\setminus B_\alpha}} = \gamma\right) g(\beta,\gamma) \\ &\overset{(9)}{=} \sum_{\beta\in A\setminus B_\alpha}\mathbb{P}_\alpha\left(H_{A\setminus B_\alpha} < +\infty,\, S_{H_{A\setminus B_\alpha}} = \beta\right) g(\alpha,\beta) \le \sup_{\beta\in A\setminus B_\alpha} g(\alpha,\beta) \\ &\le \frac{c}{(\log N)^{2(1+\epsilon)(d-2)}} \end{aligned} \tag{10}$$

by the standard estimates for the Green's function

$$c_d\|\alpha-\beta\|^{2-d} \le g(\alpha,\beta) \le C_d\|\alpha-\beta\|^{2-d} \tag{11}$$

for some $0 < c_d \le C_d < +\infty$ independent of α and β (Lawler, 1991, Theorem 1. 5. 4). Using the estimate

$$\mathsf{P}\left(|\mathcal{N}(0,1)| > a\right) \le \mathrm{e}^{-a^2/2},\ a > 0 \tag{12}$$

we get that there exists a constant $C > 0$ such that

$$\mathsf{P}\left(|\mu_\alpha| > (u_N(z))^{-1-\epsilon}\right) \le C\exp\left(-(\log N)^{(2d-5)(1+\epsilon)}\right). \tag{13}$$

Note that this quantity goes to zero since $d \ge 3$. Hence

$$\mathsf{E}\left[\left|\widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(z)) - p\right|\right] = \mathsf{E}\left[\left|\widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(z)) - p\right| \mathbb{1}_{\left\{|\mu_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}}\right]$$
$$+\mathsf{E}\left[\left|\widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(z)) - p\right| \mathbb{1}_{\left\{|\mu_\alpha| > (u_N(z))^{-1-\epsilon}\right\}}\right] =: T_1 + T_2.$$

By (13) and the fact that $d \ge 3$, we notice that $NT_2 = \mathrm{o}(1)$. Therefore it is sufficient to treat the term T_1.

$$\begin{aligned}
&\mathsf{E}\left[\left|\widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(z)) - p\right| \mathbb{1}_{\left\{|\mu_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}}\right]\\
&= \mathsf{E}\left[\left(\widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(z)) - p\right) \mathbb{1}_{\left\{|\mu_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}} \mathbb{1}_{\left\{p < \widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(z))\right\}}\right]\\
&+\mathsf{E}\left[\left(p - \widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(z))\right) \mathbb{1}_{\left\{|\mu_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}} \mathbb{1}_{\left\{p < \widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha \le u_N(z))\right\}}\right]\\
&=: T_{1,1} + T_{1,2}.
\end{aligned} \tag{14}$$

We will now deal with $T_{1,2}$. The first one can be treated with a similar calculation using an upper bound for the Mills ratio. We have on the event $\left\{|\mu_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}$

$$\begin{aligned}
&p - \widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(z))\\
&\le \frac{\sqrt{g(0)}\,\mathrm{e}^{-\frac{u_N(z)^2}{2g(0)}}}{\sqrt{2\pi}u_N(z)} - \left(1 - \left(\frac{\sqrt{g_{U_\alpha}(\alpha)}}{u_N(z) - \mu_\alpha}\right)^2\right) \frac{\sqrt{g_{U_\alpha}(\alpha)}\,\mathrm{e}^{-\frac{(u_N(z)-\mu_\alpha)^2}{2g_{U_\alpha}(\alpha)}}}{\sqrt{2\pi}(u_N(z) - \mu_\alpha)}\\
&\le \frac{\sqrt{g(0)}\,\mathrm{e}^{-\frac{u_N(z)^2}{2g(0)}}}{\sqrt{2\pi}u_N(z)} \left(1 - (1 + \mathrm{o}(1)) \frac{\sqrt{g_{U_\alpha}(\alpha)}u_N(z)\,\mathrm{e}^{\left(1 - \frac{g(0)}{g_{U_\alpha}(\alpha)}\right)\frac{u_N(z)^2}{2g(0)} + \frac{\mu_\alpha u_N(z)}{g_{U_\alpha}(\alpha)} - \frac{\mu_\alpha^2}{2g_{U_\alpha}(\alpha)}}}{\sqrt{g(0)}(u_N(z) - \mu_\alpha)}\right)\\
&= \frac{\sqrt{g(0)}\,\mathrm{e}^{-\frac{u_N(z)^2}{2g(0)}}}{\sqrt{2\pi}u_N(z)} \left(1 - (1 + \mathrm{o}(1)) \frac{\sqrt{g_{U_\alpha}(\alpha)}u_N(z)\,\mathrm{e}^{\left(1 - \frac{g(0)}{g_{U_\alpha}(\alpha)}\right)\frac{u_N(z)^2}{2g(0)} + \frac{u_N(z)^{-\epsilon}}{g_{U_\alpha}(\alpha)} - \frac{u_N(z)^{-2-2\epsilon}}{2g_{U_\alpha}(\alpha)}}}{\sqrt{g(0)}u_N(z)(1 - u_N(z)^{-2-\epsilon})}\right).
\end{aligned} \tag{15}$$

Since the bound is non random, by bounding the indicator functions by 1,

$$\mathsf{E}\left[\left(p - \widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(z))\right) \mathbb{1}_{\left\{|\mu_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}} \mathbb{1}_{\left\{p < \widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha \le u_N(z))\right\}}\right] \le (15).$$

Now

$$b_3 \le \sum_{\alpha \in A}(T_1 + T_2) \overset{(13)}{\le} \sum_{\alpha \in A} T_1 + \mathrm{o}(1) = \sum_{\alpha \in A} T_{1,1} + \sum_{\alpha \in A} T_{1,2} + \mathrm{o}(1). \tag{16}$$

Then

$$T_{1,2}=\frac{\sqrt{g(0)}\,\mathrm{e}^{-\frac{u_N(z)^2}{2g(0)}}}{\sqrt{2\pi}u_N(z)}\left(1-(1+\mathrm{o}(1))\left(\frac{\sqrt{g_{U_\alpha}(\alpha)}u_N(z)\,\mathrm{e}^{\left(1-\frac{g(0)}{g_{U_\alpha}(\alpha)}\right)\frac{u_N(z)^2}{2g(0)}+\mathrm{o}(1)}}{\sqrt{g(0)}u_N(z)(1+\mathrm{o}(1))}\right)\right).$$

Observe that $1-\frac{g(0)}{g_{U_\alpha}(\alpha)}<0$ since $g(0)>g_{U_\alpha}(\alpha)$. We observe further that (and we will prove it in a moment)

Claim 6. $\sup_{\alpha\in A}\left(1-\frac{g(0)}{g_{U_\alpha}(\alpha)}\right)u_N(z)^2=\mathrm{o}(1)$.

Hence

$$\mathrm{e}^{\left(1-\frac{g(0)}{g_{U_\alpha}(\alpha)}\right)\frac{u_N(z)^2}{2g(0)}}=\mathrm{e}^{\mathrm{o}(1)}$$

Therefore $T_{1,2}=\mathrm{o}(1)$ uniformly in α. This yields that

$$\sum_{\alpha\in\mathcal{A}}T_{1,2}\le N\frac{\sqrt{g(0)}\,\mathrm{e}^{-\frac{u_N(z)^2}{2g(0)}}}{\sqrt{2\pi}u_N(z)}\mathrm{o}(1)=\mathrm{e}^{-z+\mathrm{o}(1)}\,\mathrm{o}(1). \tag{17}$$

Analogously, $\sum_{\alpha\in\mathcal{A}}T_{1,1}=\mathrm{o}(1)$. Plugging (17) in (16), one obtains $b_3=\mathrm{o}(1)$. In particular, a standard computation yields $b_3\le c(\log N)^{-1+2(d-2)(1+\epsilon)}$ for some $c>0$.
We now only need to show Claim 6. By the Markov property we know

$$g_{U_\alpha}(\alpha)=g(0)-\sum_{\gamma\in A\setminus B_\alpha}\mathbb{P}_\alpha\left(H_{A\setminus B_\alpha}<+\infty,\,S_{H_{A\setminus B_\alpha}}=\gamma\right)g(\gamma,\alpha).$$

This shows that

$$0\le\frac{g(0)}{g_{U_\alpha}(\alpha)}-1\le\frac{\sup_{\gamma\in A\setminus B_\alpha}g(\gamma,\alpha)}{g_{U_\alpha}(\alpha)}.$$

Note that $g(\gamma,\alpha)\overset{(11)}{\le}C_d(\log N)^{-2(d-2)(1+\epsilon)}$. Also, $g_{U_\alpha}(\alpha)=\mathbb{E}_\alpha\left[\sum_{n=0}^{H_{A\setminus B_\alpha}}\mathbb{1}_{\{S_n=\alpha\}}\right]\ge 1$ and hence we have

$$0\le\frac{g(0)}{g_{U_\alpha}(\alpha)}-1\le c(\log N)^{-2(d-2)(1+\epsilon)} \tag{18}$$

from which it follows that

$$\left(1-\frac{g(0)}{g_{U_\alpha}(\alpha)}\right)u_N(z)^2\le c(\log N)^{-2(d-2)(1+\epsilon)}(\log N+z+\mathrm{o}(1))=\mathrm{o}(1). \tag{19}$$

Therefore the claim follows.

Uniformity of the estimates. We will now show our bound is uniform over $z\in\mathbb{R}$ considering separately three regions: $(-\infty,\,z_\ell)$, $[z_\ell,\,z_r]$ and $(z_r,\,+\infty)$ for some $z_\ell=z_\ell(N)$ and $z_r=z_r(N)$ to be explicited below.
In first place we consider $z_\ell:=-\log\log\log N$. We could have chosen any other function going sufficiently slowly to ∞ with N in order to accommodate our previous estimates. In particular it is important that

$$u_N(z)^2=2g(0)\log N(1+\mathrm{o}(1)),\quad z\in\{z_\ell,\,z_r\}. \tag{20}$$

We evaluate then the error bound $b_1+b_2+b_3$ at $z:=z_\ell$ considering each summand.

i. $b_1\le cN^{-1}(\log N)^{d(2+2\epsilon)}\,\mathrm{e}^{-2z_\ell+\mathrm{o}(1)}=\mathrm{o}(1)$.

ii. $b_2 \leq c' N^{-\kappa/(2-\kappa)} (\log N)^{d(2+2\epsilon)} \mathrm{e}^{-2z_\ell} = \mathrm{o}\,(1)$.

iii. b_3 requires some more care. We recall that $b_3 = \sum_{\alpha \in A} T_1 + \sum_{\alpha \in A} T_2$. (13) remains true because of (20), so $\sum_{\alpha \in A} T_2 = \mathrm{o}\,(1)$. Consequently we look at $\sum_\alpha T_1$ evaluated at $z = z_\ell$.

$$\frac{\sqrt{g(0)}\,\mathrm{e}^{-\frac{u_N(z_\ell)^2}{2g(0)}}}{\sqrt{2\pi} u_N(z_\ell)} \left(1 - (1+\mathrm{o}\,(1)) \frac{\sqrt{g_{U_\alpha}(\alpha)} u_N(z_\ell)\,\mathrm{e}^{\left(1-\frac{g(0)}{g_{U_\alpha}(\alpha)}\right)\frac{u_N(z_\ell)^2}{2g(0)} + \frac{u_N(z_\ell)^{-\epsilon}}{g_{U_\alpha}(\alpha)} - \frac{u_N(z_\ell)^{-2-2\epsilon}}{2g_{U_\alpha}(\alpha)}}}{\sqrt{g(0)} u_N(z_\ell)(1 - u_N(z_\ell)^{-2-\epsilon})} \right)$$

$$= N^{-1}\,\mathrm{e}^{-z_\ell + \mathrm{o}(1)} \left(1 - (1+\mathrm{o}\,(1))\,\mathrm{e}^{\left(1-\frac{g(0)}{g_{U_\alpha}(\alpha)}\right)\frac{u_N(z_\ell)^2}{2g(0)} + \mathrm{o}(1)} (1+\mathrm{o}\,(1)) \right).$$

The estimate (19) is still valid, therefore

$$\left(1 - \frac{g(0)}{g_{U_\alpha}(\alpha)}\right) u_N(z_\ell)^2 = \mathrm{o}(1).$$

Now the order of b_3, exactly in the same fashion as before, yields

$$b_3 \leq cN \left(N^{-1}\,\mathrm{e}^{-z_\ell + \mathrm{o}(1)} \right) (\log N)^{-1+2(d-2)(1+\epsilon)} = \mathrm{o}\,(1)\,.$$

This in turn yields that

$$\left| \mathrm{P}\left(\max_{\alpha \in A} \varphi_\alpha \leq u_N(z_\ell) \right) - \exp\left(-\mathrm{e}^{-z_\ell}\right) \right| = \mathrm{o}\,(1)\,. \tag{21}$$

For $z < z_\ell$ we notice that

$$\left| \mathrm{P}\left(\max_{\alpha \in A} \varphi_\alpha \leq u_N(z) \right) - \exp\left(-\mathrm{e}^{-z}\right) \right| \leq \mathrm{P}\left(\max_{\alpha \in A} \alpha \in A \leq u_N(z) \right) + \exp\left(-\mathrm{e}^{-z}\right)$$
$$\leq \mathrm{P}\left(\max_{\alpha \in A} \varphi_\alpha \leq u_N(z_\ell) \right) + \exp\left(-\mathrm{e}^{-z_\ell}\right)$$
$$= \left(\mathrm{P}\left(\max_{\alpha \in A} \varphi_\alpha \leq u_N(z_\ell) \right) - \exp\left(-\mathrm{e}^{-z_\ell}\right) \right) + 2\exp\left(-\mathrm{e}^{-z_\ell}\right).$$

By (21) we are able to conclude $\left|\mathrm{P}\left(\max_{\alpha \in A} \varphi_\alpha \leq u_N(z)\right) - \exp\left(-\mathrm{e}^{-z}\right)\right| \to 0$ uniformly over $z < z_\ell$.

On the other hand for $z_r := \log\log\log N$ observe that the same uniform estimates hold as above, with an obvious change in the sign in the power $\log\log\log N$. In particular b_1, b_2 and b_3 all go to 0 for $N \to +\infty$ and this entails

$$\lim_{N \to +\infty} \left| \mathrm{P}\left(\max_{\alpha \in A} \varphi_\alpha \leq u_N(z_r) \right) - \exp\left(-\mathrm{e}^{-z_r}\right) \right| = 0. \tag{22}$$

Therefore if $z > z_r$

$$\left| \mathrm{P}\left(\max_{\alpha \in A} \varphi_\alpha \leq u_N(z) \right) - \exp\left(-\mathrm{e}^{-z}\right) \right| = \left| 1 - \exp\left(-\mathrm{e}^{-z}\right) + \mathrm{P}\left(\max_{\alpha \in A} \varphi_\alpha \leq u_N(z) \right) - 1 \right|$$
$$\leq \left(1 - \exp\left(-\mathrm{e}^{-z}\right)\right) + \left(1 - \mathrm{P}\left(\max_{\alpha \in A} \varphi_\alpha \leq u_N(z) \right) \right)$$
$$\leq 2\left(1 - \exp\left(-\mathrm{e}^{-z_r}\right)\right) + \left(\exp\left(-\mathrm{e}^{-z_r}\right) - \mathrm{P}\left(\max_{\alpha \in A} \varphi_\alpha \leq u_N(z_r) \right) \right).$$

The first bracket goes to 0 and the second one, by (22), is $o(1)$ as well. We are therefore left with $z \in [z_\ell, z_r]$. The uniformity in this case holds by the inequality

$$\exp(-z) \le \exp(|z|) \le \log\log N, \quad z \in [z_\ell, z_r]. \tag{23}$$

Plugging this in (7) and (8), this shows the uniformity for the terms b_1 and b_2. As for the term b_3, we observe that (19) still holds, so that Claim 6 is valid and we are able to conclude inserting (23) in (17). □

3.2. **DGFF with boundary conditions: proof of Theorem 2.** The idea of the proof is to exploit the convergence we have obtained in the previous section. We will show a lower bound through a comparison with i. i. d. variables, and an upper bound by considering the maximum restricted to the bulk of V_N, concluding by means of a convergence-of-types result. We abbreviate $g_N(\cdot, \cdot) := g_{V_N}(\cdot, \cdot)$. For $\delta > 0$ define (recall that $V_N = [0, n-1]^d \cap \mathbb{Z}^d$, with $N = n^d$)

$$V_N^\delta := \left\{ \alpha \in V_N : \|\alpha - \gamma\| > \delta N^{1/d}, \gamma \in \mathbb{Z}^d \setminus V_N \right\}.$$

We begin with the easier lower bound.

Proof of Theorem 2: lower bound. We will need a lower and an upper bound on the limiting distribution of the maximum. Let us start with the former. We use the shortcut $\widetilde{\mathrm{P}}_N := \widetilde{\mathrm{P}}_{V_N}$. First we note that since the covariance of (ψ_α) is non-negative, we can apply Slepian's lemma for the lower bound. Let $(Z_\alpha)_{\alpha \in V_N}$ be independent mean zero Gaussian variables with variance $g_N(\alpha)$; then by Slepian's lemma it follows that

$$\widetilde{\mathrm{P}}_N \left(\max_{\alpha \in V_N} Z_\alpha \le u_N(z) \right) \le \widetilde{\mathrm{P}}_N \left(\max_{\alpha \in V_N} \psi_\alpha \le u_N(z) \right),$$

where $u_N(z) = a_N z + b_N$ as before. Then we want to analyze $\mathrm{P}(\max_{\alpha \in A} Z_\alpha \le u_N(z))$. First fix $z \in \mathbb{R}$. Take N large enough such that $-g(0) b_N^2 \le z$ (this is possible as $b_N^2 \to +\infty$). Now note that

$$\begin{aligned}
\widetilde{\mathrm{P}}_N \left(\max_{\alpha \in V_N} Z_\alpha \le u_N(z) \right) &= \prod_{\alpha \in V_N} \left(1 - \widetilde{\mathrm{P}}_N(Z_\alpha > u_N(z))\right) \\
&\overset{(6)}{\ge} \prod_{\alpha \in V_N} \left(1 - \frac{e^{-\frac{u_N(z)^2}{2 g_N(\alpha)}}}{\sqrt{2\pi} u_N(z)} \sqrt{g_N(\alpha)} \right) \ge \left(1 - \frac{e^{-\frac{u_N(z)^2}{2 g(0)}}}{\sqrt{2\pi} u_N(z)} \sqrt{g(0)} \right)^N .
\end{aligned}$$

The last term converges to $\exp(-\mathrm{e}^{-z})$ as $N \to +\infty$. This shows that for any fixed $z \in \mathbb{R}$,

$$\liminf_{N \to +\infty} \widetilde{\mathrm{P}}_N \left(\max_{\alpha \in V_N} \psi_\alpha \le u_N(z) \right) \ge \exp(-\mathrm{e}^{-z}).$$

□

We need some preliminary Lemmas for the upper bound. We begin with

Lemma 7. *For any $\delta > 0$ and $\alpha, \beta \in V_N^\delta$ one has*

$$g(\alpha, \beta) - C_d \left(\delta N^{1/d} \right)^{2-d} \le g_N(\alpha, \beta) \le g(\alpha, \beta). \tag{24}$$

In particular we have, $g_N(\alpha, \beta) = g(\alpha, \beta) \left(1 + \mathrm{O}\left(N^{(2-d)/d} \right) \right)$ uniformly in $\alpha, \beta \in V_N^\delta$.

Proof. It follows from Sznitman (2012, Proposition 1.6) that

$$g_N(\alpha,\beta) = g(\alpha,\beta) - \sum_{\gamma\in\partial V_N} \mathbb{P}_\alpha\left(H_{\mathbb{Z}^d\setminus V_N} < \infty, S_{H_{\mathbb{Z}^d\setminus V_N}} = \gamma\right) g(\gamma,\beta).$$

Note that $g_N(\alpha,\beta) \le g(\alpha,\beta)$. Take any $\alpha,\beta \in V_N^\delta$: using the bounds (11),

$$\sum_{\gamma\in\partial V_N} \mathbb{P}_\alpha\left(H_{\mathbb{Z}^d\setminus V_N} < \infty, S_{H_{\mathbb{Z}^d\setminus V_N}} = \gamma\right) g(\gamma,\beta) \le \sup_{\gamma\in\partial V_N} g(\gamma,\beta) \le C_d \sup_{\gamma\in\partial V_N} \|\gamma-\beta\|^{2-d}$$

which gives that

$$g_N(\alpha,\beta) \ge g(\alpha,\beta) - C_d\left(\delta N^{1/d}\right)^{2-d}. \tag{25}$$

Hence the proof follows. □

The next Lemma will allow us to derive the convergence of the maximum in V_N from that of the maximum in V_N^δ.

Lemma 8. *Let $N \ge 1$, F_N be a distribution function, and $m_N = (1-2\delta)^d N$. Let a_N and b_N be as in (1). If $\lim_{N\to+\infty} F_N(a_{m_N} z + b_{m_N}) = \exp(-\mathrm{e}^{-z})$, then*

$$\lim_{N\to+\infty} F_N(a_N z + b_N) = \exp\left(-\mathrm{e}^{-z+d\log(1-2\delta)}\right).$$

Proof. The proof follows from a convergence-of-types theorem (see Resnick (1987, Proposition 0.2)) if we can show that

$$\frac{a_{m_N}}{a_N} \to 1 \quad \text{and} \quad \frac{b_{m_N}-b_N}{a_N} \to d\log(1-2\delta). \tag{26}$$

It is easy to see that

$$\frac{a_{m_N}}{a_N} \sim \left(1 + \frac{d\log(1-2\delta)}{\log N}\right)^{1/2} \to 1.$$

To show the second asymptotics note that

$$\sqrt{2g(0)\log m_N} - \sqrt{2g(0)\log N} = \left[\frac{d\log(1-2\delta)}{2\log N} + \mathrm{O}\left(\frac{1}{(\log N)^2}\right)\right]\sqrt{2g(0)\log N}. \tag{27}$$

Also observe that as $N \to +\infty$ one gets

$$\begin{aligned}\sqrt{g(0)}&\left[\frac{\log\log(4\pi N)}{2\sqrt{2\log N}} - \frac{\log\log(4\pi m_N)}{2\sqrt{2\log m_N}}\right]\\ &= \frac{\sqrt{g(0)}}{2\sqrt{2\log N}}\left[-\log\left(1+\frac{d\log(1-2\delta)}{\log N}\right) + \mathrm{o}(1)\right].\end{aligned}$$

So using the above equation and (27) we get that

$$\frac{b_{m_N}-b_N}{a_N} = \frac{b_{m_N}-b_N}{g(0)}\sqrt{2g(0)\log N}(1+\mathrm{o}(1)) \to d\log(1-2\delta).$$

□

We have now the tools to finish with the upper bound.

Proof of Theorem 2: upper bound. For the upper bound, we again use Theorem 5, but this time on V_N^δ. We first observe that for any $\delta > 0$

$$\widetilde{\mathsf{P}}_N\left(\max_{\alpha\in V_N}\psi_\alpha \le u_N(z)\right) \le \widetilde{\mathsf{P}}_N\left(\max_{\alpha\in V_N^\delta}\psi_\alpha \le u_N(z)\right).$$

We claim that

Claim 9. *For any fixed $z \in \mathbb{R}$ and $\delta > 0$, set $m_N := \left|V_N^\delta\right| = (1-2\delta)^d N$. Then*

$$\lim_{N\to+\infty}\widetilde{\mathsf{P}}_N\left(\max_{\alpha\in V_N^\delta}\psi_\alpha \le u_{m_N}(z)\right) = \exp(-\mathrm{e}^{-z}).$$

Note that by Lemma 8 and Claim 9 one can conclude that

$$\widetilde{\mathsf{P}}_N\left(\max_{\alpha\in V_N^\delta}\psi_\alpha \le u_N(z)\right) = \exp\left(-\mathrm{e}^{-z+d\log(1-2\delta)}\right)$$

and thus

$$\limsup_{N\to+\infty}\widetilde{\mathsf{P}}_N\left(\max_{\alpha\in V_N}\psi_\alpha \le u_N(z)\right) \le \exp\left(-\mathrm{e}^{-z+d\log(1-2\delta)}\right).$$

Letting $\delta \longrightarrow 0$, the result will follow. To complete the proof of the Claim 9 we apply Theorem 5 and show that b_1, b_2 and $b_3 \longrightarrow 0$. To this end, define $Y_\alpha = \mathbb{1}_{\{\psi_\alpha > u_{m_N}(z)\}}$ and $\widetilde{W} = \sum_{\alpha\in V_N^\delta} Y_\alpha$. We see that using Mills ratio and Lemma 7 it follows that

$$\widetilde{\lambda} = \mathsf{E}\left[\widetilde{W}\right] \sim \mathrm{e}^{-z}.$$

As before we define for a fixed but small $\epsilon > 0$,

$$\widetilde{B}_\alpha := B\left(\alpha, (\log m_N)^{2+2\epsilon}\right), \quad \alpha \in V_N^\delta.$$

We notice that for N large, $\widetilde{B}_\alpha \subsetneq V_N$. Recall that $p_\alpha = \widetilde{\mathsf{P}}_N\left(\psi_\alpha > u_{m_N}(z)\right)$ and $b_1 = \sum_{\alpha\in V_N^\delta}\sum_{\beta\in\widetilde{B}_\alpha} p_\alpha p_\beta$. Exploiting the fact that for fixed z one can choose N large enough so that $u_N(z) > 0$, from Lemma 7 it again follows that

$$p_\alpha \le \frac{e^{-\frac{u_{m_N}(z)^2}{2g(0)}}}{\sqrt{2\pi}u_{m_N}(z)}\sqrt{g(0)}.$$

By previous calculations and $|V_N^\delta| = m_N$ we get that

$$b_1 \le m_N(\log m_N)^{d(2+2\epsilon)}\left(\frac{e^{-\frac{u_{m_N}(z)^2}{2g(0)}}}{\sqrt{2\pi}u_{m_N}(z)}\sqrt{g(0)}\right)^2 \le c m_N^{-1}(\log m_N)^{d(2+2\epsilon)}\,\mathrm{e}^{-2z} = \mathrm{o}\left(1\right).$$

As for b_2 we consider the covariance matrix of $\left(\psi_\alpha, \psi_\beta\right)$

$$\Sigma_2 = \begin{bmatrix} g_N(\alpha) & g_N(\alpha,\beta) \\ g_N(\alpha,\beta) & g_N(\beta)\end{bmatrix}$$

and hence we have

$$\mathbf{1}^t\Sigma_2^{-1}\mathbf{1} = \frac{g_N(\alpha)+g_N(\beta)-2g_N(\alpha,\beta)}{g_N(\alpha)g_N(\beta)-g_N(\alpha,\beta)^2} = \frac{2}{g(0)+g(\alpha-\beta)}\left(1+\mathrm{O}\left(N^{\frac{2-d}{d}}\right)\right).$$

Here the last line follows from an application of Lemma 7 again. Repeating the calculation of b_2 as before it follows that $b_2 \longrightarrow 0$ as $N \longrightarrow +\infty$.

For b_3 we use the Markov property which follows by the representation of g_N. Observe that $b_3 = \sum_{\alpha\in V_N^\delta}\widetilde{\mathsf{E}}_N\left[\left|\widetilde{\mathsf{E}}_N[Y_\alpha - p_\alpha|\widetilde{H}_1]\right|\right]$ where $\widetilde{H}_1 = \sigma\left(Y_\beta : \beta\in V_N^\delta\setminus\widetilde{B}_\alpha\right)$. For $\widetilde{H}_2 = \sigma\left(\psi_\beta : \beta\in V_N\setminus\widetilde{B}_\alpha\right)\supseteq\widetilde{\mathcal{H}}_1$ we can write for all $\alpha\in V_N$

$$\psi_\alpha = \left(\psi_\alpha - \widetilde{\mathsf{E}}_N\left[\psi_\alpha|\widetilde{H}_2\right]\right) + \widetilde{\mathsf{E}}_N\left[\psi_\alpha|\widetilde{H}_2\right] =: \xi_\alpha + h_\alpha$$

where ξ_α is a DGFF on $\widetilde{B}_\alpha$ and h_α is independent of ξ_α, measurable with respect to $\widetilde{H}_2$ and has the random walk representation (Sznitman, 2012, Proposition 2.3)

$$h_\alpha = \sum_{\beta\in V_N\setminus\widetilde{B}_\alpha}\mathbb{P}_\alpha\left(H_{V_N\setminus\widetilde{B}_\alpha} < +\infty,\, S_{H_{V_N\setminus\widetilde{B}_\alpha}} = \beta\right)\psi_\beta,\quad \alpha\in V_N.$$

This yields also that $\widetilde{\mathsf{P}}_N\left((\psi_\alpha)_{\alpha\in V_N}\in\cdot\,|\,\widetilde{H}_2\right) = \widetilde{\mathsf{P}}_{\widetilde{B}_\alpha}\left((\xi_\alpha + h_\alpha)_{\alpha\in V_N}\in\cdot\,\right)\,\widetilde{\mathsf{P}}_N$-a. s. and that

$$b_3 \le \sum_{\alpha\in V_N^\delta}\widetilde{\mathsf{E}}_N\left[\left|\widetilde{\mathsf{P}}_N\left(\psi_\alpha > u_{m_N}(z)\,|\,\widetilde{H}_2\right) - p_\alpha\right|\right] = \sum_{\alpha\in V_N^\delta}\widetilde{\mathsf{E}}_N\left[\left|\widetilde{\mathsf{P}}_{\widetilde{B}_\alpha}\left(\xi_\alpha + h_\alpha > u_{m_N}(z)\,\right) - p_\alpha\right|\right].$$

Mimicking the previous section, we have $b_3 \le \sum_{\alpha\in V_N^\delta}\widetilde{T}_1 + \sum_{\alpha\in V_N^\delta}\widetilde{T}_2$ with

$$\widetilde{T}_1 := \mathsf{E}\left[\left|\widetilde{\mathsf{P}}_{\widetilde{B}_\alpha}(\xi_\alpha + h_\alpha > u_{m_N}(z)) - p_\alpha\right|\mathbb{1}_{\left\{|h_\alpha|\le\left(u_{m_N}(z)\right)^{-1-\epsilon}\right\}}\right],$$

$$\widetilde{T}_2 := \mathsf{E}\left[\left|\widetilde{\mathsf{P}}_{\widetilde{B}_\alpha}(\xi_\alpha + h_\alpha > u_{m_N}(z)) - p_\alpha\right|\mathbb{1}_{\left\{|h_\alpha|>\left(u_{m_N}(z)\right)^{-1-\epsilon}\right\}}\right].$$

By means of the Markov property, one can proceed as in (10) to have

$$\mathsf{Var}\left[h_\alpha\right] \le \sup_{\beta\in V_N\setminus\widetilde{B}_\alpha} g_N(\alpha,\beta) \le \frac{C_d}{(\log m_N)^{2(1+\epsilon)(d-2)}},\tag{28}$$

thus, the conclusion that $\sum_{\alpha\in V_N^\delta}\widetilde{T}_2 = o(1)$ follows as before combining (12) with (28). As for $\widetilde{T}_1$, we again break it into two summands, $\widetilde{T}_{1,1}$ and $\widetilde{T}_{1,2}$, similarly to (14). It is possible to control these two terms by means of Mills ratio (6), Lemma 7 and following line by line the steps done for the infinite-volume case with $\widetilde{B}_\alpha$ instead of U_α. □

Acknowledgments

We kindly thank Ofer Zeitouni for several remarks to a previous version of the paper, and the suggestion to look at the DGFF with zero boundary conditions.

References

R. Adler and J. E. Taylor. *Random Fields and Geometry*. Springer, 2007.

R. Arratia, L. Goldstein, and L. Gordon. Two Moments Suffice for Poisson Approximations: The Chen-Stein Method. *Ann. Probab.*, 17(1):9–25, 01 1989. doi: 10.1214/aop/1176991491. URL http://dx.doi.org/10.1214/aop/1176991491.

M. Biskup and O. Louidor. Extreme local extrema of two-dimensional discrete Gaussian free field. *ArXiv e-prints*, June 2013. http://adsabs.harvard.edu/abs/2013arXiv1306.2602B.

M. Biskup and O. Louidor. Conformal symmetries in the extremal process of two-dimensional discrete Gaussian Free Field. *ArXiv e-prints*, Oct. 2014. http://adsabs.harvard.edu/abs/2014arXiv1410.4676B.

M. Bramson, J. Ding, and O. Zeitouni. Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *ArXiv e-prints*, Jan. 2013. http://arxiv.org/abs/1301.6669v4.

S. Chatterjee. *Superconcentration and Related Topics*. Springer Monographs in Mathematics. Springer, 2014. ISBN 9783319038865. URL http://books.google.de/books?id=s-m5BAAAQBAJ.

A. Drewitz and P.-F. Rodriguez. High-dimensional asymptotics for percolation of Gaussian free field level sets. *ArXiv e-prints*, Oct. 2013. http://adsabs.harvard.edu/abs/2013arXiv1310.1041D.

T. Funaki. Stochastic interface models. Lectures on probability theory and statistics. *Lect. Notes in Math.*, 1869:103–274, 2005.

H. O. Georgii. *Gibbs measures and Phase transitions*. de Gruyter, Berlin, 1988.

P. Hall. On the rate of convergence in the weak law of large numbers. *Ann. Probab.*, 10(2):374–381, 05 1982. doi: 10.1214/aop/1176993863. URL http://dx.doi.org/10.1214/aop/1176993863.

G. Lawler. *Intersections of random walks*. Birkhäuser, Boston, 1991.

S. Popov and B. Ráth. On decoupling inequalities and percolation of excursion sets of the Gaussian free field. *ArXiv e-prints*, July 2013. URL http://adsabs.harvard.edu/abs/2013arXiv1307.2862P.

S. Resnick. *Extreme Values, Regular Variation, and Point Processes*. Applied probability : a series of the applied probability trust. Springer, 1987. ISBN 9783540964810. URL http://books.google.it/books?id=DXi1QgAACAAJ.

P.-F. Rodriguez and A.-S. Sznitman. Phase transition and level-set percolation for the gaussian free field. *Communications in Mathematical Physics*, 320(2):571–601, 2013. ISSN 0010-3616. doi: 10.1007/s00220-012-1649-y. URL http://dx.doi.org/10.1007/s00220-012-1649-y.

I. Savage. Mill's ratio for multivariate normal distributions. *J. Res. Natl. Bur. Stand. Sec. B: Math.& Math. Phys.*, 66B:93, 1962.

A. Sznitman. *Topics in Occupation Times and Gaussian Free Fields*. Zurich Lectures in Advanced Mathematics. American Mathematical Society, 2012. ISBN 9783037191095. URL http://books.google.ch/books?id=RnENO-nQ7TIC.

A.-S. Sznitman. A lower bound on the critical parameter of interlacement percolation in high dimension. *Probability Theory and Related Fields*, 150(3-4):575–611, 2011. ISSN 0178-8051. doi: 10.1007/s00440-010-0284-9. URL http://dx.doi.org/10.1007/s00440-010-0284-9.

M. Talagrand. *Spin Glasses: A Challenge for Mathematicians: Cavity and Mean Field Models*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge A Series of Modern Surveys in Mathematics. Springer, 2003. ISBN 9783540003564. URL http://books.google.de/books?id=Ej0VrZLsgMsC.