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A boundary control problem for the pure Cahn–Hilliard equation with dynamic boundary conditions

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Abstract

A boundary control problem for the pure Cahn–Hilliard equations with possibly singular potentials and dynamic boundary conditions is studied and first-order necessary conditions for optimality are proved.

1 Introduction

The simplest form of the Cahn–Hilliard equation (see [3, 12, 13]) reads as follows

$$\partial_t y - \Delta w = 0 \quad \text{and} \quad w = -\Delta y + f'(y) \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

where Ω is the domain where the evolution takes place, and y and w denote the order parameter and the chemical potential, respectively. Moreover, f' represents the derivative of a double well potential f, and typical and important examples are the following

$$f_{reg}(r) = \tfrac{1}{4}(r^2-1)^2\,, \quad r \in \mathbb{R} \tag{1.2}$$

$$f_{log}(r) = ((1+r)\ln(1+r) + (1-r)\ln(1-r)) - cr^2\,, \quad r \in (-1,1), \tag{1.3}$$

where $c > 0$ in the latter is large enough in order that f_{log} be nonconvex. The potentials (1.2) and (1.3) are usually called the classical regular potential and the logarithmic double-well potential, respectively.

The present paper is devoted to the study of the control problem described below for the initial–boundary value problem obtained by complementing (1.1) with an initial condition like $y(0) = y_0$ and the following boundary conditions

$$\partial_n w = 0 \quad \text{and} \quad \partial_n y + \partial_t y_\Gamma - \Delta_\Gamma y_\Gamma + f'_\Gamma(y_\Gamma) = u_\Gamma \quad \text{on } \Gamma \times (0,T) \tag{1.4}$$

where Γ is the boundary of Ω. The former is very common in the literature and preserves mass conservation, i.e., it implies that the space integral of y is constant in time. The latter is an evolution equation for the trace y_Γ of the order parameter on the boundary, and the normal derivative $\partial_n y$ and u_Γ act as forcing terms. This condition enters the class of the so-called dynamic boundary conditions that have been widely used in the literature in the last twenty years, say: in particular, the study of dynamic boundary conditions with Cahn–Hilliard type equations has been taken up by some authors (let us quote [5, 9, 14, 18, 19, 24] and also refer to the recent contribution [8] in which also a forced mass constraint on the boundary is considered).

The dynamic boundary condition in (1.4) contains the Laplace-Beltrami operator Δ_Γ and a nonlinearity f'_Γ which is analogous to f' but is now acting on the boundary values u_Γ. Even though some of our results hold under weaker hypotheses, we assume from the very beginning that f'

and f'_Γ have the same domain $\mathcal{D}$. The main assumption we make is a compatibility condition between these nonlinearities. Namely, we suppose that f'_Γ dominates f' in the following sense:

$$|f'(r)| \leq \eta \, |f'_\Gamma(r)| + C \tag{1.5}$$

for some positive constants η and C and for every $r \in \mathcal{D}$. This condition, earlier introduced in [4] in relation with the Allen–Cahn equation with dynamic boundary conditions (see also [11]), is then used in [9] (as well as in [6] and [10]) to deal with the Cahn–Hilliard system. This complements [14], where some kind of an opposite inequality is assumed.

As just said, this paper deals with a control problem for the state system described above, the control being the source term u_Γ that appears in the dynamic boundary condition (1.4). Namely, the problem we want to address consists in minimizing a proper cost functional depending on both the control u_Γ and the associate state (y, y_Γ). Among several possibilities, we choose the cost functional

$$\mathcal{J}(y, y_\Gamma, u_\Gamma) := \frac{b_Q}{2} \, \|y - z_Q\|^2_{L^2(Q)} + \frac{b_\Sigma}{2} \, \|y_\Gamma - z_\Sigma\|^2_{L^2(\Sigma)} + \frac{b_0}{2} \, \|u_\Gamma\|^2_{L^2(\Sigma)} \,, \tag{1.6}$$

where the functions z_Q, z_Σ and the nonnegative constants b_Q, b_Σ, b_0 are given. The control problem then consists in minimizing (1.6) subject to the state system and to the constraint $u_\Gamma \in \mathcal{U}_{\rm ad}$, where the control box $\mathcal{U}_{\rm ad}$ is given by

$$\begin{aligned} \mathcal{U}_{\rm ad} := \quad & \bigl\{ u_\Gamma \in H^1(0,T;H_\Gamma) \cap L^\infty(\Sigma) : \\ & u_{\Gamma,\min} \leq u_\Gamma \leq u_{\Gamma,\max} \text{ a.e. on } \Sigma, \ \|\partial_t u_\Gamma\|_{L^2(\Sigma)} \leq M_0 \bigr\} \end{aligned} \tag{1.7}$$

for some given functions $u_{\Gamma,\min}, u_{\Gamma,\max} \in L^\infty(\Sigma)$ and some prescribed positive constant M_0. Of course, the control box $\mathcal{U}_{\rm ad}$ must be nonempty and this is guaranteed if, for instance, at least one of $u_{\Gamma,\min}$ or $u_{\Gamma,\max}$ is in $H^1(0,T;H_\Gamma)$ and its time derivative satisfies the above $L^2(\Sigma)$ bound.

This paper is a follow-up of the recent contributions [9] and [10] already mentioned. They deal with a similar system and a similar control problem. The paper [9] contains a number of results on the state system obtained by considering

$$w = \tau \, \partial_t y - \Delta y + f'(y) \tag{1.8}$$

in place of the second condition in (1.1). In (1.8), τ is a nonnegative parameter and the case $\tau > 0$ coupled with the first equation in (1.1) yields the well-known viscous Cahn–Hilliard equation (in contrast, we term (1.1) the pure Cahn–Hilliard system). More precisely, existence, uniqueness and regularity results are proved in [9] for general potentials that include (1.2)–(1.3), and are valid for both the viscous and pure cases, i.e., by assuming just $\tau \geq 0$. Moreover, if $\tau > 0$, further regularity and properties of the solution are ensured. These results are then used in [10], where the boundary control problem associated to a cost functional that generalizes (1.6) is addressed and both the existence of an optimal control and first-order necessary conditions for optimality are proved and expressed in terms of the solution of a proper adjoint problem.

In fact, recently Cahn–Hilliard systems have been rather investigated from the viewpoint of optimal control. In this connection, we refer to [15, 23, 27] and to [25, 26] which deal with the convective Cahn–Hilliard equation; the case with a nonlocal potential is studied in [20]. There also exist contributions addressing some discretized versions of general Cahn–Hilliard systems, cf. [16, 22]. However, about the optimal control of viscous or non-viscous Cahn–Hilliard systems with dynamic boundary conditions of the form (1.4), we only know of the papers [10] and [6] dealing with the viscous case; to the best of our knowledge, the present contribution is the first paper treating the optimal control of the pure Cahn–Hilliard system with dynamic boundary conditions.

The technique used in our approach essentially consists in starting from the known results for $\tau > 0$ and then letting the parameter τ tend to zero. In doing that, we use some of the ideas of [7] and [6], which deal with the Allen–Cahn and the viscous Cahn–Hilliard equations, respectively, and address similar control problems related to the nondifferentiable double obstacle potential by seeing it as a limit of logarithmic double-well potentials.

The paper is organized as follows. In the next section, we list our assumptions, state the problem in a precise form and present our results. The corresponding proofs are given in the last section.

2 Statement of the problem and results

In this section, we describe the problem under study and give an outline of our results. As in the Introduction, Ω is the body where the evolution takes place. We assume $\Omega \subset \mathbb{R}^3$ to be open, bounded, connected, and smooth, and we write $|\Omega|$ for its Lebesgue measure. Moreover, Γ, ∂_n, ∇_Γ and Δ_Γ stand for the boundary of Ω, the outward normal derivative, the surface gradient and the Laplace–Beltrami operator, respectively. Given a finite final time $T > 0$, we set for convenience

$$Q_t := \Omega \times (0,t) \quad \text{and} \quad \Sigma_t := \Gamma \times (0,t) \quad \text{for every } t \in (0,T] \tag{2.1}$$

$$Q := Q_T\,, \quad \text{and} \quad \Sigma := \Sigma_T\,. \tag{2.2}$$

Now, we specify the assumptions on the structure of our system. Even though some of the results we quote hold under rather mild hypotheses, we give a list of assumptions that implies the whole set of conditions required in [9]. We assume that

$$-\infty \le r_- < 0 < r_+ \le +\infty \tag{2.3}$$

$$f,\ f_\Gamma : (r_-, r_+) \to [0,+\infty) \text{ are } C^3 \text{ functions} \tag{2.4}$$

$$f(0) = f_\Gamma(0) = 0 \quad \text{and} \quad f'' \text{ and } f_\Gamma'' \text{ are bounded from below} \tag{2.5}$$

$$|f'(r)| \le \eta\,|f_\Gamma'(r)| + C \quad \text{for some } \eta,\ C > 0 \text{ and every } r \in (r_-, r_+) \tag{2.6}$$

$$\lim_{r \searrow r_-} f'(r) = \lim_{r \searrow r_-} f_\Gamma'(r) = -\infty \quad \text{and} \quad \lim_{r \nearrow r_+} f'(r) = \lim_{r \nearrow r_+} f_\Gamma'(r) = +\infty\,. \tag{2.7}$$

We note that (2.3)–(2.7) imply the possibility of splitting f' as $f' = \beta + \pi$, where β is a monotone function that diverges at $r_\pm$ and π is a perturbation with a bounded derivative. Moreover, the same is true for f_Γ, so that the assumptions of [9] are satisfied. Furthermore, the choices $f = f_{reg}$ and $f = f_{log}$ corresponding to (1.2) and (1.3) are allowed.

Next, in order to simplify notations, we set

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad H_\Gamma := L^2(\Gamma) \quad \text{and} \quad V_\Gamma := H^1(\Gamma) \tag{2.8}$$

$$\mathcal{V} := \{(v, v_\Gamma) \in V \times V_\Gamma :\ v_\Gamma = v_{|\Gamma}\} \quad \text{and} \quad \mathcal{H} := H \times H_\Gamma \tag{2.9}$$

and endow these spaces with their natural norms. If X is any Banach space, then $\|\cdot\|_X$ and X^* denote its norm and its dual space, respectively. Furthermore, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ usually stands for the duality pairing between V^* and V itself and the similar notation $\langle\,\cdot\,,\,\cdot\,\rangle_\Gamma$ refers to the spaces V_Γ^* and V_Γ. In the following, it is understood that H is identified with H^* and thus embedded in V^* in the usual way, i.e., such that we have $\langle u,v\rangle=(u,v)$ with the inner product $(\,\cdot\,,\,\cdot)$ of H, for every $u\in H$ and $v\in V$. Thus, we introduce the Hilbert triplet (V,H,V^*) and analogously behave with the boundary spaces V_Γ, H_Γ and V_Γ^*. Finally, if $u\in V^*$ and $\underline{u}\in L^1(0,T;V^*)$, we define their generalized mean values $u^\Omega\in\mathbb{R}$ and $\underline{u}^\Omega\in L^1(0,T)$ by setting

$$u^\Omega:=\frac{1}{|\Omega|}\,\langle u,1\rangle \quad\text{and}\quad \underline{u}^\Omega(t):=\big(\underline{u}(t)\big)^\Omega \quad \text{for a.a. } t\in(0,T). \tag{2.10}$$

Clearly, the relations in (2.10) give the usual mean values when applied to elements of H or $L^1(0,T;H)$.

At this point, we can describe the state problem. For the data, we assume that

$$y_0\in H^2(\Omega) \quad\text{and}\quad y_{0|_\Gamma}\in H^2(\Gamma) \tag{2.11}$$

$$r_-<y_0(x)<r_+ \quad \text{for every } x\in\overline{\Omega} \tag{2.12}$$

$$u_\Gamma\in H^1(0,T;H_\Gamma)\,. \tag{2.13}$$

We look for a triplet (y,y_Γ,w) satisfying

$$y\in H^1(0,T;V^*)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega)) \tag{2.14}$$

$$y_\Gamma\in H^1(0,T;H_\Gamma)\cap L^\infty(0,T;V_\Gamma)\cap L^2(0,T;H^2(\Gamma)) \tag{2.15}$$

$$y_\Gamma(t)=y(t)_{|_\Gamma} \quad \text{for a.a. } t\in(0,T) \tag{2.16}$$

$$w\in L^2(0,T;V)\,, \tag{2.17}$$

as well as, for almost every $t\in(0,T)$, the variational equations

$$\langle\partial_t y(t)\,v\rangle+\int_\Omega\nabla w(t)\cdot\nabla v=0 \qquad \text{for every } v\in V \tag{2.18}$$

$$\int_\Omega w(t)\,v=\int_\Omega\nabla y(t)\cdot\nabla v+\int_\Gamma\partial_t y_\Gamma(t)\,v_\Gamma+\int_\Gamma\nabla_\Gamma y_\Gamma(t)\cdot\nabla_\Gamma v_\Gamma$$
$$+\int_\Omega f'(y(t))\,v+\int_\Gamma\big(f_\Gamma'(y_\Gamma(t))-u_\Gamma(t)\big)\,v_\Gamma \qquad \text{for every } (v,v_\Gamma)\in\mathcal{V} \tag{2.19}$$

$$y(0)=y_0\,. \tag{2.20}$$

Thus, we require that the state variables satisfy the variational counterpart of the problem described in the Introduction in a strong form. We note that an equivalent formulation of (2.18)–(2.19) is given by

$$\int_0^t\langle\partial_t y(t)\,v(t)\rangle\,dt+\int_Q\nabla w\cdot\nabla v=0 \tag{2.21}$$

$$\int_Q wv=\int_Q\nabla y\cdot\nabla v+\int_\Sigma\partial_t y_\Gamma\,v_\Gamma+\int_\Sigma\nabla_\Gamma y_\Gamma\cdot\nabla_\Gamma v_\Gamma$$
$$+\int_Q f'(y)\,v+\int_\Sigma\big(f_\Gamma'(y_\Gamma)-u_\Gamma\big)\,v_\Gamma \tag{2.22}$$

for every $v \in L^2(0,T;V)$ and every $(v,v_\Gamma) \in L^2(0,T;\mathcal{V})$, respectively.

Besides, we consider the analogous state system with viscosity. Namely, for $\tau > 0$ we replace (2.19) by

$$\int_\Omega w(t)\,v = \tau \int_\Omega \partial_t y(t)\,v + \int_\Omega \nabla y(t) \cdot \nabla v + \int_\Gamma \partial_t y_\Gamma(t)\,v_\Gamma + \int_\Gamma \nabla_\Gamma y_\Gamma(t) \cdot \nabla_\Gamma v_\Gamma + \int_\Omega f'(y(t))\,v + \int_\Gamma \bigl(f'_\Gamma(y_\Gamma(t)) - u_\Gamma(t)\bigr)\,v_\Gamma \quad \text{for every } (v,v_\Gamma) \in \mathcal{V} \tag{2.23}$$

in the above system. We notice that a variational equation equivalent to (2.23) is given by the analogue of (2.22), i.e.,

$$\int_Q wv = \tau \int_Q \partial_t y\,v + \int_Q \nabla y \cdot \nabla v + \int_\Sigma \partial_t y_\Gamma\,v_\Gamma + \int_\Sigma \nabla_\Gamma y_\Gamma \cdot \nabla_\Gamma v_\Gamma + \int_Q f'(y)\,v + \int_\Sigma \bigl(f'_\Gamma(y_\Gamma) - u_\Gamma\bigr)\,v_\Gamma \quad \text{for every } (v,v_\Gamma) \in L^2(0,T;\mathcal{V}). \tag{2.24}$$

As far as existence, uniqueness, regularity and continuous dependence are concerned, we directly refer to [9]. From [9, Thms. 2.2 and 2.3] (where $\mathcal{V}$ has a slightly different meaning with respect to the present paper), we have the following results:

Theorem 2.1. *Assume* (2.3)–(2.7) *and* (2.11)–(2.13)*. Then, there exists a unique triplet* (y, y_Γ, w) *satisfying* (2.14)–(2.17) *and solving* (2.18)–(2.20).

Theorem 2.2. *Assume* (2.3)–(2.7) *and* (2.11)–(2.13)*. Then, for every* $\tau > 0$*, there exists a unique triplet* $(y^\tau, y_\Gamma^\tau, w^\tau)$ *satisfying* (2.14)–(2.17) *and solving* (2.18)*,* (2.20) *and* (2.23)*. Moreover, this solution satisfies* $\partial_t y^\tau \in L^2(0,T;H)$ *and the estimate*

$$\begin{aligned} &\|y^\tau\|_{H^1(0,T;V^*)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))} \\ &\quad + \|y_\Gamma^\tau\|_{H^1(0,T;H_\Gamma)\cap L^\infty(0,T;V_\Gamma)\cap L^2(0,T;H^2(\Gamma))} \\ &\quad + \|w^\tau\|_{L^2(0,T;V)} + \|f'(y^\tau)\|_{L^2(0,T;H)} + \|f'_\Gamma(y_\Gamma^\tau)\|_{L^2(0,T;H_\Gamma)} \\ &\quad + \tau^{1/2}\|\partial_t y^\tau\|_{L^2(0,T;H)} \le C_0 \end{aligned} \tag{2.25}$$

holds true for some constant $C_0 > 0$ *that depends only on* Ω*,* T*, the shape of the nonlinearities* f *and* f_Γ*, and the norms* $\bigl\|(y_0, y_{0|\Gamma})\bigr\|_{\mathcal{V}}$*,* $\|f'(y_0)\|_{L^1(\Omega)}$*,* $\bigl\|f'_\Gamma\bigl(y_{0|\Gamma}\bigr)\bigr\|_{L^1(\Gamma)}$*, and* $\|u_\Gamma\|_{L^2(0,T;H_\Gamma)}$.

In fact, if the data are more regular, in particular, if $u_\Gamma \in H^1(0,T;H_\Gamma) \cap L^\infty(\Sigma)$, then the solution $(y^\tau, y_\Gamma^\tau, w^\tau)$ is even smoother (see [9, Thms. 2.4 and 2.6]) and, specifically, it satisfies

$$r_-^\tau \le y^\tau \le r_+^\tau \quad \text{a.e. in } Q \tag{2.26}$$

for some constants $r_-^\tau, r_+^\tau \in (r_-, r_+)$ that depend on τ, in addition. It follows that the functions $f''(y^\tau)$ and $f''_\Gamma(y_\Gamma^\tau)$ (which will appear as coefficients in a linear system later on) are bounded. We also notice that the stability estimate (2.25) is not explicitely written in [9]. However, as the proof of the regularity (2.14)–(2.17) of the solution performed there relies on a priori estimates and compactness arguments and the dependence on τ in the whole calculation of [9] is always made explicit, (2.25) holds as well, and we stress that the corresponding constant C_0 does not depend on τ.

Once well-posedness for problem (2.18)–(2.20) is established, we can address the corresponding control problem. As in the Introduction, given two functions

$$z_Q \in L^2(Q) \quad \text{and} \quad z_\Sigma \in L^2(\Sigma) \tag{2.27}$$

and three nonnegative constants $b_Q,\ b_\Sigma,\ b_0$, we set

$$\mathcal{J}(y, y_\Gamma, u_\Gamma) := \frac{b_Q}{2}\,\|y - z_Q\|^2_{L^2(Q)} + \frac{b_\Sigma}{2}\,\|y_\Gamma - z_\Sigma\|^2_{L^2(\Sigma)} + \frac{b_0}{2}\,\|u_\Gamma\|^2_{L^2(\Sigma)} \tag{2.28}$$

for, say, $y \in L^2(0,T;H)$, $y_\Gamma \in L^2(0,T;H_\Gamma)$ and $u_\Gamma \in L^2(\Sigma)$, and consider the problem of minimizing the cost functional (2.28) subject to the constraint $u_\Gamma \in \mathcal{U}_{\rm ad}$, where the control box $\mathcal{U}_{\rm ad}$ is given by

$$\begin{aligned}\mathcal{U}_{\rm ad} := \ &\big\{u_\Gamma \in H^1(0,T;H_\Gamma) \cap L^\infty(\Sigma) :\\ &\quad u_{\Gamma,\min} \le u_\Gamma \le u_{\Gamma,\max} \text{ a.e. on } \Sigma,\ \|\partial_t u_\Gamma\|_{L^2(\Sigma)} \le M_0\big\}\end{aligned} \tag{2.29}$$

and to the state system (2.18)–(2.20). We simply assume that

$$M_0 > 0, \quad u_{\Gamma,\min},\ u_{\Gamma,\max} \in L^\infty(\Sigma) \quad \text{and} \quad \mathcal{U}_{\rm ad} \text{ is nonempty.} \tag{2.30}$$

Besides, we consider the analogous control problem of minimizing the cost functional (2.28) subject to the constraint $u_\Gamma \in \mathcal{U}_{\rm ad}$ and to the state system (2.18), (2.20) and (2.23). From [10, Thm. 2.3], we have the following result.

Theorem 2.3. *Assume* (2.3)–(2.7) *and* (2.11)–(2.13)*, and let* $\mathcal{J}$ *and* $\mathcal{U}_{\rm ad}$ *be defined by* (2.28) *and* (2.29) *under the assumptions* (2.27) *and* (2.30)*. Then, for every* $\tau > 0$*, there exists* $\overline{u}_\Gamma^\tau \in \mathcal{U}_{\rm ad}$ *such that*

$$\mathcal{J}(\overline{y}^\tau, \overline{y}_\Gamma^\tau, \overline{u}_\Gamma^\tau) \le \mathcal{J}(y^\tau, y_\Gamma^\tau, u_\Gamma) \quad \textit{for every } u_\Gamma \in \mathcal{U}_{\rm ad}\,, \tag{2.31}$$

where $\overline{y}^\tau$*,* $\overline{y}_\Gamma^\tau$*,* y^τ *and* y_Γ^τ *are the components of the solutions* $(\overline{y}^\tau, \overline{y}_\Gamma^\tau, \overline{w}^\tau)$ *and* $(y^\tau, y_\Gamma^\tau, w^\tau)$ *to the state system* (2.18)*,* (2.20) *and* (2.23) *corresponding to the controls* $\overline{u}_\Gamma^\tau$ *and* u_Γ*, respectively.*

In [10] first-order necessary conditions are obtained in terms of the solution to a proper adjoint system. More precisely, just the case $\tau = 1$ is considered there. However, by going through the paper with some care, one easily reconstructs the version of the adjoint system corresponding to an arbitrary $\tau > 0$. Even though the adjoint problem considered in [10] involves a triplet $(p^\tau, q^\tau, q_\Gamma^\tau)$ as an adjoint state, only the third component q_Γ^τ enters the necessary condition for optimality. On the other hand, q^τ and q_Γ^τ are strictly related to each other. Hence, we mention the result that deals with the pair (q^τ, q_Γ^τ). To this end, we recall a tool, the generalized Neumann problem solver $\mathcal{N}$, that is often used in connection with the Cahn–Hilliard equations. With the notation for the mean value introduced in (2.10), we define

$$\operatorname{dom}\mathcal{N} := \{v_* \in V^* :\ v_*^\Omega = 0\} \quad \text{and} \quad \mathcal{N} : \operatorname{dom}\mathcal{N} \to \{v \in V :\ v^\Omega = 0\} \tag{2.32}$$

by setting, for $v_* \in \operatorname{dom} \mathcal{N}$,

$$\mathcal{N}v_* \in V, \quad (\mathcal{N}v_*)^\Omega = 0, \quad \text{and} \quad \int_\Omega \nabla \mathcal{N}v_* \cdot \nabla z = \langle v_*, z \rangle \quad \text{for every } z \in V. \tag{2.33}$$

Thus, $\mathcal{N}v_*$ is the solution v to the generalized Neumann problem for $-\Delta$ with datum v_* that satisfies $v^\Omega = 0$. Indeed, if $v_* \in H$, the above variational equation means that $-\Delta \mathcal{N}v_* = v_*$ and $\partial_n \mathcal{N}v_* = 0$. As Ω is bounded, smooth, and connected, it turns out that (2.33) yields a well-defined isomorphism. Moreover, we have

$$\langle u_*, \mathcal{N}v_* \rangle = \langle v_*, \mathcal{N}u_* \rangle = \int_\Omega (\nabla \mathcal{N}u_*) \cdot (\nabla \mathcal{N}v_*) \quad \text{for } u_*, v_* \in \operatorname{dom} \mathcal{N}, \tag{2.34}$$

whence also

$$2\langle \partial_t v_*(t), \mathcal{N}v_*(t) \rangle = \frac{d}{dt} \int_\Omega |\nabla \mathcal{N}v_*(t)|^2 = \frac{d}{dt} \|v_*(t)\|_*^2 \quad \text{for a.a. } t \in (0,T) \tag{2.35}$$

for every $v_* \in H^1(0,T;V^*)$ satisfying $(v_*)^\Omega = 0$ a.e. in $(0,T)$, where we have set

$$\|v_*\|_* := \|\nabla \mathcal{N}v_*\|_H \quad \text{for } v_* \in V^*. \tag{2.36}$$

One easily sees that $\|\cdot\|_*$ is a norm in V^* which is equivalent to the usual dual norm.

Furthermore, we introduce the spaces $\mathcal{H}_\Omega$ and $\mathcal{V}_\Omega$ by setting

$$\mathcal{H}_\Omega := \{(v, v_\Gamma) \in \mathcal{H} : \ v^\Omega = 0\} \quad \text{and} \quad \mathcal{V}_\Omega := \mathcal{H}_\Omega \cap \mathcal{V}, \tag{2.37}$$

and endow them with their natural topologies as subspaces of $\mathcal{H}$ and $\mathcal{V}$, respectively. As in [10, Thms. 2.5 and 5.4], we have the following result.

Theorem 2.4. *Assume*

$$\lambda \in L^\infty(Q), \quad \lambda_\Gamma \in L^\infty(\Sigma), \quad \varphi_Q \in L^2(Q) \quad \textit{and} \quad \varphi_\Sigma \in L^2(\Sigma). \tag{2.38}$$

Then, for every $\tau > 0$, there exists a unique pair (q^τ, q_Γ^τ) satisfying the regularity conditions

$$q^\tau \in H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)) \quad \textit{and} \quad q_\Gamma^\tau \in H^1(0,T;H_\Gamma) \cap L^2(0,T;H^2(\Gamma)) \tag{2.39}$$

and solving the following adjoint problem:

$$(q^\tau, q_\Gamma^\tau)(t) \in \mathcal{V}_\Omega \quad \textit{for every } t \in [0,T] \tag{2.40}$$

$$\begin{aligned} &- \int_\Omega \partial_t \bigl(\mathcal{N}(q^\tau(t)) + \tau q^\tau(t) \bigr) v + \int_\Omega \nabla q^\tau(t) \cdot \nabla v + \int_\Omega \lambda(t)\, q^\tau(t)\, v \\ &\qquad - \int_\Gamma \partial_t q_\Gamma^\tau(t)\, v_\Gamma + \int_\Gamma \nabla_\Gamma q_\Gamma^\tau(t) \cdot \nabla_\Gamma v_\Gamma + \int_\Gamma \lambda_\Gamma(t)\, q_\Gamma^\tau(t)\, v_\Gamma \\ &= \int_\Omega \varphi_Q(t) v + \int_\Gamma \varphi_\Sigma(t) v_\Gamma \quad \textit{for a.a. } t \in (0,T) \textit{ and every } (v, v_\Gamma) \in \mathcal{V}_\Omega \end{aligned} \tag{2.41}$$

$$\int_\Omega \bigl(\mathcal{N}q^\tau + \tau q^\tau \bigr)(T)\, v + \int_\Gamma q_\Gamma(T)\, v_\Gamma = 0 \quad \textit{for every } (v, v_\Gamma) \in \mathcal{V}_\Omega\,. \tag{2.42}$$

More precisely, in [10] the above theorem is proved with the particular choice

$$\lambda = f''(\overline{y}^\tau), \quad \lambda_\Gamma = f''_\Gamma(\overline{y}^\tau_\Gamma), \quad \varphi_Q = b_Q(\overline{y}^\tau - z_Q) \quad \text{and} \quad \varphi_\Sigma = b_\Sigma(\overline{y}^\tau_\Gamma - z_\Sigma) \tag{2.43}$$

where $\overline{y}^\tau$ and $\overline{y}^\tau_\Gamma$ are the components of the state associated to an optimal control $\overline{u}^\tau_\Gamma$. However, the same proof is valid under assumption (2.38).

Finally, [10, Thm. 2.6] gives a necessary condition for $\overline{u}^\tau_\Gamma$ to be an optimal control in terms of the solution to the above adjoint system corresponding to (2.43). This condition reads

$$\int_\Sigma (q^\tau_\Gamma + b_0 \overline{u}^\tau_\Gamma)(v_\Gamma - \overline{u}^\tau_\Gamma) \geq 0 \quad \text{for every } v_\Gamma \in \mathcal{U}_{\mathrm{ad}}. \tag{2.44}$$

In this paper, we first show the existence of an optimal control $\overline{u}_\Gamma$. Namely, we prove the following result.

Theorem 2.5. *Assume* (2.3)–(2.7) *and* (2.11)–(2.13)*, and let* $\mathcal{J}$ *and* $\mathcal{U}_{\mathrm{ad}}$ *be defined by* (2.28) *and* (2.29) *under the assumptions* (2.27) *and* (2.30)*. Then there exists some* $\overline{u}_\Gamma \in \mathcal{U}_{\mathrm{ad}}$ *such that*

$$\mathcal{J}(\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma) \leq \mathcal{J}(y, y_\Gamma, u_\Gamma) \quad \textit{for every } u_\Gamma \in \mathcal{U}_{\mathrm{ad}}\,, \tag{2.45}$$

where $\overline{y}$, $\overline{y}_\Gamma$, y *and* y_Γ *are the components of the solutions* $(\overline{y}, \overline{y}_\Gamma, \overline{w})$ *and* (y, y_Γ, w) *to the state system* (2.18)–(2.20) *corresponding to the controls* $\overline{u}_\Gamma$ *and* u_Γ*, respectively.*

Next, for every optimal control $\overline{u}_\Gamma$, we derive a necessary optimality condition like (2.44) in terms of the solution of a generalized adjoint system. In order to make the last sentence precise, we introduce the spaces

$$\mathcal{W} := L^2(0,T;\mathcal{V}_\Omega) \cap \big(H^1(0,T;V^*) \times H^1(0,T;V^*_\Gamma)\big) \tag{2.46}$$

$$\mathcal{W}_0 := \{(v, v_\Gamma) \in \mathcal{W} : \ (v, v_\Gamma)(0) = (0,0)\} \tag{2.47}$$

and endow them with their natural topologies. Moreover, we denote by $\langle\!\langle \,\cdot\,, \,\cdot\, \rangle\!\rangle$ the duality product between $\mathcal{W}^*_0$ and $\mathcal{W}_0$. We will prove the following representation result for the elements of the dual space $\mathcal{W}^*_0$.

Proposition 2.6. *A functional* $F : \mathcal{W}_0 \to \mathbb{R}$ *belongs to* $\mathcal{W}^*_0$ *if and only if there exist* Λ *and* Λ_Γ *satisfying*

$$\Lambda \in \big(H^1(0,T;V^*) \cap L^2(0,T;V)\big)^* \quad \textit{and} \quad \Lambda_\Gamma \in \big(H^1(0,T;V^*_\Gamma) \cap L^2(0,T;V_\Gamma)\big)^* \tag{2.48}$$

$$\langle\!\langle F, (v, v_\Gamma) \rangle\!\rangle = \langle \Lambda, v \rangle_Q + \langle \Lambda_\Gamma, v_\Gamma \rangle_\Sigma \quad \textit{for every } (v, v_\Gamma) \in \mathcal{W}_0\,, \tag{2.49}$$

where the duality products $\langle \,\cdot\,, \,\cdot\, \rangle_Q$ *and* $\langle \,\cdot\,, \,\cdot\, \rangle_\Sigma$ *are related to the spaces* X^* *and* X *with* $X = H^1(0,T;V^*) \cap L^2(0,T;V)$ *and* $X = H^1(0,T;V^*_\Gamma) \cap L^2(0,T;V_\Gamma)$*, respectively.*

However, this representation is not unique, since different pairs $(\Lambda, \Lambda_\Gamma)$ satisfying (2.48) could generate the same functional F through formula (2.49).

At this point, we are ready to present our result on the necessary optimality conditions for the control problem related to the pure Cahn–Hilliard equations, i.e., the analogue of (2.44) in terms of a solution to a generalized adjoint system.

Theorem 2.7. *Assume* (2.3)–(2.7) *and* (2.11)–(2.13)*, and let* $\mathcal{J}$ *and* $\mathcal{U}_{\rm ad}$ *be defined by* (2.28) *and* (2.29) *under the assumptions* (2.27) *and* (2.30)*. Moreover, let* $\overline{u}_\Gamma$ *be any optimal control as in the statement of Theorem 2.5. Then, there exist* Λ *and* Λ_Γ *satisfying* (2.48)*, and a pair* (q, q_Γ) *satisfying*

$$q \in L^\infty(0,T;V^*) \cap L^2(0,T;V) \tag{2.50}$$

$$q_\Gamma \in L^\infty(0,T;H_\Gamma) \cap L^2(0,T;V_\Gamma) \tag{2.51}$$

$$(q,q_\Gamma)(t) \in \mathcal{V}_\Omega \quad \textit{for every } t \in [0,T]\,, \tag{2.52}$$

as well as

$$\begin{aligned} &\int_0^T \langle \partial_t v(t), \mathcal{N} q(t) \rangle \, dt + \int_0^T \langle \partial_t v_\Gamma(t), q_\Gamma(t) \rangle_\Gamma \, dt \\ &\quad + \int_Q \nabla q \cdot \nabla v + \int_\Sigma \nabla_\Gamma q_\Gamma \cdot \nabla_\Gamma v_\Gamma + \langle \Lambda, v \rangle_Q + \langle \Lambda_\Gamma, v_\Gamma \rangle_\Sigma \\ &= \int_Q \varphi_Q \, v + \int_\Sigma \varphi_\Sigma \, v_\Gamma \qquad \textit{for every } (v, v_\Gamma) \in \mathcal{W}_0\,, \end{aligned} \tag{2.53}$$

such that

$$\int_\Sigma (q_\Gamma + b_0 \overline{u}_\Gamma)(v_\Gamma - \overline{u}_\Gamma) \geq 0 \quad \textit{for every } v_\Gamma \in \mathcal{U}_{\rm ad}. \tag{2.54}$$

In particular, if $b_0 > 0$*, then the optimal control* $\overline{u}_\Gamma$ *is the* $L^2(\Sigma)$*-projection of* $-q_\Gamma/b_0$ *onto* $\mathcal{U}_{\rm ad}$*.*

One recognizes in (2.53) a problem that is analogous to (2.41)–(2.42). Indeed, if Λ, Λ_Γ and the solution (q, q_Γ) were regular functions, then its strong form should contain both a generalized backward parabolic equation like (2.41) and a final condition for $(\mathcal{N}q, q_\Gamma)$ of type (2.42), since the definition of $\mathcal{W}_0$ allows its elements to be free at $t = T$. However, the terms $\lambda^\tau q^\tau$ and $\lambda_\Gamma^\tau q_\Gamma^\tau$ are just replaced by the functionals Λ and Λ_Γ and cannot be identified as products, unfortunately.

3 Proofs

In the whole section, we assume that all of the conditions (2.3)–(2.7) and (2.11)–(2.12) on the structure and the initial datum of the state system, as well as assumptions (2.27) and (2.30) that regard the cost functional (2.28) and the control box (2.29), are satisfied. We start with an expected result.

Proposition 3.1. *Assume* $u_\Gamma^\tau \in H^1(0,T;H_\Gamma)$ *and let* $(y^\tau, y_\Gamma^\tau, w^\tau)$ *be the solution to the problem* (2.18)*,* (2.20) *and* (2.23) *associated to* u_Γ^τ*. If* u_Γ^τ *converges to* u_Γ *weakly in* $H^1(0,T;H_\Gamma)$ *as* $\tau \searrow 0$*, then*

$$\begin{aligned} y^\tau \to y \qquad & \textit{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \\ & \quad \textit{and strongly in } C^0([0,T];H) \cap L^2(0,T;V) \end{aligned} \tag{3.1}$$

$$\begin{aligned} y_\Gamma^\tau \to y_\Gamma \qquad & \textit{weakly star in } H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma) \cap L^2(0,T;H^2(\Gamma)) \\ & \quad \textit{and strongly in } C^0([0,T];H_\Gamma) \cap L^2(0,T;V_\Gamma) \end{aligned} \tag{3.2}$$

$$w^\tau \to w \qquad \textit{weakly star in } L^2(0,T;V)\,, \tag{3.3}$$

where (y, y_Γ, w) *is the solution to problem* (2.18)–(2.20) *associated with* u_Γ*.*

Proof. The family $\{u_\Gamma^\tau\}$ is bounded in $H^1(0,T;H_\Gamma)$. Thus, the solution $(y^\tau, y_\Gamma^\tau, w^\tau)$ satisfies (2.25) for some constant C_0, so that the weak or weak star convergence specified in (3.1)–(3.3) holds for a subsequence. In particular, the Cauchy condition (2.20) for y is satisfied. Moreover, we also have $\tau\,\partial_t y^\tau \to 0$ strongly in $L^2(0,T;H)$ as well as $f'(y^\tau) \to \xi$ and $f_\Gamma'(y_\Gamma^\tau) \to \xi_\Gamma$ weakly in $L^2(0,T;H)$ and in $L^2(0,T;H_\Gamma)$, respectively, for some ξ and ξ_Γ. Furthermore, y^τ and y_Γ^τ converge to their limits strongly in $L^2(0,T;H)$ and $L^2(0,T;H_\Gamma)$, respectively, thanks to the Aubin-Lions lemma (see, e.g., [17, Thm. 5.1, p. 58], which also implies a much better strong convergence [21, Sect. 8, Cor. 4]). Now, as said in Section 2, we can split f' as $f' = \beta + \pi$, where β is monotone and π is Lipschitz continuous. It follows that $\pi(y^\tau)$ converges to $\pi(y)$ strongly in $L^2(0,T;H)$, whence we obtain that also $\beta(y^\tau)$ converges to $\xi - \pi(y)$ weakly in $L^2(0,T;H)$. Then, we infer that $\xi - \pi(y) = \beta(y)$ a.e. in Q, i.e., $\xi = f'(y)$ a.e. in Q, with the help of standard monotonicity arguments (see, e.g., [1, Lemma 1.3, p. 42]). Similarly, we have $\xi_\Gamma = f_\Gamma'(y_\Gamma)$. Therefore, by starting from (2.21) and (2.24) written with u_Γ^τ in place of u_Γ, we can pass to the limit and obtain (2.21)–(2.22) associated to the limit control u_Γ. As the solution to the limit problem is unique, the whole family $(y^\tau, y_\Gamma^\tau, w^\tau)$ converges to (y, y_Γ, w) in the sense of the statement and the proof is complete. □

Corollary 3.2. *Estimate* (2.25)*, written formally with* $\tau = 0$*, holds for the solution to the pure Cahn–Hilliard system* (2.18)–(2.20)*.*

Proof. By applying the above proposition with $u_\Gamma^\tau = u_\Gamma$ and using (2.25) for the solution to the viscous problem, we immediately conclude the claim. □

Proof of Theorem 2.5. We use the direct method and start from a minimizing sequence $\{u_{\Gamma,n}\}$. Then, $u_{\Gamma,n}$ remains bounded in $H^1(0,T;H)$, whence we have $u_{\Gamma,n} \to \overline{u}_\Gamma$ weakly in $H^1(0,T;H_\Gamma)$ for a subsequence. By Corollary 3.2, the sequence of the corresponding states $(y_n, y_{\Gamma,n}, w_n)$ satisfies the analogue of (2.25). Hence, by arguing as in the proof of Proposition 3.1, we infer that the solutions $(y_n, y_{\Gamma,n}, w_n)$ converge in the proper topology to the solution $(\overline{y}, \overline{y}_\Gamma, \overline{w})$ associated to $\overline{u}_\Gamma$. In particular, there holds the strong convergence specified by the analogues of (3.1) and (3.2). Thus, by also owing to the semicontinuity of $\mathcal{J}$ and the optimality of $u_{\Gamma,n}$, we have

$$\mathcal{J}(\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma) \le \liminf_{n\to\infty} \mathcal{J}(y_n, y_{\Gamma,n}, u_{\Gamma,n}) \le \mathcal{J}(y, y_\Gamma, u_\Gamma)$$

for every $u_\Gamma \in \mathcal{U}_{\rm ad}$, where y and y_Γ are the components of the solution to the Cahn–Hilliard system associated with u_Γ. This means that $\overline{u}_\Gamma$ is an optimal control. □

Proof of Proposition 2.6. Assume that Λ and Λ_Γ satisfy (2.48). Then, formula (2.49) actually defines a functional F on $\mathcal{W}_0$. Clearly, F is linear. Moreover, we have, for every $(v, v_\Gamma) \in \mathcal{W}_0$,

$$\begin{aligned}
&|\langle \Lambda, v\rangle_Q + \langle \Lambda_\Gamma, v_\Gamma\rangle_\Sigma| \\
&\le \|\Lambda\|_{(H^1(0,T;V^*)\cap L^2(0,T;V))^*}\, \|v\|_{H^1(0,T;V^*)\cap L^2(0,T;V)} \\
&\quad + \|\Lambda_\Gamma\|_{(H^1(0,T;V_\Gamma^*)\cap L^2(0,T;V_\Gamma))^*}\, \|v\|_{H^1(0,T;V_\Gamma^*)\cap L^2(0,T;V_\Gamma)} \\
&\le \bigl(\|\Lambda\|_{(H^1(0,T;V^*)\cap L^2(0,T;V))^*} + \|\Lambda_\Gamma\|_{(H^1(0,T;V_\Gamma^*)\cap L^2(0,T;V_\Gamma))^*}\bigr) \|(v, v_\Gamma)\|_{\mathcal{W}}\,,
\end{aligned}$$

so that F is continuous. Conversely, assume that $F \in \mathcal{W}_0^*$. As $\mathcal{W}_0$ is a (closed) subspace of $\tilde{\mathcal{W}} := \big(H^1(0,T;V^*) \cap L^2(0,T;V)\big) \times (H^1(0,T;V_\Gamma^*) \cap L^2(0,T;V_\Gamma))$, we can extend F to a linear continuous functional $\tilde{F}$ on $\tilde{\mathcal{W}}$. Then, there exist Λ and Λ_Γ (take $\Lambda(v) := \tilde{F}(v,0)$ and $\Lambda_\Gamma(v_\Gamma) := \tilde{F}(0,v_\Gamma)$) satisfying (2.48) such that

$$\langle \tilde{F}, (v, v_\Gamma) \rangle = \langle \Lambda, v \rangle_Q + \langle \Lambda_\Gamma, v_\Gamma \rangle_\Sigma \quad \text{for every } (v, v_\Gamma) \in \tilde{\mathcal{W}},$$

where the duality product on the left-hand side refers to the spaces $(\tilde{\mathcal{W}})^*$ and $\tilde{\mathcal{W}}$. Since $\langle\langle F, (v, v_\Gamma) \rangle\rangle = \langle \tilde{F}, (v, v_\Gamma) \rangle$ for every $(v, v_\Gamma) \in \mathcal{W}_0$, (2.49) immediately follows. □

The rest of this section is devoted to the proof of Theorem 2.7 on the necessary optimality conditions. Therefore, besides the general assumptions, we also suppose that

$$\overline{u}_\Gamma \textit{ is any optimal control as in Theorem 2.5}, \tag{3.4}$$

that is, an arbitrary optimal control $\overline{u}_\Gamma$ is fixed once and for all. In order to arrive at the desired necessary optimality condition for $\overline{u}_\Gamma$, we follow [2] and introduce the modified functional $\tilde{\mathcal{J}}$ defined by

$$\tilde{\mathcal{J}}(y, y_\Gamma, u_\Gamma) := \mathcal{J}(y, y_\Gamma, u_\Gamma) + \frac{1}{2} \|u_\Gamma - \overline{u}_\Gamma\|^2_{L^2(\Sigma)}. \tag{3.5}$$

Then the analogue of Theorem 2.3 holds, and we have:

Theorem 3.3. *For every $\tau > 0$, there exists some $\tilde{u}_\Gamma^\tau \in \mathcal{U}_{\text{ad}}$ such that*

$$\tilde{\mathcal{J}}(\tilde{y}^\tau, \tilde{y}_\Gamma^\tau, \tilde{u}_\Gamma^\tau) \le \tilde{\mathcal{J}}(y^\tau, y_\Gamma^\tau, u_\Gamma) \quad \textit{for every } u_\Gamma \in \mathcal{U}_{\text{ad}}, \tag{3.6}$$

where $\tilde{y}^\tau$, $\tilde{y}_\Gamma^\tau$, y^τ and y_Γ^τ are the components of the solutions $(\tilde{y}^\tau, \tilde{y}_\Gamma^\tau, \tilde{w}^\tau)$ and $(y^\tau, y_\Gamma^\tau, w^\tau)$ to the state system (2.18)*,* (2.20) *and* (2.23) *corresponding to the controls $\tilde{u}_\Gamma^\tau$ and u_Γ, respectively.*

For the reader's convenience, we fix the notation just used and introduce a new one (which was already used with a different meaning earlier in this paper):

$$\tilde{u}_\Gamma^\tau \textit{ is an optimal control as in Theorem 3.3} \tag{3.7}$$

$$(\tilde{y}^\tau, \tilde{y}_\Gamma^\tau, \tilde{w}^\tau) \textit{ is the solution to (2.18), (2.20) and (2.23) corresponding to } \tilde{u}_\Gamma^\tau \tag{3.8}$$

$$(\overline{y}^\tau, \overline{y}_\Gamma^\tau, \overline{w}^\tau) \textit{ is the solution to (2.18), (2.20) and (2.23) corresponding to } \overline{u}_\Gamma. \tag{3.9}$$

The next step consists in writing the proper adjoint system and the corresponding necessary optimality condition, which can be done by repeating the argument of [10]. However, instead of just stating the corresponding result, we spend some words that can help the reader. The optimality variational inequality is derived as a condition on the Fréchet derivative of the map (defined in a proper functional framework) $u_\Gamma \mapsto \tilde{\mathcal{J}}(y, y_\Gamma, u_\Gamma)$, where the pair (y, y_Γ) is subjected to the state system. Thus, this derivative depends on the Fréchet derivative of the functional $(y, y_\Gamma, u_\Gamma) \mapsto \tilde{\mathcal{J}}(y, u_\Gamma, u_\Gamma)$, which is given by

$$[D\tilde{\mathcal{J}}(y, y_\Gamma, u_\Gamma)](k, k_\Gamma, h_\Gamma)] = b_Q \int_Q (y - z_Q) k + b_\Sigma \int_\Sigma (y_\Gamma - z_\Sigma) k_\Gamma + \int_\Sigma \big(b_0 u_\Gamma + (u_\Gamma - \overline{u}_\Gamma)\big) h_\Gamma.$$

Hence, the argument for $\tilde{\mathcal{J}}$ differs from the one for $\mathcal{J}$ only in relation to the last integral. In other words, we just have to replace $b_0 u_\Gamma$ by $b_0 u_\Gamma + (u_\Gamma - \overline{u}_\Gamma)$ in the whole argument of [10]. In particular, the adjoint system remains unchanged. Here is the conclusion.

Proposition 3.4. *With the notations* (3.7)–(3.8)*, we have*

$$\int_\Sigma \big(q_\Gamma^\tau + b_0\tilde u_\Gamma^\tau + (\tilde u_\Gamma^\tau - \overline{u}_\Gamma)\big)(v_\Gamma - \tilde u_\Gamma^\tau) \geq 0 \quad \textit{for every } v_\Gamma \in \mathcal{U}_{\rm ad}\,, \tag{3.10}$$

where q_Γ^τ is the component of the solution (q^τ, q_Γ^τ) to (2.40)–(2.42) *corresponding to $u_\Gamma = \tilde u_\Gamma^\tau$ with the choices $\lambda = \lambda^\tau$, $\lambda_\Gamma = \lambda_\Gamma^\tau$, $\varphi_Q = \varphi_Q^\tau$ and $\varphi_\Sigma = \varphi_\Sigma^\tau$ specified by*

$$\lambda^\tau = f''(\tilde y^\tau),\ \ \lambda_\Gamma^\tau = f_\Gamma''(\tilde y_\Gamma^\tau),\ \ \varphi_Q^\tau = b_Q(\tilde y^\tau - z_Q)\ \ and\ \ \varphi_\Sigma^\tau = b_\Sigma(\tilde y_\Gamma^\tau - z_\Sigma). \tag{3.11}$$

Thus, our project for the proof of Theorem 2.7 is the following: we take the limit in (3.10) and in the adjoint system mentioned in the previous statement as τ tends to zero. This will lead to the desired necessary optimality condition (2.54) provided that we prove that the optimal controls $\tilde u_\Gamma^\tau$ converge to $\overline{u}_\Gamma$. The details of this project are the following.

$i)$ There hold

$$\tilde u_\Gamma^\tau \to \overline{u}_\Gamma \qquad \text{weakly star in } H^1(0,T;H_\Gamma)\cap L^\infty(\Sigma) \text{ and strongly in } L^2(\Sigma) \tag{3.12}$$

$$\tilde y^\tau \to \overline{y} \qquad \text{weakly star in } H^1(0,T;V^*)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega)) \text{ and strongly in } C^0([0,T];H)\cap L^2(0,T;V) \tag{3.13}$$

$$\tilde y_\Gamma^\tau \to \overline{y}_\Gamma \qquad \text{weakly star in } H^1(0,T;H_\Gamma)\cap L^\infty(0,T;V_\Gamma)\cap L^2(0,T;H^2(\Gamma)) \text{ and strongly in } C^0([0,T];H_\Gamma)\cap L^2(0,T;V_\Gamma) \tag{3.14}$$

$$\tilde w^\tau \to \overline{w} \qquad \text{weakly star in } L^2(0,T;V) \tag{3.15}$$

$$q^\tau \to q \qquad \text{weakly star in } L^\infty(0,T;V^*)\cap L^2(0,T;V) \tag{3.16}$$

$$q_\Gamma^\tau \to q_\Gamma \qquad \text{weakly star in } L^\infty(0,T;H_\Gamma)\cap L^2(0,T;V_\Gamma)\,, \tag{3.17}$$

as well as

$$\tilde{\mathcal{J}}(\tilde y^\tau, \tilde y_\Gamma^\tau, \tilde u_\Gamma^\tau) \to \mathcal{J}(\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma)\,, \tag{3.18}$$

at least for a subsequence, and $(\overline{y}, \overline{y}_\Gamma, \overline{w})$ solves problem (2.18)–(2.20) with $u_\Gamma = \overline{u}_\Gamma$.

$ii)$ The functionals associated with the pair $(\lambda^\tau q^\tau, \lambda_\Gamma^\tau, q_\Gamma^\tau)$ through Proposition 2.6 converge to some functional weakly in $\mathcal{W}_0^*$, at least for a subsequence, and we then represent the limit by some pair $(\Lambda, \Lambda_\Gamma)$, so that we have

$$\langle \lambda^\tau q^\tau, v\rangle_Q + \langle \lambda_\Gamma^\tau q_\Gamma^\tau, v_\Gamma\rangle_\Sigma \to \langle \Lambda, v\rangle_Q + \langle \Lambda_\Gamma, v_\Gamma\rangle_\Sigma \quad \text{for every } (v, v_\Gamma)\in \mathcal{W}_0. \tag{3.19}$$

$iii)$ With such a choice of $(\Lambda, \Lambda_\Gamma)$, the pair (q, q_Γ) solves (2.52)–(2.53).

$iv)$ Condition (2.54) holds.

The main tool is proving a priori estimates. To this concern, we use the following rule to denote constants in order to avoid a boring notation. The small-case symbol c stands for different constants that neither depend on τ nor on the functions whose norm we want to estimate. Hence, the meaning of c might change from line to line and even in the same chain of equalities or inequalities. Similarly, a symbol like c_δ denotes different constants that depend on the parameter δ, in addition.

First a priori estimate. As $\overline{u}_\Gamma^\tau \in \mathcal{U}_{\rm ad}$ and Theorem 2.2 holds, we have

$$\begin{aligned}
&\|\tilde{u}_\Gamma^\tau\|_{H^1(0,T;H_\Gamma)} + \|\tilde{y}^\tau\|_{H^1(0,T;V^*)\cap L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))} \\
&\quad + \|\tilde{y}_\Gamma^\tau\|_{H^1(0,T;H_\Gamma)\cap L^\infty(0,T;V_\Gamma)\cap L^2(0,T;H^2(\Gamma))} + \|\tilde{w}^\tau\|_{L^2(0,T;V)} \\
&\quad + \|f'(\tilde{y}^\tau)\|_{L^2(0,T;H)} + \|f'_\Gamma(\tilde{y}_\Gamma^\tau)\|_{L^2(0,T;H_\Gamma)} + \tau^{1/2}\|\partial_t\tilde{y}^\tau\|_{L^2(0,T;H)} \le c\,. \qquad (3.20)
\end{aligned}$$

Second a priori estimate. For the reader's convenience, we explicitly write the adjoint system mentioned in Proposition 3.4, as well as the regularity of its solution,

$$q^\tau \in H^1(0,T;H)\cap L^2(0,T;H^2(\Omega)), \quad q_\Gamma^\tau \in H^1(0,T;H_\Gamma)\cap L^2(0,T;H^2(\Gamma)) \qquad (3.21)$$

$$(q^\tau, q_\Gamma^\tau)(s) \in \mathcal{V}_\Omega \quad \text{for every } s\in[0,T] \qquad (3.22)$$

$$\begin{aligned}
&-\int_\Omega \partial_t\big(\mathcal{N}(q^\tau(s)) + \tau q^\tau(s)\big)\, v + \int_\Omega \nabla q^\tau(s)\cdot\nabla v + \int_\Omega \lambda^\tau(s)\, q^\tau(s)\, v \\
&\quad - \int_\Gamma \partial_t q_\Gamma^\tau(s)\, v_\Gamma + \int_\Gamma \nabla_\Gamma q_\Gamma^\tau(s)\cdot\nabla_\Gamma v_\Gamma + \int_\Gamma \lambda_\Gamma^\tau(s)\, q_\Gamma^\tau(s)\, v_\Gamma \\
&= \int_\Omega \varphi_Q^\tau(s) v + \int_\Gamma \varphi_\Sigma^\tau(s) v_\Gamma \quad \text{for a.e. } s\in(0,T) \text{ and every } (v,v_\Gamma)\in\mathcal{V}_\Omega \qquad (3.23)
\end{aligned}$$

where $\lambda^\tau = f''(\tilde{y}^\tau)$, $\lambda_\Gamma^\tau = f''_\Gamma(\tilde{y}_\Gamma^\tau)$, $\varphi_Q^\tau = b_Q(\tilde{y}^\tau - z_Q)$ and $\varphi_\Sigma^\tau = b_\Sigma(\tilde{y}_\Gamma^\tau - z_\Sigma)$

$$\int_\Omega \big(\mathcal{N}q^\tau + \tau q^\tau\big)(T)\, v + \int_\Gamma q_\Gamma(T)\, v_\Gamma = 0 \quad \text{for every } (v,v_\Gamma)\in\mathcal{V}_\Omega. \qquad (3.24)$$

Now, we choose $v = q^\tau(s)$ and $v_\Gamma = q_\Gamma^\tau(s)$, and integrate over (t,T) with respect to s. Recalling (2.35) and now reading $Q_t := \Omega\times(t,T)$ and $\Sigma_t := \Gamma\times(t,T)$, we have

$$\begin{aligned}
&\frac12\|q^\tau(t)\|_*^2 + \frac{\tau}{2}\int_\Omega |q^\tau(t)|^2 + \int_{Q_t}|\nabla q^\tau|^2 + \int_{Q_t}\lambda^\tau|q^\tau|^2 \\
&\quad + \frac12\int_\Gamma |q_\Gamma^\tau(t)|^2 + \int_{\Sigma_t}|\nabla_\Gamma q_\Gamma^\tau|^2 + \int_{\Sigma_t}\lambda_\Gamma^\tau|q_\Gamma^\tau|^2 \\
&= \int_{Q_t}\varphi_Q^\tau\, q^\tau + \int_{\Sigma_t}\varphi_\Sigma^\tau\, q_\Gamma^\tau \le \int_Q|\varphi_Q^\tau|^2 + \int_{Q_t}|q^\tau|^2 + \int_\Sigma|\varphi_\Sigma^\tau|^2 + \int_{\Sigma_t}|q_\Gamma^\tau|^2 \\
&\le \int_{Q_t}|q^\tau|^2 + \int_{\Sigma_t}|q_\Gamma^\tau|^2 + c \qquad (3.25)
\end{aligned}$$

where the last inequality follows from (3.20). By accounting for (2.5), we also have

$$\int_{Q_t}\lambda^\tau|q^\tau|^2 \ge -c\int_{Q_t}|q^\tau|^2 \quad \text{and} \quad \int_{Q_t}\lambda_\Gamma^\tau|q_\Gamma^\tau|^2 \ge -c\int_{\Sigma_t}|q_\Gamma^\tau|^2.$$

We treat the volume integral (and the same on the right-hand side of (3.25)) invoking the compact embedding $V\subset H$. We have

$$\int_\Omega |v|^2 \le \delta\int_\Omega|\nabla v|^2 + c_\delta\|v\|_*^2 \quad \text{for every } v\in V \text{ and } \delta>0.$$

Hence, we deduce that

$$\int_{Q_t} |q^\tau|^2 \le \delta \int_{Q_t} |\nabla q^\tau|^2 + c_\delta \int_t^T \|q^\tau(s)\|_*^2 \, ds \, .$$

Therefore, by combining, choosing δ small enough and applying the backward Gronwall lemma, we conclude that

$$\|q^\tau\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)} + \|q_\Gamma^\tau\|_{L^\infty(0,T;H_\Gamma)\cap L^2(0,T;V_\Gamma)} + \tau^{1/2}\|q^\tau\|_{L^\infty(0,T;H)} \le c \, . \tag{3.26}$$

Third a priori estimate. Take an arbitrary pair $(v, v_\Gamma) \in H^1(0,T;\mathcal{H}_\Omega) \cap L^2(0,T;\mathcal{V}_\Omega)$, and test (3.23) by $v(s)$ and $v_\Gamma(s)$. Then, we sum over $s \in (0,T)$ and integrate by parts with the help of (3.24), so that no integral related to the time T appears. In particular, if $(v, v_\Gamma) \in \mathcal{W}_0$, even the terms evaluated at $t = 0$ vanish and we obtain that

$$\int_Q (\mathcal{N}q^\tau + \tau q^\tau)\partial_t v + \int_Q \nabla q^\tau \cdot \nabla v + \int_Q \lambda^\tau q^\tau v + \int_\Sigma q_\Gamma^\tau \partial_t v_\Gamma + \int_\Sigma \nabla q_\Gamma^\tau \cdot \nabla v_\Gamma + \int_\Sigma \lambda^\tau q_\Gamma^\tau v_\Gamma$$
$$= \int_Q \varphi_Q^\tau \, v + \int_\Sigma \varphi_\Sigma^\tau \, v_\Gamma \, . \tag{3.27}$$

Therefore, we have, for every $(v, v_\Gamma) \in \mathcal{W}_0$,

$$\begin{aligned} &\left| \int_Q \lambda^\tau q^\tau v + \int_\Sigma \lambda^\tau q_\Gamma^\tau v_\Gamma \right| \\ &\le \|\mathcal{N}q^\tau + \tau q^\tau\|_{L^2(0,T;V)} \, \|\partial_t v\|_{L^2(0,T;V^*)} + \|q^\tau\|_{L^2(0,T;V)} \, \|v\|_{L^2(0,T;V)} \\ &\quad + \|q_\Gamma^\tau\|_{L^2(0,T;V_\Gamma)} \, \|\partial_t v_\Gamma\|_{L^2(0,T;V_\Gamma^*)} + \|q_\Gamma^\tau\|_{L^2(0,T;V_\Gamma)} \, \|v_\Gamma\|_{L^2(0,T;V_\Gamma)} \\ &\quad + \|\varphi_Q^\tau\|_{L^2(0,T;H)} \, \|v\|_{L^2(0,T;H)} + \|\varphi_\Sigma^\tau\|_{L^2(0,T;H_\Gamma)} \, \|v_\Gamma\|_{L^2(0,T;H_\Gamma)} \, . \end{aligned}$$

Now, by assuming $\tau \le 1$, we have $\|\mathcal{N}v + \tau v\|_V \le c\|v\|_* + \tau\|v\|_V \le c\|v\|_V$ for every $v \in V$ with zero mean value (see (2.36)). Therefore, by accounting for (3.20) and (3.26), we conclude that

$$\left| \int_Q \lambda^\tau q^\tau v + \int_\Sigma \lambda^\tau q_\Gamma^\tau v_\Gamma \right| \le c \, \|(v, v_\Gamma)\|_{\mathcal{W}} \quad \text{for every } (v, v_\Gamma) \in \mathcal{W}_0. \tag{3.28}$$

Conclusion of the proof of Theorem 2.7. From the above estimates, we infer that

$$\tilde{u}_\Gamma^\tau \to u_\Gamma \qquad \text{weakly star in } H^1(0,T;H_\Gamma) \cap L^\infty(\Sigma) \tag{3.29}$$

$$\tilde{y}^\tau \to y \qquad \text{weakly star in } H^1(0,T;V^*) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \text{ and strongly in } C^0([0,T];H) \cap L^2(0,T;V) \tag{3.30}$$

$$\tilde{y}_\Gamma^\tau \to y_\Gamma \qquad \text{weakly star in } H^1(0,T;H_\Gamma) \cap L^\infty(0,T;V_\Gamma) \cap L^2(0,T;H^2(\Gamma)) \text{ and strongly in } C^0([0,T];H_\Gamma) \cap L^2(0,T;V_\Gamma) \tag{3.31}$$

$$\tilde{w}^\tau \to w \qquad \text{weakly star in } L^2(0,T;V) \tag{3.32}$$

$$q^\tau \to q \qquad \text{weakly star in } L^\infty(0,T;V^*) \cap L^2(0,T;V) \tag{3.33}$$

$$q_\Gamma^\tau \to q_\Gamma \qquad \text{weakly star in } L^\infty(0,T;H_\Gamma) \cap L^2(0,T;V_\Gamma) \tag{3.34}$$

$$\tau q^\tau \to 0 \qquad \text{strongly in } L^\infty(0,T;H) \tag{3.35}$$

at least for a subsequence, and (y, y_Γ, w) is the solution to the problem (2.14)–(2.20) corresponding to u_Γ, thanks to Proposition 3.1. Notice that (3.33)–(3.34) coincide with (3.16)–(3.17) and that (3.12)–(3.15) hold once we prove that $u_\Gamma = \overline{u}_\Gamma$ and that $\tilde{u}_\Gamma^\tau$ converges strongly in $L^2(\Sigma)$.

To this end, we recall the notations (3.7)–(3.9), and it is understood that all the limits we write are referred to the selected subsequence. By optimality, we have

$$\mathcal{J}(\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma) \le \mathcal{J}(y, y_\Gamma, u_\Gamma) \quad \text{and} \quad \tilde{\mathcal{J}}(\tilde{y}^\tau, \tilde{y}_\Gamma^\tau, \tilde{u}_\Gamma^\tau) \le \tilde{\mathcal{J}}(\overline{y}^\tau, \overline{y}_\Gamma^\tau, \overline{u}_\Gamma).$$

On the other hand, (3.29)–(3.31) and Proposition 3.1 applied with $u_\Gamma^\tau = \overline{u}_\Gamma$ yield

$$\tilde{\mathcal{J}}(y, y_\Gamma, u_\Gamma) \le \liminf \tilde{\mathcal{J}}(\tilde{y}^\tau, \tilde{y}_\Gamma^\tau, \tilde{u}_\Gamma^\tau) \quad \text{and} \quad \lim \mathcal{J}(\overline{y}^\tau, \overline{y}_\Gamma^\tau, \overline{u}_\Gamma) = \mathcal{J}(\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma).$$

By combining, we deduce that

$$\begin{aligned}
&\mathcal{J}(\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma) + \frac{1}{2}\|u_\Gamma - \overline{u}_\Gamma\|_{L^2(\Sigma)}^2 \le \mathcal{J}(y, y_\Gamma, u_\Gamma) + \frac{1}{2}\|u_\Gamma - \overline{u}_\Gamma\|_{L^2(\Sigma)}^2 \\
&= \tilde{\mathcal{J}}(y, y_\Gamma, u_\Gamma) \le \liminf \tilde{\mathcal{J}}(\tilde{y}^\tau, \tilde{y}_\Gamma^\tau, \tilde{u}_\Gamma^\tau) \le \limsup \tilde{\mathcal{J}}(\tilde{y}^\tau, \tilde{y}_\Gamma^\tau, \tilde{u}_\Gamma^\tau) \\
&\le \limsup \tilde{\mathcal{J}}(\overline{y}^\tau, \overline{y}_\Gamma^\tau, \overline{u}_\Gamma) = \limsup \mathcal{J}(\overline{y}^\tau, \overline{y}_\Gamma^\tau, \overline{u}_\Gamma) = \mathcal{J}(\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma).
\end{aligned}$$

By comparing the first and last terms of this chain, we infer that the $L^2(\Sigma)$-norm of $u_\Gamma - \overline{u}_\Gamma$ vanishes, whence $u_\Gamma = \overline{u}_\Gamma$, as desired. In order to prove the strong convergence mentioned in (3.12), we observe that the above argument also shows that

$$\liminf \tilde{\mathcal{J}}(\tilde{y}^\tau, \tilde{y}_\Gamma^\tau, \tilde{u}_\Gamma^\tau) = \limsup \tilde{\mathcal{J}}(\tilde{y}^\tau, \tilde{y}_\Gamma^\tau, \tilde{u}_\Gamma^\tau) = \mathcal{J}(\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma).$$

Notice that this coincides with (3.18). From the strong convergence given by (3.13) and (3.14), and by comparison, we deduce that

$$\lim \left(\frac{b_0}{2} \int_\Sigma |\tilde{u}_\Gamma^\tau|^2 + \frac{1}{2} \int_\Sigma |\tilde{u}_\Gamma^\tau - \overline{u}_\Gamma|^2 \right) = \frac{b_0}{2} \int_\Sigma |\overline{u}_\Gamma|^2 ,$$

whence also

$$\begin{aligned}
&\limsup \frac{b_0}{2} \int_\Sigma |\tilde{u}_\Gamma^\tau|^2 \le \limsup \left(\frac{b_0}{2} \int_\Sigma |\tilde{u}_\Gamma^\tau|^2 + \frac{1}{2} \int_\Sigma |\tilde{u}_\Gamma^\tau - \overline{u}_\Gamma|^2 \right) \\
&= \frac{b_0}{2} \int_\Sigma |\overline{u}_\Gamma|^2 \le \liminf \frac{b_0}{2} \int_\Sigma |\tilde{u}_\Gamma^\tau|^2.
\end{aligned}$$

Therefore, we have

$$\lim \frac{b_0}{2} \int_\Sigma |\tilde{u}_\Gamma^\tau|^2 = \frac{b_0}{2} \int_\Sigma |\overline{u}_\Gamma|^2 , \quad \text{whence} \quad \lim \frac{1}{2} \int_\Sigma |\tilde{u}_\Gamma^\tau - \overline{u}_\Gamma|^2 = 0 ,$$

and (3.12)–(3.15) are completely proved.

Now, we deal with the limit (q, q_Γ) given by (3.33)–(3.34), i.e., (3.16)–(3.17). Clearly, (2.52) holds as well. Furthermore, as $\|\mathcal{N} v_*\|_V \le c\|v_*\|_*$ for every $v_* \in V^*$ with zero mean value (see (2.33)), and since the convergence (3.35) holds, we also have

$$\mathcal{N} q^\tau + \tau q^\tau \to \mathcal{N} q \quad \text{weakly star in } L^\infty(0, T; H).$$

On the other hand, (3.28) implies that the functionals $F^\tau \in \mathcal{W}_0^*$ defined by

$$\langle\!\langle F^\tau, (v, v_\Gamma) \rangle\!\rangle := \langle \lambda^\tau q^\tau, v \rangle_Q + \langle \lambda_\Gamma^\tau q_\Gamma^\tau, v_\Gamma \rangle_\Sigma \,,$$

i.e., the functionals associated with $(\lambda^\tau q^\tau, \lambda_\Gamma^\tau, q_\Gamma^\tau)$ as in Proposition 2.6, are bounded in $\mathcal{W}_0^*$. Therefore, for a subsequence, we have $F^\tau \to F$ weakly star in $\mathcal{W}_0^*$, where F is some element of $\mathcal{W}_0^*$. Hence, if we represent F as stated in Proposition 2.6, we find Λ and Λ_Γ satisfying (2.48) and (3.19). At this point, it is straightforward to pass to the limit in (3.27) and in (3.10) to obtain both (2.53) and (2.54). This completes the proof of Theorem 2.7. □

Remark 3.5. The above proof can be repeated without any change starting from any sequence $\tau_n \searrow 0$. By doing that, we obtain: there exists a subsequence $\{\tau_{n_k}\}$ such that (3.12)–(3.18) hold along the selected subsequence. As the limits $\overline{u}_\Gamma$, $\overline{y}$, $\overline{y}_\Gamma$, $\overline{w}$ and $\mathcal{J}(\overline{y}, \overline{y}_\Gamma, \overline{u}_\Gamma)$ are always the same, this proves that in fact (3.12)–(3.15) as well as (3.18) hold for the whole family. On the contrary, the limits q and q_Γ might depend on the selected subsequence since no uniqueness result for the adjoint problem is known. Nevertheless, the necessary optimality condition (2.54) holds for every solution (q, q_Γ) to the adjoint problem that can be found as a limit of pairs (q^τ, q_Γ^τ) as specified in the above proof.

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