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**Spectral properties of limiting solitons in optical fibers**

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## Abstract

It seems to be self-evident that stable optical pulses cannot be considerably shorter than a single oscillation of the carrier field. From the mathematical point of view the solitary solutions of pulse propagation equations should lose stability or demonstrate some kind of singular behavior. Typically, an unphysical cusp develops at the soliton top, preventing the soliton from being too short. Consequently, the power spectrum of the limiting solution has a special behavior: the standard exponential decay is replaced by an algebraic one. We derive the shortest soliton and explicitly calculate its spectrum for the so-called short pulse equation. The latter applies to ultra-short solitons in transparent materials like fused silica that are relevant for optical fibers.

## 1 Introduction

Description of short optical pulses is usually based on the generalized nonlinear Schrödinger equation (GNLSE) [1] which yields dynamics of single- and even sub-cycle pulses [2, 3]. Another attractive option is to use a special non-envelope propagation model, one of the so-called short pulse equations (SPEs). The latter are derived using a simplified dispersion law. For instance, Schäfer and Wayne [4] noticed that refractive index of bulk fused silica can be precisely approximated by an extremely simple expression

$$\epsilon(\omega) = n_s^2(1 - \omega_p^2/\omega^2), \quad (1)$$

for all frequencies that are relevant for optical solitons. As opposed by the Drude dispersion law,  $n_s$  and  $\omega_p$  are just fit parameters. For instance, selecting them as

$$n_s = 1.4538 \quad \text{and} \quad \omega_p = 136.85 \text{ THz}, \quad (2)$$

one gains the relative accuracy of  $\lesssim 0.15\%$  between  $0.9 \mu\text{m}$  and  $3.6 \mu\text{m}$ , i.e., for two octaves in fused silica. As opposed by the Taylor expansion of the dispersion relation in the GNLSE, i.e., a local approach that is very accurate but only within the convergence radius at the carrier frequency [5], Eq. (1) pretends to be a global approximation. For instance, one can apply the Kramers Kronig relations and derive that the imaginary part of  $\epsilon(\omega)$  is proportional to  $\delta'(\omega)$ . Therefore the low-frequency components of the electric field should dissipate and the electric area  $\int_{-\infty}^{\infty} E(z, t) dt = 0$  of any solitary solution must vanish.

An useful feature of the simplified propagation equations for short pulses is that they yield analytic solutions for optical solitons [6, 7]. In some cases even the full solution is available [8]. In particular, one can address the following fundamental question: what happens with an optical soliton when its duration approaches a single oscillation of the carrier field? The known

analytical [9, 10, 11] and numerical solutions [12] suggest the following feature: a singular cusp develops at the soliton top, preventing the soliton from being too short. Such peaked solitons have originally been found outside optics, for the shallow water waves [13].

In the frequency domain the limiting shortest soliton is characterized by the following property: an exponential decay of the spectral power is replaced by a rational one. This behavior is of special interest because pulse spectrum is easier to measure than the actual value of the electric field inside the pulse. In what follows we first revisit the derivation of the SPE and its limiting soliton [10, 14] and then explicitly calculate corresponding spectra.

## 2 Complex short pulse equation

In this section we briefly outline the derivation of the real [4] and complex [10, 14] SPEs. For the first time we will demonstrate that transition to a more simple complex SPE is actually dictated by the dispersion law (1).

Propagation of a short pulse in a one-dimensional setting, e.g., in a single-mode polarization-preserving optical fiber, is approximated by a nonlinear wave equation for the pulse electric field

$$\partial_t^2 (\hat{\epsilon}E + \chi^{(3)}E^3) - c^2 \partial_z^2 E = 0, \quad \text{where} \quad \hat{\epsilon} \left[ \sum_{\omega} E_{\omega} e^{-i\omega t} \right] = \sum_{\omega} \epsilon(\omega) E_{\omega} e^{-i\omega t}.$$

The nonlinear susceptibility of the third order  $\chi^{(3)}$  is taken constant. Using Eq. (1) one derives the following self-consistent equation

$$n_s^2 (\partial_t^2 E + \omega_p^2 E) - c^2 \partial_z^2 E + \chi^{(3)} \partial_t^2 (E^3) = 0, \quad (3)$$

with the following dispersion law  $\beta(\omega) = (n_s/c)(\omega^2 - \omega_p^2)^{1/2}$  for a linear  $e^{i(\beta z - \omega t)}$  wave.

Note, that typical carrier frequencies of optical solitons are considerably larger than the fit parameter  $\omega_p$  from Eq. (2). For instance, a standard  $1.4 \mu\text{m}$  wavelength corresponds to  $10\omega_p$ . Therefore  $\beta(\omega) \approx (n_s/c)[\omega - \omega_p^2/(2\omega)]$  and one concludes that the third harmonics generation (THG) process is a non-resonant one for our system. Indeed, the corresponding resonance conditions

$$\omega_1 + \omega_2 + \omega_3 = \omega \quad \text{and} \quad \frac{1}{\omega_1} + \frac{1}{\omega_2} + \frac{1}{\omega_3} = \frac{1}{\omega},$$

are incompatible for any four positive frequencies.

The SPE results from the asymptotic expansion of Eq. (3) with respect to  $\mu = \omega_p/\omega_0 \ll 1$ . Actually, we will keep  $O(\mu^2)$  terms and neglect  $O(\mu^4)$  ones. To derive the SPE we first consider a reference carrier wave (the slowly varying envelope approximation is not assumed)  $e^{i(\beta_0 z - \omega_0 t)}$  and calculate the phase

$$\omega_0 t - \beta_0 z = \omega_0 \left( t - \frac{n_s}{c} z \sqrt{1 - \mu^2} \right) = \omega_0 \left( t - \frac{n_s}{c} z \right) + \mu^2 \frac{n_s \omega_0}{2c} z + O(\mu^4).$$

Motivated by this expression we introduce the normalized field  $F(\zeta, \tau)$  and assume the following scaling of the new variables

$$\tau = \omega_0 \left( t - \frac{n_s}{c} z \right), \quad \zeta = \mu^2 \frac{n_s \omega_0}{c} z, \quad \sqrt{\frac{3\chi^{(3)}}{8}} E(z, t) = \mu n_s F(\zeta, \tau).$$

A simple calculation yields that Eq. (3) takes the following form

$$2\partial_\zeta \partial_\tau F + F + \frac{8}{3} \partial_\tau^2 (F^3) = \mu^2 \partial_\zeta^2 F.$$

After neglecting the last term one obtains the real SPE [4]. To get the complex one we introduce a complex-valued electric field  $\mathcal{F}(\zeta, \tau)$  such that by construction

$$\partial_\zeta \partial_\tau \mathcal{F} + \frac{1}{2} \mathcal{F} + \frac{1}{3} \partial_\tau^2 (3|\mathcal{F}|^2 \mathcal{F} + \mathcal{F}^3) = 0.$$

It is easy to see that  $F = (\mathcal{F} + \mathcal{F}^*)/2$  is a valid solution of the real SPE. The clear advantage of the complex representation is that the self-phase modulation (SPM) term and the THG term are now separated. Neglecting the non-resonant THG term we obtain the complex SPE

$$\partial_\zeta \partial_\tau \mathcal{F} + \frac{1}{2} \mathcal{F} + \partial_\tau^2 (|\mathcal{F}|^2 \mathcal{F}) = 0. \quad (4)$$

Another physical situation that directly leads to the complex SPE appears for a wave with circular polarization. Two components of the real electric field are just combined into a single complex field. Thereafter one obtains the complex SPE (4) directly, without referring to the real equation [10, 14].

To conclude this section we note that strictly speaking Eq. (4) presupposes that  $\mathcal{F}$  contains mainly positive (and  $\mathcal{F}^*$  mainly negative) frequencies. That is, the initial condition  $\mathcal{F}(0, \tau)$  is chosen to be exactly the positive frequency part of  $F(0, \tau)$  in accord with the most general definition of the complex envelope [15, 1]. In the course of pulse propagation, the complex field  $\mathcal{F}(z, \tau)$  accrues some small negative-frequency part generated by the SPM term, however, this process is non-resonant such that the negative frequencies are not further amplified.

### 3 Solitary solutions of the complex SPE

For a linear  $\mathcal{F} \sim \exp[i(\kappa\zeta - \nu\tau)]$  wave in which the dimensionless frequency  $\nu$  corresponds to the physical frequency  $\nu\omega_0$  and the reference wave corresponds to  $\nu = 1$ , Eq. (4) yields  $\kappa = -1/(2\nu)$  such that  $V_{\text{ph}} = -2\nu^2$  and  $V_{\text{gr}} = 2\nu^2$ . In particular,  $V_{\text{ph}} + V_{\text{gr}} = 0$ . Although both velocities slightly change for a solitary wave, they still have equal magnitudes. These nonlinear phase and group velocities will be denoted by  $-2\gamma\nu^2$  and  $2\gamma\nu^2$ , respectively. Parameter  $\gamma$  defines soliton's shape for a given carrier frequency  $\nu$ . We will try the following substitution

$$\mathcal{F}(\zeta, \tau) = f \left( \nu\tau - \frac{\zeta}{2\gamma\nu} \right) \exp \left[ -i \left( \nu\tau + \frac{\zeta}{2\gamma\nu} \right) \right],$$

where  $f(\xi)$  is the complex shape function that depends also on  $\gamma$  and  $\nu$ . Substituting  $\mathcal{F}(\zeta, \tau)$  into Eq. (4) we obtain the following complex ordinary differential equation for the shape function  $f(\xi)$

$$(f - 2\gamma\nu^2|f|^2f)'' - (\gamma - 1)f + 4i\gamma\nu^2(|f|^2f)' + 2\gamma\nu^2|f|^2f = 0,$$

where prime denotes derivative with respect to  $\xi$ . At the soliton tails, where  $f(\xi) \rightarrow 0$ , we have  $f'' = (\gamma - 1)f$  such that localized solitary solutions require  $\gamma > 1$  and soliton duration is proportional to  $(\gamma - 1)^{-1/2}$ .

To proceed we split the amplitude and the phase

$$f(\xi) = \frac{a(\xi)}{\nu\sqrt{\gamma}} e^{i\phi(\xi)}$$

and obtain two real ordinary differential equations for  $a(\xi)$  and  $\phi(\xi)$ . The phase equation can be integrated and yields

$$\phi' = -\frac{(3 - 4a^2)a^2}{(1 - 2a^2)^2}. \quad (5)$$

The amplitude equation

$$A'' - (\gamma - 1)a - \phi'^2 A - 4\phi' a^3 + 2a^3 = 0 \quad \text{with} \quad A \equiv a - 2a^3,$$

can be multiplied with  $A'$  and integrated as well. Restricting ourselves with the localized solutions we obtain a “mechanical” equation for some effective potential  $U(a)$

$$a'^2 + U(a) = 0, \quad U(a) = -(\gamma - 1)\frac{(1 - 3a^2)a^2}{(1 - 6a^2)^2} + \frac{(1 - 7a^2 + 12a^4)a^4}{(1 - 2a^2)^2(1 - 6a^2)^2}. \quad (6)$$

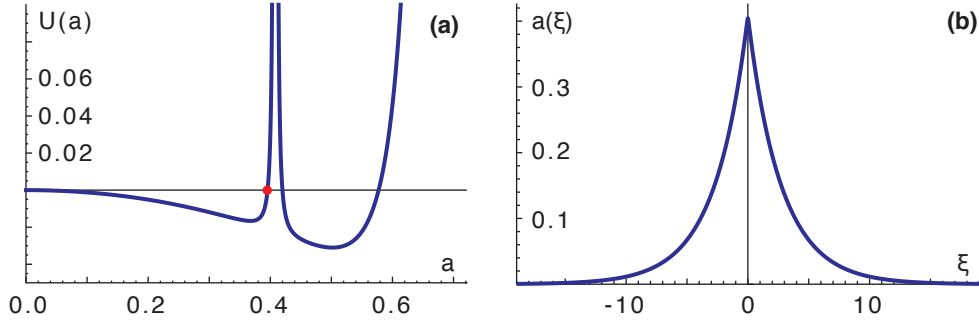


Figure 1: (a) An exemplary potential  $U(a)$  from Eq. (6) for  $\gamma = 9/8 - \delta$  with  $\delta = 10^{-3}$ . The red point labels the upper value of  $a(\xi)$  from the inequality (7). As  $\delta \rightarrow 0$ , an infinite wall is formed at  $a = 1/\sqrt{6}$ , resulting in cusp formation at the top of the soliton. (b) Shape of the shortest soliton calculated from Eq. (8).

An exemplary shape of  $U(a)$  for  $\gamma$  slightly smaller than  $9/8$  is shown in Fig. 1a. The target soliton starts from the equilibrium position  $a = 0$  for  $\xi \rightarrow -\infty$  and returns to it for  $\xi \rightarrow +\infty$ .

Physically acceptable solutions require  $U(a) \leq 0$ . The solution of the latter inequality that is “connected” to the  $a = 0$  state is given by

$$0 \leq a \leq \sqrt{\frac{1}{2} - \frac{1}{3 - \sqrt{9 - 8\gamma}}}, \quad (7)$$

the right limit is labelled by the red point in Fig. 1a. The target solitary solution exists for  $1 \leq \gamma \leq 9/8$ . Classical solitons appear for  $\gamma = 1 + \varepsilon^2$  with  $\varepsilon \ll 1$ , then  $a = O(\varepsilon)$ . It is easy to see that in the leading term  $a(\xi) = \varepsilon / \cosh(\varepsilon\xi)$ . As  $\gamma$  increases the soliton becomes shorter, the final limiting soliton is obtained for  $\gamma \rightarrow 9/8$  and is yielded by the equation

$$a'^2 = \frac{a^2(1 - 3a^2)}{8(1 - 2a^2)^2} \quad \text{with} \quad a(0) = \frac{1}{\sqrt{6}} \quad \Rightarrow \quad ae^{\mathfrak{B}(a)} = \Lambda e^{-\frac{|\xi|}{2\sqrt{2}}}, \quad (8)$$

where

$$\mathfrak{B}(a) \equiv \left[ \frac{2}{3} \sqrt{1 - 3a^2} - \ln \left( 1 + \sqrt{1 - 3a^2} \right) \right]_0^a, \quad \Lambda \equiv \frac{e^{\mathfrak{B}(1/\sqrt{6})}}{\sqrt{6}} \approx 0.3935.$$

The phase dependence of the limiting soliton is calculated from Eq. (5). Without loss of generality we take  $\phi(0) = 0$ , then  $\phi(-\xi) = -\phi(\xi)$ . The shape of  $\phi(\xi)$  is shown in Fig. 2a. It is of interest that the limiting values  $\phi(\pm\infty) = \pm\phi_\infty$  can be calculated analytically from (5) and (8)

$$\phi_\infty = \int_0^\infty \phi'(\xi) d\xi = - \int_0^\infty \frac{(3 - 4a^2)a^2}{(1 - 2a^2)^2} \left( \frac{d\xi}{da} da \right) = - \frac{4(\sqrt{2} - 1)}{3} - \arcsin \frac{1}{3} \approx -0.8921,$$

the quantity  $\phi_\infty$  appears in the equation for the soliton spectrum that will be derived in the next section.

## 4 The limiting spectrum

In this section we explicitly calculate the spectrum of the shortest soliton. At first glance, the key Eq. (8) looks too complicated and numerical treatment of the soliton spectrum seems to be the only choice. Actually, this is not the case. Our analytic approach is based on the observation that for all relevant amplitudes,  $0 \leq a \leq 1/\sqrt{6}$ , the factor  $e^{\mathfrak{B}(a)}$  from Eq. (8) is a slow function that is close to 1. This happens just because  $\mathfrak{B}(0) = 0$  and  $\mathfrak{B}(1/\sqrt{6}) \approx -0.037$ . Actually  $a(\xi) \approx \Lambda e^{-|\xi|/(2\sqrt{2})}$  is already a very good approximation to the full solution of Eq. (8) that is shown in Fig. 1b. As a natural generalization we consider  $\xi > 0$  and expand the amplitude  $a(\xi) = a(-\xi)$  and phase  $\phi(\xi) = -\phi(-\xi)$  in a power series with respect to  $X = \Lambda e^{-\xi/(2\sqrt{2})}$ .

With the help of any computer algebra system it is easy to obtain the equation

$$X = ae^{\mathfrak{B}(a)} = a - \frac{a^3}{4} + \frac{a^5}{8} + \frac{49a^7}{192} + \dots$$

and to verify that the later equation is inverted as

$$a = X + \frac{X^3}{4} + \frac{X^5}{16} - \frac{61X^7}{192} + \dots$$

The last equation is inserted into (5) and after direct integration one obtains the phase

$$\phi = \phi_\infty + 3\sqrt{2}X^2 + \frac{19X^4}{2\sqrt{2}} + \frac{457X^6}{24\sqrt{2}} + \dots,$$

and the full solution for  $\xi > 0$

$$a(\xi)e^{i\phi(\xi)} = e^{i\phi_\infty} \left( X + \frac{1 + 12i\sqrt{2}}{4}X^3 + \frac{-143 + 88i\sqrt{2}}{16}X^5 + \dots \right), \quad (9)$$

where  $\phi_\infty$  was calculated in the end of the previous section. Recall that the solution for  $\xi < 0$  immediately follows from the fact that  $a(\xi)$  and  $\phi(\xi)$  are even and odd functions respectively.

Finally the expansion (9), which can be extended to any power, is used to obtain the spectrum

$$(ae^{i\phi})_\Omega = \int_{-\infty}^{\infty} a(\xi)e^{i\phi(\xi)}e^{i\Omega\xi}d\xi = \int_0^{\infty} a(\xi)e^{i\phi(\xi)}e^{i\Omega\xi}d\xi + \text{c.c.},$$

where  $\Omega$  refers to the normalized frequency. The result reads

$$(ae^{i\phi})_\Omega = \Lambda \frac{R_1 + S_1\Omega}{1 + 8\Omega^2} + \Lambda^3 \frac{R_3 + S_3\Omega}{9 + 8\Omega^2} + \Lambda^5 \frac{R_5 + S_5\Omega}{25 + 8\Omega^2} + \dots, \quad (10)$$

where

$$\begin{aligned} R_1 &= 4\sqrt{2}\cos\phi_\infty, & S_1 &= -16\sin\phi_\infty, \\ R_3 &= 3\sqrt{2}\cos\phi_\infty - 72\sin\phi_\infty, & S_3 &= -48\sqrt{2}\cos\phi_\infty - 4\sin\phi_\infty, \\ R_5 &= -(715/4)\sqrt{2}\cos\phi_\infty - 220\sin\phi_\infty, & S_5 &= -88\sqrt{2}\cos\phi_\infty + 143\sin\phi_\infty. \end{aligned}$$

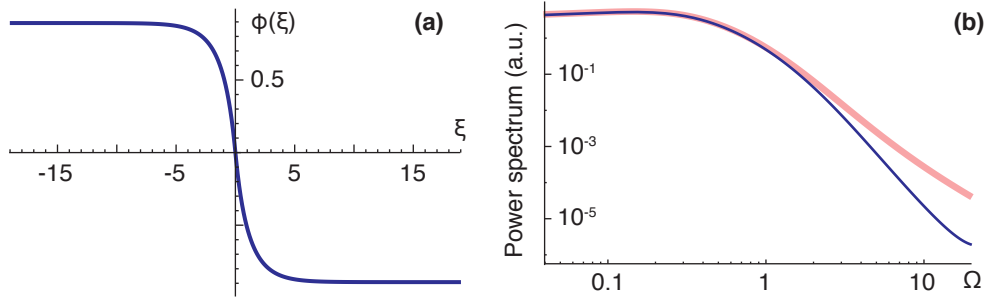


Figure 2: (a) Phase of the limiting soliton shown in Fig. 1b. (b) Power spectrum of the limiting soliton. Blue line shows numerical solution, red line is the asymptotic expression (10).

## 5 Conclusions

Our main result is a simple explicit expression (10) for the spectrum of the shortest soliton. As expected, the spectral power has an algebraic decay, as opposed by the exponential one for ordinary solitons. An important point is that shape of the spectrum is non-trivial, such a function is difficult to reconstruct from the purely numerical solution. Figure 2 shows predictions of Eq. (10) and compares them with the numerical results for the soliton spectrum. The agreement is reasonably good and can be easily improved by calculating further terms in Eq. (10).



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