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Hölder estimates for second-order operators with mixed boundary conditions

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Abstract

In this paper we investigate linear elliptic, second-order boundary value problems with mixed boundary conditions. Assuming only boundedness/ellipticity on the coefficient function and very mild conditions on the geometry of the domain – including a very weak compatibility condition between the Dirichlet boundary part and its complement – we prove first Hölder continuity of the solution. Secondly, Gaussian Hölder estimates for the corresponding heat kernel are derived. The essential instruments are De Giorgi and Morrey-Campanato estimates.

1 Introduction

Hölder continuity is one of the classical features in the theory of elliptic and parabolic equations. Based on the pioneering ideas of De Giorgi, Nash, Stampacchia and Morrey, the Dirichlet problem for second-order divergence operators with real coefficients was elaborated in clarity and beauty: under general and traceable conditions, Hölder continuity of the solution is proved, see [KS, Chapter II.B.4] or [LU, Chapter III.14], as long as the boundary condition is pure Dirichlet. Since in recent years it became manifest that the appearance of mixed boundary conditions is not an exception when modelling real world problems (see e.g. [Sel] or [HMRR]) one should, of course, also treat this case. Here the situation is less satisfactory: on the one hand, there is the fundamental paper of Stampacchia [Sta1], where under very general conditions also Hölder continuity is proved. Unfortunately, the conditions of the main theorem are very implicit and extremely difficult to control in examples if the geometry of the underlying domain becomes complicated. On the other hand, there are several articles ([CV], [Ibr], [Nov], [Fio], [Lie1], [HMRS]) where Hölder continuity for the solution is proved under different assumptions on the geometry of the domain and the Dirichlet boundary part, if mixed boundary conditions are imposed. In this paper, the conditions on the domain Ω and the Dirichlet boundary part are purely geometric in nature, so that one can decide 'at a glance' whether a concrete setting falls into this class or not. In particular, the Dirichlet points are subject to the outer volume condition (see e.g. [KS, Chapter II Theorem B.4]) as is classical in the pure Dirichlet case. Our second basic assumption demands bi-Lipschitz charts around points in the closure of the Neumann boundary part. Finally, for boundary points from the border between the Dirichlet and Neumann boundary part we replace the geometrical condition in [Grö] (compare also [HMRS, Chapter 5]) by a measure theoretic one (see Theorem 1.1 below). Roughly speaking, this states that, in balls around such points, the set of inner points from the Dirichlet boundary part is not rare (in a certain quantitative sense) with respect to the boundary measure, see (1) below. This is, not accidentally, again in correspondence with the fact that Hölder continuity for the Dirichlet problem also needs only a certain measure theoretic requirement. The resulting framework is then much broader, it is easy to verify in examples and it should cover nearly everything what is needed for the treatment of real-world problems – as long as the domain does not include cracks or things like that. In particular, the Dirichlet boundary part need not be (part of) a continuous boundary in the sense of [Gri, Definition 1.2.1.1], i.e. the domain is not forced to 'lie on one side of the Dirichlet boundary part' so that, among others, the following example is included.

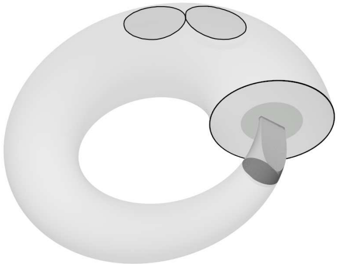

Figure 1: A geometric non-Lipschitz setting which fulfills our assumptions, if the grey apex and the three shaded circles carry the Dirichlet condition.

The first main result of this paper is that Hölder continuity for the solution of the mixed boundary value problem can be obtained even within this extremely wide concept, cf. Theorem 1.1 below. We emphasize that also unbounded domains are admissible, see Theorem 6.8. The initial instrument in the derivation of the Hölder properties are De Giorgi estimates. These are afterwards transferred into Morrey–Campanato estimates, which are a common instrument for deriving Hölder estimates, compare [Cam], [Gia], [Lie2] and references therein. The second main result are Gaussian Hölder estimates for the heat kernel of the corresponding semigroup, cf. Theorem 1.3. We use and extend techniques developed in [Aus], [AT], [ERo1], [Sta2] and [ERo2]. We emphasize that the principal coefficients of the elliptic operator are real bounded and merely measurable, whilst the lower-order coefficients can be complex measurable and bounded.

It is well-known that Hölder continuity is often the decisive instrument for the application of Schauder's fixed point theorem within the investigation of nonlinear equations, since it provides the required compactness.

In order to present the main results of this paper we introduce some notation and definitions. Fix $d \in \{2, 3, \ldots\}$. Throughout this paper the field is $\mathbb{C}$, although in Section 2 we mainly work with real valued functions.

Let $\Omega \subset \mathbb{R}^d$ be open and let Γ be an open subset of the boundary $\partial\Omega$ (with relative topology). We define

$$C_\Gamma^\infty(\Omega) := \{w|_\Omega : w \in C_c^\infty(\mathbb{R}^d) \text{ and } \operatorname{supp} w \cap (\partial\Omega \setminus \Gamma) = \emptyset\}.$$

Note that if $\Gamma = \emptyset$, then $C_c^\infty(\Omega) = C_\emptyset^\infty(\Omega)$. Moreover, for all $p \in [1, \infty)$, we denote the closure of $C_\Gamma^\infty(\Omega)$ in $W^{1,p}(\Omega)$ by $W_\Gamma^{1,p}(\Omega)$. If $\Gamma = \emptyset$, then we write, as usual, $W_0^{1,p}(\Omega) = W_\emptyset^{1,p}(\Omega)$. If $p \in (1, \infty]$ then the space $W_\Gamma^{-1,p}(\Omega)$ is the anti-dual of $W_\Gamma^{1,p'}(\Omega)$ in $L^p(\Omega)$. We denote the anti-dual of $W_0^{1,p'}(\Omega)$ by $W^{-1,p}(\Omega)$. Here and in the remainder of this paper p'

is the conjugate index for p, so $\frac{1}{p} + \frac{1}{p'} = 1$. Obviously, if $\Gamma_1 \subset \Gamma_2$, then $C^\infty_{\Gamma_1}(\Omega) \subset C^\infty_{\Gamma_2}(\Omega)$. Hence $W^{1,p}_{\Gamma_1}(\Omega) \subset W^{1,p}_{\Gamma_2}(\Omega)$ and consequently $W^{-1,p}_{\Gamma_2}(\Omega) \subset W^{-1,p}_{\Gamma_1}(\Omega)$. If $f_0 \in L_2(\Omega)$ and $f \in L_2(\Omega)^d$, then define $f_0 - \operatorname{div} f \in W^{-1,2}_\Gamma(\Omega)$ by

$$\langle f_0 - \operatorname{div} f, v\rangle = \int_\Omega f_0 \, \overline{v} + \sum_{i=1}^d \int_\Omega f_i \, \overline{\partial_i v}$$

for all $v \in W^{1,2}_\Gamma(\Omega)$.

Let $\Omega \subset \mathbb{R}^d$ be open and let $\mu, M > 0$. Define $\mathcal{A}(\Omega, \mu, M)$ to be the set of all measurable $A \colon \Omega \to \mathbb{C}^{d\times d}$ such that

$$\operatorname{Re} \sum_{i,j=1}^d a_{ij}(x) \, \varsigma_i \, \overline{\varsigma_j} \geq \mu \, |\varsigma|^2$$

and

$$\|A(x)\| \leq M$$

for almost all $x \in \Omega$ and $\varsigma \in \mathbb{C}^d$. Here and in the sequel $a_{ij}(x)$ is the appropriate matrix coefficient of $A(x)$ and $\|\cdot\|$ is the norm on $\mathcal{L}(\mathbb{C}^d)$, where $\mathbb{C}^d$ has the Euclidean norm. Let

$$\mathcal{A}_{\mathrm{r}}(\Omega, \mu, M) = \{A \in \mathcal{A}(\Omega, \mu, M) : A(x) \in \mathbb{R}^{d\times d} \text{ for all } x \in \Omega\}$$

be the subset with real coefficients. Further, set $\mathcal{A}(\Omega) = \bigcup_{\mu,M>0} \mathcal{A}(\Omega, \mu, M)$ and $\mathcal{A}_{\mathrm{r}}(\Omega) = \bigcup_{\mu,M>0} \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$

If $A \in \mathcal{A}(\Omega)$, then define the form $\mathfrak{l}_A \colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{l}_A(u, v) = \int_\Omega \sum_{i,j=1}^d a_{ij} \, (\partial_i u) \, \overline{(\partial_j v)}.$$

Then $\mathfrak{l}_A$ is a closed sectorial form. Let L_A be the associated operator. If no confusion is possible then we drop the subscript A and write $\mathfrak{l} = \mathfrak{l}_A(u, v)$ and $L = L_A$. If Γ is a (relatively) open subset of $\partial\Omega$, then we denote by $\mathfrak{l}_{A,\Gamma}$ the restriction of $\mathfrak{l}_A$ to the space $W^{1,2}_\Gamma(\Omega) \times W^{1,2}_\Gamma(\Omega)$. Then again $\mathfrak{l}_{A,\Gamma}$ is a closed sectorial form. Let $L_{A,\Gamma}$ be the associated m-sectorial operator in $L_2(\Omega)$.

Next, define $\mathcal{L}_{A,\Gamma} \colon W^{1,2}_\Gamma(\Omega) \to W^{-1,2}_\Gamma(\Omega)$ by

$$\langle \mathcal{L}_{A,\Gamma} u, v\rangle = \mathfrak{l}_{A,\Gamma}(u, v)$$

for all $v \in W^{1,2}_\Gamma(\Omega)$. It follows from the Lax–Milgram theorem that for all $f \in W^{-1,2}_\Gamma(\Omega)$ there exists a unique $u \in W^{1,2}_\Gamma(\Omega)$ such that $(\mathcal{L}_{A,\Gamma} + I)u = f$.

If Ω satisfies suitable regularity conditions, then one obtains an operator L_A for which the elements u of its domain satisfy the conditions $u|_{\partial\Omega\setminus\Gamma} = 0$ in the sense of traces and $\nu\cdot(A\nabla u) = 0$ on Γ in a generalized sense, where ν is the outward unit normal of Ω, compare [Cia, Chapter 1.2] or [GGZ, Chapter II.2]). Thus the operator $\mathcal{L}_{A,\Gamma}$ can be understood as one with mixed boundary conditions – as announced in the title. In general we consider Γ as the Neumann part and $\partial\Omega \setminus \Gamma$ as the Dirichlet part of the boundary.

We also need various balls and cylinders on $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$. For any $x \in \mathbb{R}^d$ and $r \in (0, \infty)$, we denote by $B(x, r)$ the ball in $\mathbb{R}^d$ with radius r and centre x. We denote by

$$E = \{x = (\tilde{x}, x_d) : -1 < x_d < 1 \text{ and } \|\tilde{x}\|_{\mathbb{R}^{d-1}} < 1\}$$

the open cylinder in $\mathbb{R}^d$, its lower half $E^- = \{x \in E : x_d < 0\}$ and its midplate

$$P = E \cap \{x \in \mathbb{R}^d : x_d = 0\}.$$

Further, $\widetilde{B}_r(\tilde{x})$ denotes the ball in $\mathbb{R}^{d-1}$ with radius r and centre $\tilde{x}$. We denote the volume of a measurable subset $A \subset \mathbb{R}^d$ by $|A|$ and the volume of a measurable subset $A \subset \mathbb{R}^{d-1}$ by $\mathrm{mes}_{d-1}(A)$. Let ω_d denote the volume of the unit ball in $\mathbb{R}^d$.

If $M \subset \mathbb{R}^d$ is non-empty and $x \in \mathbb{R}^d$ then we denote by

$$\mathrm{dist}(x, M) := \inf_{z \in M} \|x - z\|$$

the **distance** between x and M.

Let $\Omega \subset \mathbb{R}^d$ be open, $\Upsilon \subset \partial\Omega$ and $\alpha \in (0, 1]$. Then, following Definition II.C.1 in [KS] and Section 1.1 in [LU], we say that Υ is of class $(\mathbf{A}_\alpha)$ if

$$|B(x, r) \setminus \Omega| \geq \alpha \, |B(x, r)|$$

for all $r \in (0, 1]$ and $x \in \Upsilon$. It is not hard to see that the boundary of any Lipschitz domain is of class $(\mathbf{A}_\alpha)$ for a suitable $\alpha > 0$.

The first result is a global Hölder estimate in case of bounded domains and mixed boundary conditions. Note that Condition (III) below imposes a comparably simple, purely geometrical condition for the points from the border between Dirichlet and Neumann boundary part which may be viewed as a certain, extremely weak compatibility condition between Dirichlet and Neumann part of the boundary.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and Γ a relatively open subset of the boundary $\partial\Omega$. Moreover, let $A \in \mathcal{A}_{\mathrm{r}}(\Omega)$ and consider the operator $\mathcal{L}_{A,\Gamma}$. Assume the following conditions.*

(I) *For all $x \in \overline{\Gamma}$ there is an open neighbourhood U and a bi-Lipschitz map ϕ from an neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$, such that $\phi(U) = E$, $\phi(\Omega \cap U) = E^-$, $\phi(\partial\Omega \cap U) = P$ and $\phi(x) = 0$.*

(II) *There is an $\alpha > 0$ such that the set $\partial\Omega \setminus \Gamma$ is of class $(\mathbf{A}_\alpha)$.*

(III) *For all $x \in \partial\Gamma$ there are $c_0 \in (0, 1)$ and $c_1 > 0$ such that*

$$\mathrm{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_s(\tilde{y}) : \mathrm{dist}(\tilde{z}, \phi(\Gamma \cap U)) > c_0 \, s\} \geq c_1 \, s^{d-1} \qquad (1)$$

for all $s \in (0, 1]$ and $\tilde{y} \in \mathbb{R}^{d-1}$ with $(\tilde{y}, 0) \in \phi(\partial\Gamma \cap U)$, where U and ϕ are as in Condition (I). (*Here and in the sequel, the set $\partial\Gamma$ denotes the boundary of Γ where Γ is viewed as a subset of the topological space $\partial\Omega$.*)

Then for all $q \in (d, \infty)$ there exists a $\kappa > 0$ such that for all $f \in W^{-1,q}_\Gamma(\Omega)$ the (unique) solution $u \in W^{1,2}_\Gamma(\Omega)$ of the equation

$$(\mathcal{L}_{A,\Gamma} + I)u = f$$

belongs to the Hölder space $C^\kappa(\Omega)$. Moreover, the map $f \mapsto u$ from $W^{-1,q}_\Gamma(\Omega)$ into $C^\kappa(\Omega)$ is continuous.

We shall prove Theorem 1.1 at the end of Section 6. Moreover, we provide in Theorem 6.8 a quantitative version of Theorem 1.1 which allows Ω to be unbounded.

Condition (I) of the above theorem implies the (essential) boundedness of the solution u, and, in addition, the continuity of the map $W^{-1,q}_{\Gamma}(\Omega) \ni f \mapsto u \in L^\infty(\Omega)$. This was proved in case the coefficients are symmetric in [ERe] and if the coefficients are not symmetric, then it follows from a variation of the techniques in that paper. In general the condition $q > d$ is already necessary for the boundedness of the solution, see [LU, Section I.2].

Special cases of Theorem 1.1 are the pure Dirichlet case if $\Gamma = \emptyset$ (see [KS, Section II.C]) and the pure Neumann case if $\Gamma = \partial\Omega$ (see [Nit]).

Our geometrical concept excludes cracks (or other lacking lower dimensional objects): assume that a crack is included in Ω in form of a lacking hypersurface. Of course, all points of the crack are then boundary points of Ω. But no point x can be a Dirichlet point, due to our requirement that the set $\partial\Omega \setminus \Gamma$ is of class $(\mathbf{A}_\alpha)$ On the other hand, around x there is no chart which satisfies the requirements in Condition (I) in Theorem 1.1.

The reader should carefully notice that the conditions in Theorem 1.1 are *not* symmetric with respect to the Dirichlet and the Neumann boundary part. Interchanging for example in Figure 1 the role of Dirichlet and Neumann boundary part, then the new version does neither satisfy Condition (I) nor Condition (III).

Condition (III) implies the 'lower bound' in the Ahlfors–David condition (see also [JW, Chapter II]), that is, there is a $\check{c}_1 > 0$ such that

$$\mathcal{H}_{d-1}((\Omega \setminus \Gamma) \cap B(x,r) \geq \check{c}_1 r^{d-1}$$

for all $x \in \partial\Omega \setminus \Gamma$ and $r \in (0,1]$, where $\mathcal{H}_{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. This may be concluded from Lemma 5.4.

It is known from the Nash and De Giorgi theory that the solution u is Hölder continuous on every subset with positive distance to the boundary of Ω, cf. [LU, Section III.14], [GT, Section 8.9] and [KS, Section II.C]. Obviously, u is then continuous on Ω.

A principal tool in the proof of Theorem 1.1 is a rephrasement of a result of Ladyshenskaya–Ural'zeva. In Theorem 3.14.1 in [LU] Hölder continuity for weak solutions on subdomains which have a positive distance to the complement of the Dirichlet part of the boundary was proved, but with Hölder continuity understood as a suitable boundedness of the oscillation only over the connected components of the intersection of the domain with balls. In this paper, however, we consider the usual Hölder spaces. Hence we prove here that the solution indeed is Hölder continuous in the classical sense. The precise statement is given in the next theorem.

Before we can state the theorem, we need one more definition. Let $A \in \mathcal{A}(\Omega)$, $f \in L_2(\Omega)^d$, $f_0 \in L_2(\Omega)$, $u \in W^{1,2}(\Omega)$ and $V \subset \Omega$ open. Then we say that $Lu = f_0 - \operatorname{div} f$ **weakly on** V if

$$\mathfrak{l}_A(u,v) = \langle f_0 - \operatorname{div} f, v \rangle \tag{2}$$

for all $v \in C_c^\infty(V)$. Note that this notion is independent of Γ. Then by density (2) is valid for all $v \in W_0^{1,2}(V)$. If $q \in [1,\infty]$ and $f \in L_q(\Omega)^d$ then set $\|f\|_{L_q(\Omega)^d} = \sum_{i=1}^d \|f_i\|_{L_q(\Omega)}$.

Our version of Theorem 3.14.1 in [LU] with 'normal' Hölder spaces is as follows. Note that we do not require Ω to be bounded in Theorem 1.2.

Theorem 1.2. *For all* $\mu, M, \alpha, \zeta > 0$, $q \in (d, \infty)$ *and* $q_0 \in (\frac{d}{2}, \infty)$ *with* $q_0 \geq 2$ *there exist* $\kappa \in (0,1)$ *and* $c > 0$ *such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ *be open,* $\Gamma \subset \partial\Omega$ *relatively open and* $\Upsilon \subset \Omega$ *open. Suppose* $d(\Gamma, \Upsilon) \geq \zeta$ *and* $\{z \in \partial\Omega : d(z, \Upsilon) < \zeta\}$ *is of class* $(\mathbf{A}_\alpha)$. *Let* $A \in \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$, $f \in L_q(\Omega)^d \cap L_2(\Omega)^d$, $f_0 \in L_{q_0}(\Omega) \cap L_2(\Omega)$ *and* $u \in W^{1,2}_\Gamma(\Omega)$. *Suppose that* $Lu = f_0 - \operatorname{div} f$ *weakly on* Ω. *Then* u *is bounded on* Υ *and*

$$|u(x)| \leq c\Big(\|u\|_{W^{1,2}(\Omega)} + \|f_0\|_{L_{q_0}(\Omega)} + \|f\|_{L_q(\Omega)^d}\Big)$$

for all $x \in \Upsilon$. *Moreover, the restriction* $u|_\Upsilon$ *is Hölder continuous of order* κ *and*

$$|u(x) - u(y)| \leq c\,|x-y|^\kappa \Big(\|\nabla u\|_{L_2(\Omega)} + \|f_0\|_{L_{q_0}(\Omega)} + \|f\|_{L_q(\Omega)^d}\Big) \tag{3}$$

for all $x, y \in \Upsilon$ *with* $|x - y| \leq \frac{\zeta}{2}$.

Our last main theorem is that the kernel of the semigroup generated by $-L_{A,\Gamma}$ satisfies Gaussian Hölder bounds.

Theorem 1.3. *Adopt the assumptions of Theorem* 1.1. *Then there exist* $\kappa \in (0,1)$ *and* $b, c, \omega > 0$ *such that*

$$|K_t(x,y) - K_t(x',y')| \leq c\,t^{-d/2}\Big(\frac{|x-x'| + |y-y'|}{t^{1/2}}\Big)^\kappa e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t}$$

for all $x, x', y, y' \in \Omega$ *and* $t > 0$ *with* $|x - x'| + |y - y'| \leq t^{1/2}$, *where* $(K_t)_{t>0}$ *is the kernel of the semigroup generated by* $-L_{A,\Gamma}$.

We prove Theorem 1.3 in Section 7, where we also provide a version for unbounded Ω and operators with complex lower-order terms (see Theorem 7.5). For pure Dirichlet boundary conditions Theorem 7.5 has the following special case. Note that $L_{A,\emptyset}$ is the operator with Dirichlet boundary conditions.

Corollary 1.4. *For all* $\alpha, \mu, M > 0$ *there exist* $\kappa \in (0,1)$ *and* $b, c, \omega > 0$ *such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ *be an open set,* $A \in \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$ *and* $(K_t)_{t>0}$ *the kernel of the semigroup generated by* $-L_{A,\emptyset}$. *Then*

$$|K_t(x,y) - K_t(x',y')| \leq c\,t^{-d/2}\Big(\frac{|x-x'| + |y-y'|}{t^{1/2}}\Big)^\kappa e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t}$$

for all $x, x', y, y' \in \Omega$ *and* $t > 0$ *with* $|x - x'| + |y - y'| \leq t^{1/2}$.

The proofs of Theorems 1.2 and 1.3 involve Morrey and Campanato spaces, together with De Giorgi estimates. The important estimates near the Dirichlet part of the boundary are a variation on the estimates obtained by Stampacchia [Sta2]. (See also the appendix to Chapter 2 in the book of Kinderlehrer–Stampacchia [KS].) This then gives De Giorgi estimates on all points in Ω which have positive distance to the complement of the Dirichlet part of the boundary. Using localization, a bi-Lipschitz transformation and a reflection argument we deduce Hölder continuity near the Neumann part Γ of the boundary. Then Theorem 1.1 follows since Hölder continuity is a local property.

The outline of the paper is as follows: In Section 2 we prove De Giorgi estimates near the Dirichlet part of the boundary. In Section 3 we revisit the classical Ladyshenskaya/Ural'zeva result on Hölder continuity and prove Theorem 1.2. In Section 4 we investigate how De Giorgi estimates behave under bi-Lipschitz transformations of the domain and afterwards establish a reflection principle in this context, while in Section 5 consequences of this are drawn with respect to points near the Neumann part of the boundary. The Hölder continuity for the solution of the elliptic equation is proved in Section 6. Section 7 contains the Hölder bounds for the heat kernel of the corresponding semigroup.

It is well known that certain Morrey and Campanato spaces coincide, with equivalent norms if the domain satisfies an inner volume condition. Moreover, a Neumann type Poincaré inequality gives an estimate for the Campanato norm of a function in terms of the Morrey norm of the gradient of that function. Then a regularity theorem enables to bootstrap along the scale of Morrey–Campanato spaces to obtain Hölder regularity of the solution. Unfortunately, our boundary conditions do not give the required Neumann type Poincaré inequality on the full domain Ω. The way around this, is to introduce pointwise Morrey and Campanato type seminorms. For points near the Dirichlet boundary we obtain (pointwise) the estimates for the Campanato seminorm on $\mathbb{R}^d$ in terms of the Morrey seminorm for the gradient on Ω, cf. Lemma 3.4. Luckily, the already mentioned equivalence of the norms on the Morrey and Campanato spaces allows a similar pointwise estimate for the seminorms, see Lemma 3.1. For the convenience of the reader we give a proof of Lemma 3.1 in the appendix, with explicit constants.

2 De Giorgi estimates for weak solutions

In this section we prove De Giorgi estimates for weak solutions away from the Neumann part of the boundary.

Let $\Omega \subset \mathbb{R}^d$ be open, $A \in \mathcal{A}(\Omega)$ and write $L = L_A$. Let $u \in W^{1,2}(\Omega)$ and $V \subset \Omega$ open. Recall that we say that $Lu = 0$ **weakly on** V if

$$\mathfrak{t}_A(u, v) = 0 \tag{4}$$

for all $v \in C_c^\infty(V)$. Then by density (4) is valid for all $v \in W_0^{1,2}(V)$.

Let $\Omega \subset \mathbb{R}^d$ be open, Γ a relatively open subset of $\partial\Omega$, $A \in \mathcal{A}(\Omega)$, $\kappa_0 \in (0,1)$, $c_{DG} > 0$ and $\Upsilon \subset \overline{\Omega}$ a set. Then we say that $L_{A,\Gamma}$ **satisfies** (κ_0, c_{DG})**-De Giorgi estimates on** Υ if

$$\int_{\Omega(x,r)} |\nabla u|^2 \le c_{DG} \left(\frac{r}{R}\right)^{d-2+2\kappa_0} \int_{\Omega(x,R)} |\nabla u|^2$$

for all $x \in \Upsilon$, $0 < r \le R \le 1$ and $u \in W_\Gamma^{1,2}(\Omega)$ satisfying $Lu = 0$ weakly on $\Omega(x,R)$. Here and in the sequel we set $\Omega(x,r) = \Omega \cap B(x,r)$ for all $x \in \mathbb{R}^d$ and $r > 0$. Note the dependence on Γ, since we require that $u \in W_\Gamma^{1,2}(\Omega)$.

The main aim of this section is to prove the following estimates.

Proposition 2.1. *For all* $\mu, M, \alpha, \zeta > 0$ *there exist* $\kappa_0 \in (0,1)$ *and* $c_{DG} > 0$ *such that for every open set* $\Omega \subset \mathbb{R}^d$*, relatively open* $\Gamma \subset \partial\Omega$ *and subset* $\Upsilon \subset \overline{\Omega}$ *satisfying* $d(\Gamma, \Upsilon) \ge \zeta$ *and* $\{z \in \partial\Omega : d(z, \Upsilon) < \zeta\}$ *is of class* $(\mathbf{A}_\alpha)$ *it follows that* $L_{A,\Gamma}$ *satisfies* (κ_0, c_{DG})*-De Giorgi estimates on* Υ *for all* $A \in \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$*.*

The proof of the proposition needs several nontrivial prerequisites and at the end of this section we prove Proposition 2.1.

In all what follows we exploit repeatedly (without further comment) the following topological fact: If U, V are open sets in a metric space, then

$$(\partial U \cap V) \cup (U \cap \partial V) \subset \partial (U \cap V) \subset \partial U \cup \partial V.$$

Generally, we frequently need the following lemma.

Lemma 2.2. *Let $p \in (1,\infty)$, let $\Omega \subset \mathbb{R}^d$ be open and let Γ be a relatively open subset of $\partial\Omega$. Moreover, let $u \in W^{1,p}_\Gamma(\Omega)$.*

(a) *If u is real valued and $k \in [0,\infty)$, then $(u-k)^+ \in W^{1,p}_\Gamma(\Omega)$. If $\delta \geq 0$, then $u \wedge \delta \in W^{1,p}_\Gamma(\Omega)$.*

(b) *Let $x_0 \in \overline{\Omega}$ and $R > 0$. Suppose that $\overline{B(x_0,R)} \cap \Gamma = \emptyset$. Define $\tilde{u} \colon B(x_0,R) \to \mathbb{C}$ by*

$$(\tilde{u})(x) = \begin{cases} u(x) & \text{if } x \in \Omega(x_0,R), \\ 0 & \text{if } x \in B(x_0,R) \setminus \Omega. \end{cases} \tag{5}$$

Then $\tilde{u} \in W^{1,p}(B(x_0,R))$.

(c) *Let $x_0 \in \partial\Omega$ and $R > 0$. Suppose that $\overline{B(x_0,R)} \cap \Gamma = \emptyset$. Let $\eta \in C_c^\infty(B(x_0,R))$. Then $\eta\, u \in W^{1,p}_0(\Omega(x_0,R))$.*

(d) *Let $U \subset \mathbb{R}^d$ be open and define $\Lambda := \Omega \cap U$. Set $\Delta := \overline{(\partial\Omega \setminus \Gamma) \cap U}$. Then $\Delta \subset \partial\Lambda$ and $\Delta \cap \Gamma = \emptyset$. Moreover, $u|_\Lambda \in W^{1,p}_{\partial\Lambda \setminus \Delta}(\Lambda)$.*

(e) *Let $x_0 \in \partial\Omega$ and $R > 0$. Suppose that $\overline{B(x_0,R)} \cap \Gamma = \emptyset$ and $|B(x_0,R) \setminus \Omega| > 0$. Then $\operatorname{ess\,inf}_{\Omega(x_0,R)} |u| = 0$.*

Proof. '(a)'. Let $k \in [0,\infty)$. Let $w \in C_c^\infty(\mathbb{R},\mathbb{R})$ and suppose that $\operatorname{supp} w \cap (\partial\Omega \setminus \Gamma) = \emptyset$. Then $(w-k)^+ \in W^{1,p}(\Omega)$ and $\operatorname{supp}((w-k)^+) \cap (\partial\Omega \setminus \Gamma) = \emptyset$. Let $\tau \in C_c^\infty(\Omega,\mathbb{R})$ and suppose that $\int \tau = 1$. For all $n \in \mathbb{N}$ define $\tau_n \in C_c^\infty(\mathbb{R}^d)$ by $\tau_n(x) = n^d\, \tau(n\,x)$. Then $\lim \tau_n * ((w-k)^+) = (w-k)^+$ in $W^{1,p}(\mathbb{R}^d)$. But $(\tau_n * ((w-k)^+))|_\Omega \in C_\Gamma^\infty(\Omega)$ if $n \in \mathbb{N}$ is large enough. So $(w|_\Omega - k)^+ \in W^{1,p}_\Gamma(\Omega)$. Hence $(u-k)^+ \in W^{1,p}_\Gamma(\Omega)$ for all real valued $u \in C_\Gamma^\infty(\Omega)$. Finally, it follows from [MM] that the map $u \mapsto (u-k)^+$ is continuous from $W^{1,p}(\Omega,\mathbb{R})$ into $W^{1,p}(\Omega)$ and the first Statement of (a) follows. If $\delta \in [0,\infty)$ then $u \wedge \delta = u - (u-\delta)^+ \in W^{1,2}_\Gamma(\Omega)$.

'(b)'. Let $w \in C_c^\infty(\mathbb{R}^d)$ be such that $\operatorname{supp} w \cap (\partial\Omega \setminus \Gamma) = \emptyset$ and consider $u = w|_\Omega$. Since $\overline{B(x_0,R)} \cap \Gamma = \emptyset$ it follows that $\operatorname{supp} w \cap \partial\Omega \cap \overline{B(x_0,R)} = \emptyset$. Hence there exists an open $U \subset \mathbb{R}^d$ such that $\partial\Omega \cap \overline{B(x_0,R)} \subset U$ and $\operatorname{supp} w \cap U = \emptyset$. Then $\tilde{u}(x) = w(x)$ for all $x \in B(x_0,R) \cap U$. In particular $\tilde{u} \in C^\infty(B(x_0,R))$ and obviously $\|\tilde{u}\|_{W^{1,p}(B(x_0,R))} = \|u\|_{W^{1,p}(\Omega(x_0,R))}$. The rest follows by density.

'(c)'. Let $w \in C_c^\infty(\mathbb{R}^d)$ and suppose that $\operatorname{supp} w \cap (\partial\Omega \setminus \Gamma) = \emptyset$. Then $\eta\,(w|_\Omega) = (\eta\, w)|_{\Omega(x_0,R)}$. Since

$$\partial\Omega(x_0,R) \subset \partial B(x_0,R) \cup \left(\overline{B(x_0,R)} \cap \partial\Omega\right) \subset \partial B(x_0,R) \cup (\partial\Omega \setminus \Gamma)$$

it follows that $(\eta\, w)|_{\Omega(x_0,R)} \in C_c^\infty(\Omega(x_0,R)) \subset W_0^{1,p}(\Omega(x_0,R))$. So $\eta\, u \in W_0^{1,p}(\Omega(x_0,R))$ for all $u \in C_\Gamma^\infty(\Omega)$. Then the statement follows by continuity of the map $u \mapsto \eta\, u$ from $W^{1,p}(\Omega)$ into $W^{1,p}(\Omega(x_0,R))$.

'(d)'. Observe that $(\partial\Omega \setminus \Gamma) \cap U \subset \partial\Omega \cap U \subset \partial\Lambda$. Since $\partial\Lambda$ is closed, this gives $\Delta \subset \partial\Lambda$. On the other hand,

$$\Delta = \overline{(\partial\Omega \setminus \Gamma) \cap U} \subset \overline{\partial\Omega \setminus \Gamma} = \partial\Omega \setminus \Gamma,$$

since $\partial\Omega \setminus \Gamma$ is closed. Hence if $u \in C_\Gamma^\infty(\Omega)$, then the restriction $u|_\Lambda \in C_{\partial\Lambda\setminus\Delta}^\infty(\Lambda)$. Thus the restriction operator from Ω to Λ is continuous from $W_\Gamma^{1,p}(\Omega)$ into $W_{\partial\Lambda\setminus\Delta}^{1,p}(\Lambda)$.

'(e)'. We may assume that u is real valued. Let $\delta = \operatorname{ess\,inf}_{x\in\Omega(x_0,R)} |u(x)|$. Set $v = |u| \wedge \delta$. Then $v \in W_\Gamma^{1,p}(\Omega)$ by Statement (a). Let $\tilde v$ be the extension of v to $B(x_0,R)$ which is given by (5). Since $|B(x_0,R) \setminus \Omega| > 0$ by assumption, there exists by Theorem 4.4.2 in [Zie] a $c > 0$ such that $\|v\|_{L_p(B(x_0,R))} \le c\, \|\nabla v\|_{L_p(B(x_0,R))}$. Hence

$$\delta\, |\Omega(x_0,R)|^{1/p} \le \|v\|_{L_p(\Omega(x_0,R))} = \|v\|_{L_p(B(x_0,R))} \le c\, \|\nabla v\|_{L_p(B(x_0,R))} = \|\nabla v\|_{L_p(\Omega(x_0,R))} = 0.$$

So $\delta = 0$. □

We continue with Caccioppoli inequalities.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^d$ be open and let Γ be a relatively open subset of $\partial\Omega$. Let $\mu, M > 0$ and $A \in \mathcal{A}(\Omega,\mu,M)$. Then*

$$\int_{\Omega(x_0,r)} |\nabla((u-k)^+)|^2 \le \frac{b_1}{(R-r)^2} \int_{\Omega(x_0,R)} |(u-k)^+|^2$$

for all $x_0 \in \partial\Omega$, $0 < r < R < \infty$, $k \in [0,\infty)$ and real valued $u \in W_\Gamma^{1,2}(\Omega)$ such that $\overline{B(x_0,R)} \cap \Gamma = \emptyset$ and $Lu = 0$ weakly on $\Omega(x_0,R)$, where $b_1 = 16 M^2 \mu^{-2}$.

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ be such that $0 \le \eta \le \mathbb{1}$, $\eta|_{B(x_0,r)} = \mathbb{1}$, $\operatorname{supp}\eta \subset B(x_0,R)$ and $\|\nabla\eta\|_\infty \le \frac{2}{R-r}$. Then $v = \eta^2\,(u-k)^+ \in W_0^{1,2}(\Omega(x_0,R))$ by Lemma 2.2. Note that $\partial_l((u-k)^+) = \mathbb{1}_{[u>k]}\,\partial_l u$ for all $l \in \{1,\ldots,d\}$. Therefore

$$\begin{aligned} 0 &= \int_{\Omega(x_0,R)} \sum a_{ij}\,(\partial_i u)\,\partial_j v \\ &= \int_{\Omega(x_0,R)} \sum a_{ij}\,(\partial_i u)\,\eta^2\,\partial_j((u-k)^+) + 2\int_{\Omega(x_0,R)} \sum a_{ij}\,(\eta\,\partial_i u)\,(u-k)^+\,\partial_j\eta \\ &= \int_{\Omega(x_0,R)} \sum a_{ij}\,(\partial_i((u-k)^+))\,\eta^2\,\partial_j((u-k)^+) \\ &\qquad\qquad + 2\int_{\Omega(x_0,R)} \sum a_{ij}\,(\eta\,\partial_i((u-k)^+))\,(u-k)^+\,\partial_j\eta. \end{aligned}$$

So

$$\mu \int_{\Omega(x_0,R)} |\eta\, \nabla((u-k)^+)|^2$$

$$\leq \operatorname{Re} \int_{\Omega(x_0,R)} \sum a_{ij}\, (\partial_i((u-k)^+))\, \eta^2\, \partial_j((u-k)^+)$$

$$\leq 2 \Big| \int_{\Omega(x_0,R)} \sum a_{ij}\, (\eta\, \partial_i((u-k)^+)\, (u-k)^+\, \partial_j \eta \Big|$$

$$\leq 2M \Big(\int_{\Omega(x_0,R)} |\eta\, \nabla((u-k)^+)|^2 \Big)^{1/2} \Big(\int_{\Omega(x_0,R)} |(u-k)^+\, \nabla\eta|^2 \Big)^{1/2}.$$

Hence

$$\int_{\Omega(x_0,r)} |\nabla((u-k)^+)|^2 \leq \int_{\Omega(x_0,R)} |\eta\, \nabla((u-k)^+)|^2$$

$$\leq \frac{4M^2}{\mu^2} \int_{\Omega(x_0,R)} |(u-k)^+\, \nabla\eta|^2$$

$$\leq \frac{16M^2}{\mu^2}\, \frac{1}{(R-r)^2} \int_{\Omega(x_0,R)\setminus\Omega(x_0,r)} |(u-k)^+|^2$$

as required. □

Set $\theta = \frac{1}{2} + (\frac{1}{4} + \frac{2}{d})^{1/2} > 1$. Then $\theta^2 - \theta - \frac{2}{d} = 0$. If $u \in W^{1,2}(\Omega)$ is real valued, $x_0 \in \overline{\Omega}$, $k \in [0,\infty)$ and $R \in (0,\infty)$, then we define

$$A(k,R) = \{x \in \Omega(x_0,R) : u(x) > k\}.$$

In the notation of $A(k,R)$ we deleted the dependence of the function u and the point x_0, since it will not give any confusion.

Proposition 2.4. *For all $M, \mu > 0$ there exists a $b_2 > 0$ such that*

$$\operatorname*{ess\,sup}_{x \in \Omega(x_0,R/2)} u(x) \leq k + b_2 \Big(R^{-d} \int_{A(k,R)} |u-k|^2 \Big)^{1/2} \Big(R^{-d} |A(k,R)| \Big)^{(\theta-1)/2}$$

uniformly for every open set $\Omega \subset \mathbb{R}^d$, relatively open subset Γ of $\partial\Omega$, $A \in \mathcal{A}_{\rm r}(\Omega,\mu,M)$, $x_0 \in \partial\Omega$, $R \in (0,1]$, $k \in [0,\infty)$ and real-valued $u \in W^{1,2}_\Gamma(\Omega)$ such that $\overline{B(x_0,R)} \cap \Gamma = \emptyset$ and $Lu = 0$ weakly on $\Omega(x_0,R)$.

Proof. The proof is almost the same as the proof of Lemma 8.12 and Proposition 8.13 in [GM05].

For all $0 < r < R < \infty$ there exists an $\eta_{r,R} \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \eta_{r,R} \leq \mathbb{1}$, $\eta_{r,R}|_{B(x_0,r)} = \mathbb{1}$, $\operatorname{supp} \eta_{r,R} \subset B(x_0,R)$ and $\|\nabla \eta_{r,R}\|_\infty \leq \frac{2}{R-r}$. Let b_1 be as in Lemma 2.3. Let $A \in \mathcal{A}_{\rm r}(\Omega,\mu,M)$, Γ a relatively open subset of $\partial\Omega$, $S \in (0,1]$ and $u \in W^{1,2}_\Gamma(\Omega)$ be a real-valued function such that $\overline{B(x_0,S)} \cap \Gamma = \emptyset$ and $Lu = 0$ weakly on $\Omega(x_0,S)$. Then for all $0 < r < R \leq S$ and $k \in [0, \infty)$ one deduces from the Caccioppoli inequality of Lemma 2.3 that

$$\begin{aligned}\int_{\Omega(x_0,(r+R)/2)} |\nabla(\eta_{r,(r+R)/2}(u-k)^+)|^2 &\leq 2 \int_{\Omega(x_0,(r+R)/2)} |\nabla \eta_{r,(r+R)/2}|^2 \, |(u-k)^+)|^2 \\ &\quad + 2 \int_{\Omega(x_0,(r+R)/2)} |\eta_{r,(r+R)/2}|^2 \, |\nabla((u-k)^+)|^2 \\ &\leq \frac{32 + 8b_1}{(R-r)^2} \int_{\Omega(x_0,R)} |(u-k)^+|^2.\end{aligned}$$

Next, by the Sobolev inequality there exists a $b > 0$ such that

$$\Big(\int_{\mathbb{R}^d} |v|^{2d/(d-2)} \Big)^{(d-2)/d} \leq b \int_{\mathbb{R}^d} |\nabla v|^2 + b \int_{\mathbb{R}^d} |v|^2 \tag{6}$$

uniformly for all $v \in W^{1,2}(\mathbb{R}^d)$. It is a consequence of Lemma 2.2 that $\eta_{r,(r+R)/2}\,(u-k)^+ \in W_0^{1,2}(\Omega(x_0,R))$. Let v be the extension by 0 of the function $\eta_{r,(r+R)/2}\,(u-k)^+$ to $\mathbb{R}^d$ and use (6). Then

$$\begin{aligned}&\Big(\int_{\Omega(x_0,r)} |(u-k)^+|^{2d/(d-2)} \Big)^{(d-2)/d} \\ &\quad\leq \Big(\int_{\Omega(x_0,(r+R)/2)} |\eta_{r,(r+R)/2}(u-k)^+|^{2d/(d-2)} \Big)^{(d-2)/d} \\ &\quad\leq b \int_{\mathbb{R}^d} |\nabla v|^2 + b \int_{\mathbb{R}^d} |v|^2 \\ &\quad= b \int_{\Omega(x_0,(r+R)/2)} |\nabla(\eta_{r,(r+R)/2}(u-k)^+)|^2 + b \int_{\Omega(x_0,(r+R)/2)} |\eta_{r,(r+R)/2}(u-k)^+|^2 \\ &\quad\leq b \, \frac{32 + 8b_1 + 1}{(R-r)^2} \int_{\Omega(x_0,R)} |(u-k)^+|^2.\end{aligned}$$

The Hölder inequality gives

$$\int_{\Omega(x_0,r)} |(u-k)^+|^2 \leq \Big(\int_{\Omega(x_0,r)} |(u-k)^+|^{2d/(d-2)} \Big)^{(d-2)/d} |A(k,R)|^{2/d}.$$

Hence

$$\int_{A(k,r)} |u-k|^2 = \int_{\Omega(x_0,r)} |(u-k)^+|^2 \leq b \, \frac{32 + 8b_1 + 1}{(R-r)^2} \, |A(k,R)|^{2/d} \int_{A(k,R)} |u-k|^2.$$

This inequality can be iterated as in the proof of Proposition 8.13 in [GM05] and the proposition follows with $b_2 = 2^{d\theta/2} 2^{(d\theta+2)\theta/(2\theta-2)} \, (b\,(32 + 8b_1 + 1))^{d\theta/4}$. □

Corollary 2.5. *For all $\mu, M > 0$ there exists a $b_3 > 0$ such that*

$$\operatorname*{ess\,sup}_{x \in \Omega(x_0,R/2)} |u(x)| \leq b_3 \Big(R^{-d} \int_{\Omega(x_0,R)} |u|^2 \Big)^{1/2}$$

uniformly for every open set $\Omega \subset \mathbb{R}^d$, relatively open subset Γ of $\partial\Omega$, $A \in \mathcal{A}_{\rm r}(\Omega, \mu, M)$, $x_0 \in \partial\Omega$, $R \in (0,1]$ and real-valued $u \in W_\Gamma^{1,2}(\Omega)$ such that $\overline{B(x_0,R)} \cap \Gamma = \emptyset$ and $Lu = 0$ weakly on $\Omega(x_0,R)$.

Proof. Let b_2 be as in Proposition 2.4. Then it follows from Proposition 2.4 applied with $k = 0$ that

$$\operatorname*{ess\,sup}_{x\in\Omega(x_0,R/2)} u(x) \le b_2 \Big(R^{-d} \int_{\Omega(x_0,R)} |u|^2 \Big)^{1/2} \Big(R^{-d} \, |B(x_0,R)| \Big)^{(\theta-1)/2}$$
$$\le b_2 \, \omega_d^{(\theta-1)/2} \Big(R^{-d} \int_{\Omega(x_0,R)} |u|^2 \Big)^{1/2} .$$

The same estimate is valid for $-u$ instead of u and the corollary follows. □

We next need a Dirichlet-type Poincaré inequality and a Poincaré–Sobolev inequality. In both cases we need that a relevant part of the boundary of Ω is of class $(\mathbf{A}_\alpha)$.

Proposition 2.6. *Let $\alpha > 0$. Then there exist $c_D, c_S > 0$ such that*

$$\int_{\Omega(x_0,r)} |u|^2 \le c_D \, r^2 \int_{\Omega(x_0,r)} |\nabla u|^2 \quad \text{and} \quad r^{-d} \int_{\Omega(x_0,r)} |u|^p \le c_S \Big(r^{-d} \int_{\Omega(x_0,r)} r \, |\nabla u| \Big)^p$$

for every open set $\Omega \subset \mathbb{R}^d$ and relatively open subset Γ of $\partial\Omega$, all $r \in (0,1]$, $x_0 \in \partial\Omega$ and $u \in W^{1,2}_\Gamma(\Omega)$ such that $\overline{B(x_0,r)} \cap \Gamma = \emptyset$ and $\{x_0\}$ is of class $(\mathbf{A}_\alpha)$, where $1 = \frac{1}{p} + \frac{1}{d}$.

Proof. Without loss of generality we may assume $x_0 = 0$. By the Sobolev embedding theorem one has $W^{1,1}(B(0,1)) \subset L_p(B(0,1))$ and the inclusion map is continuous. Let $c_1 > 0$ be such that $\|v\|_{L_p(B(0,1))} \le c_1 \, \|v\|_{W^{1,1}(B(0,1))}$ for all $v \in W^{1,1}(B(0,1))$. By the Poincaré inequality in [Zie] Theorem 4.4.2 there exists a $c_2 > 0$ such that $\|v\|_{L_1(B(0,1))} \le c_2 \, \|\nabla v\|_{L_1(B(0,1))}$ for all $v \in W^{1,1}(B(0,1))$ such that there exists a measurable $E \subset B(0,1)$ with $|E| \ge \alpha \, |B(0,1)|$ and $\int_E v = 0$. Then

$$\|v\|^p_{L_p(B(0,1))} \le c_1^p \, \|v\|^p_{W^{1,1}(B(0,1))} \le c_1^p \, (1+c_2)^p \, \|\nabla v\|^p_{L_1(B(0,1))}$$

for all such v. Hence by scaling

$$r^{-d} \, \|v\|^p_{L_p(B(0,r))} \le c_1^p \, (1+c_2)^p \, \Big(r^{-d} \, \|r \, \nabla v\|_{L_1(B(0,r))} \Big)^p$$

for all $r \in (0,\infty)$ and $v \in W^{1,1}(B(0,r))$ such that there exists a measurable $E \subset B(0,r)$ with $|E| \ge \alpha \, |B(0,r)|$ and $\int_E v = 0$. Finally, let $r \in (0,1]$, $u \in W^{1,2}_\Gamma(\Omega)$ and suppose that $\overline{B(0,r)} \cap \Gamma = \emptyset$. Let v be the extension by zero of $u|_{\Omega(0,r)}$ to an element of $W^{1,p}(B(0,r))$, which exists by Lemma 2.2(b). Let $E = B(0,r) \setminus \Omega(0,r)$. Then $\int_E v = 0$ and $|E| \ge \alpha \, |B(0,r)|$. So

$$r^{-d} \, \|u\|^p_{L_p(\Omega(0,r))} = r^{-d} \, \|v\|^p_{L_p(B(0,r))} \le c_1^p \, (1+c_2)^p \, \Big(r^{-d} \, \|r \, \nabla v\|_{L_1(B(0,r))} \Big)^p$$
$$= c_1^p \, (1+c_2)^p \, \Big(r^{-d} \, \|r \, \nabla u\|_{L_1(\Omega(0,r))} \Big)^p .$$

This gives the second inequality.

The first inequality can be proved by the same scaling argument and again an application of Poincaré's inequality. □

Lemma 2.7. *There exists a $\beta > 0$ such that for all $\mu, M, \alpha > 0$ there exists a $b_4 > 0$ such that*

$$R^{-d}\,|A(k_n,R)| \le b_4\, n^{-\beta}$$

uniformly for all open $\Omega \subset \mathbb{R}^d$ and relatively open subset Γ of $\partial\Omega$, all $A \in \mathcal{A}_{\mathrm{r}}(\Omega,\mu,M)$, $x_0 \in \partial\Omega$, $R \in (0,1/4]$, $n \in \mathbb{N}_0$ and real-valued $u \in W^{1,2}_{\Gamma}(\Omega)$ such that $\overline{B(x_0,4R)} \cap \Gamma = \emptyset$, the set $\{x_0\}$ is of class $(\mathbf{A}_\alpha)$ and $Lu = 0$ weakly on $\Omega(x_0,4R)$, where

$$k_n = \operatorname*{ess\,sup}_{y\in\Omega(x_0,2R)} u(y) - 2^{-(n+1)}\Big(\operatorname*{ess\,sup}_{y\in\Omega(x_0,2R)} u(y) - \operatorname*{ess\,inf}_{y\in\Omega(x_0,2R)} u(y)\Big).$$

Proof. First note that the essential suprema and infima are finite by Corollary 2.5. Secondly, if $\operatorname{ess\,sup}_{y\in B(x_0,2R)} u(y) = \operatorname{ess\,inf}_{y\in B(x_0,2R)} u(y)$ then $|A(k_n,R)| = 0$ and the lemma is trivial. So we may assume that $\operatorname{ess\,sup}_{y\in B(x_0,2R)} u(y) \neq \operatorname{ess\,inf}_{y\in B(x_0,2R)} u(y)$.

Let $h > k \ge k_0$. Set $v = u \wedge h - u \wedge k$. Then $v = (u-k)^+ - (u-h)^+ \in W^{1,2}_{\Gamma}(\Omega)$ by Lemma 2.2(a).

Fix $p \in (1,\infty)$ such that $1 = \frac{1}{p} + \frac{1}{d}$. Let $c_S > 0$ be as in Proposition 2.6. Using the definition of v one then deduces that

$$\begin{aligned}
&|h-k|^p\, R^{-d}\, |A(h,R)| \\
&= R^{-d} \int_{A(h,R)} |v|^p \le R^{-d} \int_{\Omega(x_0,R)} |v|^p \\
&\le c_S \Big(R^{-d} \int_{\Omega(x_0,R)} R\,|\nabla v| \Big)^p = c_S \Big(R^{-d} \int_{A(k,R)\setminus A(h,R)} (R\,|\nabla u|) \Big)^p \\
&\le c_S \Big(\Big(R^{-d}\, |A(k,R)\setminus A(h,R)| \Big)^{1/2} \Big(R^{-d} \int_{A(k,R)\setminus A(h,R)} (R\,|\nabla u|)^2 \Big)^{1/2} \Big)^p \\
&\le c_S \Big(R^{-d}\, |A(k,R)\setminus A(h,R)| \Big)^{p/2} \Big(R^{-d} \int_{A(k,R)} R^2 |\nabla u|^2 \Big)^{p/2}, \qquad (7)
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality. But by the Caccioppoli inequality, Lemma 2.3, one estimates

$$\begin{aligned}
R^{-d} \int_{A(k,R)} R^2\, |\nabla u|^2 &= R^{-d} \int_{\Omega(x_0,R)} R^2\, |\nabla((u-k)^+)|^2 \\
&\le b_1\, R^{-d} \int_{\Omega(x_0,2R)} |(u-k)^+|^2 \\
&\le b_1\, R^{-d} \int_{\Omega(x_0,2R)} |(M(2R)-k)^+|^2 \le b_1\, \omega_d\, 2^d (M(2R)-k)^2,
\end{aligned}$$

where $M(2R) = \operatorname{ess\,sup}_{y\in\Omega(x_0,2R)} u(y)$ and b_1 is as in Lemma 2.3. Together with (7) this gives

$$|h-k|^2 \Big(R^{-d}\, |A(k_i,R)| \Big)^\gamma \le b'\, R^{-d}\, |A(k,R)\setminus A(h,R)|\, \big|M(2R)-k\big|^2,$$

where $\gamma = 2/p$ and $b' = 2^d c_S^\gamma\, b_1\, \omega_d$. Next apply these estimates with $h = k_i$ and $k = k_{i-1}$, where $i \in \mathbb{N}$. Then

$$\Big(R^{-d}\, |A(k_i,R)| \Big)^\gamma \le 4b'\, R^{-d} \Big(|A(k_{i-1},R)| - |A(k_i,R)| \Big),$$

where we used that $\operatorname{ess\,sup}_{y\in B(x_0,2R)} u(y) \neq \operatorname{ess\,inf}_{y\in B(x_0,2R)} u(y)$. Thus one obtains

$$\begin{aligned} n\Big(R^{-d}\,|A(k_n,R)|\Big)^\gamma &\le \sum_{i=1}^n \Big(R^{-d}\,|A(k_i,R)|\Big)^\gamma \\ &\le 4b'\,R^{-d}\Big(|A(k_0,R)| - |A(k_n,R)|\Big) \\ &\le 4b'\,R^{-d}\,|A(k_0,R)| \le 4b'\,\omega_d \end{aligned}$$

for all $n \in \mathbb{N}$, where we have used $|A(k_0,R)| \le |B(x_0,R)| \le \omega_d\,R^d$. Therefore

$$R^{-d}\,|A(k_n,R)| \le (4b'\,\omega_d)^\beta\,n^{-\beta}$$

with $\beta = 1/\gamma$ and the proof of the lemma is complete. □

If $x_0 \in \overline{\Omega}$, $r > 0$ and $u\colon \Omega(x_0,r) \to \mathbb{R}$ is a bounded function, then we define the **oscillation** of u on $\Omega(x_0,r)$ by

$$\operatorname{osc}_{u,x_0}(r) = \operatorname*{ess\,sup}_{y\in\Omega(x_0,r)} u(y) - \operatorname*{ess\,inf}_{y\in\Omega(x_0,r)} u(y).$$

Note that one always has $\operatorname{osc}_{u,x_0}(r) \le 2\operatorname{ess\,sup}_{\Omega(x_0,r)} |u|$. In case of $\overline{B(x_0,r)} \cap \Gamma = \emptyset$ and $|B(x_0,r) \setminus \Omega| > 0$ one can easily deduce from Lemma 2.2(e) that $\operatorname{ess\,sup}_{\Omega(x_0,r)} |u| \le \operatorname{osc}_{u,x_0}(r)$, which we need in the proof of Proposition 2.9.

Proposition 2.8. *For all $\mu, M, \alpha > 0$ there exists a $\kappa_0 \in (0,1)$ such that*

$$\operatorname{osc}_{u,x_0}(r) \le 4\left(\frac{r}{R}\right)^{\kappa_0} \operatorname{osc}_{u,x_0}(R/2)$$

uniformly for every open set $\Omega \subset \mathbb{R}^d$ and relatively open subset Γ of $\partial\Omega$, all $A \in \mathcal{A}_{\mathrm{r}}(\Omega,\mu,M)$, $x_0 \in \partial\Omega$, $R \in (0,1]$, $r \in (0,R/2]$ and real valued $u \in W^{1,2}_\Gamma(\Omega)$ such that $\overline{B(x_0,R)} \cap \Gamma = \emptyset$, the set $\{x_0\}$ is of class $(\mathbf{A}_\alpha)$ and $Lu = 0$ weakly on $\Omega(x_0,R)$.

Proof. Let b_2, b_4 and β be as in Proposition 2.4 and Lemma 2.7. For all $r \in (0,R/2]$ define

$$m(r) = \operatorname*{ess\,inf}_{y\in\Omega(x_0,r)} u(y) \quad \text{and} \quad M(r) = \operatorname*{ess\,sup}_{y\in\Omega(x_0,r)} u(y).$$

Now suppose $r \in (0,R/4]$. Set $k_0 = \frac{1}{2}\Big(M(2r)+m(2r)\Big)$. Replacing u by $-u$ if necessary, we may assume that $k_0 \ge 0$. Next, for all $n \in \mathbb{N}$ we set $k_n = M(2r) - 2^{-(n+1)}\Big(M(2r)-m(2r)\Big)$. Then it follows from Proposition 2.4 that

$$\begin{aligned} M(r/2) &\le k_n + b_2\Big(r^{-d}\int_{A(k_n,r)} |M(r)-k_n|^2\Big)^{1/2}\Big(r^{-d}\,|A(k_n,r)|\Big)^{(\theta-1)/2} \\ &\le k_n + b_2\,\omega_d^{1/2}\Big(M(2r)-k_n\Big)(b_4\,n^{-\beta})^{(\theta-1)/2} \end{aligned}$$

uniformly for all $n \in \mathbb{N}$, where Lemma 2.7 is used in the last inequality. Next fix $N \in \mathbb{N}$ such that

$$b_2\,\omega_d^{1/2}\,(b_4 N^{-\beta})^{(\theta-1)/2} \le \tfrac{1}{2}.$$

Note that N depends only on μ, M and α. Then

$$\begin{aligned} M(r/2) &\leq M(2r) - 2^{-(N+1)}\Big(M(2r) - m(2r)\Big) + 2^{-(N+2)}\Big(M(2r) - m(2r)\Big) \\ &= M(2r) - 2^{-(N+2)}\Big(M(2r) - m(2r)\Big). \end{aligned}$$

Hence

$$\begin{aligned} M(r/2) - m(r/2) &\leq M(2r) - m(2r) - 2^{-(N+2)}\Big(M(2r) - m(2r)\Big) \\ &= (1 - 2^{-(N+2)})\Big(M(2r) - m(2r)\Big). \end{aligned}$$

This is valid for all $r \in (0, R/4]$. Therefore one deduces by induction that

$$M(2^{-(2n+1)}r) - m(2^{-(2n+1)}r) \leq (1 - 2^{-(N+2)})^n \Big(M(r/2) - m(r/2)\Big)$$

for all $n \in \mathbb{N}_0$ and

$$M(r) - m(r) \leq 4^{\kappa_0} \left(\frac{r}{R}\right)^{\kappa_0} \Big(M(R/2) - m(R/2)\Big)$$

for all $r \in (0, R/2]$, where $\kappa_0 = -(2 \log 2)^{-1} \log(1 - 2^{-(N+2)}) > 0$. □

Another consequence of Proposition 2.8 are De Giorgi estimates.

Proposition 2.9. *For all $\mu, M, \alpha, \zeta > 0$ there exist $\kappa_0 \in (0, 1)$ and $c_{DG} > 0$ such that*

$$\int_{\Omega(x_0,r)} |\nabla u|^2 \leq c_{DG} \left(\frac{r}{R}\right)^{d-2+2\kappa_0} \int_{\Omega(x_0,R)} |\nabla u|^2$$

uniformly for every open set $\Omega \subset \mathbb{R}^d$ and relatively open subset Γ of $\partial\Omega$, all $A \in \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$, $x_0 \in \partial\Omega$, $0 < r \leq R \leq 1$ and $u \in W^{1,2}_\Gamma(\Omega)$ such that $\overline{B(x_0, \zeta)} \cap \Gamma = \emptyset$, the set $\{x_0\}$ is of class $(\mathbf{A}_\alpha)$ and $Lu = 0$ weakly on $\Omega(x_0, R)$.

Proof. Let b_1 and b_3 be as in Lemma 2.3 and Corollary 2.5. Let $\kappa_0 \in (0, 1)$ be as in Proposition 2.8.

Without loss of generality we may assume that $\zeta \leq 1$. Fix $u \in W^{1,2}_\Gamma(\Omega)$ and $x_0 \in \partial\Omega$. Suppose that $\overline{B(x_0, \zeta)} \cap \Gamma = \emptyset$, the set $\{x_0\}$ is of class $(\mathbf{A}_\alpha)$ and $Lu = 0$ weakly on $\Omega(x_0, \zeta)$. Since A is a real operator, it suffices to prove the inequality for real-valued u.

It follows from Lemma 2.2(e) that $m(r) \leq 0 \leq M(r)$ for all $r \in (0, \zeta]$ and hence $|u(y)| \leq |M(r) - m(r)|$ for almost all $y \in \Omega(x_0, r)$. Let $R \in (0, \zeta]$ and $r \in (0, R/4]$. Apply

Lemma 2.3 to the function u and $-u$. Then

$$\begin{aligned}
\int_{\Omega(x_0,r)} |\nabla u|^2 &\le b_1\, r^{-2} \int_{\Omega(x_0,2r)} |u|^2 \\
&\le b_1\, r^{-2} \int_{\Omega(x_0,2r)} |\mathrm{osc}_{u,x_0}(2r)|^2 \\
&\le 16 b_1\, \omega_d\, (2r)^{d-2} \Big(\frac{2r}{R}\Big)^{2\kappa_0} \Big(\mathrm{osc}_{u,x_0}(R/2)\Big)^2 \\
&\le 16 b_1\, \omega_d\, (2r)^{d+2\kappa_0-2} R^{-2\kappa_0} \Big(2 \operatorname*{ess\,sup}_{y\in\Omega(x_0,R/2)} |u(y)|\Big)^2 \\
&\le 64 b_1\, b_3^2\, \omega_d\, (2r)^{d+2\kappa_0-2} R^{-(d+2\kappa_0)} \int_{\Omega(x_0,R)} |u|^2 \\
&\le 64 b_1\, b_3^2\, \omega_d\, c_D\, 2^{d+2\kappa_0-2} \Big(\frac{r}{R}\Big)^{d+2\kappa_0-2} \int_{\Omega(x_0,R)} |\nabla u|^2,
\end{aligned}$$

where we have used the Dirichlet-type Poincaré inequality of Proposition 2.6 in the last step. □

Corollary 2.10. *For all $\mu, M, \alpha, \zeta > 0$ there exist $\kappa_0 \in (0,1)$ and $c_{DG} > 0$ such that*

$$\int_{\Omega(x_0,r)} |\nabla u|^2 \le c_{DG} \Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{\Omega(x_0,R)} |\nabla u|^2$$

uniformly for every open set $\Omega \subset \mathbb{R}^d$ and relatively open subset Γ of $\partial\Omega$, every subset $\Upsilon \subset \partial\Omega$ satisfying $d(\Gamma, \Upsilon) \ge \zeta$ and the set Υ is of class $(\mathbf{A}_\alpha)$, all $A \in \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$, $x_0 \in \Upsilon$, $0 < r \le R \le 1$ and $u \in W^{1,2}_\Gamma(\Omega)$ such that $Lu = 0$ weakly on $\Omega(x_0, R)$.

We also need interior De Giorgi estimates.

Proposition 2.11. *For all $\mu, M > 0$ there exist $\kappa_0 \in (0,1)$ and $c_{DG} > 0$ such that*

$$\int_{\Omega(x,r)} |\nabla u|^2 \le c_{DG} \Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{\Omega(x,R)} |\nabla u|^2 \qquad (8)$$

uniformly for every open set $\Omega \subset \mathbb{R}^d$, $A \in \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$, $x \in \Omega$, $0 < r \le R \le 1$ and $u \in W^{1,2}(B(x,R))$ such that $B(x,R) \subset \Omega$ and $Lu = 0$ weakly on $B(x,R)$.

Proof. This is the basic inequality of De Giorgi. See for example (8.12) in [GM05]. □

Knowing De Giorgi estimates for points on $\Upsilon \subset \partial\Omega$ one can deduce De Giorgi estimates for interior points away from $\partial\Omega \setminus \Upsilon$. The next proof is inspired by Step 4 in the proof of Theorem 5.19 in [GM05].

Proof of Proposition 2.1. Without loss of generality we may assume that $\zeta \le 1$. Let

$$\Upsilon_0 = \{z \in \partial\Omega : d(z, \Upsilon) < \tfrac{\zeta}{2}\}.$$

Then $d(\Gamma, \Upsilon_0) \ge \frac{\zeta}{2}$ and the set Υ_0 is of class $(\mathbf{A}_\alpha)$.

By Corollary 2.10 there exist $c_{DG} \geq 1$ and $\kappa_0 \in (0,1)$ such that $L_{A,\Gamma}$ satisfies (κ_0, c_{DG})-De Giorgi estimates on Υ_0 for all $A \in \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$. Without loss of generality we may assume that (8) is valid for all $A \in \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$, $x \in \Omega$, $0 < r \leq R \leq 1$ and $u \in W^{1,2}(B(x,R))$ such that $B(x,R) \subset \Omega$ and $Lu = 0$ weakly on $B(x,R)$.

Let $x \in \Upsilon$, $0 < r \leq R \leq 1$ and $A \in \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$. First assume that $4r \leq R \leq 2\zeta$. Let $u \in W^{1,2}_\Gamma(\Omega)$ and suppose that $Lu = 0$ weakly on $\Omega(x,R)$. Set $\rho = d(x, \partial\Omega)$.

Case 1. Suppose $R/4 \leq \rho$.
Then $B(x, R/4) \subset \Omega$ and

$$\int_{\Omega(x,r)} |\nabla u|^2 \leq c_{DG} \Big(\frac{r}{R/4}\Big)^{d-2+2\kappa_0} \int_{\Omega(x,R/4)} |\nabla u|^2 \leq 4^d \, c_{DG} \Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{\Omega(x,R)} |\nabla u|^2$$

by (8).

Case 2. Suppose $r \leq \rho \leq R/4$.
Let $x_0 \in \partial\Omega$ be such that $|x - x_0| = \rho$. Then $|x - x_0| = \rho \leq R/4 < \frac{\zeta}{2}$. Since $x \in \Upsilon$ it follows that $x_0 \in \Upsilon_0$. Moreover, $B(x,\rho) \subset \Omega(x_0, 2\rho)$. Hence by Corollary 2.10 applied to Υ_0 it follows that

$$\begin{aligned}
\int_{\Omega(x,r)} |\nabla u|^2 &\leq c_{DG} \Big(\frac{r}{\rho}\Big)^{d-2+2\kappa_0} \int_{\Omega(x,\rho)} |\nabla u|^2 \\
&\leq c_{DG} \Big(\frac{r}{\rho}\Big)^{d-2+2\kappa_0} \int_{\Omega(x_0,2\rho)} |\nabla u|^2 \\
&\leq c_{DG}^2 \Big(\frac{r}{\rho}\Big)^{d-2+2\kappa_0} \Big(\frac{2\rho}{R/2}\Big)^{d-2+2\kappa_0} \int_{\Omega(x_0,R/2)} |\nabla u|^2 \\
&\leq 4^d \, c_{DG}^2 \Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{\Omega(x,R)} |\nabla u|^2
\end{aligned}$$

as required.

Case 3. Suppose $\rho \leq r \leq R/4$.
As in Case 2, there exists an $x_0 \in \Upsilon_0$ such that $|x - x_0| = \rho$. Then $\Omega(x,r) \subset \Omega(x_0, 2r)$ and

$$\begin{aligned}
\int_{\Omega(x,r)} |\nabla u|^2 &\leq \int_{\Omega(x_0,2r)} |\nabla u|^2 \\
&\leq c_{DG} \Big(\frac{2r}{R/2}\Big)^{d-2+2\kappa_0} \int_{\Omega(x_0,R/2)} |\nabla u|^2 \\
&\leq 4^d \, c_{DG} \Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{\Omega(x,R)} |\nabla u|^2
\end{aligned}$$

as required.

So $L_{A,\Gamma}$ satisfies (κ_0, c'_{DG})-De Giorgi estimates on Υ, where $c'_{DG} = 8^d c_{DG}^2 \, \zeta^{-d}$. □

The De Giorgi estimates will be used in the proof of every theorem in this paper.

3 Ladyshenskaya–Ural'zeva revisited

The aim in this section is to provide a proof of the Ladyshenskaya–Ural'zeva theorem, Theorem 3.14.1 in [LU], with the classical Hölder spaces, that is Theorem 1.2. For good

Hölder estimates we use a variation of Morrey and Campanato spaces. We modify the definition of Morrey and Campanato space from [Gia] in two ways. First, we change the range of the radius for the balls from diam(Ω) to a to be determined endpoint R_e, where $R_e \in (0, 1]$. Secondly, we consider pointwise estimates and are not interested in the global spaces.

For all $\gamma \in [0, d]$, $R_e \in (0, 1]$ and $x \in \Omega$ define $\| \cdot \|_{M,\gamma,x,\Omega,R_e} \colon L_2(\Omega) \to [0, \infty]$ by

$$\|u\|_{M,\gamma,x,\Omega,R_e} = \sup_{r \in (0,R_e]} \Big(r^{-\gamma} \int_{\Omega(x,r)} |u|^2 \Big)^{1/2}.$$

Then the **Morrey space** $M_\gamma(\Omega)$ is defined by $M_\gamma(\Omega) = \{u \in L_2(\Omega) : \sup_{x\in\Omega} \|u\|_{M,\gamma,x,\Omega,1} < \infty\}$. The space $M_\gamma(\Omega)$ is a Banach space under the natural norm.

Next, for all $\gamma \in [0, d+2]$, $R_e \in (0, 1]$ and $x \in \Omega$ define $|||\cdot|||_{\mathcal{M},\gamma,x,\Omega,R_e} \colon L_2(\Omega) \to [0, \infty]$ by

$$|||u|||_{\mathcal{M},\gamma,x,\Omega,R_e} = \sup_{r\in(0,R_e]} \Big(r^{-\gamma} \int_{\Omega(x,r)} |u - \langle u\rangle_{\Omega(x,r)}|^2 \Big)^{1/2},$$

where for an L_2 function v we denote by $\langle v\rangle_D = \frac{1}{|D|} \int_D v$ the average of v over a bounded measurable subset D of the domain of u with $|D| > 0$. Define the **Campanato space** $\mathcal{M}_\gamma(\Omega)$ by $\mathcal{M}_\gamma(\Omega) = \{u \in L_2(\Omega) : \sup_{x\in\Omega} |||u|||_{\mathcal{M},\gamma,x,\Omega,1} < \infty\}$. Then $\mathcal{M}_\gamma(\Omega)$ is a Banach space with the norm $u \mapsto \Big(\|u\|^2_{L_2(\Omega)} + \sup_{x\in\Omega} |||u|||^2_{\mathcal{M},\gamma,x,\Omega,1} \Big)^{1/2}$. If no confusion is possible, then we drop the dependence of Ω and write $\|u\|_{M,\gamma,x,R_e} = \|u\|_{M,\gamma,x,\Omega,R_e}$ and $|||u|||_{\mathcal{M},\gamma,x,R_e} = |||u|||_{\mathcal{M},\gamma,x,\Omega,R_e}$.

For all $\kappa \in (0, 1)$ define $|||\cdot|||_{C^\kappa(\Omega)} \colon C(\Omega) \to [0, \infty]$ by

$$|||u|||_{C^\kappa(\Omega)} = \sup_{\substack{x,y\in\Omega \\ 0<|x-y|\le 1}} \frac{|u(x) - u(y)|}{|x - y|^\kappa}.$$

Let $C^\kappa(\Omega) = \{u \in C(\Omega) : |||u|||_{C^\kappa(\Omega)} < \infty\}$.

It follows from [Gia] Proposition III.1.2 and Theorem III.1.2 that $\mathcal{M}_\gamma(\Omega) = M_\gamma(\Omega)$ for all $\gamma \in [0, d)$ and $\mathcal{M}_\gamma(\Omega) = C^{(\gamma-d)/2}(\Omega) \cap L_2(\Omega)$ for all $\gamma \in (d, d+2)$ if Ω satisfies a uniform interior volume estimate. Moreover, then also the norms on the spaces are equivalent. The proofs give pointwise estimates, which we need in the sequel. In fact, we do not need the (global) spaces $M_\gamma(\Omega)$ and $\mathcal{M}_\gamma(\Omega)$ at all. Explicitly, one has the following estimates.

Lemma 3.1.

(a) *For all $\gamma \in [0, d)$, $\tilde{c} > 0$ and $R_e \in (0, 1]$ there exist $c_1, c_2 > 0$ such that*

$$\|u\|^2_{\mathcal{M},\gamma,x,R_e} \le \|u\|^2_{M,\gamma,x,R_e} \le c_1 \|u\|^2_{\mathcal{M},\gamma,x,R_e} + c_2 \int_{\Omega(x,R_e)} |u|^2$$

for all open $\Omega \subset \mathbb{R}^d$, $x \in \Omega$, $R_e \in (0, 1]$ and $u \in L_2(\Omega)$ such that $|\Omega(x, r)| \ge \tilde{c}\, r^d$ for all $r \in (0, R_e]$.

(b) *Let $\Omega \subset \mathbb{R}^d$ be open, $\gamma \in (d, d+2)$, $\tilde{c} > 0$, $x \in \Omega$, $u \in L_2(\Omega)$ and $R_e \in (0, 1]$. Assume that $\|u\|_{\mathcal{M},\gamma,x,R_e} < \infty$ and $|\Omega(x, r)| \ge \tilde{c}\, r^d$ for all $r \in (0, R_e]$. Then $\lim_{R\downarrow 0} \langle u\rangle_{\Omega(x,R)}$ exists. Write $\hat{u}(x) = \lim_{R\downarrow 0} \langle u\rangle_{\Omega(x,R)}$. Then*

$$|\langle u\rangle_{\Omega(x,R)} - \hat{u}(x)| \le \frac{2^{1+d/2}}{\sqrt{\tilde{c}}(1 - 2^{-(\gamma-d)/2})} R^{(\gamma-d)/2} \|u\|_{\mathcal{M},\gamma,x,R_e}$$

for all $R \in (0, R_e]$.

(c) *Let* $\gamma \in (d, d+2)$ *and* $\tilde{c} > 0$. *Then there exists a* $c > 0$ *such that*

$$|\hat{u}(x) - \hat{u}(y)| \leq c \, (\|u\|_{\mathcal{M},\gamma,x,R_e} + \|u\|_{\mathcal{M},\gamma,y,R_e}) \, |x-y|^{(\gamma-d)/2}$$

for all open $\Omega \subset \mathbb{R}^d$, $x, y \in \Omega$, $R_e \in (0,1]$ *and* $u \in L_2(\Omega)$ *such that* $\|u\|_{\mathcal{M},\gamma,x,R_e} < \infty$, $\|u\|_{\mathcal{M},\gamma,y,R_e} < \infty$, $|x-y| \leq \frac{R_e}{2}$ *and, in addition,* $|\Omega(x,r)| \geq \tilde{c}\, r^d$ *and* $|\Omega(y,r)| \geq \tilde{c}\, r^d$ *for all* $r \in (0, R_e]$, *where* $\hat{u}(x)$ *and* $\hat{u}(y)$ *are as in* (b).

Proof. The proof is as in the proof of [Gia] Proposition III.1.2 and Theorem III.1.2. For the convenience of the reader we included the details in the appendix. □

Note that $\hat{u}(x) = u(x)$ if u is continuous.

The next proposition is stated in more generality than that we currently need, so that it can be used again in Section 7 to prove semigroup kernel bounds. The proposition is a modification of a proposition which appears at many places in the literature ([Mor], [GM05] Theorem 5.13, [Aus] Theorem 3.6, [AT] Lemma 1.12, [ERo1] Proposition 4.2, [DER] Proposition A.3.1.)

Proposition 3.2. *For all* $\mu, M, \alpha > 0$ *and* $\zeta \in (0,1]$ *there exists a* $\kappa_0 \in (0,1)$ *such that for all* $\gamma \in [0,d)$ *and* $\delta \in (0,2]$ *with* $\gamma + \delta < d - 2 + 2\kappa_0$ *there exists an* $a_1 > 0$, *such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ *be open, let* Γ *be a relatively open subset of* $\partial\Omega$, *let* $\Upsilon \subset \Omega$ *and suppose that* $d(\Gamma, \Upsilon) \geq \zeta$ *and* $\{z \in \partial\Omega : d(z, \Upsilon) < \zeta\}$ *is of class* $(\mathbf{A}_\alpha)$. *Let* $A \in \mathcal{A}_{\rm r}(\Omega, \mu, M)$, $u \in W^{1,2}_\Gamma(\Omega)$ *and* $\xi, \xi_1, \ldots, \xi_d \in L_2(\Omega)$ *be such that*

$$\mathfrak{l}_{A,\Gamma}(u,v) = (\xi, v)_{L_2(\Omega)} - \sum_{i=1}^d (\xi_i, \partial_i v)_{L_2(\Omega)} \tag{9}$$

for all $v \in W^{1,2}_0(\Omega)$. *Then*

$$\|\nabla u\|_{M,\gamma+\delta,x,\Omega,\zeta} \leq a_1 \Big(\varepsilon^{2-\delta} \|\xi\|_{M,\gamma,x,\Omega,\zeta} + \sum_{i=1}^d \|\xi_i\|_{M,\gamma+\delta,x,\Omega,\zeta} + \varepsilon^{-(\gamma+\delta)} \|\nabla u\|_{L_2(\Omega)} \Big)$$

for all $\varepsilon \in (0,1]$ *and* $x \in \Upsilon$.

Proof. By Proposition 2.1 there are $c_{DG} > 0$ and $\kappa_0 \in (0,1)$ such that $L_{A,\Gamma}$ satisfies (κ_0, c_{DG})-De Giorgi estimates on Υ for all $A \in \mathcal{A}_{\rm r}(\Omega, \mu, M)$.

Let $A \in \mathcal{A}_{\rm r}(\Omega, \mu, M)$, $u \in W^{1,2}_\Gamma(\Omega)$ and $\xi, \xi_1, \ldots, \xi_d \in L_2(\Omega)$. Suppose that (9) is valid for all $v \in W^{1,2}_0(\Omega)$. Let $0 < r \leq R \leq \zeta$ and $x \in \Upsilon$. By the Lax–Milgram theorem there exists a unique $v \in W^{1,2}_0(\Omega(x,R))$ such that

$$\sum_{i,j=1}^d \int_{\Omega(x,R)} a_{ij} \, (\partial_i v) \, \overline{\partial_j \varphi} = \sum_{i,j=1}^d \int_{\Omega(x,R)} a_{ij} \, (\partial_i u) \, \overline{\partial_j \varphi} \tag{10}$$

for all $\varphi \in W^{1,2}_0(\Omega(x,R))$. Extend v by zero to a function $\tilde{v} \colon \Omega \to \mathbb{C}$. Then $\tilde{v} \in W^{1,2}_0(\Omega)$. Set $w = u - \tilde{v}$. Then $w \in W^{1,2}_\Gamma(\Omega)$. Moreover,

$$\sum_{i,j=1}^d \int_{\Omega(x,R)} a_{ij} \, (\partial_i w) \, \overline{\partial_j \varphi} = 0$$

for all $\varphi \in W_0^{1,2}(\Omega(x,R))$. The De Giorgi inequalities applied to the function w imply

$$\begin{aligned}
\int_{\Omega(x,r)} |\nabla u|^2 &\leq 2 \int_{\Omega(x,r)} |\nabla w|^2 + 2 \int_{\Omega(x,r)} |\nabla v|^2 \\
&\leq 2c_{DG}\Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{\Omega(x,R)} |\nabla w|^2 + 2 \int_{\Omega(x,r)} |\nabla v|^2 \\
&\leq 4c_{DG}\Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{\Omega(x,R)} |\nabla u|^2 + (2+4c_{DG}) \int_{\Omega(x,R)} |\nabla v|^2. \qquad (11)
\end{aligned}$$

Choose $\varphi = v$ in (10). Then

$$\begin{aligned}
\sum_{i,j=1}^d \int_{\Omega(x,R)} a_{ij}\,(\partial_i v)\,\overline{\partial_j v} &= \sum_{i,j=1}^d \int_{\Omega(x,R)} a_{ij}\,(\partial_i u)\,\overline{\partial_j v} \\
&= \sum_{i,j=1}^d \int_{\Omega} a_{ij}\,(\partial_i u)\,\overline{\partial_j \tilde{v}} \\
&= \mathfrak{l}_{A,\Gamma}(u,\tilde{v}) = (\xi,\tilde{v})_{L_2(\Omega)} - \sum_{i=1}^d (\xi_i, \partial_i \tilde{v})_{L_2(\Omega)}.
\end{aligned}$$

Hence, by ellipticity and the Cauchy–Schwarz inequality, one estimates

$$\begin{aligned}
\mu \int_{\Omega(x,R)} |\nabla v|^2 &\leq \Big(\int_{\Omega(x,R)} |\xi|^2\Big)^{1/2} \Big(\int_{\Omega(x,R)} |v|^2\Big)^{1/2} \\
&\quad + \sum_{i=1}^d \Big(\int_{\Omega(x,R)} |\xi_i|^2\Big)^{1/2} \Big(\int_{\Omega(x,R)} |\partial_i v|^2\Big)^{1/2} \\
&\leq \|\xi\|_{M,\gamma,x,\Omega,\zeta} R^{\gamma/2} \Big(\int_{\Omega(x,R)} |v|^2\Big)^{1/2} \\
&\quad + \sum_{i=1}^d \|\xi_i\|_{M,\gamma+\delta,x,\Omega,\zeta} R^{(\gamma+\delta)/2} \Big(\int_{\Omega(x,R)} |\nabla v|^2\Big)^{1/2}.
\end{aligned}$$

Then the Dirichlet type Poincaré inequality in Theorem V.3.22 in [EE] gives

$$\int_{\Omega(x,R)} |v|^2 \leq 4\,R^2 \int_{\Omega(x,R)} |\nabla v|^2.$$

So

$$\int_{\Omega(x,R)} |\nabla v|^2 \leq \mu^{-2} \Big(2R^{(2-\delta)/2} \|\xi\|_{M,\gamma,x,\Omega,\zeta} + \sum_{i=1}^d \|\xi_i\|_{M,\gamma+\delta,x,\Omega,\zeta}\Big)^2 R^{\gamma+\delta}.$$

Now we can combine these bounds with (11) to obtain

$$\begin{aligned}
\int_{\Omega(x,r)} |\nabla u|^2 &\leq 4c_{DG}\Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{\Omega(x,R)} |\nabla u|^2 \\
&\quad + a_0 \Big(2R^{(2-\delta)/2} \|\xi\|_{M,\gamma,x,\Omega,\zeta} + \sum_{i=1}^d \|\xi_i\|_{M,\gamma+\delta,x,\Omega,\zeta}\Big)^2 R^{\gamma+\delta}
\end{aligned}$$

uniformly for all $0 < r \leq R \leq \zeta$, where $a_0 = \mu^{-2}(2 + 4c_{DG})$. These bounds can be improved immediately by use of Lemma III.2.1 of [Gia]. It follows that there exists an $a > 0$, depending only on c_{DG}, $\gamma + \delta$ and κ_0, such that

$$\int_{\Omega(x,r)} |\nabla u|^2 \leq a\Big(\Big(\frac{r}{R}\Big)^{\gamma+\delta} \int_{\Omega(x,R)} |\nabla u|^2 + a_0\Big(2\varepsilon^{2-\delta}\|\xi\|_{M,\gamma,x,\Omega,\zeta} + \sum_{i=1}^d \|\xi_i\|_{M,\gamma+\delta,x,\Omega,\zeta}\Big)^2 r^{\gamma+\delta}\Big)$$

uniformly for all $x \in \Upsilon$, $\varepsilon \in (0,1]$ and $0 < r \leq R \leq \zeta\,\varepsilon^2$. Choosing $R = \zeta\,\varepsilon^2$ it follows that

$$\int_{\Omega(x,r)} |\nabla u|^2 \leq a\Big(\zeta^{-(\gamma+\delta)}\,(\varepsilon^{-(\gamma+\delta)}\,\|\nabla u\|_{L_2(\Omega)})^2 + a_0\Big(2\varepsilon^{2-\delta}\|\xi\|_{M,\gamma,x,\Omega,\zeta} + \sum_{i=1}^d \|\xi_i\|_{M,\gamma+\delta,x,\Omega,\zeta}\Big)^2\Big) r^{\gamma+\delta}$$

for all $x \in \Upsilon$ and $0 < r \leq \zeta\,\varepsilon^2$. Alternatively, if $\zeta\,\varepsilon^2 \leq r \leq \zeta$ then

$$\int_{\Omega(x,r)} |\nabla u|^2 \leq \zeta^{-(\gamma+\delta)}\,(\varepsilon^{-(\gamma+\delta)}\|\nabla u\|_{L_2(\Omega)})^2\, r^{\gamma+\delta}$$

and a combination of the last two inequalities completes the proof of the proposition. □

By the Neumann type Poincaré inequality there exists a $c_N > 0$ such that

$$\int_{B(x,R)} |v - \langle v\rangle_{B(x,R)}|^2 \leq c_N\, R^2 \int_{B(x,R)} |\nabla v|^2 \tag{12}$$

uniformly for all $x \in \mathbb{R}^d$, $R \in (0,\infty)$ and $v \in W^{1,2}(B(x,R))$. The next lemma is a Neumann type Poincaré inequality for truncated balls away from Γ. Note that the integral on the left hand side is over a ball in $\mathbb{R}^d$, whilst the integral on the right hand side it is over a (truncated) ball in Ω.

Lemma 3.3. *Let $c_N > 0$ be as in* (12). *Let $\Omega \subset \mathbb{R}^d$ be open and Γ a relatively open subset of $\partial\Omega$. Then*

$$\int_{B(x,R)} |\tilde{u} - \langle \tilde{u}\rangle_{B(x,R)}|^2 \leq c_N\, R^2 \int_{\Omega(x,R)} |\nabla u|^2 \tag{13}$$

for every $x \in \Omega$, $R \in (0,\infty)$ and $u \in W^{1,2}_\Gamma(\Omega)$ such that $\overline{B(x,R)} \cap \Gamma = \emptyset$, where $\tilde{u} \colon B(x,R) \to \mathbb{C}$ is the extension of u with zero.

Proof. Let $x \in \Omega$, $R \in (0,\infty)$, $u \in W^{1,2}_\Gamma(\Omega)$ and suppose that $\overline{B(x,R)} \cap \Gamma = \emptyset$. Then $\tilde{u} \in W^{1,2}(B(x,R))$ by Lemma 2.2(b). Hence

$$\int_{B(x,R)} |\tilde{u} - \langle \tilde{u}\rangle_{B(x,R)}|^2 \leq c_N\, R^2 \int_{B(x,R)} |\nabla \tilde{u}|^2 = c_N\, R^2 \int_{\Omega(x,R)} |\nabla u|^2$$

as required. □

This inequality can be used to deduce a kind of Sobolev embedding for Morrey–Campanato spaces. We emphasize that the Campanato seminorm in the next lemma is with respect to $\mathbb{R}^d$, thus not with Ω.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^d$ be open, Γ a relatively open subset of $\partial\Omega$, $u \in W^{1,2}_\Gamma(\Omega)$, $x \in \Omega$, $R_e \in (0,1]$ and $\gamma \in [0,d)$. Suppose $\overline{B(x,R_e)} \cap \Gamma = \emptyset$. Then*

$$\|\tilde{u}\|_{\mathcal{M},\gamma+2,x,\mathbb{R}^d,R_e} \le \sqrt{c_N}\, \|\nabla u\|_{M,\gamma,x,\Omega,R_e},$$

where c_N is as in (12) and $\tilde{u}\colon \mathbb{R}^d \to \mathbb{C}$ is the extension by zero of u.

Proof. This follows from (13). □

Now we have enough preparation to prove the Ladyshenskaya–Ural'zeva type result of Theorem 1.2.

Proof of Theorem 1.2. Without loss of generality we may assume that $\zeta \le 1$. It follows from the Hölder inequality that

$$\|g\|_{M,d-\frac{2d}{p},x,\Omega,R_e} \le \sqrt{\omega_d}\, \|g\|_{L_p(\Omega)}$$

for all $p \in [2,\infty)$ and $g \in L_p(\Omega)$, $x \in \Omega$ and $R_e \in (0,1]$. Let $\kappa_0 \in (0,1)$ be as in Proposition 3.2. We distinguish two cases.

Case 1. Suppose that $d \ge 4$.
Observe that $d - \frac{2d}{q} > d-2$ and $d - \frac{2d}{q_0} > d-4$. Choose

$$\kappa = \tfrac{1}{2} \min\Big(d - \frac{2d}{q} - (d-2), d - \frac{2d}{q_0} - (d-4), \tfrac{1}{2}\,\kappa_0\Big).$$

Then $\kappa \in (0,1)$. Choose $\delta = 2$ and $\gamma = d-4+2\kappa$. Note that $\gamma \in [0,d)$ since $d \ge 4$. Then $\gamma + \delta = d - 2 + 2\kappa \le d - \frac{2d}{q}$ and $\|f_0\|_{M,\gamma,x,\Omega,\zeta} \le \|f_0\|_{M,d-\frac{2d}{q_0},x,\Omega,\zeta} \le \sqrt{\omega_d}\, \|f_0\|_{L_{q_0}(\Omega)}$.

Case 2. Suppose that $d \in \{2,3\}$.
Since $q > d$, there exists a $\kappa \in (0,\kappa_0)$ such that $d - 2 + 2\kappa \le d - \frac{2d}{q}$ and $\kappa < \frac{1}{2}$. Moreover, $d - \frac{2d}{q_0} \ge 0$. Choose $\gamma = 0$ and $\delta = d - 2 + 2\kappa$. Then $\delta \in (0,2]$, $\gamma + \delta = d - 2 + 2\kappa$ and $\|f_0\|_{M,\gamma,x,\Omega,\zeta} \le \|f_0\|_{M,d-\frac{2d}{q_0},x,\Omega,\zeta} \le \sqrt{\omega_d}\, \|f_0\|_{L_{q_0}(\Omega)}$.

In both cases, let $a_1 > 0$ be as in Proposition 3.2 and choose $\varepsilon = 1$. Let $u \in W^{1,2}_\Gamma(\Omega)$ and suppose that $Lu = f_0 - \operatorname{div} f$ weakly on Ω. Let $x \in \Upsilon$. Then

$$\begin{aligned}\|\nabla u\|_{M,d-2+2\kappa,x,\Omega,\zeta} &\le a_1 \Big(\|f_0\|_{M,\gamma,x,\Omega,\zeta} + \sum_{i=1}^d \|f_i\|_{M,d-2+2\kappa,x,\Omega,\zeta} + \|\nabla u\|_{L_2(\Omega)}\Big) \\ &\le a_1 \sqrt{\omega_d} \Big(\|f_0\|_{L_{q_0}(\Omega)} + \|f\|_{L_q(\Omega)^d} + \|\nabla u\|_{L_2(\Omega)}\Big)\end{aligned}$$

by Proposition 3.2. So

$$\|\tilde{u}\|_{\mathcal{M},d+2\kappa,x,\mathbb{R}^d,\zeta} \le a_1 \sqrt{c_N\, \omega_d} \Big(\|f_0\|_{L_{q_0}(\Omega)} + \|f\|_{L_q(\Omega)^d} + \|\nabla u\|_{L_2(\Omega)}\Big)$$

by Lemma 3.4, where $\tilde{u} \colon \mathbb{R}^d \to \mathbb{C}$ is the extension by zero of u. Hence

$$
\begin{aligned}
|u(x)| &\leq \frac{1}{\sqrt{\omega_d}} \frac{2^{1+d/2}}{1-2^{-\kappa}} \zeta^\kappa \, \|\tilde{u}\|_{\mathcal{M},d+2\kappa,x,\mathbb{R}^d,\zeta} + |\langle \tilde{u} \rangle_{B(x,\zeta)}| \\
&\leq \frac{2^{1+d/2}}{1-2^{-\kappa}} \, a_1 \sqrt{c_N} \Big(\|f_0\|_{L_{q_0}(\Omega)} + \|f\|_{L_q(\Omega)^d} + \|\nabla u\|_{L_2(\Omega)} \Big) + (\omega_d \, \zeta^d)^{-1/2} \, \|u\|_{L_2(\Omega)}
\end{aligned}
$$

by Lemma 3.1(b) and $u|_\Upsilon$ is bounded. Finally, let $y \in \Upsilon$ and suppose that $|x-y| < \frac{\zeta}{2}$. By Lemma 3.1(c) there exists a $c > 0$, depending only on κ (and d) such that

$$
\begin{aligned}
|u(x) - u(y)| &\leq c \, (\|\tilde{u}\|_{\mathcal{M},d+2\kappa,x,\mathbb{R}^d,\zeta} + \|\tilde{u}\|_{\mathcal{M},d+2\kappa,y,\mathbb{R}^d,\zeta}) \, |x-y|^\kappa \\
&\leq 2 a_1 \, c \sqrt{c_N \, \omega_d} \Big(\|f_0\|_{L_{q_0}(\Omega)} + \|f\|_{L_q(\Omega)^d} + \|\nabla u\|_{L_2(\Omega)} \Big) \, |x-y|^\kappa.
\end{aligned}
$$

The proof of Theorem 1.2 is complete. □

4 Transformations

In this section we prove two transformation lemmas for Sobolev spaces and second-order differential equations with boundary conditions. The first is the transformation under a bi-Lipschitz map, the second is an even reflection. We will use them to transfer De Giorgi estimates near the Dirichlet part of the boundary to another type of De Giorgi estimates near the Neumann part of the boundary. We first introduce the appropriate version of De Giorgi estimates near the Neumann part of the boundary.

The space $W^{1,2}_\Gamma(\Omega)$ is defined relative to the Neumann part Γ of the boundary. It is convenient to have a counter part with respect to the Dirichlet part, in order to avoid many complements. Let $\Omega \subset \mathbb{R}^d$ be open and Δ a closed subset of $\partial\Omega$. Then define

$$
\widetilde{W}^{1,2}_\Delta(\Omega) = W^{1,2}_{\partial\Omega \setminus \Delta}(\Omega).
$$

Let $\Omega \subset \mathbb{R}^d$ be open, Γ a relatively open subset of $\partial\Omega$, Δ a closed subset of $\partial\Omega$, $A \in \mathcal{A}(\Omega)$, $\kappa_0 \in (0,1)$, $c_{DG} > 0$ and $\Upsilon \subset \overline{\Omega}$ a set. Then we say that L_A **satisfies** (κ_0, c_{DG})**-De Giorgi estimates on** Υ **for functions vanishing on** Δ **and Neumann boundary conditions on** Γ if

$$
\int_{\Omega(x,r)} |\nabla u|^2 \leq c_{DG} \left(\frac{r}{R} \right)^{d-2+2\kappa_0} \int_{\Omega(x,R)} |\nabla u|^2
$$

for all $x \in \Upsilon$, $0 < r \leq R \leq 1$ and $u \in \widetilde{W}^{1,2}_\Delta(\Omega)$ satisfying

$$
\sum_{i,j=1}^d \int_{\Omega(x,R)} a_{ij} \, (\partial_i u) \, \overline{\partial_j v} = 0 \tag{14}
$$

for all $v \in W^{1,2}_{\Gamma(x,R)}(\Omega(x,R))$. Here (and below) we write $\Gamma(x,R) = \Gamma \cap B(x,R)$.

Remark 4.1.

(a) The phrase ‘vanishing on Δ’ is related to the condition $u \in \widetilde{W}^{1,2}_\Delta(\Omega)$, which implies $u|_\Delta = 0$ in the sense of trace (if the latter exists).

(b) The phrase 'Neumann conditions' is motivated by the fact that the right hand side in (14) is zero, and hence an element of L^2. Therefore a (generalized) Gauss' theorem provides $\nu \cdot A\nabla u = 0$ on $\Gamma(x, R)$. This gives a (generalized) Neumann condition on the corresponding part of Γ, cf. [Cia, Chapter 1.2] or [GGZ, Chapter II.2]).

(c) The cases $\Delta = \emptyset$ and/or $\Gamma = \emptyset$ are explicitly allowed (and will appear in the sequel).

We rephrase Proposition 2.1.

Proposition 4.2. *For all $\mu, M, \alpha, \zeta > 0$ there exist $\kappa_0 \in (0,1)$ and $c_{DG} > 0$ such that for every open set Ω and subset $\Upsilon \subset \overline{\Omega}$ satisfying $\Delta := \{z \in \partial\Omega : d(z, \Upsilon) \le \zeta\}$ is of class $(\mathbf{A}_\alpha)$ it follows that for all $A \in \mathcal{A}_{\rm r}(\Omega, \mu, M)$ the operator L_A satisfies (κ_0, c_{DG})-De Giorgi estimates on Υ for functions vanishing on Δ and Neumann boundary conditions on $\emptyset$.*

Proof. Apply Proposition 2.1 with ζ replaced by $\frac{\zeta}{2}$ and $\Gamma = \{z \in \partial\Omega : d(z, \Upsilon) > \frac{\zeta}{2}\}$ □

Our first transformation is for bi-Lipschitz maps.

Proposition 4.3. *Let Ω_1 and Ω_2 be two open subsets of $\mathbb{R}^d$ and $\phi \colon \Omega_1 \to \Omega_2$ a bi-Lipschitz map. Let $K \ge 1$ be such that K is larger than both the Lipschitz constant for ϕ and ϕ^{-1}. Assume that ϕ admits an extension, also denoted by ϕ, to an open neighbourhood of $\overline{\Omega_1}$ that is again bi-Lipschitz. Let $\Gamma_1 \subset \partial\Omega_1$ be a relatively open set and define $\Gamma_2 = \phi(\Gamma_1)$ Finally, for every measurable $u \colon \Omega_1 \to \mathbb{C}$ define $\Phi u \colon \Omega_2 \to \mathbb{C}$ by $\Phi u = u \circ \phi^{-1}$. Then the following is valid.*

(a) *If $p \in (1, \infty)$, then the restriction of Φ to the space $W^{1,p}_{\Gamma_1}(\Omega_1)$ induces a linear, topological isomorphism from $W^{1,p}_{\Gamma_1}(\Omega_1)$ onto $W^{1,p}_{\Gamma_2}(\Omega_2)$.*

(b) *Let $\mu, M > 0$ and $A \in \mathcal{A}(\Omega_1, \mu, M)$. Define $A^\phi \colon \Omega_2 \to \mathbb{C}^{d\times d}$ by*

$$A^\phi(y) = \frac{1}{\big|\det(D\phi)(\phi^{-1}(y))\big|}\,(D\phi)(\phi^{-1}(y))\; A(\phi^{-1}(y))\; \big(D\phi\big)^T(\phi^{-1}(y)), \tag{15}$$

where $D\phi$ denotes the derivative of ϕ and $\det(D\phi)$ the corresponding determinant. Then $A^\phi \in \mathcal{A}(\Omega_2, (d!\,K^{d+2})^{-1}\mu, d!\,K^{d+2}\,M)$. In addition, one has

$$\mathfrak{l}_A(u,v) = \mathfrak{l}_{A^\phi}(\Phi u, \Phi v) \tag{16}$$

for all $u, v \in W^{1,2}_{\Gamma_1}(\Omega_1)$.

(c) *Let $\Delta_1 \subset \partial\Omega_1$ be a closed set. Let $\Upsilon_1 \subset \overline{\Omega_1}$ be a set and define $\Delta_2 = \phi(\Delta_1)$ and $\Upsilon_2 = \phi(\Upsilon_1)$. Moreover, let $A \in \mathcal{A}(\Omega_1)$, $\kappa_0 \in (0,1)$ and $c_{DG} > 0$. Suppose that the operator L_A satisfies (κ_0, c_{DG})-De Giorgi estimates on Υ_1 for functions vanishing on Δ_1 and Neumann boundary conditions on Γ_1. Then the operator L_{A^ϕ} satisfies (κ_0, c'_{DG})-De Giorgi estimates on Υ_2 for functions vanishing on Δ_2 and Neumann boundary conditions on Γ_2, where $c'_{DG} = d!^2\,K^{4d+4}\,c_{DG} + c_{DG}\,K^{2d}$.*

Proof. The proof of (a) is contained in [GGKR, Theorem 2.10)]. Statement (b) is well known, see [HKR] Proposition 16 for an explicit verification or [AT, Section 0.8]. For the bi-Lipschitz ϕ the derivative $D\phi$ and its inverse $(D\phi)^{-1}$ are essentially bounded by K (see [EG, Section 3.1]).

For the proof of Statement (c) first note that $K^{-1}\,|x-y| \le |\phi(x)-\phi(y)| \le K\,|x-y|$ for all $x,y \in \Omega_1$. Hence

$$\Omega_2(\phi(x), K^{-1}\,r) \subset \phi(\Omega_1(x,r)) \subset \Omega_2(\phi(x), K\,r)$$

for all $x \in \overline{\Omega_1}$ and $r > 0$.

If $V \subset \Omega_1$ is an open set, then it follows by a change of variables that

$$\int_V |\nabla(w \circ \phi)|^2 \le (d!\,K^d)\,K^2 \int_{\phi(V)} |\nabla w|^2$$

for all $w \in C_c^\infty(\mathbb{R}^d)$. Hence by density

$$\int_V |\nabla(u \circ \phi)|^2 \le (d!\,K^d)\,K^2 \int_{\phi(V)} |\nabla u|^2$$

for all $u \in \widetilde{W}^{1,2}_{\Delta_2}(\Omega_2)$.

Let $y \in \Upsilon_2$, $R \in (0,1]$ and $u \in \widetilde{W}^{1,2}_{\Delta_2}(\Omega_2)$. Suppose that

$$\sum_{i,j=1}^d \int_{\Omega_2(y,R)} (A^\phi)_{ij}\,(\partial_i u)\,\overline{\partial_j v} = 0$$

for all $v \in W^{1,2}_{\Gamma_2(y,R)}(\Omega_2(y,R))$. Then $u \circ \phi \in \widetilde{W}^{1,2}_{\Delta_1}(\Omega_1)$ and

$$\sum_{i,j=1}^d \int_{\Omega_1(\phi^{-1}(y),K^{-1}\,R)} a_{ij}\,(\partial_i(u \circ \phi)\,\overline{\partial_j v} = 0$$

for all $v \in W^{1,2}_{\Gamma_1(\phi^{-1}(y),K^{-1}R)}(\Omega_1(\phi^{-1}(y),K^{-1}\,R))$ by (16). So if $r \in (0,R]$ then

$$\int_{\Omega_1(\phi^{-1}(y),K^{-1}\,r)} |\nabla(u \circ \phi)|^2 \le c_{DG} \left(\frac{r}{R}\right)^{d-2+2\kappa_0} \int_{\Omega_1(\phi^{-1}(y),K^{-1}\,R)} |\nabla(u \circ \phi)|^2.$$

Therefore

$$\begin{aligned}
\int_{\Omega_2(y,K^{-2}\,r)} |\nabla u|^2 &\le d!\,K^{d+2} \int_{\Omega_1(\phi^{-1}(y),K^{-1}\,r)} |\nabla(u \circ \phi)|^2 \\
&\le d!\,c_{DG}\,K^{d+2} \left(\frac{r}{R}\right)^{d-2+2\kappa_0} \int_{\Omega_1(\phi^{-1}(y),K^{-1}\,R)} |\nabla(u \circ \phi)|^2 \\
&\le (d!\,K^{d+2})^2\,c_{DG} \left(\frac{r}{R}\right)^{d-2+2\kappa_0} \int_{\Omega_2(y,R)} |\nabla u|^2.
\end{aligned}$$

Now Statement (c) follows easily. □

We next need a simple mirror argument. Let $\pi_d \colon \mathbb{R}^d \to \mathbb{R}$ be the projection onto the last coordinate.

Proposition 4.4. *Define the linear map $\phi \colon \mathbb{R}^d \to \mathbb{R}^d$ by $\phi(y_1,\ldots,y_d) = (y_1,\ldots,y_{d-1},-y_d)$. Let $\Omega \subset \{x \in \mathbb{R}^d : \pi_d(x) < 0\}$ be open. Let $\Sigma \subset \partial\Omega \cap \{x \in \mathbb{R}^d : \pi_d(x) = 0\}$ be a non-empty* *set which is open in $\partial\Omega$. Let $\Delta \subset \partial\Omega \setminus \Sigma$ be a closed set and define $\widehat{\Delta} = \Delta \cup \phi(\Delta)$. Moreover, set*

$$\widehat{\Omega} = \Omega \cup \Sigma \cup \phi(\Omega).$$

For every $u \in W^{1,2}(\Omega)$ define $\hat{u} \colon \widehat{\Omega} \to \mathbb{R}$ by

$$\hat{u}(y) = \begin{cases} u(y) & \text{if } y \in \Omega, \\ (\operatorname{Tr} u)(y) & \text{if } y \in \Sigma, \\ u(\phi(y)) & \text{if } \phi(y) \in \Omega. \end{cases}$$

Further, for all $A \colon \Omega \to \mathbb{C}^{d\times d}$ define $\widehat{A} \colon \widehat{\Omega} \to \mathbb{C}^{d\times d}$ by

$$\widehat{A}(y) = \begin{cases} A(y) & \text{if } y \in \Omega, \\ I & \text{if } y \in \Sigma, \\ \phi\, A(\phi(y))\, \phi & \text{if } y \in \phi(\Omega). \end{cases}$$

(*Note that ϕ is a linear, symmetric, idempotent map.*)

Then one has the following.

(a) *The set $\widehat{\Omega}$ is open. In particular, the points in Σ are* inner *points of $\widehat{\Omega}$.*

(b) *$\widehat{\Delta}$ is closed.*

(c) *If $\mu, M > 0$ and $A \in \mathcal{A}(\Omega, \mu, M)$, then $\widehat{A} \in \mathcal{A}(\widehat{\Omega}, \mu, M)$.*

(d) *If $u \in W^{1,2}(\Omega)$, then $\hat{u} \in W^{1,2}(\widehat{\Omega})$ and $\|\hat{u}\|_{W^{1,2}(\widehat{\Omega})} = 2\|u\|_{W^{1,2}(\Omega)}$.*

(e) *If $u \in \widetilde{W}^{1,2}_{\Delta}(\Omega)$, then $\hat{u} \in \widetilde{W}^{1,2}_{\widehat{\Delta}}(\widehat{\Omega})$.*

(f) *Let $x \in \overline{\Omega}$, $R > 0$ and $u \in W^{1,2}_0(\widehat{\Omega}(x,R))$. Extend u by zero to an element $\tilde{u} \in W^{1,2}(\mathbb{R}^d)$. Then $u|_{\Omega(x,R)} \in W^{1,2}_{\Sigma(x,R)}(\Omega(x,R))$ and $\tilde{u}|_{\phi(\Omega(x,R))} \in W^{1,2}_{\Sigma(x,R)}(\phi(\Omega(x,R)))$.*

(g) *Let $A \in \mathcal{A}(\Omega)$, $\kappa_0 \in (0,1)$ and $c_{DG} > 0$. Let $\Upsilon \subset \overline{\widehat{\Omega}}$ be a set. Suppose the operator $L_{\widehat{A}}$ satisfies (κ_0, c_{DG})-De Giorgi estimates on Υ for functions vanishing on $\widehat{\Delta}$ and Neumann boundary conditions on $\emptyset$. Then the operator L_A satisfies $(\kappa_0, 2c_{DG})$-De Giorgi estimates on $\Upsilon \cap \overline{\Omega}$ for functions vanishing on Δ and Neumann boundary conditions on Σ.*

Proof. '(a)'. Let $x_0 \in \Sigma$. Since Σ is open in $\partial\Omega$ there exists an $r > 0$ such that $B(x_0,r) \cap \partial\Omega \subset \Sigma$. Then the connected set $B(x_0,r) \cap [\pi_d < 0]$ is the disjoint union of the open sets $B(x_0,r) \cap [\pi_d < 0] \cap \Omega$ and $B(x_0,r) \cap [\pi_d < 0] \cap \overline{\Omega}^{\mathrm{c}}$. So $B(x_0,r) \cap [\pi_d < 0] \subset \Omega$. Then $B(x_0,r) \cap [\pi_d = 0] \subset \overline{\Omega} \setminus \Omega = \partial\Omega$. So $B(x_0,r) \cap [\pi_d = 0] \subset B(x_0,r) \cap \partial\Omega \subset \Sigma$. Now it is easy to see that $B(x_0,r) \subset \widehat{\Omega}$.

Statements (b) and (c) are trivial.

'(d)'. Primarily, one has to show the weak differentiability of the extended function. This is a *local* property and is, hence, achieved by the classical reflection argument [Giu, Lemma 3.4]. The equality $\|\hat{u}\|_{W^{1,2}(\widehat{\Omega})} = 2\|u\|_{W^{1,2}(\Omega)}$ is obvious.

'(e)'. Let $w \in C_c^\infty(\mathbb{R}^d)$. Define the function $\breve{w}$ by

$$\breve{w}(y) := \begin{cases} w(y) & \text{if } y_d \le 0, \\ w(\phi(y)) & \text{if } y_d > 0. \end{cases}$$

Suppose $\operatorname{supp} w \cap \Delta = \emptyset$ and set $u := w|_\Omega$. Then, on the one hand, $\hat{u} = \check{w}|_{\widehat{\Omega}}$. On the other hand, $\operatorname{supp} w \cap \Delta = \emptyset$ implies $\operatorname{supp} \check{w} \cap \widehat{\Delta} = \emptyset$. Hence, for all $\varepsilon > 0$, one finds a suitable mollifier χ such that $\operatorname{supp}(\chi * \check{w}) \cap \widehat{\Delta} = \emptyset$ and, in addition, $\|\check{w} - \chi * \check{w}\|_{W^{1,2}(\mathbb{R}^d)} \le \varepsilon$. This gives $\|\hat{u} - (\chi * \check{w})|_{\widehat{\Omega}}\|_{W^{1,2}(\widehat{\Omega})} \le \|\check{w} - \chi * \check{w}\|_{W^{1,2}(\mathbb{R}^d)} \le \varepsilon$. Now the statement follows from (d).

'(f)'. Consider first $u \in C_c^\infty(\widehat{\Omega}(x,R))$. Extending such u by zero to all of $\mathbb{R}^d$, one obtains a function $\tilde{u} \in C_c^\infty(\mathbb{R}^d)$. Moreover, simple geometric considerations show that $\operatorname{supp} \tilde{u} \subset \widehat{\Omega(x,R)}$, where $\widehat{\Omega(x,R)} = \Omega(x,R) \cup \phi(\Omega(x,R)) \cup \Sigma((x,R)$. Therefore the restriction $w := \tilde{u}|_{\widehat{\Omega(x,R)}} \in C_c^\infty(\widehat{\Omega(x,R)})$. Hence, the claim for $u \in C_c^\infty(\widehat{\Omega}(x,R))$ is proved if we can show that $w \in C_c^\infty(\widehat{\Omega(x,R)})$ implies

$$w|_{\Omega(x,R)} \in W^{1,2}_{\Sigma(x,R)}(\Omega(x,R)) \quad \text{and} \quad w|_{\phi(\Omega(x,R))} \in W^{1,2}_{\Sigma(x,R)}(\phi(\Omega(x,R))). \tag{17}$$

In order to prove the first statement in (17) it suffices to show

$$\operatorname{supp} w \cap \partial(\Omega(x,R)) \subset \Sigma(x,R); \tag{18}$$

namely this implies

$$\operatorname{supp} w \cap \partial(\Omega(x,R)) \setminus \Sigma(x,R) = \emptyset,$$

therefore $w|_{\Omega(x,R)} \in C^\infty_{\Sigma(x,R)}(\Omega(x,R)) \subset W^{1,2}_{\Sigma(x,R)}(\Omega(x,R))$. Let us show (18). One clearly has

$$\Omega(x,R) = \widehat{\Omega(x,R)} \cap [\pi_d < 0]$$

and, consequently,

$$\partial(\Omega(x,R)) \subset \Big(\partial\widehat{\Omega(x,R)} \cap [\pi_d < 0]\Big) \cup [\pi_d = 0] = \Big(\partial\widehat{\Omega}(x,R) \cap [\pi_d < 0]\Big) \cup [\pi_d = 0].$$

But $\operatorname{supp} w \cap \Big(\partial\widehat{\Omega}(x,R) \cap [\pi_d < 0]\Big) = \emptyset$ and $\operatorname{supp} w \cap [\pi_d = 0] \subset \Sigma(x,R)$, what proves (18). In order to show the second statement in (17), one considers the function $w \circ \phi$ and proceeds as before. For general $u \in W_0^{1,2}(\widehat{\Omega}(x,R))$ the statement follows by density.

'(g)'. Let $x \in \Upsilon \cap \overline{\Omega}$, $0 < r \le R \le 1$, $u \in \widetilde{W}^{1,2}_\Delta(\Omega)$ and suppose that

$$\sum_{i,j=1}^d \int_{\Omega(x,R)} a_{ij}\,(\partial_i u)\,\overline{\partial_j v} = 0 \tag{19}$$

for all $v \in W^{1,2}_{\Sigma(x,R)}(\Omega(x,R))$. Then $\hat{u} \in \widetilde{W}^{1,2}_{\widehat{\Delta}}(\widehat{\Omega})$, thanks to Statement (e).

Let $v \in W_0^{1,2}(\widehat{\Omega}(x,R))$. Then $v|_{\Omega(x,R)} \in W^{1,2}_{\Sigma(x,R)}(\Omega(x,R))$ by Statement (f), so

$$\sum_{i,j=1}^d \int_{\Omega(x,R) \cap [\pi_d < 0]} \hat{a}_{ij}\,(\partial_i \hat{u})\,\overline{\partial_j v} = 0 \tag{20}$$

by (19). Extend v by zero to an element $\tilde{v} \in W^{1,2}(\mathbb{R}^d)$. Then Statement (f) implies that also $\tilde{v}|_{\phi(\Omega(x,R))} \in W^{1,2}_{\Sigma(x,R)}(\phi(\Omega(x,R)))$. So $(\tilde{v} \circ \phi)|_{\Omega(x,R)} \in W^{1,2}_{\Sigma(x,R)}(\Omega(x,R))$ by Proposition 4.3(a). Hence

$$\sum_{i,j=1}^d \int_{\Omega(x,R)} a_{ij}\,(\partial_i u)\,\overline{\partial_j (\tilde{v} \circ \phi)} = 0$$

by (19). Then (16) and the inclusions $\operatorname{supp}(\tilde{v} \circ \phi) \cap [\pi_d < 0] \subset \phi(\widehat{\Omega}(x, R)) \cap [\pi_d < 0] \subset \Omega(x, R)$ give

$$\sum_{i,j=1}^{d} \int_{\Omega(x,R) \cap [\pi_d > 0]} \hat{a}_{ij} \, (\partial_i \hat{u}) \, \overline{\partial_j v} = \sum_{i,j=1}^{d} \int_{\phi(\Omega(x,R)) \cap [\pi_d < 0]} a_{ij} \, (\partial_i u) \, \overline{\partial_j (\tilde{v} \circ \phi)}$$

$$= \sum_{i,j=1}^{d} \int_{\Omega(x,R)} a_{ij} \, (\partial_i u) \, \overline{\partial_j (\tilde{v} \circ \phi)} = 0 \qquad (21)$$

Adding (20) and (21) gives

$$\sum_{i,j=1}^{d} \int_{\widehat{\Omega}(x,R)} \hat{a}_{ij} \, (\partial_i \hat{u}) \, \overline{\partial_j v} = 0.$$

So

$$\int_{\widehat{\Omega}(x,r)} |\nabla \hat{u}|^2 \leq c_{DG} \left(\frac{r}{R} \right)^{d-2+2\kappa_0} \int_{\widehat{\Omega}(x,R)} |\nabla \hat{u}|^2.$$

Therefore

$$\int_{\Omega(x,r)} |\nabla u|^2 \leq \int_{\widehat{\Omega}(x,r)} |\nabla \hat{u}|^2 \leq c_{DG} \left(\frac{r}{R} \right)^{d-2+2\kappa_0} \int_{\widehat{\Omega}(x,R)} |\nabla \hat{u}|^2$$

$$\leq 2 c_{DG} \left(\frac{r}{R} \right)^{d-2+2\kappa_0} \int_{\Omega(x,R)} |\nabla u|^2$$

and the proposition follows. □

5 De Giorgi estimates near the Neumann part of the boundary

Proposition 2.1 provides De Giorgi estimates for boundary points away from the Neumann part Γ under the assumption of Theorem 1.1 The aim of this section is to obtain De Giorgi estimates at any point in $\frac{1}{2} E^-$ after a bi-Lipschitz transformation as in Theorem 1.1.

If $\Omega, U \subset \mathbb{R}^d$ are open and ϕ is a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\phi(U) = E$ and $\phi(\Omega \cap U) = E^-$, then for all $A \in \mathcal{A}(\Omega)$ define $A^\phi \colon E^- \to \mathbb{C}^{d \times d}$ by

$$(A^\phi)(y) = \frac{1}{\big| \det(D\phi)(\phi^{-1}(y)) \big|} \, (D\phi)(\phi^{-1}(y)) \, A(\phi^{-1}(y)) \, \big(D\phi\big)^T (\phi^{-1}(y)).$$

(Cf. (15).)

We first consider a quantitive version of the case $\partial\Omega \cap U \subset \Gamma$ in Theorem 1.1. Note that in the next lemma the functions are vanishing on $\phi((\partial\Omega \setminus \Gamma) \cap U)$ and there are Neumann boundary conditions on $\phi(\Gamma \cap U)$.

Lemma 5.1. *For all $K \geq 1$ and $\mu, M > 0$ there exist $c_{DG} > 0$ and $\kappa_0 \in (0,1)$ such that the following is valid.*

Let $\Omega, U \subset \mathbb{R}^d$ be open, $\Gamma \subset \partial\Omega$ relatively open, $A \in \mathcal{A}_{\rm r}(\Omega, \mu, M)$ and ϕ a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\phi(U) = E$, $\phi(\Omega \cap U) = E^-$, $\partial\Omega \cap U \subset \Gamma$ and $\phi(\partial\Omega \cap U) = P$, and with K larger than the Lipschitz constant for $\phi|_{\Omega\cap U}$ and $\phi^{-1}|_{E^-}$. Then the operator L_{A^ϕ} satisfies (κ_0, c_{DG})-De Giorgi estimates on $\frac{1}{2} E^-$ for functions vanishing on $\emptyset$ with Neumann boundary conditions on P.

Proof. It follows from Proposition 4.2 that there exist $\kappa_0 \in (0, 1)$ and $c_{DG} > 0$ such that for all $A \in \mathcal{A}_{\rm r}(E, (d!K^d)^{-1}\mu, d!\, K^{d+2}\, M)$ the operator L_A satisfies (κ_0, c_{DG})-De Giorgi estimates on $\frac{1}{2} E^-$ for functions vanishing on $\emptyset$ and Neumann boundary conditions on $\emptyset$, cf. Remark 4.1(c). Now let $A \in \mathcal{A}_{\rm r}(\Omega, \mu, M)$. Then $\widehat{A^\phi} \in \mathcal{A}_{\rm r}(E, (d!\, K^{d+2})^{-1}\mu, d!\, K^{d+2}\, M)$ by a combination of Propositions 4.3(b) and 4.4(c), where we use the notation as in Proposition 4.4. Hence the operator $L_{\widehat{A^\phi}}$ satisfies (κ_0, c_{DG})-De Giorgi estimates on $\frac{1}{2} E^-$ for functions vanishing on $\emptyset$ and Neumann boundary conditions on $\emptyset$. Therefore by Proposition 4.4(g) with $\Sigma = P$, the operator L_{A^ϕ} satisfies $(\kappa_0, 2c_{DG})$-De Giorgi estimates on $\frac{1}{2} E^-$ for functions vanishing on $\emptyset$ with Neumann boundary conditions on P. □

If $\phi^{-1}(0) \in \Gamma$ then the condition $\partial\Omega \cap U \subset \Gamma$ in Lemma 5.1 can be arranged by a simple scaling. We state this in the next lemma in order to obtain uniformity of the constants.

Lemma 5.2. *Let $\Omega, U \subset \mathbb{R}^d$ be open and $\Gamma \subset \partial\Omega$ relatively open. Let ϕ be a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\phi(U) = E$, $\phi(\Omega \cap U) = E^-$ and $\phi(\partial\Omega \cap U) = P$. Let $K \geq 1$ be larger than the Lipschitz constant for $\phi|_{\Omega\cap U}$ and $\phi^{-1}|_{E^-}$. Let $x_0 = \phi^{-1}(0)$. Suppose that $x_0 \in \Gamma$ and $U \cap (\partial\Omega \setminus \Gamma) \neq \emptyset$. Let $\lambda = d(x_0, \partial\Gamma) \wedge 1$. Define $V = \{x \in U : \frac{K}{\lambda}\, \phi(x) \in E\}$ and define $\check{\phi}\colon V \to E$ by $\check{\phi}(x) = \frac{K}{\lambda}\, \phi(x)$. Then $\check{\phi}$ is a bi-Lipschitz map from an open neighbourhood of $\overline{V}$ onto an open subset of $\mathbb{R}^d$ such that $\check{\phi}(V) = E$, $\check{\phi}(\Omega \cap V) = E^-$ and $\check{\phi}(\partial\Omega \cap V) = P$. Moreover, $\partial\Omega \cap V \subset \Gamma$ and $\frac{K^2}{\lambda}$ is larger than the Lipschitz constant for $\check{\phi}|_{\Omega\cap V}$ and $\check{\phi}^{-1}|_{E^-}$.*

Proof. Easy. □

In Theorem 6.8 we will use Lemmas 5.1 and 5.2 to cover the case $x \in \Gamma$ in Theorem 1.1. The next aim is to have a similar result if $x \in \partial\Gamma$. Recall that we write $\partial\Gamma$ for the boundary of Γ in $\partial\Omega$. This requires delicate estimates.

Proposition 5.3. *For all $K, \mu, M, c_1 > 0$ and $c_0 \in (0, 1)$ there are $\kappa_0 \in (0, 1)$ and $c_{DG} > 0$ such that the following is valid.*

Let $\Omega, U \subset \mathbb{R}^d$ be open, $\Gamma \subset \partial\Omega$ relatively open, ϕ a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\phi(U) = E$, $\phi(\Omega \cap U) = E^-$, $\phi(\partial\Omega \cap U) = P$ and $\phi^{-1}(0) \in (\partial\Gamma) \cap U$. Moreover, let $A \in \mathcal{A}_{\rm r}(\Omega, \mu, M)$. Suppose that K is larger than the Lipschitz constant for $\phi|_{\Omega\cap U}$ and $\phi^{-1}|_{E^-}$, and

$$\operatorname{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_s(\tilde{y}) : \operatorname{dist}((\tilde{z}, 0), \phi(\Gamma \cap U)) > c_0\, s\} \geq c_1\, s^{d-1} \tag{22}$$

for all $s \in (0, 1]$ and $\tilde{y} \in \mathbb{R}^{d-1}$ with $(\tilde{y}, 0) \in \phi((\partial\Gamma) \cap U)$. Then the operator L_{A^ϕ} satisfies (κ_0, c_{DG})-De Giorgi estimates on $\frac{1}{2} E^-$ for functions vanishing on $\overline{\phi((\partial\Omega \setminus \Gamma) \cap U)}$ and Neumann boundary conditions on $\phi(\Gamma \cap U)$.

For the proof of the proposition we need a couple of lemmas. Adopt the assumptions and notation of Proposition 5.3. In the sequel all bars over sets denote the closure in $\mathbb{R}^d$. Define $\Delta = \overline{(\partial\Omega \setminus \Gamma) \cap U} = \overline{(\partial\Omega \cap U) \setminus (\Gamma \cap U)}$. Then $\Delta \cap \Gamma = \emptyset$ by Lemma 2.2(d). Since $\partial\Omega$ is closed in $\mathbb{R}^d$ it follows that the closure of Γ in $\partial\Omega$ is equal to $\overline{\Gamma}$, the closure of Γ in $\mathbb{R}^d$. Recall that we write $\partial\Gamma$ for the boundary of Γ in $\partial\Omega$. Then $\partial\Gamma = \overline{\Gamma} \setminus \Gamma$. Clearly $\Gamma \cap U$ is open in $\partial\Omega$, hence also in $\partial\Omega \cap U$. Since U is open, it follows that $\overline{\Gamma} \cap U$ is the closure of $\Gamma \cap U$ in $\partial\Omega \cap U$. Therefore the boundary of $\Gamma \cap U$ in $\partial\Omega \cap U$ is the set $(\overline{\Gamma} \cap U) \setminus (\Gamma \cap U) = (\overline{\Gamma} \setminus \Gamma) \cap U = (\partial\Gamma) \cap U$.

Next we apply the transformation ϕ. Write

$$\Sigma = \phi(\Gamma \cap U) \quad \text{and} \quad \Delta_1 = \phi(\Delta).$$

Then Σ is open in P and hence Σ is open in ∂E^-. Moreover, $\Delta_1 \cap \Sigma = \emptyset$ and since $\phi(\partial\Omega \cap U) = P$, it follows that $\Sigma \subset P$ and $\Delta_1 \subset \overline{P}$ and Δ_1 is closed in P. Also, since ϕ is a homeomorphism, it follows that the boundary of $\Sigma = \phi(\Gamma \cap U)$ in $P = \phi(\partial\Omega \cap U)$ is equal to $\phi((\partial\Gamma) \cap U)$.

Condition (22) is valid for all $\tilde{y} \in \mathbb{R}^{d-1}$ with $(\tilde{y}, 0)$ is an element of the boundary of Σ in P. We next show that Condition (22) carries over to all $\tilde{y} \in \mathbb{R}^{d-1}$ with $(\tilde{y}, 0) \in P \setminus \Sigma$. It is in essence this subsequent lemma which allows us to formulate (22) merely for boundary points of Γ and not for all points in the transformed Dirichlet part.

Lemma 5.4. *If $(\tilde{y}, 0) \in P \setminus \Sigma$ and $s \in (0, 1]$, then*

$$\mathrm{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_s(\tilde{y}) : \mathrm{dist}((\tilde{z}, 0), \Sigma) > \hat{c}_0\, s\} \geq \hat{c}_1 s^{d-1}, \tag{23}$$

where $\hat{c}_0 := \min\{\frac{1}{4}, \frac{c_0}{2}\}$ and $\hat{c}_1 := \min\{\frac{\omega_{d-1}}{4^{d-1}}, \frac{c_1}{2^{d-1}}\}$. Here c_0 and c_1 are the constants in (22).

Proof. Obviously if $(\tilde{y}, 0)$ is an element of the boundary of Σ in P, then the estimate (23) immediately follows from (22).

Note that $\overline{\Sigma} \cap P$ is the closure of Σ in P, so $\overline{\Sigma} \cap P = \Sigma \cup (\partial\Sigma)$. Next fix $(\tilde{y}, 0) \in P \setminus \overline{\Sigma}$. Then

$$\varepsilon := \mathrm{dist}((\tilde{y}, 0), \Sigma) = \mathrm{dist}((\tilde{y}, 0), \overline{\Sigma}) > 0.$$

We distinguish the three cases $0 < s \leq \varepsilon/2$, $\varepsilon/2 < s \leq 2\varepsilon$ and $2\varepsilon < s \leq 1$.

Case 1. Suppose $0 < s \leq \varepsilon/2$.

Since

$$\{\tilde{z} \in \widetilde{B}_s(\tilde{y}) : \mathrm{dist}((\tilde{z}, 0), \Sigma) > s\} = \widetilde{B}_s(\tilde{y})$$

it follows that

$$\mathrm{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_s(\tilde{y}) : \mathrm{dist}((\tilde{z}, 0), \Sigma) > s\} = \mathrm{mes}_{d-1}\, \widetilde{B}_s(\tilde{y}) = \omega_{d-1}\, s^{d-1}$$

as required.

Case 2. Suppose $\varepsilon/2 < s \leq 2\varepsilon$.

Since $s/4 \leq \varepsilon/2$, we infer from the first case

$$\mathrm{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_s(\tilde{y}) : \mathrm{dist}((\tilde{z}, 0), \Sigma) > \frac{s}{4}\} \geq \mathrm{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_{\frac{s}{4}}(\tilde{y}) : \mathrm{dist}((\tilde{z}, 0), \Sigma) > \frac{s}{4}\}$$
$$\geq \omega_{d-1}\, \frac{s^{d-1}}{4^{d-1}}.$$

Case 3. Suppose $2\varepsilon < s \leq 1$.
Because P is convex there exists an element $\tilde{y}^*$ of the boundary of Σ in P such that $\varepsilon \leq \|\tilde{y} - \tilde{y}^*\| < \frac{s}{2}$. Since $\widetilde{B}_{\frac{s}{2}}(\tilde{y}^*) \subset \widetilde{B}_s(\tilde{y})$, this yields

$$\begin{aligned}\operatorname{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_s(\tilde{y}) : \operatorname{dist}((\tilde{z},0),\Sigma) > \frac{c_0}{2}\, s\} &\geq \operatorname{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_{\frac{s}{2}}(\tilde{y}^*) : \operatorname{dist}((\tilde{z},0),\Sigma) > \frac{c_0}{2}\, s\} \\ &\geq c_1\Big(\frac{s}{2}\Big)^{d-1}\end{aligned}$$

as required

This completes the proof of the lemma. □

Unfortunately, both Σ and Δ_1 are in the same hyperplane through P. Reflection in Σ as in the proof of Lemma 5.1 then gives problems with Δ_1, where Dirichlet conditions are assumed. Therefore we apply a second transformation to shift down the points in Δ_1. For all $\tau \in \mathbb{R}$ define $\psi_\tau \colon \mathbb{R}^d \to \mathbb{R}^d$ by

$$\psi_\tau(\tilde{x}, x_d) := (\tilde{x}, x_d - \tau \operatorname{dist}((\tilde{x},0),\Sigma)).$$

In order to justify the application of ψ_τ, we need a lemma.

Lemma 5.5. *Let $\tau \in \mathbb{R}$.*

(a) *The function $(\tilde{x}, x_d) \mapsto \operatorname{dist}((\tilde{x},0),\Sigma)$ from $\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$ into $\mathbb{R}$ is Lipschitz with Lipschitz constant 1.*

(b) *The function ψ_τ is Lipschitz with Lipschitz constant $1 + |\tau|$. Its inverse is $\psi_{-\tau}$, so ψ_τ is bi-Lipschitz.*

(c) *The function ψ_τ is volume preserving.*

Proof. '(a)'. If $x = (\tilde{x}, x_d) \in \mathbb{R}^d$ and $y = (\tilde{y}, y_d) \in \mathbb{R}^d$ then

$$|\operatorname{dist}((\tilde{x},0),\Sigma) - \operatorname{dist}((\tilde{y},0),\Sigma)| \leq \|\tilde{x} - \tilde{y}\| \leq \|x - y\|.$$

'(b)'. The first assertion follows from Statement (a). The second is straightforward to verify.

'(c)'. It is clear that the Jacobian of ψ_τ is identical 1, thus the assertion follows from Theorem 3.3.3.2 in [EG]. □

We emphasize that the transformation ψ_τ only modifies the last component of a point $y = (\tilde{y}, y_d)$, but keeps $\tilde{y}$ fixed.

Let $\hat{c}_0, \hat{c}_1$ be as in Lemma 5.4. Choose $\tau := \frac{3}{\hat{c}_0}$ and abbreviate $\psi := \psi_\tau$. Set $\Lambda := \psi(E^-)$ and observe that Σ remains invariant under ψ. Moreover, set $\Delta_2 = \psi(\Delta_1)$. Then $\Delta_2 \cap \Sigma = \emptyset$.

Finally, we reflect Λ across Σ, using Proposition 4.4. We end up with the open set $\widehat{\Lambda}$ and a closed set $\widehat{\Delta}_2$, where we use the notation as in Proposition 4.4.

Lemma 5.6. *The set $\psi(\frac{1}{2}E^-)$ has a distance to $\partial\widehat{\Lambda} \setminus \widehat{\Delta}_2$ of at least $\frac{1}{2\sqrt{1+\tau^2}}$.*

Proof. This is an elementary geometric exercise. □

Lemma 5.7. *The set $\widehat{\Delta}_2$, viewed as a subset of the boundary of $\widehat{\Lambda}$, belongs to a class $(\mathbf{A}_\beta)$, where* $\beta = \min\Big(\frac{1}{2(1+\tau)^d}, \frac{\hat{c}_1}{d\,\omega_d}\Big(\big(\frac{1}{2}\big)^{\frac{d}{2}} - \big(\frac{1}{2}\big)^d\Big)\Big)$.

Proof. We show that the set $\psi(P \setminus \Sigma)$ belongs to the class $(\mathbf{A}_\beta)$. The proof for the the reflected part follows by the fact that $\widehat{\Lambda}$ is invariant under the map $(\tilde{y}, y_d) \mapsto (\tilde{y}, -y_d)$.

Let $x \in \psi(P \setminus \Sigma)$. Let $\tilde{y} \in \mathbb{R}^{d-1}$ be such that $x = \psi(\tilde{y}, 0)$ and $(\tilde{y}, 0) \in P$. Then

$$x = (\tilde{y}, -\tau \operatorname{dist}((\tilde{y}, 0), \Sigma)).$$

Let $r \in (0, 1]$ and let $B(x, r)$ be the corresponding ball around x. We distinguish two cases.

Case 1. Suppose $r < \tau \operatorname{dist}((\tilde{y}, 0), \Sigma)$.

Then $B(x, r) \subset \{z \in \mathbb{R}^d : z_d < 0\}$. This gives

$$B(x, r) \setminus \widehat{\Lambda} = B(x, r) \setminus \Lambda = B(x, r) \setminus \psi(E^-).$$

Apply ψ^{-1} and note that ψ^{-1} is volume preserving by Lemma 5.5(c). It follows that

$$\big|B(x, r) \setminus \widehat{\Lambda}\big| = \Big|\psi^{-1}\big(B(x, r)\big) \setminus E^-\Big| \geq \Big|\psi^{-1}\big(B(x, r)\big) \cap \{z : z_d > 0\}\Big|.$$

But $\psi^{-1}\big(B(x, r)\big)$ contains the ball $B((\tilde{y}, 0), \frac{r}{L})$, where $L = 1 + \tau$ is the Lipschitz constant of ψ. Thus,

$$\big|B(x, r) \setminus \widehat{\Lambda}\big| \geq \frac{1}{2} \frac{1}{L^d} \omega_d \, r^d,$$

as required.

Case 2. Suppose $r \geq \tau \operatorname{dist}((\tilde{y}, 0), \Sigma)$.

Define

$$B^-(x, r) = \{z \in B(x, r) : z_d \leq -\tau \operatorname{dist}((\tilde{y}, 0), \Sigma)\}.$$

By construction of $\widehat{\Lambda}$, one has

$$B(x, r) \setminus \widehat{\Lambda} \supset B^-(x, r) \setminus \widehat{\Lambda} = B^-(x, r) \setminus \Lambda. \tag{24}$$

It is clear that $\Lambda \subset \{(\tilde{z}, z_d) \in \mathbb{R}^d : z_d < -\tau \operatorname{dist}((\tilde{z}, 0), \Sigma)\}$. Therefore

$$\begin{aligned} B^-(x, r) \setminus \Lambda &\supset B^-(x, r) \setminus \{(\tilde{z}, z_d) : z_d < -\tau \operatorname{dist}((\tilde{z}, 0), \Sigma)\} \\ &= B^-(x, r) \cap \{(\tilde{z}, z_d) : z_d \geq -\tau \operatorname{dist}((\tilde{z}, 0), \Sigma)\}. \end{aligned} \tag{25}$$

For all $s \in (0, r]$ let $\mathcal{H}_s$ be the hyperplane $\mathcal{H}_s := \{z : z_d + \tau \operatorname{dist}((\tilde{y}, 0), \Sigma) = -s\}$. If $z \in B(x, r)$ then $|z_d + \tau \operatorname{dist}((\tilde{y}, 0), \Sigma)| < r$. So if $z \in B^-(x, r)$ then

$$-r < z_d + \tau \operatorname{dist}((\tilde{y}, 0), \Sigma) \leq 0.$$

Hence (24) and (25) give

$$\begin{aligned} B(x, r) \setminus \widehat{\Lambda} &\supset B^-(x, r) \cap \{(\tilde{z}, z_d) : z_d \geq -\tau \operatorname{dist}((\tilde{z}, 0), \Sigma)\} \\ &= B^-(x, r) \cap \Big(\bigcup_{s \in [0, r]} \mathcal{H}_s\Big) \cap \{(\tilde{z}, z_d) \in \mathbb{R}^d : z_d \geq -\tau \operatorname{dist}((\tilde{z}, 0), \Sigma)\} \\ &= \bigcup_{s \in [0, r]} B^-(x, r) \cap \mathcal{H}_s \cap \{(\tilde{z}, z_d) \in \mathbb{R}^d : z_d \geq -\tau \operatorname{dist}((\tilde{z}, 0), \Sigma)\}\big) \\ &= \bigcup_{s \in [0, r]} \{(\tilde{z}, z_d) \in \mathbb{R}^d : z_d = -\tau \operatorname{dist}((\tilde{y}, 0), \Sigma) - s, \ \|\tilde{y} - \tilde{z}\|_{\mathbb{R}^{d-1}} < \sqrt{r^2 - s^2} \\ &\qquad\qquad \text{and } \tau \operatorname{dist}((\tilde{z}, 0), \Sigma) \geq \tau \operatorname{dist}((\tilde{y}, 0), \Sigma) + s\}. \end{aligned}$$

Recall that $\widetilde{B}_r(\tilde{y}) = \{\tilde{z} \in \mathbb{R}^{d-1} : \|\tilde{y} - \tilde{z}\| < r\}$. An application of Cavalieri's principle yields

$$
\begin{aligned}
\big|B(x,r) \setminus \widehat{\Lambda}\big| \geq \int_0^r \operatorname{mes}_{d-1}\{\tilde{z} \in \mathbb{R}^{d-1} : \|\tilde{y} - \tilde{z}\|_{\mathbb{R}^{d-1}} < \sqrt{r^2 - s^2} \text{ and } \qquad (26) \\
\tau \operatorname{dist}((\tilde{z},0),\Sigma) \geq \tau \operatorname{dist}((\tilde{y},0),\Sigma) + s\}\, ds \\
= \int_0^r \operatorname{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_{\sqrt{r^2-s^2}}(\tilde{y}) : \tau \operatorname{dist}((\tilde{z},0),\Sigma) \geq \tau \operatorname{dist}(\tilde{y},\Sigma) + s\}\, ds \\
\geq \int_{\frac{r}{2}}^{\frac{r}{\sqrt{2}}} \operatorname{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_{\sqrt{r^2-s^2}}(\tilde{y}) : \tau \operatorname{dist}((\tilde{z},0),\Sigma) \geq \tau \operatorname{dist}((\tilde{y},0),\Sigma) + s\}\, ds
\end{aligned}
$$

If $s \in (0, \frac{r}{\sqrt{2}}]$ then $\widetilde{B}_s(\tilde{y}) \subset \widetilde{B}_{\sqrt{r^2-s^2}}(\tilde{y})$. Recall that by assumption $r \geq \tau \operatorname{dist}((\tilde{y},0),\Sigma)$. Hence if $s \geq \frac{r}{2}$, then $3s \geq \tau \operatorname{dist}((\tilde{y},0),\Sigma) + s$. Therefore for all $s \in [\frac{r}{2}, \frac{r}{\sqrt{2}}]$ one obtains the inclusion

$$
\begin{aligned}
\{\tilde{z} \in \widetilde{B}_{\sqrt{r^2-s^2}}(\tilde{y}) : \tau \operatorname{dist}((\tilde{z},0),\Sigma) \geq \tau \operatorname{dist}((\tilde{y},0),\Sigma) + s\} \\
\supset \{\tilde{z} \in \widetilde{B}_s(\tilde{y}) : \tau \operatorname{dist}((\tilde{z},0),\Sigma) \geq 3s\} \\
= \{\tilde{z} \in \widetilde{B}_s(\tilde{y}) : \operatorname{dist}((\tilde{z},0),\Sigma) \geq \hat{c}_0 s\}
\end{aligned}
$$

since $\tau = \frac{3}{\hat{c}_0}$. Thus (26) may be further estimated from below by

$$
\begin{aligned}
\big|B(x,r) \setminus \widehat{\Lambda}\big| &\geq \int_{\frac{r}{2}}^{\frac{r}{\sqrt{2}}} \operatorname{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_s(\tilde{y}) : \operatorname{dist}((\tilde{z},0),\Sigma) \geq \hat{c}_0 s\}\, ds \\
&\geq \hat{c}_1 \int_{\frac{r}{2}}^{\frac{r}{\sqrt{2}}} s^{d-1} ds = \frac{\hat{c}_1}{d}\Big(\big(\frac{1}{2}\big)^{\frac{d}{2}} - \big(\frac{1}{2}\big)^{d}\Big)\, r^d
\end{aligned}
$$

by Lemma 5.4. □

Now we have enough preparation to prove Proposition 5.3.

Proof of Proposition 5.3. Using Lemmas 5.6 and 5.7 it follows from Proposition 4.2 that there exist $\kappa_0 \in (0,1)$ and $c_{DG} > 0$, depending only on K, μ, M, c_1 and c_0, such that for all $A \in \mathcal{A}_{\mathrm{r}}(\widehat{\Lambda}, (d!^2\, K^{d+2}\, (1+\tau)^{d+2})^{-1}\mu, d!^2\, K^{d+2}\, (1+\tau)^{d+2}\, M)$ the operator L_A satisfies (κ_0, c_{DG})-De Giorgi estimates on $\psi(\frac{1}{2}\, E^-)$ for functions vanishing on $\widehat{\Delta_2}$ and Neumann boundary conditions on $\emptyset$.

Now let $A \in \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$. Then $\widehat{(A^\phi)^\psi} \in \mathcal{A}_{\mathrm{r}}(\widehat{\Lambda}, (d!^2\, K^{d+2}\, (1+\tau)^{d+2})^{-1}\mu, d!^2\, K^{d+2}\, (1+\tau)^{d+2}\, M)$ by Propositions 4.3(b) and 4.4(c). Hence the operator $L_{\widehat{(A^\phi)^\psi}}$ satisfies (κ_0, c_{DG})-De Giorgi estimates on $\psi(\frac{1}{2}\, E^-)$ for functions vanishing on $\widehat{\Delta_2}$ and Neumann boundary conditions on $\emptyset$. Therefore by Proposition 4.4(g) the operator $L_{(A^\phi)^\psi}$ satisfies $(\kappa_0, 2c_{DG})$-De Giorgi estimates on $\psi(\frac{1}{2}\, E^-)$ for functions vanishing on Δ_2 and Neumann boundary conditions on Σ. Hence by Proposition 4.3(c) the operator L_{A^ϕ} satisfies (κ_0, c'_{DG})-De Giorgi estimates on $\frac{1}{2}\, E^-$ for functions vanishing on $\Delta_1 = \overline{\phi((\partial\Omega \setminus \Gamma) \cap U)}$ and Neumann boundary conditions on $\Sigma = \phi(\Gamma \cap U)$, where $c'_{DG} = 2(d!)^2\, K^{4d+4}\, c_{DG} + 2c_{DG}\, K^{2d}$. □

6 Hölder continuity of solutions

We aim to prove Theorem 1.1 and an extension for unbounded Ω in this section. First we need two Poincaré inequalities on the cube E^- for truncated balls with centre in $x_0 \in \frac{1}{2}\,E^-$ and radius $R \in (0, \frac{1}{2}]$. Note that $B(x_0, R) \cap \partial E^- \subset P$ and that at least half of the ball $B(x_0, R)$ is in E^-.

Lemma 6.1. *Let $c_N > 0$ be as in* (12).

(a) *If $x_0 \in \frac{1}{2}\,E^-$, $R \in (0, \frac{1}{2}]$ and $u \in W^{1,2}(E^-)$, then*

$$\int_{E^-(x_0,R)} |u - \langle u \rangle_{E^-(x_0,R)}|^2 \leq 2c_N\, R^2 \int_{E^-(x_0,R)} |\nabla u|^2 \tag{27}$$

(b) *If $\Gamma \subset P$ is relatively open, then*

$$\int_{E^-(x_0,R)} |u|^2 \leq 4\, R^2 \int_{E^-(x_0,R)} |\nabla u|^2.$$

for all $x_0 \in \frac{1}{2}\,E^-$, $R \in (0, \frac{1}{2}]$ and $u \in W^{1,2}_{\Gamma(x_0,R)}(E^-(x_0, R))$.

Proof. '(a)'. If $\pi_d(x_0) \leq -R$ then (27) follows from (12). Alternatively, if $\pi_d(x_0) > -R$, then define $\tilde{u} \colon B(x_0, R) \to \mathbb{C}$ by

$$\tilde{u}(y) = \begin{cases} u(y) & \text{if } \pi_d(y) < 0, \\ (\operatorname{Tr} u)(y) & \text{if } \pi_d(y) = 0, \\ u(y - 2\pi_d(y)\, e_d) & \text{if } \pi_d(y) > 0. \end{cases}$$

Then $\tilde{u} \in W^{1,2}(B(x_0, R))$ by Proposition 4.4(d). Moreover,

$$\begin{aligned} \int_{E^-(x_0,R)} |u - \langle u \rangle_{E^-(x_0,R)}|^2 &\leq \int_{E^-(x_0,R)} |u - \langle \tilde{u} \rangle_{B(x_0,R)}|^2 \\ &\leq \int_{B(x_0,R)} |\tilde{u} - \langle \tilde{u} \rangle_{B(x_0,R)}|^2 \\ &\leq c_N\, R^2 \int_{B(x_0,R)} |\nabla \tilde{u}|^2 \leq 2c_N\, R^2 \int_{E^-(x_0,R)} |\nabla u|^2 \end{aligned}$$

and Statement (a) follows.

'(b)'. This follows by an adaption of the proof of Theorem V.3.22 in [EE]). □

The Neumann type Poincaré inequality implies a kind of Sobolev embedding between Morrey and Campanato spaces.

Lemma 6.2. *Let $c_N > 0$ be as in* (12). *Then*

$$\|u\|_{\mathcal{M},\gamma+2,x,E^-,\frac{1}{2}} \leq \sqrt{2c_N}\, \|\nabla u\|_{M,\gamma,x,E^-,\frac{1}{2}} \tag{28}$$

and

$$\|u\|_{\mathcal{M},\gamma+\delta,x,E^-,\frac{1}{2}} \leq c_1\, (\varepsilon^{2-\delta} \|\nabla u\|_{M,\gamma,x,E^-,\frac{1}{2}} + \varepsilon^{-(\gamma+\delta)} \|u\|_{L_2(E^-)})$$

for all $\gamma \in [0, d)$ and $\delta \in (0, 2]$, $\varepsilon \in (0, 1]$, $u \in W^{1,2}(E^-)$ and $x \in \frac{1}{2}\,E^-$, where $c_1 = 2^{d+2} + 2c_N$.

Proof. The first inequality follows from (27). If $r \in (0, \frac{1}{2}\,\varepsilon^2]$ then

$$r^{-(\gamma+\delta)} \int_{E^1(x,r)} |u - \langle u \rangle_{E^1(x,r)}|^2 \le 2c_N\, r^{2-\delta} r^{-\gamma} \int_{E^1(x,r)} |\nabla u|^2 \le 2c_N \varepsilon^{2(2-\delta)} \|\nabla u\|^2_{M,\gamma,x,E^-,\frac{1}{2}}$$

for all $x \in \frac{1}{2}\,E^-$ by the Poincaré inequality (27). Alternatively,

$$\int_{E^1(x,r)} |u - \langle u \rangle_{E^1(x,r)}|^2 \le \int_{E^1(x,r)} |u|^2 \le 2^{\gamma+\delta}\,\varepsilon^{-2(\gamma+\delta)} \|u\|^2_{L_2(E^-)}\, r^{\gamma+\delta}$$

if $r \in [\frac{1}{2}\,\varepsilon^2, \frac{1}{2}]$, from which the lemma follows. □

It is well known that elements in $W^{1,2}_0(\Omega)$ can be extended by zero to elements in $W^{1,2}(\mathbb{R}^d)$. We next need a variation of this extension property.

Lemma 6.3. *Let $\Omega \subset \mathbb{R}^d$ be open and $\Gamma \subset \partial\Omega$ a relatively open subset. Let $U \subset \mathbb{R}^d$ be open and define $\Lambda := U \cap \Omega$. Set $\Upsilon := \Gamma \cap U$. Then Υ is open in $\partial\Lambda$. Let $p \in [1, \infty)$. Then there exists a unique isometric map $\mathfrak{E}\colon W^{1,p}_\Upsilon(\Lambda) \to W^{1,p}_\Gamma(\Omega)$ such that $\mathfrak{E}u$ is the extension of u to Ω by 0 for all $u \in C^\infty_\Upsilon(\Lambda)$.*

Proof. There exists an open $V \subset \mathbb{R}^d$ such that $\Gamma = V \cap \partial\Omega$. Then $\Upsilon = \Gamma \cap U = U \cap V \cap \partial\Omega \subset U \cap V \cap \partial\Lambda$. But $\partial\Lambda \subset \partial\Omega \cup \partial U$. Therefore $U \cap V \cap \partial\Lambda \subset U \cap V \cap \partial\Omega$. So $(U \cap V) \cap \partial\Lambda = U \cap V \cap \partial\Omega = \Upsilon$ and Υ is open in $\partial\Lambda$.

Let $w \in C^\infty_c(\mathbb{R}^d)$ with $\operatorname{supp} w \cap (\partial\Lambda \setminus \Upsilon) = \emptyset$. Since

$$\overline{\Lambda} = \Lambda \cup \partial\Lambda = \Lambda \cup \Upsilon \cup (\partial\Lambda \setminus \Upsilon) \subset U \cup (\partial\Lambda \setminus \Upsilon)$$

and $(\partial\Lambda \setminus \Upsilon) \cap \operatorname{supp} w = \emptyset$ it follows that $\overline{\Lambda} \cap \operatorname{supp} w \subset U$. Therefore there exists an $\eta \in C^\infty_c(\mathbb{R}^d)$ such that $\eta|_{\overline{\Lambda} \cap \operatorname{supp} w} = \mathbb{1}$ and $\operatorname{supp} \eta \subset U$. Consider the function ηw. First, observe that

$$U \cap (\partial\Omega \setminus \Gamma) = (U \cap \partial\Omega) \setminus \Upsilon \subset \partial\Lambda \setminus \Upsilon.$$

Hence $\operatorname{supp}(\eta w) \cap (\partial\Omega \setminus \Gamma) = \emptyset$ and $\eta w \in C^\infty_\Gamma(\Omega)$. Secondly, one has $(\eta w)|_\Lambda = w|_\Lambda$. Moreover, if $x \in \Omega \setminus \Lambda$, then $x \in U^c$ and $\eta(x) = 0$. So

$$(\eta w)|_\Omega(x) = \begin{cases} w|_\Lambda(x) & \text{if } x \in \Lambda \\ 0 & \text{if } x \notin \Lambda \end{cases}$$

for all $x \in \Omega$. Hence

$$\|(\eta w)|_\Omega\|_{W^{1,p}_\Gamma(\Omega)} = \|(\eta w)|_\Omega\|_{W^{1,p}(\Omega)} = \|w|_\Lambda\|_{W^{1,p}(\Lambda)} = \|w|_\Lambda\|_{W^{1,p}_\Upsilon(\Lambda)}.$$

Therefore there exists a unique isometric map $\mathfrak{E}\colon W^{1,p}_\Upsilon(\Lambda) \to W^{1,p}_\Gamma(\Omega)$ such that $\mathfrak{E}u$ is the extension of u to Ω by 0 for all $u \in C^\infty_\Upsilon(\Lambda)$. □

The Dirichlet boundary conditions on a particular part of the boundary in the next lemma allow that the function (after transformation) can be extended by zero to obtain an element of $W^{1,2}_\Gamma(\Omega)$.

Lemma 6.4. *Let $\Omega, U \subset \mathbb{R}^d$ be open and Γ a relatively open subset of $\partial\Omega$. Let ϕ be a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\phi(U) = E$, $\phi(\Omega \cap U) = E^-$ and $\phi(\partial\Omega \cap U) = P$. Set $\Gamma_E = \phi(\Gamma \cap U)$. Let $x \in \frac{1}{2} E^-$, $R \in (0, \frac{1}{2}]$ and $\tilde{u} \in W^{1,2}_{\Gamma_E(x,R)}(E^-(x,R))$. Define $u \colon \Omega \to \mathbb{C}$ by*

$$u(y) = \begin{cases} \tilde{u}(\phi(y)) & \text{if } y \in \phi^{-1}(E^-(x,R)), \\ 0 & \text{if } y \in \Omega \setminus \phi^{-1}(E^-(x,R)). \end{cases}$$

Then $u \in W^{1,2}_\Gamma(\Omega)$.

Proof. Define $\hat{u} \colon E^- \to \mathbb{C}$ by

$$(\hat{u})(y) = \begin{cases} u(y) & \text{if } y \in E^-(x,R), \\ 0 & \text{if } y \in E^- \setminus E^-(x,R). \end{cases}$$

Thanks to Lemma 6.3 one has $\hat{u} \in W^{1,2}_{\Gamma_E}(E^-)$. Then it follows from Proposition 4.3(a) that $u|_{U\cap\Omega} \in W^{1,2}_{\Gamma\cap U}(\Omega \cap U)$. Another application of Lemma 6.3 gives the result. □

The next proposition is a version of Proposition 3.2 with mixed boundary conditions.

Proposition 6.5. *For all $c_{DG}, K, \mu > 0$, $\kappa_0 \in (0,1)$, $\gamma \in [0,d)$ and $\delta \in [0,2)$ with $\gamma + \delta < d - 2 + 2\kappa_0$ there exists an $a_1 > 0$ such that the following is valid.*

Let $\Omega, U \subset \mathbb{R}^d$ be open, $\Gamma \subset \partial\Omega$ relatively open, ϕ a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\phi(U) = E$, $\phi(\Omega \cap U) = E^-$ and $\phi(\partial\Omega \cap U) = P$. Moreover, let $A \in \bigcup_{M\in(0,\infty)} \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$. Suppose that K is larger than the Lipschitz constant for $\phi|_{\Omega\cap U}$ and $\phi^{-1}|_{E^-}$ and the operator L_{A^ϕ} satisfies (κ_0, c_{DG})-De Giorgi estimates on $\frac{1}{2} E^-$ for functions vanishing on $\overline{\phi((\partial\Omega \setminus \Gamma) \cap U)}$ and Neumann boundary conditions on $\phi(\Gamma \cap U)$. Then

$$\|\nabla(u \circ \phi^{-1})\|_{M,\gamma+\delta,x,E^-,\frac{1}{2}}$$

$$\leq a_1 \Big(\varepsilon^{2-\delta} \|\xi \circ \phi^{-1}\|_{M,\gamma,x,E^-,\frac{1}{2}} + \sum_{i=1}^d \|\xi_i \circ \phi^{-1}\|_{M,\gamma+\delta,x,E^-,\frac{1}{2}} + \varepsilon^{-(\gamma+\delta)} \|\nabla u\|_{L_2(\Omega)} \Big)$$

for all $u \in W^{1,2}_\Gamma(\Omega)$, $x \in \frac{1}{2} E^-$, $\varepsilon \in (0,1]$ and $\xi, \xi_1, \ldots, \xi_d \in L_2(\Omega)$ such that

$$\mathfrak{l}_{A,\Gamma}(u,v) = (\xi, v)_{L_2(\Omega)} - \sum_{i=1}^d (\xi_i, \partial_i v)_{L_2(\Omega)}$$

for all $v \in W^{1,2}_\Gamma(\Omega)$.

Proof. Set $\Gamma_E = \phi(\Gamma \cap U)$ and $\Delta_E = \overline{\phi((\partial\Omega \setminus \Gamma) \cap U)}$.

Let $x \in \frac{1}{2} E^-$ and $0 < r \leq R \leq \frac{1}{2}$. By Lemma 6.1(b) and the Lax–Milgram theorem there exists a unique $\tilde{v} \in W^{1,2}_{\Gamma_E(x,R)}(E^-(x,R))$ such that

$$\sum_{i,j=1}^d \int_{E^-(x,R)} (A^\phi)_{ij} \, (\partial_i \tilde{v}) \, \overline{\partial_j \varphi} = \sum_{i,j=1}^d \int_{E^-(x,R)} (A^\phi)_{ij} \, (\partial_i (u \circ \phi^{-1})) \, \overline{\partial_j \varphi} \tag{29}$$

for all $\varphi \in W^{1,2}_{\Gamma_E(x,R)}(E^-(x,R))$. Define $v\colon \Omega \to \mathbb{C}$ by

$$v(y) = \begin{cases} \tilde{v}(\phi(y)) & \text{if } y \in \phi^{-1}(E^-(x,R))), \\ 0 & \text{if } y \in \Omega \setminus \phi^{-1}(E^-(x,R))). \end{cases}$$

Then $v \in W^{1,2}_{\Gamma}(\Omega)$ by Lemma 6.4. Set $w = u - v$. Then $w \in W^{1,2}_{\Gamma}(\Omega)$ and $w \circ \phi^{-1} \in \widetilde{W}^{1,2}_{\Delta_E}(E^-)$ by Lemma 2.2(d) and Proposition 4.3(a). Moreover,

$$\sum_{i,j=1}^{d} \int_{E^-(x,R)} (A^\phi)_{ij} \, (\partial_i(w \circ \phi^{-1})) \, \overline{\partial_j \varphi} = 0$$

for all $\varphi \in W^{1,2}_{\Gamma_E(x,R)}(E^-(x,R))$ by (29). The De Giorgi inequalities applied to the function $w \circ \phi^{-1}$ imply

$$\begin{aligned}
&\int_{E^-(x,r)} |\nabla(u \circ \phi^{-1})|^2 \\
&\quad \leq 2 \int_{E^-(x,r)} |\nabla(w \circ \phi^{-1})|^2 + 2 \int_{E^-(x,r)} |\nabla \tilde{v}|^2 \\
&\quad \leq 2c_{DG} \Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{E^-(x,R)} |\nabla(w \circ \phi^{-1})|^2 + 2 \int_{E^-(x,r)} |\nabla \tilde{v}|^2 \\
&\quad \leq 4c_{DG} \Big(\frac{r}{R}\Big)^{d-2+2\kappa_0} \int_{E^-(x,R)} |\nabla(u \circ \phi^{-1})|^2 + (2 + 4c_{DG}) \int_{E^-(x,R)} |\nabla \tilde{v}|^2.
\end{aligned}$$

Choose $\varphi = \tilde{v}$ in (29). Then (16) gives

$$\begin{aligned}
\sum_{i,j=1}^{d} \int_{E^-(x,R)} (A^\phi)_{ij} \, (\partial_i \tilde{v}) \, \overline{\partial_j \tilde{v}} &= \sum_{i,j=1}^{d} \int_{E^-(x,R)} (A^\phi)_{ij} \, (\partial_i(u \circ \phi^{-1})) \, \overline{\partial_j \tilde{v}} \\
&= \mathfrak{l}_{A^\phi}(u \circ \phi^{-1}, v \circ \phi^{-1}) \\
&= \mathfrak{l}_{A|_{\Omega \cap U}}(u|_{\Omega \cap U}, v|_{\Omega \cap U}) \\
&= \mathfrak{l}_{A,\Gamma}(u, v) = (\xi, v)_{L_2(\Omega)} - \sum_{i=1}^{d} (\xi_i, \partial_i v)_{L_2(\Omega)}.
\end{aligned}$$

Hence, by ellipticity on E^- (see Proposition 4.3(b)) and the Cauchy–Schwarz inequality, one estimates

$$\begin{aligned}
&(d! \, K^{d+2})^{-1} \, \mu \int_{E^{-1}(x,R)} |\nabla \tilde{v}|^2 \\
&\quad \leq d! \, K^d \Big(\int_{E^-(x,R)} |\xi \circ \phi^{-1}|^2 \Big)^{1/2} \Big(\int_{E^-(x,R)} |\tilde{v}|^2 \Big)^{1/2} \\
&\qquad + d! \, K^{d+1} \sum_{i=1}^{d} \Big(\int_{E^-(x,R)} |\xi_i \circ \phi^{-1}|^2 \Big)^{1/2} \Big(\int_{E^-(x,R)} |\partial_i \tilde{v}|^2 \Big)^{1/2}
\end{aligned}$$

$$\leq 2d!\,K^d\,\|\xi\circ\phi^{-1}\|_{M,\gamma,x,E^-,\frac12} R^{(\gamma+2)/2}\Big(\int_{E^-(x,R)}|\nabla\tilde v|^2\Big)^{1/2}$$

$$+\,d!\,K^{d+1}\sum_{i=1}^d\|\xi_i\circ\phi^{-1}\|_{M,\gamma+\delta,x,E^-,\frac12} R^{(\gamma+\delta)/2}\Big(\int_{E^-(x,R)}|\nabla\tilde v|^2\Big)^{1/2},$$

where we used Lemma 6.1(b) in the last step. So

$$\int_{E^-(x,R)}|\nabla\tilde v|^2$$

$$\leq 4(d!\,K^{d+3})^4\,\mu^{-2}\Big(R^{(2-\delta)/2}\|\xi\circ\phi^{-1}\|_{M,\gamma,x,E^-,\frac12}+\sum_{i=1}^d\|\xi_i\circ\phi^{-1}\|_{M,\gamma+\delta,x,E^-,\frac12}\Big)^2 R^{\gamma+\delta}.$$

Now the rest of the proof is similar to the end of the proof of Proposition 3.2 □

Before we apply Proposition 6.5 we state a lemma for which the proof is well known.

Lemma 6.6. *Let $\Omega\subset\mathbb{R}^d$ be an open set, $\mu>0$, $A\in\bigcup_{M\in(0,\infty)}\mathcal{A}_{\rm r}(\Omega,\mu,M)$, $\Gamma\subset\partial\Omega$ relatively open, $u\in W^{1,2}_\Gamma(\Omega)$, $f_0\in L_2(\Omega)$ and $f\in L_2(\Omega)^d$. Suppose that*

$$(\mathcal{L}_{A,\Gamma}+I)u=f_0-\operatorname{div}f.$$

Then

$$\|u\|_{W^{1,2}(\Omega)}\leq\frac{1}{\mu\wedge 1}\Big(\|f_0\|_{L_2(\Omega)}+\|f\|_{L_2(\Omega)^d}\Big).$$

The next lemma gives the missing Hölder continuity for Theorem 1.1 for points near Γ.

Lemma 6.7. *For all $K\geq 1$, $\mu>0$, $\kappa_0\in(0,1)$, $c_{DG}>0$ and $q\in(d,\infty)$ there are $\kappa\in(0,1)$ and $c>0$ such that the following is valid.*

Let $\Omega,U\subset\mathbb{R}^d$ be open, $\Gamma\subset\partial\Omega$ relatively open, ϕ a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\phi(U)=E$, $\phi(\Omega\cap U)=E^-$ and $\phi(\partial\Omega\cap U)=P$. Moreover, let $A\in\bigcup_{M\in(0,\infty)}\mathcal{A}_{\rm r}(\Omega,\mu,M)$. Suppose that K is larger than the Lipschitz constant for $\phi|_{\Omega\cap U}$ and $\phi^{-1}|_{E^-}$ and the operator L_{A^ϕ} satisfies (κ_0,c_{DG})-De Giorgi estimates on $\frac12 E^-$ for functions vanishing on $\overline{\phi((\partial\Omega\setminus\Gamma)\cap U)}$ and Neumann boundary conditions on $\phi(\Gamma\cap U)$. Let $u\in W^{1,2}_\Gamma(\Omega)$, $f_0\in L_q(\Omega)\cap L_2(\Omega)$, $f\in L_q(\Omega)^d\cap L_2(\Omega)^d$ and suppose that

$$(\mathcal{L}_{A,\Gamma}+I)u=f_0-\operatorname{div}f.$$

Then

$$|u(x)-u(y)|\leq c\,|x-y|^\kappa\Big(\|f_0\|_{L_q(\Omega)}+\|f\|_{L_q(\Omega)^d}+\|f_0\|_{L_2(\Omega)}+\|f\|_{L_2(\Omega)^d}\Big)$$

for all $x,y\in\Omega(\phi^{-1}(0),\frac{1}{8K})$.

Proof. We distinguish two cases.

Case 1. Suppose that $d\geq 4$.
Note that $d-\frac{2d}{q}>d-2$ since $q>d$. Choose

$$\kappa=\tfrac12\min\Big(d-\frac{2d}{q}-(d-2),\tfrac12\,\kappa_0\Big).$$

Then $\kappa \in (0,1)$. Choose $\delta = 2$ and $\gamma = d - 4 + 2\kappa$. Then $\gamma \in [0,d)$ since $d \geq 4$. Moreover, $\gamma + \delta = d - 2 + 2\kappa \leq d - \frac{2d}{q}$ and $\|f_0 \circ \phi^{-1}\|_{M,\gamma,x,E^-,\frac{1}{2}} \leq \|f_0 \circ \phi^{-1}\|_{M,d-\frac{2d}{q},x,E^-,\frac{1}{2}} \leq \sqrt{\omega_d}\, \|f_0 \circ \phi^{-1}\|_{L_q(E^-)} \leq d!\, K^d \sqrt{\omega_d}\, \|f_0\|_{L_q(\Omega)}$ for all $x \in \frac{1}{2}\, E^-$ by the Hölder inequality.

Case 2. Suppose that $d \in \{2,3\}$.

Since $q > d$, there exists a $\kappa \in (0,\kappa_0)$ such that $d - 2 + 2\kappa \leq d - \frac{2d}{q}$ and $\kappa < \frac{1}{2}$. Choose $\gamma = 0$ and $\delta = d - 2 + 2\kappa$. Then $\delta \in (0,2]$, $\gamma + \delta = d - 2 + 2\kappa$ and $\|f_0\|_{M,\gamma,x,E^-,\frac{1}{2}} \leq \|f_0\|_{M,d-\frac{2d}{q},x,E^-,\frac{1}{2}} \leq d!\, K^d \sqrt{\omega_d}\, \|f_0\|_{L_q(\Omega)}$ for all $x \in \frac{1}{2}\, E^-$.

In both cases, let $a_1 > 0$ be as in Proposition 6.5. Then

$$\|\nabla(u \circ \phi^{-1})\|_{M,d-2+2\kappa,x,E^-,\frac{1}{2}} \leq d!\, K^d\, a_1 \sqrt{\omega_d} \Big(\|f_0\|_{L_q(\Omega)} + \|f\|_{L_q(\Omega)^d} + \|\nabla u\|_{L_2(\Omega)} \Big)$$

for all $x \in \frac{1}{2}\, E^-$ by Proposition 6.5. Hence (28) implies

$$\|u \circ \phi^{-1}\|_{\mathcal{M},d+2\kappa,x,E^-,\frac{1}{2}} \leq 2c_N\, d!\, K^d\, a_1 \sqrt{\omega_d} \Big(\|f_0\|_{L_q(\Omega)} + \|f\|_{L_q(\Omega)^d} + \|\nabla u\|_{L_2(\Omega)} \Big)$$

So by Lemma 3.1(c) one deduces that there exists a suitable $c > 0$ such that

$$|(u \circ \phi^{-1})(x) - (u \circ \phi^{-1})(y)| \leq c\, |x - y|^\kappa \Big(\|f_0\|_{L_q(\Omega)} + \|f\|_{L_q(\Omega)^d} + \|\nabla u\|_{L_2(\Omega)} \Big)$$

for all $x, y \in \frac{1}{2}\, E^-$ with $|x - y| \leq \frac{1}{4}$. The term $\|\nabla u\|_{L_2(\Omega)}$ can be estimated by Lemma 6.6. Since ϕ is bi-Lipschitz the lemma follows. □

We next present a quantitative version of Theorem 1.1, which does not require that the domain is bounded. We emphasize that the constant K in Condition (I) is uniform in $x \in \overline{\Gamma}$.

Theorem 6.8. *For all $K \geq 1$, $\alpha > 0$, $c_0 \in (0,1)$, $c_1 > 0$, $q \in (d,\infty)$ and $\mu, M > 0$ there exist $c > 0$, $\kappa \in (0,1)$ and $\eta > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be open and Γ a relatively open subset of the boundary $\partial\Omega$. Moreover, let $A \in \mathcal{A}_{\mathrm{r}}(\Omega, \mu, M)$. Assume the following conditions.

- (I) *For all $x \in \overline{\Gamma}$ there is an open neighbourhood U and a bi-Lipschitz map ϕ from an neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$, such that $\phi(U) = E$, $\phi(\Omega \cap U) = E^-$, $\phi(\partial\Omega \cap U) = P$ and $\phi(x) = 0$. Moreover, K is larger than the Lipschitz constant for $\phi|_{\Omega \cap U}$ and $\phi^{-1}|_{E^-}$.*
- (II) *The set $\partial\Omega \setminus \Gamma$ is of class $(\mathbf{A}_\alpha)$.*
- (III) *If $x \in \partial\Gamma$, then*

$$\mathrm{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_s(\tilde{y}) : \mathrm{dist}(\tilde{z}, \phi(\Gamma \cap U)) > c_0\, s\} \geq c_1\, s^{d-1}$$

for all $s \in (0,1]$ and $\tilde{y} \in \mathbb{R}^{d-1}$ with $(\tilde{y}, 0) \in \phi(\partial\Gamma \cap U)$, where U and ϕ are as in Condition (I).

Let $u \in W^{1,2}_\Gamma(\Omega)$, $f_0 \in L_q(\Omega) \cap L_2(\Omega)$, $f \in L_q(\Omega)^d \cap L_2(\Omega)^d$ and suppose that

$$(\mathcal{L}_{A,\Gamma} + I)u = f_0 - \mathrm{div}\, f.$$

Then

$$|u(x)| \le c\Big(\|f_0\|_{L_q(\Omega)} + \|f\|_{L_q(\Omega)^d} + \|f_0\|_{L_2(\Omega)} + \|f\|_{L_2(\Omega)^d} \Big)$$

and

$$|u(x) - u(y)| \le c\,|x-y|^\kappa \Big(\|f_0\|_{L_q(\Omega)} + \|f\|_{L_q(\Omega)^d} + \|f_0\|_{L_2(\Omega)} + \|f\|_{L_2(\Omega)^d} \Big) \tag{30}$$

for all $x, y \in \Omega$ *with* $|x-y| \le \eta$.

Proof. For simplicity write

$$K_1 = \|f_0\|_{L_q(\Omega)} + \|f\|_{L_q(\Omega)^d} + \|f_0\|_{L_2(\Omega)} + \|f\|_{L_2(\Omega)^d}.$$

First, it follows from the assumptions, Proposition 5.3 and Lemma 6.7 that there are suitable $\tilde c_1 > 0$ and $\kappa_1 \in (0,1)$ such that $|u(x) - u(y)| \le \tilde c_1\, K_1\, |x-y|^{\kappa_1}$ uniformly for all $x_0 \in \partial\Gamma$ and $x, y \in \Omega(x_0, \frac{1}{8K})$.

Secondly, we consider all $x_0 \in \Gamma$ with $d(x_0, \partial\Gamma) \ge \frac{1}{16K}$. By Lemma 5.2 and Condition (I) one can construct a bi-Lipschitz map $\check\phi$ with Lipschitz constant for both $\check\phi$ and $\check\phi^{-1}$ bounded by $16K^3$. Using this map $\check\phi$ in Lemmas 5.1 and 6.7 it follows that there are suitable $\tilde c_2 > 0$ and $\kappa_2 \in (0,1)$ such that $|u(x) - u(y)| \le \tilde c_2\, K_1\, |x-y|^{\kappa_2}$ uniformly for all $x_0 \in \Gamma$ and $x, y \in \Omega(x_0, \frac{1}{128K^3})$ with $d(x_0, \partial\Gamma) \ge \frac{1}{16K}$.

Thirdly apply Theorem 1.2 with $\zeta = \frac{1}{256K^3}$ and $\Upsilon = \{x \in \Omega : d(x, \Gamma) > \zeta\}$. It follows that there are suitable $\tilde c_3 > 0$ and $\kappa_3 \in (0,1)$ such that $|u(x) - u(y)| \le \tilde c_3\, K_1\, |x-y|^{\kappa_3}$ for all $x, y \in \Upsilon$ with $|x-y| < \frac{1}{512K^3}$. Here we also used Lemma 6.6 to estimate the term $\|\nabla u\|_{L_2(\Omega)}$ in (3). Set $\eta = \frac{1}{512K^3}$, $\tilde c = \max(\tilde c_1, \tilde c_2, \tilde c_3)$ and $\kappa = \min(\kappa_1, \kappa_2, \kappa_3)$. Then $|u(x) - u(y)| \le \tilde c\, K_1\, |x-y|^\kappa$ for all $x, y \in \Omega$ with $|x-y| \le \eta$. This proves (30).

Finally, let $c > 0$ be as in Theorem 1.2 applied to ζ and Υ. For simplicity write $K_2 = \|u\|_{W^{1,2}(\Omega)} + \|f_0\|_{L_{q_0}(\Omega)} + \|f\|_{L_q(\Omega)^d}$. Then $|u(x)| \le c\, K_2$ for all $x \in \Upsilon$. Now let $x_0 \in \Omega \setminus \Upsilon$. Then $d(x_0, \Gamma) \le \zeta$. Hence there exists an $x \in \Gamma$ such that $|x - x_0| \le 2\zeta$. Let U and ϕ be as in Condition (I) with respect to x. Let $y = \phi^{-1}(-\frac{1}{2}e_d)$. Since $y \in U$, one has on the one hand

$$K\, d(y, \partial\Omega \cap (\mathbb{R}^d \setminus U)) \ge K\, d(y, \mathbb{R}^d \setminus U) = K\, d(y, \partial U) \ge d(\phi(y), \partial E) = d(-\tfrac{1}{2}e_d, \partial E) = \tfrac{1}{2}.$$

On the other hand, one estimates

$$K\, d(y, \partial\Omega \cap U) \ge d(\phi(y), P) = \tfrac{1}{2}.$$

Consequently, $d(y, \partial\Omega) \ge \frac{1}{2K}$. Hence $d(y, \Gamma) \ge d(y, \partial\Omega) \ge \frac{1}{2K} > \zeta$. So $y \in \Upsilon$ and $|u(y)| \le c\, K_2$. Moreover, $|x_0 - y| \le \frac{K}{2} + 2\zeta$. Using a telescope argument, it follows that

$$|u(x_0)| \le c\, K_2 + \left\lceil \frac{\frac{K}{2} + 2\zeta}{\eta} \right\rceil \tilde c\, K_1\, \eta^\kappa.$$

Using again Lemma 6.6 one can replace the term $\|u\|_{W^{1,2}(\Omega)}$ in K_2 by a suitable multiple of $\|f_0\|_{L_2(\Omega)} + \|f\|_{L_2(\Omega)^d}$. The proof is complete. □

Proof of Theorem 1.1. This follows from a compactness argument with obvious modification to the proof of Theorem 6.8. We leave the details to the reader. □

Theorem 1.1 has the following consequence.

Corollary 6.9. *Hölder continuity of the solution carries over to the elliptic problem when combined with mixed Dirichlet/Robin boundary conditions.*

Proof. In what follows, the symbol σ shall denote the boundary measure on Γ. (Since there are bi-Lipschitz charts around the boundary points in $\overline{\Gamma}$, the boundary measure is well defined and admits the usual properties, see [EG, Subsection 3.3.4.C].) Let $q \in (d, \infty)$ and $b \in L_q(\Gamma; \sigma)$. In case of mixed Dirichlet/Robin boundary conditions the operator is the sum of $\mathcal{L}_{A,\Gamma}$ and the operator $B \colon L_\infty(\Gamma; \sigma) \to W_\Gamma^{-1,q}$ defined by $\langle Bv, w \rangle = \int_\Gamma b\, v\, \overline{w}\, d\sigma$, where $w \in W_\Gamma^{1,q'}$ and $v \in L_\infty(\Gamma; \sigma)$. It is straight forward that B is continuous from $L_\infty(\Gamma; \sigma)$ into $W_\Gamma^{-1,q}$. Hence, it is relatively compact with respect to $\mathcal{L}_{A,\Gamma}$ if the domain of the latter continuously embeds into a Hölder space. Hence, the domains of $\mathcal{L}_{A,\Gamma}$ and $\mathcal{L}_{A,\Gamma} + B$ coincide in this case by [Kat, Subsection IV.1.3]. □

7 Hölder kernel bounds

In this section we prove Gaussian Hölder kernel bounds for the semigroup generated by an elliptic second-order differential operator with real principal coefficients and complex lower-order coefficients. First we give a precise definition of these operators.

Let $\Omega \subset \mathbb{R}^d$ be open and $\Gamma \subset \partial\Omega$ be relatively open. Let $\mu, M > 0$, $A \in \mathcal{A}(\Omega, \mu, M)$, $a, b \in L_\infty(\Omega)^d$ and $a_0 \in L_\infty(\Omega)$. Consider the form

$$\mathfrak{l}(u, v) = \int_\Omega \sum_{i,j=1}^d a_{ij}\, (\partial_i u)\, \overline{(\partial_j v)} + \int_\Omega \sum_{i=1}^d \Big(a_i\, (\partial_i u)\, \overline{v} + b_i\, u\, \overline{(\partial_i v)} \Big) + \int_\Omega a_0\, u\, \overline{v}$$

with form domain $D(\mathfrak{l}) = W_\Gamma^{1,2}(\Omega)$. Then $\mathfrak{l}$ is a closed sectorial form. Let L be the m-sectorial operator associated with $\mathfrak{l}$. We denote by $\mathcal{E}_\Gamma^{\mathrm{op}}(\Omega, \mu, M)$ the set of all such m-sectorial operators with $A \in \mathcal{A}(\Omega, \mu, M)$, $a, b \in L_\infty(\Omega)^d$ and $a_0 \in L_\infty(\Omega)$ with bounds

$$\sum_{i=1}^d \|a_i\|_\infty \le M \quad , \quad \sum_{i=1}^d \|b_i\|_\infty \le M \quad \text{and} \quad \|a_0\|_\infty \le M.$$

We say that A, a, b and a_0 are the **coefficients** of L. Let S be the semigroup generated by $-L$. We denote by $\mathcal{E}_{\mathrm{r},\Gamma}^{\mathrm{op}}(\Omega, \mu, M)$ the set of all $L \in \mathcal{E}_\Gamma^{\mathrm{op}}(\Omega, \mu, M)$ such that A is real valued, where A, a, b and a_0 are the coefficients of L. We emphasize that a, b and a_0 can be complex.

We also need the Davies perturbation. Let

$$\mathcal{D} = \{\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}) : \|\nabla \psi\|_\infty \le 1\}.$$

For all $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$ define the multiplication operator U_ρ by $U_\rho u = e^{-\rho\psi} u$. Note that $U_\rho u \in W_\Gamma^{1,2}(\Omega)$ for all $u \in W_\Gamma^{1,2}(\Omega)$. Let $S_t^\rho = U_\rho\, S_t\, U_{-\rho}$ be the Davies perturbation for all $t > 0$. Let $-L^{(\rho)}$ the generator of $(S_t^\rho)_{t>0}$. Then $L^{(\rho)}$ is the operator associated with the form $\mathfrak{l}^{(\rho)}$ with form domain $D(\mathfrak{l}^{(\rho)}) = W_\Gamma^{1,2}(\Omega)$ and

$$\mathfrak{l}^{(\rho)}(u, v) = \mathfrak{l}_A(u, v) + \int_\Omega \sum_{i=1}^d \Big(a_i^{(\rho)}\, (\partial_i u)\, \overline{v} + b_i^{(\rho)}\, u\, \overline{(\partial_i v)} \Big) + \int_\Omega a_0^{(\rho)}\, u\, \overline{v} \tag{31}$$

with

$$a_i^{(\rho)} = a_i - \rho \sum_{j=1}^d a_{ij}\,\partial_j \psi \quad , \quad b_i^{(\rho)} = b_i + \rho \sum_{j=1}^d a_{ji}\,\partial_j \psi$$

and

$$a_0^{(\rho)} = a_0 - \rho^2 \sum_{i,j=1}^d a_{ij}\,(\partial_i \psi)\,\partial_j \psi + \rho \sum_{i=1}^d a_i\,\partial_i \psi - \rho \sum_{i=1}^d b_i\,\partial_i \psi.$$

We start with L_2-estimates for the perturbed semigroup.

Lemma 7.1. *For all $\mu, M > 0$ there exist $c_0, \omega_0 > 0$ such that*

$$\|S_t^\rho u\|_{L_2(\Omega)} \le e^{\omega_0 (1+\rho^2) t}\,\|u\|_{L_2(\Omega)} \quad , \quad \|\nabla S_t^\rho u\|_{L_2(\Omega)} \le c_0\, t^{-1/2}\, e^{\omega_0 (1+\rho^2) t}\,\|u\|_{L_2(\Omega)} \tag{32}$$

and

$$\|L^{(\rho)}\, S_t^\rho u\|_{L_2(\Omega)} \le c_0\, t^{-1}\, e^{\omega_0 (1+\rho^2) t}\,\|u\|_{L_2(\Omega)}$$

for all open $\Omega \subset \mathbb{R}^d$, relatively open $\Gamma \subset \partial\Omega$, $L \in \mathcal{E}_\Gamma^{\rm op}(\Omega, \mu, M)$, $u \in L_2(\Omega)$, $t > 0$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$, where S^ρ is the Davies perturbation of the semigroup generated by $-L$.

Proof. Without loss of generality we may assume that $\mu \le 1$. Let $u \in L_2(\Omega)$. It follows from (31) that

$$\begin{aligned}
\mu\,\|\nabla S_t^\rho u\|_{L_2(\Omega)}^2 &\le \operatorname{Re} \mathfrak{l}_A(S_t^\rho u) \\
&\le \operatorname{Re} \mathfrak{l}^{(\rho)}(S_t^\rho u) + 2d\,M\,(1+|\rho|)\,\|\nabla S_t^\rho u\|_{L_2(\Omega)}\,\|S_t^\rho u\|_{L_2(\Omega)} \\
&\qquad + d\,M(1+2|\rho|+\rho^2)\|S_t^\rho u\|_{L_2(\Omega)}^2 \\
&\le \operatorname{Re} \mathfrak{l}^{(\rho)}(S_t^\rho u) + \tfrac{1}{2}\,\mu\|\nabla S_t^\rho u\|_{L_2(\Omega)}^2 + \frac{2d^2\,M^2(1+|\rho|)^2}{\mu}\,\|S_t^\rho u\|_{L_2(\Omega)}^2 \\
&\qquad + d\,M(1+|\rho|)^2\,\|S_t^\rho u\|_{L_2(\Omega)}^2
\end{aligned}$$

for all $t > 0$. So

$$\tfrac{1}{2}\,\mu\,\|\nabla S_t^\rho u\|_{L_2(\Omega)}^2 \le \operatorname{Re} \mathfrak{l}^{(\rho)}(S_t^\rho u) + \omega_1\,(1+\rho^2)\,\|S_t^\rho u\|_{L_2(\Omega)}^2, \tag{33}$$

where $\omega_1 = 2d\,M(\frac{2dM}{\mu} + 1)$. Hence

$$\frac{d}{dt}\|S_t^\rho u\|_{L_2(\Omega)}^2 = -2\operatorname{Re}(L^{(\rho)} S_t^\rho u, S_t^\rho u)_{L_2(\Omega)} = -2\operatorname{Re} \mathfrak{l}^{(\rho)}(S_t^\rho u) \le 2\omega_1\,(1+\rho^2)\,\|S_t^\rho u\|_{L_2(\Omega)}^2$$

for all $t > 0$. This implies by Gronwall's lemma that

$$\|S_t^\rho u\|_{L_2(\Omega)} \le e^{\omega_1 (1+\rho^2) t}\,\|u\|_{L_2(\Omega)}$$

for all $t > 0$.

Next we rotate the coefficients of the operator L in the complex plane. Let $\varphi_0 \in (0, \frac{\pi}{2})$ be such that $\mu \cos\varphi_0 - M \sin\varphi_0 = \frac{1}{2}\,\mu$. Then $e^{i\varphi}\,L \in \mathcal{E}_\Gamma^{\rm op}(\Omega, \frac{1}{2}\,\mu, M)$ for all $\varphi \in [-\varphi_0, \varphi_0]$. Consequently, by the above it follows that

$$\|S_{t\,e^{i\varphi}}^\rho u\|_{L_2(\Omega)} \le e^{2\omega_1 (1+\rho^2) t}\,\|u\|_{L_2(\Omega)}$$

for all $u \in L_2(\Omega)$, $\varphi \in [-\varphi_0, \varphi_0]$ and $t > 0$. Hence the semigroup S^ρ is holomorphic with semiangle of holomorphy at least φ_0 by [Kat] Theorem IX.1.23. The Cauchy formula then gives

$$L^{(\rho)} S_t^\rho = -\frac{1}{2\pi i} \int_{C(t)} \frac{1}{(z-t)^2} S_z^\rho \, dz,$$

where $C(t)$ is the circle with centre t and radius $t \sin \varphi_0$. Hence

$$\|L^{(\rho)} S_t^\rho u\|_{L_2(\Omega)} \leq \frac{1}{t \sin \varphi_0} e^{4\omega_1 (1+\rho^2) t} \|u\|_{L_2(\Omega)}$$

for all $u \in L_2(\Omega)$ and $t > 0$. Finally (33) gives

$$\begin{aligned}
\tfrac{1}{2} \mu \|\nabla S_t^\rho u\|_{L_2(\Omega)}^2 &\leq \mathrm{Re}(L^{(\rho)} S_t^\rho u, S_t^\rho u)_{L_2(\Omega)} + \omega_1 (1+\rho^2) \|S_t^\rho u\|_{L_2(\Omega)}^2 \\
&\leq \frac{1}{t \sin \varphi_0} e^{5\omega_1 (1+\rho^2) t} \|u\|_{L_2(\Omega)}^2 + \omega_1 (1+\rho^2) e^{2\omega_1 (1+\rho^2) t} \|u\|_{L_2(\Omega)}^2 \\
&\leq \frac{2}{t \sin \varphi_0} e^{5\omega_1 (1+\rho^2) t} \|u\|_{L_2(\Omega)}^2,
\end{aligned}$$

from which the lemma follows. □

Next we consider $L_2 \to L_\infty$ and Hölder estimates for the perturbed semigroup near Γ.

Proposition 7.2. *For all $K \geq 1$, $\mu, M > 0$, $\kappa_0 \in (0,1)$, $c_{DG} > 0$ and $\kappa \in (0, \kappa_0)$ there exist $c, \omega > 0$ such that such that the following is valid.*

Let $\Omega, U \subset \mathbb{R}^d$ be open, $\Gamma \subset \partial\Omega$ relatively open, ϕ a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\phi(U) = E$, $\phi(\Omega \cap U) = E^-$ and $\phi(\partial\Omega \cap U) = P$. Let $L \in \mathcal{E}^{\mathrm{op}}_{\mathrm{r},\Gamma}(\Omega, \mu, M)$ with coefficients A, a, b and a_0. Suppose that K is larger than the Lipschitz constant for $\phi|_{\Omega \cap U}$ and $\phi^{-1}|_{E^-}$ and the operator L_{A^ϕ} satisfies (κ_0, c_{DG})-De Giorgi estimates on $\frac{1}{2} E^-$ for functions vanishing on $\overline{\phi((\partial\Omega \setminus \Gamma) \cap U)}$ and Neumann boundary conditions on $\phi(\Gamma \cap U)$. Then

$$\|S_t^\rho u\|_{L_\infty(\phi^{-1}(\frac{1}{2} E^-))} \leq c\, t^{-d/4} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)} \tag{34}$$

and

$$|(S_t^\rho u)(x) - (S_t^\rho u)(y)| \leq c\, t^{-d/4} t^{-\kappa/2} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)} |x-y|^\kappa \tag{35}$$

for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x, y \in \phi^{-1}(\frac{1}{2} E^-)$ with $|x - y| \leq \frac{1}{4K}$, where S^ρ is the Davies perturbation of the semigroup generated by $-L$.

Proof. For all $\gamma \in [0, d-2+2\kappa)$ let $P(\gamma)$ be the hypothesis

> There exist $c, \omega > 0$, depending only on K, μ, M, κ and c_{DG}, such that
>
> $$\|(S_t^\rho u) \circ \phi^{-1}\|_{M,\gamma,x,E^-,\frac{1}{2}} \leq c\, t^{-\gamma/4} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)} \tag{36}$$
>
> and
>
> $$\|\nabla((S_t^\rho u) \circ \phi^{-1})\|_{M,\gamma,x,E^-,\frac{1}{2}} \leq c\, t^{-\gamma/4} t^{-1/2} e^{\omega(1+\rho^2)t} \|u\|_{L_2(\Omega)} \tag{37}$$
>
> for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x \in \frac{1}{2} E^-$.

Clearly $P(0)$ is valid by Lemma 7.1.

Lemma 7.3. *Let $\gamma \in [0, d-2+2\kappa)$ and suppose that $P(\gamma)$ is valid. Let $\delta \in (0,2]$ and suppose that $\gamma + \delta < d - 2 + 2\kappa$. Then $P(\gamma+\delta)$ is valid.*

Proof. Let $c_0, \omega_0 > 0$ be as in Lemma 7.1. Let $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x \in \frac{1}{2} E^-$. Note that

$$\|(S_t^\rho u) \circ \phi^{-1}\|_{L_2(E^-)} \leq d!\, K^d \, \|S_t^\rho u\|_{L_2(\Omega)} \leq d!\, K^d \, e^{\omega_0 (1+\rho^2) t} \, \|u\|_{L_2(\Omega)} \tag{38}$$

by Lemma 7.1.

Choose $\varepsilon = t^{1/4} e^{-t} \in (0,1]$. Let c_1 be as in Lemma 6.2. Then it follows from Lemma 6.2, (37) and (38) that

$$\begin{aligned}\|(S_t^\rho u) \circ \phi^{-1}\|_{\mathcal{M},\gamma+\delta,x,E^-,\frac{1}{2}} &\leq c_1 \, (\varepsilon^{2-\delta} \, c \, t^{-\gamma/4} \, t^{-1/2} \, e^{\omega(1+\rho^2)t} + \varepsilon^{-(\gamma+\delta)} d!\, K^d \, e^{\omega_0(1+\rho^2)t}) \|u\|_{L_2(\Omega)} \\ &\leq c' \, t^{-(\gamma+\delta)/4} \, e^{\omega'(1+\rho^2)t} \|u\|_{L_2(\Omega)}\end{aligned}$$

where $c' = c_1 (c + d!\, K^d)$ and $\omega' = \omega_0 + \omega + \gamma + \delta$. By Lemma 3.1(a) there exist $c_2, c_3 > 0$ such that

$$\|v\|_{M,\gamma+\delta,x,E^-,\frac{1}{2}} \leq c_2 \, \|v\|_{\mathcal{M},\gamma+\delta,x,E^-,\frac{1}{2}} + c_3 \, \|v\|_{L_2(E^-)}$$

for all $x \in \frac{1}{2} E^-$ and $v \in L_2(E^-)$. Hence

$$\begin{aligned}\|(S_t^\rho u) \circ \phi^{-1}\|_{M,\gamma+\delta,x,E^-,\frac{1}{2}} &\leq c_2 \, c' \, t^{-(\gamma+\delta)/4} \, e^{\omega'(1+\rho^2)t} \|u\|_{L_2(\Omega)} + c_3 \, (d!) \, K^d \, e^{\omega_0(1+\rho^2)t} \, \|u\|_{L_2(\Omega)} \\ &\leq c'' \, t^{-(\gamma+\delta)/4} \, e^{\omega''(1+\rho^2)t} \|u\|_{L_2(\Omega)},\end{aligned} \tag{39}$$

where $c'' = c' \, c_2 + c_3 (d!)$ and $\omega'' = \omega_0 + \omega' + d + 2$. This gives the bound (36) for $P(\gamma+\delta)$.

Next we wish to use Proposition 6.5. Note that $\mathfrak{l}^{(\rho)}(S_t^\rho u, v) = (S_{t/2}^\rho \, L^{(\rho)} \, S_{t/2}^\rho u, v)_{L_2(\Omega)}$ for all $v \in W_\Gamma^{1,2}(\Omega)$. Hence with (31) it follows that $\mathfrak{l}_{A,\Gamma}(S_t^\rho u, v) = (\xi, v)_{L_2(\Omega)} - \sum_{i=1}^d (\xi_i, \partial_i \, v)_{L_2(\Omega)}$ for all $v \in W_\Gamma^{1,2}(\Omega)$, where $\xi_i = b_i^{(\rho)} \, S_t^\rho u$ and

$$\xi = S_{t/2}^\rho \, L^{(\rho)} \, S_{t/2}^\rho u - a_0^{(\rho)} \, S_t^\rho u - \sum_{i=1}^d a_i^{(\rho)} \, \partial_i S_t^\rho u.$$

We apply Proposition 6.5 with $\varepsilon = t^{1/4} e^{-t} \in (0,1]$. We estimate the three terms in ξ and then the term with ξ_i separately. First with Lemma 7.1 we obtain

$$\begin{aligned}&\varepsilon^{2-\delta} \, \|(S_{t/2}^\rho \, L^{(\rho)} \, S_{t/2}^\rho u) \circ \phi^{-1}\|_{M,\gamma,x,E^-,\frac{1}{2}} \\ &\quad\leq t^{(2-\delta)/4} \, c \, (t/2)^{-\gamma/4} \, e^{\omega(1+\rho^2)t/2} \, \|L^{(\rho)} \, S_{t/2}^\rho u\|_{L_2(\Omega)} \\ &\quad\leq c_0 \, (t/2)^{-1} \, e^{\omega_0(1+\rho^2)t/2} \, t^{(2-\delta)/4} \, c \, (t/2)^{-\gamma/4} \, e^{\omega(1+\rho^2)t/2} \, \|u\|_{L_2(\Omega)} \\ &\quad\leq 2^{1+\gamma/4} c_0 \, c \, t^{-(\gamma+\delta)/4} \, t^{-1/2} \, e^{(\omega_0+\omega)(1+\rho^2)t} \, \|u\|_{L_2(\Omega)}.\end{aligned}$$

Secondly,

$$\begin{aligned}\varepsilon^{2-\delta} \, \|(a_0^{(\rho)} \, S_t^\rho u) \circ \phi^{-1}\|_{M,\gamma,x,E^-,\frac{1}{2}} &\leq t^{(2-\delta)/4} \, 4M \, (1+\rho^2) \, c \, t^{-\gamma/4} \, e^{\omega(1+\rho^2)t} \, \|u\|_{L_2(\Omega)} \\ &\leq 4c \, M \, t^{-(\gamma+\delta)/4} \, t^{-1/2} \, e^{(\omega+1)(1+\rho^2)t} \, \|u\|_{L_2(\Omega)}.\end{aligned}$$

Thirdly,

$$
\begin{aligned}
\varepsilon^{2-\delta} \, &\|(\sum_{i=1}^{d} a_i^{(\rho)} \, \partial_i S_t^\rho u) \circ \phi^{-1}\|_{M,\gamma,x,E^-,\frac{1}{2}} \\
&\leq t^{(2-\delta)/4} \, M \, (1+|\rho|) \|(\nabla S_t^\rho u) \circ \phi^{-1}\|_{M,\gamma,x,E^-,\frac{1}{2}} \\
&\leq 2t^{(2-\delta)/4} \, M \, t^{-1/2} \, e^{(1+\rho^2)t} \, K \, \|\nabla((S_t^\rho u) \circ \phi^{-1})\|_{M,\gamma,x,E^-,\frac{1}{2}} \\
&\leq 2c \, K \, M \, t^{-(\gamma+\delta)/4} \, t^{-1/2} \, e^{(\omega+1)(1+\rho^2)t} \, \|u\|_{L_2(\Omega)}.
\end{aligned}
$$

Fourthly,

$$
\begin{aligned}
\sum_{i=1}^{d} \|(b_i^{(\rho)} \, S_t^\rho u) \circ \phi^{-1}\|_{M,\gamma+\delta,x,E^-,\frac{1}{2}} &\leq M \, (1+|\rho|) \, \|(S_t^\rho u) \circ \phi^{-1}\|_{M,\gamma+\delta,x,E^-,\frac{1}{2}} \\
&\leq 2M \, c'' \, t^{-(\gamma+\delta)/4} \, t^{-1/2} \, e^{(1+\rho^2)t} \, e^{\omega''(1+\rho^2)t} \|u\|_{L_2(\Omega)},
\end{aligned}
$$

where we used (39) in the last step. Finally,

$$
\begin{aligned}
\varepsilon^{-(\gamma+\delta)} \|\nabla S_t^\rho u\|_{L_2(\Omega)} &\leq t^{-(\gamma+\delta)/4} \, e^{(\gamma+\delta)t} \, c_0 \, t^{-1/2} e^{\omega_0(1+\rho^2)t} \, \|u\|_{L_2(\Omega)} \\
&\leq c_0 \, t^{-(\gamma+\delta)/4} \, t^{-1/2} \, e^{(\omega_0+d+2)(1+\rho^2)t} \, \|u\|_{L_2(\Omega)}
\end{aligned}
$$

by (32). Hence (37) for $P(\gamma+\delta)$ follows from Proposition 6.5 and $P(\gamma+\delta)$ is valid. □

End of proof of Proposition 7.2. It follows by induction from Lemma 7.3 that there are $c, \omega > 0$ such that

$$
\|\nabla((S_t^\rho u) \circ \phi^{-1})\|_{M,d-2+2\kappa,x,E^-,\frac{1}{2}} \leq c \, t^{-(d-2+2\kappa)/4} \, t^{-1/2} \, e^{\omega(1+\rho^2)t} \, \|u\|_{L_2(\Omega)}
$$

for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x \in \frac{1}{2} E^-$. Hence by Lemma 6.2 one deduces that

$$
\|(S_t^\rho u) \circ \phi^{-1}\|_{\mathcal{M},d+2\kappa,x,E^-,\frac{1}{2}} \leq c' \, t^{-(d+2\kappa)/4} \, e^{\omega(1+\rho^2)t} \, \|u\|_{L_2(\Omega)}, \tag{40}
$$

where $c' = c\sqrt{2c_N}$, with c_N as in (12).

Let $t > 0$ and $x \in \frac{1}{2} E^-$. Choose $R = t^{1/2} e^{-t}$. Then $R \leq \frac{1}{2}$. It follows from Lemma 3.1(b) that there exists a $c'' > 0$, depending only on κ and d, such that

$$
\begin{aligned}
&|((S_t^\rho u) \circ \phi^{-1})(x)| \\
&\quad \leq c'' \, R^\kappa \, \|(S_t^\rho u) \circ \phi^{-1}\|_{\mathcal{M},d+2\kappa,x,E^-,\frac{1}{2}} + |\langle (S_t^\rho u) \circ \phi^{-1} \rangle_{E^-(x,R)}| \\
&\quad \leq c'' \, t^{\kappa/2} \, e^{-\kappa t} \, c' \, t^{-(d+2\kappa)/4} \, e^{\omega(1+\rho^2)t} \, \|u\|_{L_2(\Omega)} + \omega_d^{-1/2} \, R^{-d/2} \, \|(S_t^\rho u) \circ \phi^{-1}\|_{L_2(E^-)}.
\end{aligned}
$$

Then (34) follows from (38).

Finally, by (40) and Lemma 3.1(c) there exists a $c''' > 0$ such that

$$
|(S_t^\rho u)(\phi^{-1}(x)) - (S_t^\rho u)(\phi^{-1}(y))| \leq c''' \, t^{-d/4} \, t^{-\kappa/2} \, e^{\omega(1+\rho^2)t} \, \|u\|_{L_2(\Omega)} \, |x-y|^\kappa
$$

for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x, y \in \frac{1}{2} E^-$ with $|x-y| < \frac{1}{4}$. Since ϕ is bi-Lipschitz with Lipschitz constant K the inequality (35) follows. □

Next we turn to the part of Ω away from Γ.

Proposition 7.4. *For all $\mu, M, \alpha, \zeta > 0$ there exist $\kappa \in (0,1)$ and $c, \omega > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be open, let Γ be a relatively open subset of $\partial\Omega$, let $\Upsilon \subset \Omega$ and suppose that $d(\Gamma, \Upsilon) \geq \zeta$ and $\{z \in \partial\Omega : d(z, \Upsilon) < \zeta\}$ is of class $(\mathbf{A}_\alpha)$. Let $L \in \mathcal{E}^{\rm op}_{\rm r,\Gamma}(\Omega, \mu, M)$. Then

$$\|S^\rho_t u\|_{L_\infty(\Upsilon)} \leq c\, t^{-d/4}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}$$

and

$$|(S^\rho_t u)(x) - (S^\rho_t u)(y)| \leq c\, t^{-d/4}\, t^{-\kappa/2}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}\, |x-y|^\kappa$$

for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x, y \in \Upsilon$ with $|x - y| \leq \frac{\zeta}{4}$, where S^ρ is the Davies perturbation of the semigroup generated by $-L$.

Proof. It follows as in the proof of Proposition 7.2, using Proposition 3.2 instead of Proposition 6.5, that there exist suitable $c, \omega > 0$ such that

$$\|\nabla S^\rho_t u\|_{M,d-2+2\kappa,x,\Omega,\zeta} \leq c\, t^{-d/4}\, t^{-1/2}\, t^{-\kappa/2}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}$$

for all $x \in \Upsilon$. If $c_N > 0$ is as in (12), then if follows from Lemma 3.4 that

$$\|\widetilde{S^\rho_t u}\|_{\mathcal{M},d-2+2\kappa,x,\mathbb{R}^d,\zeta} \leq c\, \sqrt{c_N}\, t^{-d/4}\, t^{-\kappa/2}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}$$

for all $x \in \Upsilon$, where for every $v \in L_2(\Omega)$ we define by $\tilde{v} \colon \mathbb{R}^d \to \mathbb{C}$ the extension by zero of v. We emphasize that the Campanato seminorm is with respect to $\mathbb{R}^d$, thus not with Ω. Note that the extension $\widetilde{S^\rho_t u}$ is continuous on $\Omega(x, \varepsilon)$ if $\varepsilon > 0$ is small enough. Then the proposition follows as at the end of the proof of Proposition 7.2, but now with Lemma 3.1 applied to the Campanato seminorm on $\mathbb{R}^d$. □

We can now prove Gaussian Hölder kernel bounds for second-order operators with complex lower-order coefficients. Note that the set Ω may be unbounded in the next theorem.

Theorem 7.5. *For all $K \geq 1$, $\alpha > 0$, $c_0 \in (0,1)$, $c_1 > 0$ and $\mu, M > 0$ there exist $\kappa \in (0,1)$ and $b, c, \omega > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be an open set and Γ a relatively open subset of the boundary $\partial\Omega$. Assume the following conditions.

- (I) *For all $x \in \overline{\Gamma}$ there is an open neighbourhood U and a bi-Lipschitz map ϕ from an neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$, such that $\phi(U) = E$, $\phi(\Omega \cap U) = E^-$, $\phi(\partial\Omega \cap U) = P$ and $\phi(x) = 0$. Moreover, K is larger than the Lipschitz constant for $\phi|_{\Omega\cap U}$ and $\phi^{-1}|_{E^-}$.*
- (II) *The set $\partial\Omega \setminus \Gamma$ is of class $(\mathbf{A}_\alpha)$.*
- (III) *If $x \in \partial\Gamma$, then*

$$\text{mes}_{d-1}\{\tilde{z} \in \widetilde{B}_s(\tilde{y}) : \text{dist}(\tilde{z}, \phi(\Gamma \cap U)) > c_0\, s\} \geq c_1\, s^{d-1}$$

for all $s \in (0,1]$ and $\tilde{y} \in \mathbb{R}^{d-1}$ with $(\tilde{y}, 0) \in \phi(\partial\Gamma \cap U)$, where U and ϕ are as in Condition (I).

Let $L \in \mathcal{E}^{\rm op}_{\rm r,\Gamma}(\Omega,\mu,M)$ and let $(K_t)_{t>0}$ be the kernel of the semigroup generated by $-L$. Then

$$|K_t(x,y)| \le c\, t^{-d/2}\, e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t}$$

and

$$|K_t(x,y) - K_t(x',y')| \le c\, t^{-d/2} \Big(\frac{|x-x'| + |y-y'|}{t^{1/2}} \Big)^{\kappa} e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t}$$

for all $x,x',y,y' \in \Omega$ and $t > 0$ with $|x-x'| + |y-y'| \le t^{1/2}$.

Proof. We decompose Ω similarly as in the proof of Theorem 6.7. We first apply Propositions 5.3 and 7.2 to points $x_0 \in \partial\Gamma$ to get estimates for all $x,y \in \Omega(x_0, \frac{1}{8K})$. Secondly, let $x_0 \in \partial\Gamma$ and suppose that $d(x_0,\partial\Gamma) \ge \frac{1}{16K}$. We apply Lemma 5.2 and Condition (I) to construct a bi-Lipschitz map $\check\phi$ with Lipschitz constant for both $\check\phi$ and $\check\phi^{-1}$ bounded by $16K^3$. Then Lemma 5.1 and Proposition 7.2 give estimates for all $x_0 \in \Gamma$ and $x,y \in \Omega(x_0, \frac{1}{128K^3})$ with $d(x_0,\partial\Gamma) \ge \frac{1}{16K}$. Finally set $\zeta = \frac{1}{256K^3}$ and $\Upsilon = \{x \in \Omega : d(x,\Gamma) > \zeta\}$. Apply Proposition 7.4 to get estimates for all $x,y \in \Upsilon$ with $|x-y| \le \frac{1}{1024K^3}$. It follows that there exists a $\kappa \in (0,1)$ and $c, \omega > 0$ such that

$$|(S^\rho_t u)(x)| \le c\, t^{-d/4}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} \tag{41}$$

and

$$|(S^\rho_t u)(x) - (S^\rho_t u)(y)| \le c\, t^{-d/4}\, t^{-\kappa/2}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}\, |x-y|^\kappa \tag{42}$$

for all $u \in L_2(\Omega)$, $t > 0$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$ in the following cases:

- $x_0 \in \partial\Gamma$ and $x,y \in \Omega(x_0, \frac{1}{8K})$,
- $x_0 \in \Gamma$, $d(x_0,\partial\Gamma) \ge \frac{1}{16K}$ and $x,y \in \Omega(x_0, \frac{1}{128K^3})$,
- $x,y \in \Upsilon$ and $|x-y| \le \frac{1}{1024K^3}$.

Hence for all $u \in L_2(\Omega)$, $t > 0$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$ it follows that (41) is valid for all $x \in \Omega$ and (42) is valid for all $x,y \in \Omega$ with $|x-y| \le \frac{1}{1024K^3}$. So

$$\|S^\rho_t\|_{L_2(\Omega)\to L_\infty(\Omega)} \le c\, t^{-d/4}\, e^{\omega(1+\rho^2)t} \tag{43}$$

for all $t > 0$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$. Using duality and minimising over ψ and ρ gives

$$|K_t(x,y)| \le 2^{d/2}\, c^2\, t^{-d/2}\, e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t} \tag{44}$$

for all $t > 0$ and $x,y \in \Omega$, where $b = \frac{1}{4\omega}$.

Next, choose $\rho = 0$ in (42) and (43) and use duality in (43). It follows that

$$|(S_t u)(x) - (S_t u)(x')| \le 2^{(d+\kappa)/2}\, c^2\, t^{-d/2}\, t^{-\kappa/2}\, e^{\omega t}\, \|u\|_{L_1(\Omega)}\, |x-x'|^\kappa$$

for all $t > 0$, $u \in L_1(\Omega)$ and $x,x' \in \Omega$ with $|x-x'| \le r$, where $r = \frac{1}{1024K^3}$. Then

$$|K_t(x,y) - K_t(x',y)| \le 2^{(d+\kappa)/2}\, c^2\, t^{-d/2}\, t^{-\kappa/2}\, e^{\omega t}\, |x-x'|^\kappa$$

for all $t > 0$ and $x,x',y \in \Omega$ with $|x-x'| \le r$. Alternatively, using (44) it follows that

$$|K_t(x,y) - K_t(x',y)| \le 2^{(d+2)/2}\, c^2\, t^{-d/2}\, e^{\omega t} \le 2^{(d+2)/2}\, c^2\, r^{-\kappa}\, t^{-d/2}\, t^{-\kappa/2}\, e^{(\omega+1)t}\, |x-x'|^\kappa$$

for all $t > 0$ and $x, x', y \in \Omega$ with $|x - x'| \geq r$. So

$$|K_t(x,y) - K_t(x',y)| \leq c_1 \, t^{-d/2} \, t^{-\kappa/2} \, e^{(\omega+1)t} \, |x - x'|^\kappa \tag{45}$$

for all $t > 0$ and $x, x', y \in \Omega$, where $c_1 = 2^{(d+2)/2} \, c^2 \, r^{-\kappa}$.

If $x, x', y \in \Omega$, then

$$|x - y|^2 \leq 2|x - x'|^2 + 2|x' - y|^2 \leq 2t + 2|x' - y|^2$$

for all $t > 0$ with $|x - x'| \leq t^{1/2}$. Hence with (44) it follows that

$$|K_t(x,y) - K_t(x',y)| \leq 2^{(d+2)/2} \, c^2 \, t^{-d/2} \, e^{-\frac{b}{2} \, \frac{|x-y|^2}{t}} \, e^{(\omega+1)t} \tag{46}$$

for all $x, x', y \in \Omega$ and $t > 0$ with $|x - x'| \leq t^{1/2}$.

Let $\varepsilon \in (0,1)$. We interpolate between the bounds (45) and (46). Then

$$|K_t(x,y) - K_t(x',y)| \leq c_1^{1-\varepsilon} (2^{(d+2)/2} \, c^2)^\varepsilon \, t^{-d/2} \Big(\frac{|x - x'|}{t^{1/2}} \Big)^{\kappa(1-\varepsilon)} \, e^{-\frac{b\varepsilon}{2} \, \frac{|x-y|^2}{t}} \, e^{(\omega+1)t}$$

for all $x, x', y \in \Omega$ and $t > 0$ with $|x - x'| \leq t^{1/2}$. By duality similar bounds are valid for all $x, y, y' \in \Omega$ and $t > 0$ with $|y - y'| \leq t^{1/2}$. Then the theorem follows. □

Proof of Theorem 1.3. This is now obvious. □

A Appendix, proof of Lemma 3.1

'(a)'. Let $\gamma \in [0, d)$, $x \in \Omega$ and $u \in L_2(\Omega)$. Clearly $\|u\|^2_{\mathcal{M},\gamma,x,R_e} \leq \|u\|^2_{M,\gamma,x,R_e}$. Let $r \in (0, R_e/2]$. Then

$$\frac{1}{r^\gamma} \int_{\Omega(x,r)} |u|^2 \leq \frac{2}{r^\gamma} \int_{\Omega(x,r)} |u - \langle u \rangle_{\Omega(x,r)}|^2 + \frac{2}{r^\gamma} \, |\Omega(x,r)| \, |\langle u \rangle_{\Omega(x,r)}|^2. \tag{47}$$

If $R \in [r, R_e]$ then

$$|\langle u \rangle_{\Omega(x,r)} - \langle u \rangle_{\Omega(x,R)}|^2 \leq 2|u - \langle u \rangle_{\Omega(x,r)}|^2 + 2|u - \langle u \rangle_{\Omega(x,R)}|^2.$$

Integrate over $\Omega(x,r)$. Then

$$\begin{aligned} \tilde{c} \, r^d \, |\langle u \rangle_{\Omega(x,R)} - \langle u \rangle_{\Omega(x,r)}|^2 &\leq 2 \int_{\Omega(x,r)} |u - \langle u \rangle_{\Omega(x,r)}|^2 + 2 \int_{\Omega(x,R)} |u - \langle u \rangle_{\Omega(x,R)}|^2 \\ &\leq 2(r^\gamma + R^\gamma) \, \|u\|^2_{\mathcal{M},\gamma,x,R_e} \\ &\leq 4R^\gamma \, \|u\|^2_{\mathcal{M},\gamma,x,R_e}. \end{aligned} \tag{48}$$

Hence

$$|\langle u \rangle_{\Omega(x,r)} - \langle u \rangle_{\Omega(x,R)}| \leq \frac{2}{\sqrt{\tilde{c}}} \, \frac{R^{\gamma/2}}{r^{d/2}} \, \|u\|_{\mathcal{M},\gamma,x,R_e}.$$

For all $k \in \mathbb{N}_0$ define $R_k = 2^{-k} \, R$. Then

$$|\langle u \rangle_{\Omega(x,R_k)} - \langle u \rangle_{\Omega(x,R_{k+1})}| \leq \frac{2^{1+d/2}}{\sqrt{\tilde{c}}} \, 2^{k(d-\gamma)/2} \, R^{-(d-\gamma)/2} \, \|u\|_{\mathcal{M},\gamma,x,R_e}. \tag{49}$$

Hence for all $N \in \mathbb{N}_0$ one obtains

$$|\langle u\rangle_{\Omega(x,R)} - \langle u\rangle_{\Omega(x,R_{N+1})}| \le \frac{2^{1+d/2}}{\sqrt{\tilde{c}}} \, \frac{2^{(N+1)(d-\gamma)/2} \, R^{-(d-\gamma)/2}}{2^{(d-\gamma)/2}-1} \, \|u\|_{\mathcal{M},\gamma,x,R_e}$$
$$= \frac{2^{1+d/2}}{\sqrt{\tilde{c}}} \, \frac{1}{2^{(d-\gamma)/2}-1} \, R_{N+1}^{-(d-\gamma)/2} \, \|u\|_{\mathcal{M},\gamma,x,R_e}.$$

Choose $R \in [\frac{R_e}{2}, R_e]$ and $N \in \mathbb{N}_0$ such that $2^{-(N+1)} R = r$. Then

$$|\langle u\rangle_{\Omega(x,r)}|^2 \le 2|\langle u\rangle_{\Omega(x,R)}|^2 + 2|\langle u\rangle_{\Omega(x,r)} - \langle u\rangle_{\Omega(x,R)}|^2$$
$$\le 2|\langle u\rangle_{\Omega(x,R)}|^2 + \frac{2^{d+3}}{\tilde{c}} \, \frac{1}{(2^{(d-\gamma)/2}-1)^2} \, r^{-(d-\gamma)} \, \|u\|^2_{\mathcal{M},\gamma,x,R_e}.$$

Therefore with (47) one deduces that

$$\frac{1}{r^\gamma} \int_{\Omega(x,r)} |u|^2 \le 2\|u\|^2_{\mathcal{M},\gamma,x,R_e} + \frac{2^{d+4}\omega_d}{\tilde{c}\,(2^{(d-\gamma)/2}-1)^2} \, \|u\|^2_{\mathcal{M},\gamma,x,R_e} + \frac{2^{d+2}}{\tilde{c}\,R_e^d} \int_{\Omega(x,R_e)} |u|^2.$$

This is for all $r \in (0, R_e/2]$. Together with an obvious estimate for all $r \in [R_e/2, R_e]$ one deduces that

$$\|u\|^2_{M,\gamma,x,R_e} \le \Big(2 + \frac{2^{d+4}\omega_d}{\tilde{c}\,(2^{(d-\gamma)/2}-1)^2}\Big) \|u\|^2_{\mathcal{M},\gamma,x,R_e} + \frac{2^{d+2}}{\tilde{c}\,R_e^d} \int_{\Omega(x,R_e)} |u|^2.$$

This completes the proof of Statement (a).

'(b)'. Let $\gamma \in (d, d+2)$, $x \in \Omega$ and $u \in L_2(\Omega)$. Let $R \in (0, R_e]$ and set $R_k = 2^{-k} R$ for all $k \in \mathbb{N}_0$. As in (49) it follows that

$$|\langle u\rangle_{\Omega(x,R_k)} - \langle u\rangle_{\Omega(x,R_{k+1})}| \le \frac{2^{1+d/2}}{\sqrt{\tilde{c}}} \, 2^{-k(\gamma-d)/2} \, R^{(\gamma-d)/2} \, \|u\|_{\mathcal{M},\gamma,x,R_e}$$

for all $k \in \mathbb{N}_0$. Let $h, k \in \mathbb{N}_0$ and suppose that $k < h$. Then

$$|\langle u\rangle_{\Omega(x,R_k)} - \langle u\rangle_{\Omega(x,R_h)}| \le \frac{2^{1+d/2}}{\sqrt{\tilde{c}}} \, \frac{2^{-h(\gamma-d)/2} \, R^{(\gamma-d)/2}}{1-2^{-(\gamma-d)/2}} \, \|u\|_{\mathcal{M},\gamma,x,R_e}$$
$$= \frac{2^{1+d/2}}{\sqrt{\tilde{c}}(1-2^{-(\gamma-d)/2})} \, R_k^{(\gamma-d)/2} \, \|u\|_{\mathcal{M},\gamma,x,R_e}. \qquad (50)$$

Hence $(\langle u\rangle_{\Omega(x,R_k)})_{k\in\mathbb{N}}$ is a Cauchy sequence. Set $\hat{u}(x) = \lim_{k\to\infty} \langle u\rangle_{\Omega(x,R_k)}$. Let $r \in (0, R]$ and for all $k \in \mathbb{N}$ define $r_k = 2^{-k} r$. Let $j \in \mathbb{N}_0$ be such that $R_{j+1} \le r \le R_j$. Then $R_{k+j+1} \le r_k \le R_{k+j}$ for all $k \in \mathbb{N}$. As in (48) one deduces that

$$|\langle u\rangle_{\Omega(x,R_k)} - \langle u\rangle_{\Omega(x,r_k)}| \le |\langle u\rangle_{\Omega(x,R_k)} - \langle u\rangle_{\Omega(x,R_{k+j})}| + |\langle u\rangle_{\Omega(x,R_{k+j})} - \langle u\rangle_{\Omega(x,r_k)}|$$
$$\le |\langle u\rangle_{\Omega(x,R_k)} - \langle u\rangle_{\Omega(x,R_{k+j})}| + \frac{2}{\sqrt{\tilde{c}}} \, \frac{R_{k+j}^{\gamma/2}}{R_{k+j+1}^{d/2}} \, \|u\|_{\mathcal{M},\gamma,x,R_e}$$
$$= |\langle u\rangle_{\Omega(x,R_k)} - \langle u\rangle_{\Omega(x,R_{k+j})}| + \frac{2^{1+d/2}}{\sqrt{\tilde{c}}} \, R_{k+j}^{(\gamma-d)/2} \, \|u\|_{\mathcal{M},\gamma,x,R_e}.$$

So $\lim_{k\to\infty} |\langle u\rangle_{\Omega(x,R_k)} - \langle u\rangle_{\Omega(x,r_k)}| = 0$ and $\hat{u}(x)$ does not depend on R.

Choose $k = 0$ and take the limit $h \to \infty$ in (50). Then

$$|\langle u\rangle_{\Omega(x,R)} - \hat{u}(x)| \le \frac{2^{1+d/2}}{\sqrt{\tilde{c}}(1-2^{-(\gamma-d)/2})} R^{(\gamma-d)/2} \|u\|_{\mathcal{M},\gamma,x,R_e} \tag{51}$$

for all $R \in (0,1]$. Then clearly $\lim_{R\downarrow 0}\langle u\rangle_{\Omega(x,R)} = \hat{u}(x)$.

'(c)'. Next let $x, y \in \Omega$ with $|x-y| \le \frac{R_e}{2}$. Set $R = |x-y|$. Then (51) gives

$$\begin{aligned}
|\hat{u}(x) - \hat{u}(y)| &\le |\hat{u}(x) - \langle u\rangle_{\Omega(x,2R)}| + |\langle u\rangle_{\Omega(x,2R)} - \langle u\rangle_{\Omega(y,2R)}| + |\langle u\rangle_{\Omega(y,2R)} - \hat{u}(y)| \\
&\le \frac{2^{1+d/2}}{\sqrt{\tilde{c}}(1-2^{-(\gamma-d)/2})} (2R)^{(\gamma-d)/2} \Big(\|u\|_{\mathcal{M},\gamma,x,R_e} + \|u\|_{\mathcal{M},\gamma,y,R_e} \Big) \\
&\qquad + |\langle u\rangle_{\Omega(x,2R)} - \langle u\rangle_{\Omega(y,2R)}|.
\end{aligned}$$

Consider the last term. Obviously

$$|\langle u\rangle_{\Omega(x,2R)} - \langle u\rangle_{\Omega(y,2R)}|^2 \le 2|\langle u\rangle_{\Omega(x,2R)} - u|^2 + 2|u - \langle u\rangle_{\Omega(y,2R)}|^2.$$

Integration over $\Omega(x,R)$ and using that $\Omega(x,R) \subset \Omega(x,2R) \cap \Omega(y,2R)$ gives

$$\begin{aligned}
\tilde{c}\, R^d\, |\langle u\rangle_{\Omega(x,2R)} - \langle u\rangle_{\Omega(y,2R)}|^2 &\le 2\int_{\Omega(x,2R)} |u - \langle u\rangle_{\Omega(x,2R)}|^2 + 2\int_{\Omega(y,2R)} |u - \langle u\rangle_{\Omega(y,2R)}|^2 \\
&\le 2(2R)^\gamma \Big(\|u\|^2_{\mathcal{M},\gamma,x,R_e} + \|u\|^2_{\mathcal{M},\gamma,y,R_e} \Big).
\end{aligned}$$

So

$$|\langle u\rangle_{\Omega(x,2R)} - \langle u\rangle_{\Omega(y,2R)}|^2 \le \frac{2^{\gamma+1}}{\tilde{c}} R^{\gamma-d} \Big(\|u\|^2_{\mathcal{M},\gamma,x,R_e} + \|u\|^2_{\mathcal{M},\gamma,y,R_e} \Big)$$

and

$$|\hat{u}(x) - \hat{u}(y)| \le \left(\frac{2^{1+d/2}}{\sqrt{\tilde{c}}(1-2^{-(\gamma-d)/2})} + \frac{2^{(\gamma+1)/2}}{\sqrt{\tilde{c}}} \right) R^{(\gamma-d)/2} \big(\|u\|_{\mathcal{M},\gamma,x,R_e} + \|u\|_{\mathcal{M},\gamma,y,R_e} \big).$$

The proof of the lemma is complete. □

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