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Optimal stopping via path-wise dual empirical maximization

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Abstract

The optimal stopping problem arising in the pricing of American options can be tackled by the so called dual martingale approach. In this approach, a dual problem is formulated over the space of martingales. A feasible solution of the dual problem yields an upper bound for the solution of the original primal problem. In practice, the optimization is performed over a finite-dimensional subspace of martingales. A sample of paths of the underlying stochastic process is produced by a Monte-Carlo simulation, and the expectation is replaced by the empirical mean. As a rule the resulting optimization problem, which can be written as a linear program, yields a martingale for which the obtained estimator may have a tight upward bias, but its variance can still be large. In order to decrease this variance, a penalizing term can be added to the objective function of the path-wise optimization problem. In this paper, we provide a rigorous analysis of the optimization problems obtained by adding different penalty functions. In particular, a convergence analysis implies that it is better to minimize the empirical maximum instead of the empirical mean. Numerical simulations confirm the variance reduction effect of the new approach.

1 Introduction

In this paper we consider the following optimal stopping problem. Let $\mathcal{F}_t$ be a filtration, and let Z_t, $t = 0, \dots, T$ be a discrete-time adapted process with bounded variance. We wish to maximize

$$Y^* = \max_{\tau \in \mathcal{T}} \mathbb{E}[Z_\tau], \tag{1}$$

where $\mathcal{T}$ is the set of stopping times on $\{0, \dots, T\}$. Desai et al. [3] introduced a *pathwise optimization* method for solving this kind of optimal stopping problems. It is based on the *dual martingale* approach, which was developed in [7] and [4], see also [2] which in fact contained the dual approach in germ. The dual problem to the original optimal stopping problem can be written as an optimization problem over the space of martingales M with zero initial value. More precisely,

$$Y^* = \inf_M \mathbb{E} \max_{t=0,\dots,T} (Z_t - M_t) = \max_{t=0,\dots,T} (Z_t - M_t^*) \quad \text{a.s.} \tag{2}$$

where M^* is the martingale part of the Doob-Meyer decomposition of the Snell envelope of Z_t, and the optimal values of the primal and dual problem coincide.

As the optimal stopping problem, the dual problem is infinite-dimensional. In order to reduce it to a finite-dimensional one, it was proposed in [3] to optimize over a finite-dimensional section of the space of martingales. In other words, a finite number of basis martingales is chosen and the optimization is performed over all linear combinations of these basis functions, the coefficients playing the role of the decision variables. Since the optimal martingale M^* is as a rule not contained in the linear span of the chosen basis functions, the procedure yields a suboptimal solution leading to an upper bound of the optimal value of the original optimal stopping problem. Another practical problem in solving dual problem (2) is the presence of the expectation operator. This can be circumvented by replacing the expectation by the empirical mean over a sample of N paths [3]. As a consequence, we obtain the optimization problem

$$\inf_{\alpha} \frac{1}{N} \sum_{n=1}^{N} \max_{t=0,\ldots,T} (Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)}). \tag{3}$$

Here $\alpha = (\alpha_1, \ldots, \alpha_K)^T$ is the vector of decision variables, $Z_t^{(n)}$ are the sample paths of the process Z_t, and $M_t^{k,(n)}$ are the paths of the basis martingales. The index $k = 1, \ldots, K$ denotes the index of the basis martingale, and $n = 1, \ldots, N$ denotes the index of the path. If the minimum in (3) is attained in, say α^*, then the estimate

$$\frac{1}{\tilde{N}} \sum_{\tilde{n}=1}^{\tilde{N}} \max_{t=0,\ldots,T} \left(Z_t^{(\tilde{n})} - \sum_{k=1}^{K} \alpha_k M_t^{k,(\tilde{n})} \right)$$

based on an independent new simulation gives an upper biased estimate of (2).

As mentioned before, the martingale part M^* of the Doob-Meyer decomposition of the Snell envelope of Z_t yields an optimal solution to (2). While this solution may not be unique, it has the distinguished property that the random variable $Z(M^*) = \max_{t=0,\ldots,T}(Z_t - M_t^*)$ has variance zero (see [1]). Martingales having this property have been named *surely optimal* and have been characterized in [10]. When seeking a solution to the dual problem (2), one is not only interested in a martingale M that gives a tight upper bound on the optimal value of the optimal stopping problem, but also in a martingale that is close to a surely optimal one, in the sense that the random variable $Z(M) = \max_{t=0,\ldots,T}(Z_t - M_t)$ has a low variance. This second condition, however, is usually not met by the optimal solutions of the approximated problem (3), as evidenced in [10].

To counter this problem, Belomestny proposed in [1] to add the empirical standard deviation as a penalty term to the objective in (3), leading to the optimization problem

$$\inf_{\alpha} \left(\frac{1}{N} \sum_{n=1}^{N} Z^{(n)}(M(\alpha)) + \lambda \sqrt{\frac{1}{N-1} \sum_{n=1}^{N} \left(Z^{(n)}(M(\alpha)) - \frac{1}{N} \sum_{l=1}^{N} Z^{(l)}(M(\alpha)) \right)^2} \right), \tag{4}$$

where $M(\alpha) = \sum_{j=1}^{K} \alpha_j M^j$ and $Z^{(n)}(M(\alpha)) = \max_{t=0,\ldots,T}(Z_t^{(n)} - \sum_{j=1}^{K} \alpha_j M_t^{j,(n)})$. Here $\lambda \geq 0$ is a scalar determining the weight of the penalty term. However, while problem (3) can be cast as a linear program, problem (4) may fail to be convex if λ is too large, and thus difficult to solve to a global optimum.

The subject of this paper is to analyze the effect of the penalty term in augmented problems of type (4). As a penalty we assume a general convex homogeneous function F of degree 1 of the vector $\hat{Z}(M(\alpha)) = (\hat{Z}^{(n)}(M(\alpha)))_{n=1,\ldots,N}$, where $\hat{Z}^{(n)}(M(\alpha)) = Z^{(n)}(M(\alpha)) - \frac{1}{N} \sum_{l=1}^{N} Z^{(l)}(M(\alpha))$. Note that the vector $\hat{Z}(M(\alpha))$ resides in a subspace of codimension

1, and hence F is effectively a function from $\mathbb{R}^{N-1}$ to $\mathbb{R}$. We will, however, consider convex homogeneous functions $F : \mathbb{R}^N \to \mathbb{R}$ and have in mind that only the values of F on the $(N-1)$-dimensional subspace are relevant. As F should penalize deviations of the vector $\hat{Z}(M(\alpha))$ from zero, we shall assume that $F(x) > 0$ for all nonzero vectors in this subspace. The problem considered in this paper is hence

$$\inf_{\alpha} \left(\frac{1}{N} \sum_{k=1}^{N} Z^{(k)}(M(\alpha)) + \lambda F(\hat{Z}(M(\alpha))) \right). \tag{5}$$

In problem (4) we therefore have $F(x_1, \dots, x_N) = \sqrt{\frac{1}{N-1} \sum_{n=1}^{N} x_n^2}$.

We will construct a convex conic relaxation to this augmented problem and, for given $\lambda > 0$, provide a sufficient condition on the function F such that this relaxation is exact, i.e., yields the same optimal objective value as the original problem (5).

The first aim of the paper is to show that there is a largest function F satisfying this condition, namely the function given by $\mathbb{R}^N \ni x \mapsto \lambda^{-1} \max_{n=1,\dots,N} x_n$. The corresponding augmented optimization problem

$$\inf_{\alpha} \max_{n=1,\dots,N} Z^{(n)}(M(\alpha)) \tag{6}$$

can also be cast as a linear program. Next we analyze the convergence of (6) to Y^* as $N, K \to \infty$.

2 Preliminaries

In this section we introduce some notions from convex analysis and conic optimization. The dual of the real vector space $\mathbb{R}^n$ will be denoted by $\mathbb{R}_n$, and the scalar product between $y \in \mathbb{R}_n$ and $x \in \mathbb{R}^n$ will be denoted by $\langle y, x \rangle$. Let $\mathbf{1}^n, \mathbf{1}_n = (1, \dots, 1)^T$ be the all-ones vector in $\mathbb{R}^n$ and in the dual space $\mathbb{R}_n$.

2.1 Conic programs

In this subsection we provide some basic material on convex conic programming. This is a generalization of linear programming where the ordinary inequality constraints are replaced by a more general notion of inequality defined by a convex cone.

Definition 1 *A closed convex cone $K \subset \mathbb{R}^n$, containing no lines, and with nonempty interior, is called regular.*

A *conic program* over a regular convex cone $K \subset \mathbb{R}^n$ is an optimization problem of the form

$$\inf_{x \in K} \langle c, x \rangle \ : \ Ax = b, \tag{7}$$

where $c \in \mathbb{R}_n$ is a vector defining the linear cost function of the problem, A is an $m \times n$ matrix, and $b \in \mathbb{R}^m$. Here A, b define the linear constraints of the problem.

The availability of algorithms for solving a conic program depends on the nature of the cone K. For example, if K is the positive orthant $\mathbb{R}^n_+$, then (7) can be written as a linear program and easily solved. Efficient solution algorithms are also available if K is a second order cone or a cone of positive semi-definite matrices, or a direct product of such cones. Conic programs with second order cone constraints are called *conic quadratic programs*, programs with linear matrix inequality constraints *semi-definite programs*.

2.2 Exposed and extreme points

In this subsection we introduce the notions of exposed and extreme points of a closed convex set and consider the relations between them.

Definition 2 *[9, p.162] Let $C \subset \mathbb{R}^n$ be a closed convex set. A point $x \in C$ is called extreme point if there does not exist an open line segment $L \subset C$ such that $x \in L$.*

Lemma 3 *[9, Corollary 18.5.1] A closed bounded convex set is the convex hull of its extreme points.*

Definition 4 *[9, pp.162–163] Let $C \subset \mathbb{R}^n$ be a closed convex set. A point $x \in C$ is called exposed point if there exists a supporting affine hyperplane $H \subset \mathbb{R}^n$ to C such that $H \cap C = \{x\}$.*

Lemma 5 *[9, Theorem 18.6] Let $C \subset \mathbb{R}^n$ be a closed convex set. Then the set of exposed points of C is dense in the set of extreme points of C.*

Corollary 6 *Let $C \subset \mathbb{R}^n$ be a bounded closed convex set and E its set of exposed points. Then C is the convex hull of the closure of E.*

Proof. The corollary follows immediately from the two lemmas above. ■

2.3 Convex functions and subgradients

In this subsection we introduce the notion of a subgradient.

Definition 7 *[9, pp.214–215] Let $D \subset \mathbb{R}^n$ be a convex set and $f : D \to \mathbb{R}$ a convex function. A subgradient of f at $x \in D$ is a dual vector $y \in \mathbb{R}_n$ such that $f(z) \geq f(x) + \langle y, z - x \rangle$ for all $z \in D$. The set of all subgradients at $x \in D$ is called subdifferential at $x \in D$ and denoted by $\partial f(x)$.*

The subdifferential is a closed convex set [9, p.215]. If f is differentiable at x, then the gradient $f'(x)$ is the only subgradient [9, p.216].

Lemma 8 *[5, p.261] Let $D \subset \mathbb{R}^n$ be a convex domain, $F : D \to \mathbb{R}$ a convex function, and $h : D \to \mathbb{R}$ a convex C^2 function. For $x \in D$ and $\lambda \geq 0$ we then have $\partial(\lambda F + h)(x) = \lambda\partial F(x) + h'(x)$.*

Lemma 9 *[9, Theorem 23.9] Let $D \subset \mathbb{R}^m$ be a convex domain, and $H : \mathbb{R}^m \to \mathbb{R}^n$ an affine map given by $H(x) = A(x) + b$, with A, b the linear part of H and the shift, respectively. Let further $F : H[D] \to \mathbb{R}$ be a convex function. Then for $x \in D$ we have $\partial(F \circ H)(x) = A^*[\partial F(Ax + b)]$, where A^* is the adjoint map of A.*

2.4 Convex sets and polars

In this subsection we introduce the notion of a polar and study its properties.

Definition 10 *[9, p.125] Let $C \subset \mathbb{R}^n$ be a closed convex set containing the origin of $\mathbb{R}^n$. The set $C^o = \{y \in \mathbb{R}_n \mid \langle y, x\rangle \leq 1 \ \forall\, x \in C\}$ is called the polar of the set C.*

The set C^o is also closed, convex, and contains the origin [9, p.125]. It is bounded if and only if the origin of $\mathbb{R}^n$ is contained in the interior of C [9, Corollary 14.5.1]. Moreover, the polar of C^o is again C [9, Theorem 14.5]. If C, C' are two closed convex sets containing the origin and satisfying $C \subset C'$, then their polars satisfy $(C')^o \subset C^o$ [9, p.125].

Let now $L \subset \mathbb{R}^n$ be a linear subspace, and let $L^\perp \subset \mathbb{R}_n$ be the orthogonal subspace. Then the dual space L^* of L can be identified with the quotient $\mathbb{R}_n/L^\perp$. Let $\Pi : \mathbb{R}_n \to \mathbb{R}_n/L^\perp$ be the corresponding projection. Let $C \subset \mathbb{R}^n$ be a closed convex set containing the origin. Then the intersection $C \cap L \subset L$ is a closed convex set in L, containing the origin of L. The next result gives a convenient description of the polar of $C \cap L$ as a subset of L in terms of the polar C^o.

Lemma 11 *Assume the notations of the previous paragraph. Then the polar $(C \cap L)^o$ is given by the closure of the projection $\Pi[C^o]$.*

Proof. Let $y \in C^o$ be an arbitrary point in the polar of C and $\Pi(y) = y + L^\perp$ its projection on the quotient $\mathbb{R}_n/L^\perp$. Then we have $\langle y, x\rangle \leq 1$ for all $x \in C$. In particular, we have $\langle y, x\rangle \leq 1$ for all $x \in C \cap L$. Hence $\Pi(y) \in (C \cap L)^o$. It follows that $\Pi[C^o] \subset (C \cap L)^o$. However, $(C \cap L)^o$ is closed, and hence the closure of $\Pi[C^o]$ is also a subset of $(C \cap L)^o$.

Let now $y \in \mathbb{R}_n$ such that $\Pi(y) = y + L^\perp$ is not contained in the closure of $\Pi[C^o]$. Then there exists a hyperplane $H \subset \mathbb{R}_n/L^\perp$ which separates $\Pi(y)$ from $\Pi[C^o]$, and such that $\Pi(y) \notin H$. Then the hyperplane $\Pi^{-1}[H] \subset \mathbb{R}_n$ separates y from C^o, and $y \notin \Pi^{-1}[H]$. Note also that $y \neq 0$. It follows that there exists a vector $z \in L$ such that $\langle y, z\rangle > 1$ and $\langle w, z\rangle \leq 1$ for all $w \in C^o$. Hence $z \in C \cap L$. But then $y + L^\perp \notin (C \cap L)^o$. This proves the converse inclusion and completes the proof. ■

3 An auxiliary result

Let $L \subset \mathbb{R}^n$ be the linear subspace of vectors x defined by the equation $\langle \mathbf{1}_n, x\rangle = 0$, and let Π be the orthogonal projection onto L, $\Pi x = x - \frac{1}{n}\langle \mathbf{1}_n, x\rangle \cdot \mathbf{1}^n$. Let further $\Pi^* : \mathbb{R}_n \to \mathbb{R}_n$ be the adjoint of Π, $\Pi^* w = w - \frac{1}{n}\langle w, \mathbf{1}^n\rangle \cdot \mathbf{1}_n$. In this section we prove the following result.

Theorem 12 *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a convex homogeneous function of degree 1, and set $F_1 = \{x \in \mathbb{R}^n \,|\, F(x) \leq 1\}$. Suppose that for every nonzero vector $x \in L$ we have $F(x) > 0$. Let $\lambda > 0$ and let $g : \mathbb{R}^n \to \mathbb{R}$ be the function defined by $g(x) = \frac{1}{n}\langle \mathbf{1}_n, x\rangle + \lambda F(\Pi x)$. Then the following conditions are equivalent.*

(i) For every $x \in \mathbb{R}^n$ there exists a subgradient $y \in \partial g(x)$ whose elements are all nonnegative.

(ii) The set $\frac{1}{n}\mathbf{1}_n + \lambda \Pi^*[F_1^o]$ is contained in the nonnegative orthant.

We shall prove the two directions of the equivalence relation separately.

3.1 (ii) $\Rightarrow$ (i)

First we consider condition (i) the case $\Pi x = 0$. We shall show that the dual vector $y = \frac{1}{n}\mathbf{1}_n \geq 0$ is a subgradient of g at x. We have $F(\Pi x) = 0$ and hence $g(x) = \frac{1}{n}\langle \mathbf{1}_n, x\rangle$. For every $z \in \mathbb{R}^n$ we have $F(\Pi z) \geq 0$ and hence $g(z) \geq \frac{1}{n}\langle \mathbf{1}_n, z\rangle = g(x) + \langle \frac{1}{n}\mathbf{1}_n, z - x\rangle$. This proves $\frac{1}{n}\mathbf{1}_n \in \partial g(x)$.

Now let $x \in \mathbb{R}^n$ be such that $L \ni \Pi x \neq 0$. Then $F(\Pi x) > 0$, and we can define $\tilde{x} = \frac{\Pi x}{F(\Pi x)} \in L$. By definition, we have $F(\tilde{x}) = 1$. It follows that $\tilde{x}$ is on the boundary of the set F_1. Hence there exists an element $w \in F_1^o$ such that $\langle \tilde{w}, x\rangle = 1$, and hence $\langle w, \Pi x\rangle = F(\Pi x)$. By assumption we have $y = \frac{1}{n}\mathbf{1}_n + \lambda \Pi^* w \geq 0$. We shall show that $y \in \partial g(x)$.

Indeed, let $z \in \mathbb{R}^n$. Then we have $g(z) - g(x) - \langle y, z - x\rangle = \lambda(F(\Pi z) - F(\Pi x) - \langle \Pi^* w, z - x\rangle) = \lambda(F(\Pi z) - \langle w, \Pi z\rangle)$. If $\Pi z = 0$, then $F(\Pi z) - \langle w, \Pi z\rangle = 0$. Let us assume that $\Pi z \neq 0$. Then $F(\Pi z) > 0$, and we may define $\tilde{z} = \frac{\Pi z}{F(\Pi z)}$. We get $F(\tilde{z}) = 1$, and $\tilde{z} \in F_1$. It follows that $\langle w, \tilde{z}\rangle \leq 1$, because $w \in F_1^o$. But then $\langle w, \Pi z\rangle \leq F(\Pi z)$, which proves $g(z) - g(x) - \langle y, z - x\rangle \geq 0$. Hence $y \in \partial g(x)$, which yields (i).

3.2 (i) $\Rightarrow$ (ii)

First we shall prove an auxiliary result.

Lemma 13 *Let $\tilde{F}_1 = F_1 \cap L$. Then the polar $\tilde{F}_1^o$ of $\tilde{F}_1$ in L is given by the projection $\Pi^*[F_1^o]$.*

Proof. By Lemma 11 the polar $\tilde{F}_1^o$ is given by the closure of $\Pi^*[F_1^o]$. It remains to show that $\Pi^*[F_1^o]$ is closed. We have $F(0) = 0$, and hence F_1 contains a ball around the origin with positive radius r. It follows that the polar F_1^o is contained in a ball with radius r^{-1}, and is hence compact. But projections of compact sets are compact, and in particular closed. ■

We now come to the implication (i) ⇒ (ii). Assume (i) and consider first an exposed point $w \in \tilde{F}_1^o$. Our aim is to show that $\frac{1}{n}\mathbf{1}_n + \lambda w \geq 0$. By definition, there exists $x \in \tilde{F}_1$ such that $\langle w, x\rangle = 1$, $\langle v, x\rangle \leq 1$ for all $v \in \tilde{F}_1^o$, and $\{v \in \tilde{F}_1^o \,|\, \langle v, x\rangle = 1\} = \{w\}$. Note that $x \neq 0$, hence $F(x) > 0$, and $\tilde{x} = \frac{x}{F(x)} \in \tilde{F}_1$. Therefore $\langle w, \tilde{x}\rangle \leq 1$ and $1 = \langle w, x\rangle \leq F(x)$. It follows that $F(x) = 1$.

Let $y \geq 0$ be a subgradient of g at x. By Lemmas 8 and 9 there exists $v \in \partial F(x)$ such that $y = \frac{1}{n}\mathbf{1}_n + \lambda \Pi^* v$. By definition, for all z we have $F(z) - F(x) - \langle v, z - x\rangle \geq 0$. Inserting $z = \alpha x$ for $\alpha \geq 0$, we obtain $(\alpha - 1)F(x) \geq (\alpha - 1)\langle v, x\rangle$. Since $\alpha - 1$ assumes positive as well as negative values for $\alpha \geq 0$, it follows that $1 = F(x) = \langle v, x\rangle = \langle v, \Pi x\rangle = \langle \Pi^* v, x\rangle$. Thus we get for all z that $F(z) - \langle v, z\rangle \geq 0$. In particular, for $z \in \tilde{F}_1$ we have $1 \geq F(z) \geq \langle v, z\rangle = \langle \Pi^* v, z\rangle$, and $\Pi^* v \in \tilde{F}_1^o$. From $\langle \Pi^* v, x\rangle = 1$ it follows that $\Pi^* v = w$, and $y = \frac{1}{n}\mathbf{1}_n + \lambda w \geq 0$.

Thus $\frac{1}{n}\mathbf{1}_n + \lambda w \geq 0$ for all exposed points $w \in \tilde{F}_1^o$. By Corollary 6 we get that $\frac{1}{n}\mathbf{1}_n + \lambda w \geq 0$ for all $w \in \tilde{F}_1^o = \Pi^*[F_1^o]$. This shows (ii).

4 Penalties in the optimal stopping problem

In this section we investigate under which conditions problem (5) can be written as a conic program (7). We shall identify the vector space $\mathbb{R}^N$ with its dual by means of the standard Euclidean scalar product. Then the orthogonal projection operator Π onto the subspace $L = \{x \in \mathbb{R}^N \,|\, \langle \mathbf{1}_N, x\rangle = 0\}$ can be identified with its adjoint and is given by the matrix $\Pi = I - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$.

First we relax (5) to the following problem:

$$\inf_{\alpha,c} \frac{1}{N}\sum_{n=1}^{N} c_n + \lambda F(\hat{c}) \;:\; c_n \geq Z_t^{(n)} - \sum_{j=1}^{K} \alpha_j M_t^{j,(n)},\; n = 1, \dots, N,\; t = 0, \dots, T, \quad (8)$$

where $c = (c_1, \dots, c_N)^T$ is a vector of auxiliary variables and $\hat{c} = (\hat{c}_1, \dots, \hat{c}_N)^T$ with $\hat{c}_n = c_n - \frac{1}{N}\sum_{l=1}^{N} c_l$.

Lemma 14 *The optimal value of problem* (8) *is not greater than the optimal value of* (5)*.*

Proof. Let $\alpha \in \mathbb{R}^m$ be arbitrary, and set $c_k = Z^{(k)}(M(\alpha))$. Then the pair (α, c) is a feasible point for problem (8). Moreover, we have $\hat{c}_k = \hat{Z}^{(k)}(M(\alpha))$, and hence (α, c) gives the same value to the cost function in (8) as α gives to the cost function in (5). This proves our claim. ■

Define the set $K \subset \mathbb{R}^{N+1}$ by $K = \{(a, v) \in \mathbb{R} \times \mathbb{R}^N \,|\, a \geq F(v)\}$.

Lemma 15 *The set K is a regular convex cone if and only if for all $v \neq 0$ we have $F(v) + F(-v) > 0$.*

Proof. From the homogeneity property of F it follows that K is a cone, and from convexity of F that K is convex. Clearly K is also closed. The point $(1, 0)$ is contained in the interior of K. Now K contains a line if and only if $F(v) + F(-v) \leq 0$ for some $v \neq 0$. This completes the proof. ■

Relaxation (8) can be rewritten as

$$\inf_{\alpha,c,a} \frac{1}{N}\sum_{n=1}^{N} c_n + \lambda a \;:\; (a, \hat{c}) \in K,\; c_n \geq Z_t^{(n)} + \sum_{j=1}^{K} \alpha_j M_t^{j,(n)},\; n = 1, \dots, N,\; t = 0, \dots, T, \tag{9}$$

where a is an additional scalar auxiliary variable. Indeed, since $\lambda \geq 0$, it is for fixed α, c optimal to set $a = F(\hat{c})$, which recovers the formulation (8).

Problem (9) has a linear objective function and involves the conic constraint $(a, \hat{c}) \in K$ as well as $N \cdot (T + 1)$ linear inequalities in quantities which are linear in the decision variables. Therefore, if the condition in Lemma 15 is fulfilled, problem (9) can be written as a conic program over the regular convex cone $K \times \mathbb{R}_+^{N(T+1)}$. If K is polyhedral, it can be written as a linear program.

Lemma 16 *Suppose the function $g : \mathbb{R}^N \to \mathbb{R}$ defined by $g(c) = \frac{1}{N}\sum_{n=1}^{N} c_n + \lambda F(c - (\frac{1}{N}\sum_{l=1}^{N} c_l)\mathbf{1}_N)$ is such that for every $c \in \mathbb{R}^N$ there exists a subgradient $y \in \partial g(c)$ which is nonnegative element-wise. Then relaxation* (8) *is exact, i.e., it yields the same optimal value as the original problem* (5)*.*

Proof. Assume the conditions of the lemma. Fix a vector $\hat{c}$ and a subgradient $y \geq 0$ of g at $\hat{c}$. Then for every c such that $c \geq \hat{c}$ element-wise, we have $g(c) \geq g(\hat{c}) + \langle y, c - \hat{c}\rangle \geq g(\hat{c})$. Hence in (8) it is optimal to set c_k to the minimal value allowed by the constraint, namely $c_k = Z^{(k)}(M(\alpha))$. It follows that (9) is equivalent to (5). ■

The converse of the lemma is in general not true. If g is differentiable and g' is not nonnegative at c^* defined by $c_n^* = Z^{(n)}(M(\alpha^*))$, where α^* is optimal for (5), then relaxation (8) gives a strictly lower optimal value than (5). In the Appendix we show that relaxation (8) may be considered optimal, as every other convex relaxation may also fail to be exact in the absence of the condition in Lemma 16.

Lemma 17 *Suppose that $F(x) > 0$ for all $x \neq 0$ such that $\langle \mathbf{1}_N, x\rangle = 0$. Then the condition in Lemma 16 is equivalent to the following condition:*

For every $y \in F_1^o$ we have $\lambda \Pi y \geq -\frac{1}{N}\mathbf{1}_N$, where $F_1 = \{c \in \mathbb{R}^N \mid F(c) \leq 1\}$.

Proof. The lemma follows immediately from Theorem 12. ■

Hence the condition in Lemma 16 becomes $\Pi y \geq -\frac{1}{\lambda N}\mathbf{1}_N$ for all $y \in F_1^o$. In other words, the largest allowed value for λ is such that a shift of the projection $\Pi[F_1^o]$ by the vector $\frac{1}{\lambda N}\mathbf{1}_N$ still moves it into the nonnegative orthant.

We shall now consider different examples of penalty functions F. Set $\tilde{F}_1 = F_1 \cap L$.

1. $F(x) = \max_n x_n$. The set F_1 is given by $\{x \mid x_n \leq 1\}$. Its polar is given by $F_1^o = \{y \geq 0 \mid \langle y, \mathbf{1}^N\rangle \leq 1\}$. The polar $\tilde{F}_1^o = \Pi[F_1^o]$ is then spanned by the projections Πe_n of the unit

vectors, namely the vector $(-\frac{1}{N},\dots,-\frac{1}{N},\frac{N-1}{N})$ and the vectors obtained by permutation of the elements from it. The condition on λ becomes $(-\frac{1}{N},\dots,-\frac{1}{N},\frac{N-1}{N})^T \geq -\frac{1}{\lambda N}\mathbf{1}_N$, which yields $\lambda \leq 1$.

2. $F(x) = \max_n |x_n|$. The set F_1 is the unit cube, its polar F_1^o the unit hyper-octahedron. The polar $\tilde{F}_1^o$ is then spanned by the projections $\pm\Pi e_n$, namely the vectors $\pm(-\frac{1}{N},\dots,-\frac{1}{N},\frac{N-1}{N})$ and the vectors obtained by permutation of the elements from it. The condition on λ becomes $\lambda \leq \frac{1}{N-1}$.

3. $F(x) = \sum_n |x_n|$. The set F_1 is the unit hyper-octahedron, its polar the unit cube. The set $\tilde{F}_1^o$ is then spanned by the projections of the vertices of the cube. These projections are given by $(2-\frac{2n}{N},\dots,2-\frac{2n}{N},-\frac{2n}{N},\dots,-\frac{2n}{N})$ and their permutations, where the first number appears n times and the second one $N-n$ times, $n=0,\dots,N$. The condition on λ becomes $-2N+2n \leq \frac{1}{\lambda}$ for all $n=1,\dots,N$, and $2n \leq \frac{1}{\lambda}$ for all $n=0,\dots,N-1$, yielding $\lambda \leq \frac{1}{2(N-1)}$.

4. $F(x) = ||x||_2$. Then both F_1 and F_1^o are the unit ball, and $\tilde{F}_1^o$ is the intersection of the unit ball with L. The condition on λ is determined by the unit length vector in L with the smallest element, which is $(\frac{1}{\sqrt{N(N-1)}},\dots,\frac{1}{\sqrt{N(N-1)}},-\sqrt{\frac{N-1}{N}})$. We hence get $-\sqrt{\frac{N-1}{N}} \geq -\frac{1}{\lambda N}$, yielding $\lambda \leq \frac{1}{\sqrt{N(N-1)}}$.

While relaxation (9) with F given by cases 1 – 3 is a linear program, it is a conic quadratic program with one conic quadratic constraint with F given by 4. In general, (9) is a linear program if and only if K is a polyhedral cone.

Finally we shall show that among the penalty functions F which allow a weighting value of $\lambda = 1$, the function $F(x) = \max_n x_n$ is maximal.

Lemma 18 *Suppose the function F satisfies the condition in Lemma 17 with $\lambda = 1$. Then for every $x \in \mathbb{R}^n$ such that $\langle \mathbf{1}_N, x\rangle = 0$ we have $F(x) \leq \max_n x_n$.*

Proof. Define the set $C = \{x \in \mathbb{R}^N \mid \max_k x_k \leq 1\}$ and the set $\tilde{C} = \{x \in C \mid \langle \mathbf{1}_N, x\rangle = 0\}$. From case 1 above we have that the polar $\tilde{C}^o$ is given by $\{y \geq -\frac{1}{N} \mid \langle y, \mathbf{1}^N\rangle = 0\}$. By assumption we have that $\frac{1}{N}\mathbf{1}_N^T + \Pi[F_1^o]$ is contained in the intersection of the subspace $\{y \mid \langle y, \mathbf{1}^N\rangle = 1\}$ with the nonnegative orthant, i.e., in the convex hull of the unit vectors. Hence we have the inclusion $\Pi[F_1^o] = \tilde{F}_1^o \subset \tilde{C}^o$. It follows that $\tilde{C} \subset \tilde{F}_1$. From this the claim of the lemma easily follows. ■

In this sequel we henceforth concentrate on the case where $\lambda = 1$ and $F(x) = \max_n x_n$, i.e., we consider the problem

$$\inf_{\alpha} \max_{n=1,\dots,N} \sup_{t\in[0,T]} \left[Z_t^{(n)} - \sum_{j=1}^{K} \alpha_j M_t^{j,(n)} \right]. \tag{10}$$

5 Convergence analysis of the maximally penalized problem

In this section we analyze the convergence of the solution of the sequence of problems (10) if both the dimension K of the subspace of martingales as well as the number N of paths in the Monte-Carlo simulation tend to infinity. We establish bounds on the growth rate of N in dependence on K in order for the solution sequence to converge to a surely optimal solution of the original problem (2).

Consider for a sequence of basis martingales $M^k,\, k = 1, 2, ...$ with $M_0^k = 0$, the linear span

$$\Lambda_K := \left\{ M(\alpha) = \sum_{k=1}^{K} \alpha_k M^j : \ \alpha_1, ..., \alpha_K \in \mathbb{R} \right\}$$

for $K \in \mathbb{N}_+$, and then study the convex optimization problem

$$\begin{aligned}\alpha^{K,N} &:= \underset{\alpha:\, M(\alpha)\in\Lambda_K}{\arg\inf} \max_{n=1,\dots,N} \mathcal{Z}^{(n)}(M(\alpha)) \\ &:= \underset{\alpha\in\mathbb{R}^K}{\arg\inf} \max_{n=1,\dots,N} \max_{t=0,\dots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} \right).\end{aligned} \tag{11}$$

The following result is proved in the Appendix.

Theorem 19 *Suppose that an almost surely optimal martingale M_t^* is square integrable and has a representation*

$$M_t^* := \sum_{k=1}^{\infty} \alpha_k^* \, M_t^k, \quad t \in [0, T]$$

in L^2 satisfying

$$\mathbb{E}\left[\left| \sum_{k=K+1}^{\infty} \alpha_k^* M_T^k \right|^p \right] \le \eta K^{-\rho}, \quad \forall\, K > K_0 \tag{12}$$

for some $p > 1,\, K_0 > 0,\, \eta > 0,$ and $\rho > 0.$ Let t_* be a random variable satisfying

$$Y^* = (Z_{t_*} - M_{t_*}^*), \quad \textit{a.s.}$$

Then it holds for any $c > 0,\, \epsilon > 0$ that

$$\mathbb{P}\Big(\big\{ \|\alpha^{*K} - \alpha^{K,N}\| \ge \varepsilon \big\} \cap \mathcal{E}_{c,K,N} \Big) \le A_{p,\eta}\, N K^{-\rho} / (c\varepsilon)^p, \tag{13}$$

where $\alpha^{*K} := (\alpha_1^*, ..., \alpha_K^*)$, $A_{p,\eta}$ is constant depending on p and η, and

$$\mathcal{E}_{c,K,N} := \left\{ \max_{n=1,\dots,N} \sum_{k=1}^{K} \delta_k M_{t_*^{(n)}}^{k,(n)} \ge c\, \|\delta\| \ \text{ for all } \delta \in \mathbb{R}^K \right\}.$$

Remark 20 *Suppose that $Z_t = G_t(X_t)$, where $G_t : \mathbb{R}^d \to \mathbb{R}$ is a Hölder function on $[0,T] \times \mathbb{R}$ and X_t is a d-dimensional Markov process solving the following system of SDE's:*

$$dX_t \;\; = \;\; \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \quad X_0 = x. \tag{14}$$

The coefficient functions $\mu : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are supposed to be Lipschitz in space and $1/2$-Hölder continuous in time, with m denoting the dimension of the Brownian motion $W = (W^1, \ldots, W^m)^\top$. It is well-known that under the assumption that a martingale M_t is square integrable and is adapted to the filtration generated by W_t, there is a square integrable (row vector valued) process $H_t = (H_t^1, \ldots, H_t^m)$ satisfying

$$M_t = \int_0^t H_s dW_s. \tag{15}$$

It is not hard to see that in the Markovian setting and under some rather weak assumptions, the optimal Doob martingale M^* can be represented as

$$M_t^* = \int_0^t u(s, X_s) dW_s. \tag{16}$$

for some vector function $u(s,x) = (u_1(s,x), \ldots, u_m(s,x))$ satisfying

$$\int_0^T \mathbb{E}[|u(s,X_s)|^2]\,ds < \infty.$$

In such a situation, we can consider a class of adapted square-integrable martingales which can be "parameterized" by the set $L_{2,P}([0,T] \times \mathbb{R}^d)$ of square-integrable m-dimensional vector functions ψ on $[0,T] \times \mathbb{R}^d$ that satisfy $\|\psi\|_{2,P}^2 := \int_0^T \mathbb{E}[|\psi(s,X_s)|^2]\,ds < \infty$. Choose a family of finite-dimensional linear models of functions, called sieves, with good approximation properties with respect to u. We can consider, for example, linear sieves of the form:

$$\Psi_K := \{\alpha_1\phi_1 + \ldots + \alpha_K\phi_K : \alpha_1, \ldots, \alpha_K \in \mathbb{R}\},$$

where $\phi_1, \ldots, \phi_K$ are some given vector basis functions from $L_{2,P}([0,T] \times \mathbb{R}^d)$. Then the basis martingales in (10) *can be defined via $M_t^k = M_t(\phi_k)$, $k = 1, \ldots, K$. In this case the condition* (12) *can transformed (by using Burkholder-Davis-Gundy's inequalities) to the following one*

$$\mathbb{E}\left[\left|\int_0^T (u(s,X_s) - \alpha_1^*\phi_1(s,X_s) - \ldots - \alpha_K^*\phi_K(s,X_s))^2\,ds\right|^{p/2}\right] \le \eta K^{-\rho}$$

which measures the quality of "best projection" of u on the linear subspace Ψ_K.

One of the main issues is the estimation of the probability of the event $\mathcal{E}_{c,K,N}$. We have the following result, whose proof can be found in the Appendix.

Theorem 21 *Assume that the density p of the vector $x = \left(M^k_{t_*}\right)_{k=1,\dots,K}$ satisfies*

$$p(x) \geq \frac{a}{\left(\sqrt{2\pi}\right)^K} e^{-\frac{1}{2}\|x\|^2}, \quad x \in \mathbb{R}^K, \tag{17}$$

for some $a > 0$. Set $\gamma := \frac{N-K+1}{(K-1)\log(K-1)}$ and suppose N is such that

$$\gamma > -\frac{1}{2\log\left(1 - \frac{a}{\sqrt{2\pi}}\int_c^\infty e^{-t^2/2}\,dt\right)}. \tag{18}$$

Then

$$\begin{aligned}\log \mathbb{P}\left(\Omega\backslash\mathcal{E}_{c,K,N}\right) < (K-1)\Bigg\{&\frac{3-\log 2 - c^2}{2} + \log(1+\gamma\log(K-1)) + \log a \\ &+ \log(K-1)\left(\gamma \log\left(1 - \frac{a}{\sqrt{2\pi}}\int_c^\infty e^{-t^2/2}\,dt\right) + \frac{1}{2}\right)\Bigg\}.\end{aligned}$$

Note that the main term in the upper bound in Theorem 21 is given by the last summand in the brackets. We obtain the following corollary.

Corollary 22 *Choose $\gamma > -\frac{1}{2\log\left(1-\frac{a}{\sqrt{2\pi}}\int_c^\infty e^{-t^2/2}\,dt\right)}$ and set*

$$N = (K-1)(1+\gamma\log(K-1)),$$

then

$$\mathbb{P}\left(\Omega\backslash\mathcal{E}_{c,K,N}\right) < (K-1)^{-C(K-1)} \quad \text{for some } C > 0.$$

Corollary 23 *Under the choice $N = (K-1)(1+\gamma\log(K-1))$, we have by combining Corollary 22 with the bound (13) that*

$$\begin{aligned}\mathbb{P}\left(\left\|\alpha^{*K} - \alpha^{K,N}\right\| \geq \varepsilon\right) &\leq \mathbb{P}\left(\left\|\alpha^{*K} - \alpha^{K,N}\right\| \geq \varepsilon \middle| \mathcal{E}_{c,K,N}\right) + \mathbb{P}\left(\Omega \setminus \mathcal{E}_{c,K,N}\right) \\ &\leq A_{p,\eta}\, N K^{-\rho}/(c\varepsilon)^p + e^{-C(K-1)\log(K-1)} \\ &\lesssim (c\varepsilon)^{-p}\, K^{-\rho+1}\log(K)\end{aligned}$$

for all $\varepsilon > 0$.

Suppose now we have a new set of trajectories (independent of those used to construct $\alpha^{K,N}$) $(Z^{(n)}_\cdot, M^{(n)}_\cdot)$, $n = 1, \dots, N_1$. Consider the estimate

$$Y_{K,N,N_1} := \frac{1}{N_1}\sum_{n=1}^{N_1} \max_{t=0,\dots,T}\left(Z^{(n)}_t - \sum_{k=1}^K \alpha^{K,N}_k M^{k,(n)}_t\right).$$

Using the Doob inequality, we get

$$\mathbb{E}\Big[|Y_{K,N,N_1} - Y^*|^2\Big] \leq \frac{8}{N_1}\left[\mathbb{E}\left(\left|\sum_{k=1}^K \left(\alpha^{K,N}_k - \alpha^*_k\right) M^k_T\right|^2 + \left|\sum_{k=K+1}^\infty \alpha^*_k M^k_T\right|^2\right)\right]$$

Suppose now that M_t^k is a sequence of continuous square integrable martingales satisfying $\mathbb{E}[M_T^i\, M_T^k] = \delta_{ik}$ for all $i, k \in \{1, \ldots, K\}$, then

$$\begin{aligned}\mathbb{E}\Big[|Y_{K,N,N_1} - Y^*|^2\Big] &\leq \frac{8}{N_1}\left[\mathbb{E}\left\|\alpha^{K,N} - \alpha^{*,K}\right\|^2 + \sum_{k=K+1}^{\infty} (\alpha_k^*)^2\right] \\ &\lesssim \frac{K^{-\rho+1}\log(K)}{N_1}, \quad K \to \infty,\end{aligned}$$

provided $\sum_{k=K+1}^{\infty} (\alpha_k^*)^2 = O(K^{1-\rho})$, $K \to \infty$. So if $\rho > 1$ we have a variance reduction effect by using the maximal penalty in (5).

6 Simulation example

Consider the example given in [10, Section 8]. We have $T = 2$, $Z_0 = 0$, $Z_2 = 1$, and $Z_1 = \xi$ is a random variable which is uniformly distributed on the interval $[0, 2]$. The optimal stopping time τ^* is given by

$$\tau^* = \begin{cases} 1, & \xi \geq 1, \\ 2, & \xi < 1. \end{cases}$$

and the optimal value of problems (1) and (2) by $Y^* = \mathbb{E}\max(\xi, 1) = \frac{5}{4}$.

The martingale M in problem (2) can be assumed of the general form $M_0 = 0$, $M_1 = M_2 = h(\xi)$, where $h : [0, 2] \to \mathbb{R}$ is a function satisfying $\mathbb{E}_\xi h(\xi) = 0$. It follows that

$$\max_{t=0,1,2}(Z_t - M_t) = \max(h(\xi), \xi, 1) - h(\xi)$$

and hence $\mathbb{E}\max_{t=0,1,2}(Z_t - M_t) = \mathbb{E}\max(h(\xi), \xi, 1) \geq \mathbb{E}\max(\xi, 1) = \frac{5}{4}$. Any martingale given by a function h satisfying $h(\xi) \leq \max(\xi, 1)$ almost surely is hence an optimal solution for problem (2). Such an optimal solution then yields $\max_{t=0,1,2}(Z_t - M_t) = \max(\xi, 1) - h(\xi)$ almost surely.

However, not every such martingale is surely optimal in the sense defined in [10]. A surely optimal martingale is defined by a function $h(\xi)$ satisfying $\max_{t=0,1,2}(Z_t - M_t) = \max(\xi, 1) - h(\xi) = \frac{5}{4}$ almost surely, which gives $h(\xi) = \max(\xi, 1) - \frac{5}{4}$ almost surely. Define the function $h^*(\xi) = \max(\xi, 1) - \frac{5}{4}$ and denote the martingale defined by this function by M^*.

We shall now try to find the function h^* by Monte-Carlo methods. We search over a finite-dimensional subspace L_K of functions $h(\xi)$, which will depend on an even integer parameter K. Namely, L_K consists of functions of the form

$$h(\xi) = \sum_{k=1}^{K/2} c_k \cos(k\xi\pi) + s_k \sin(k\xi\pi),$$

where $c_k, s_k, k = 1, \ldots, K/2$ are real coefficients. The dimension of the subspace L_K equals K. Note that $h^*(\xi)$ is not contained in L_K for any K. We rather have

$$h^*(\xi) = \sum_{k=1}^{\infty} c_k^* \cos(k\xi\pi) + s_k^* \sin(k\xi\pi)$$

with $s_k^* = -\frac{1}{k\pi}$, $c_k^* = 0$ for even k, and $c_k^* = \frac{2}{k^2\pi^2}$ for odd k.

Note further that for fixed K we have $\mathbb{P}(\mathcal{E}_{c,K,N}) > 0$ for $c > 0$ small enough and N large enough, and that for fixed $c > 0$ small enough we have $\lim_{N\to\infty} \mathbb{P}(\mathcal{E}_{c,K,N}) = 1$. The condition (17) in Theorem 21, however, is not satisfied for this example, since the basis martingales are all correlated with each other and the vector built from these martingales has a distribution which is concentrated on a 1-dimensional curve.

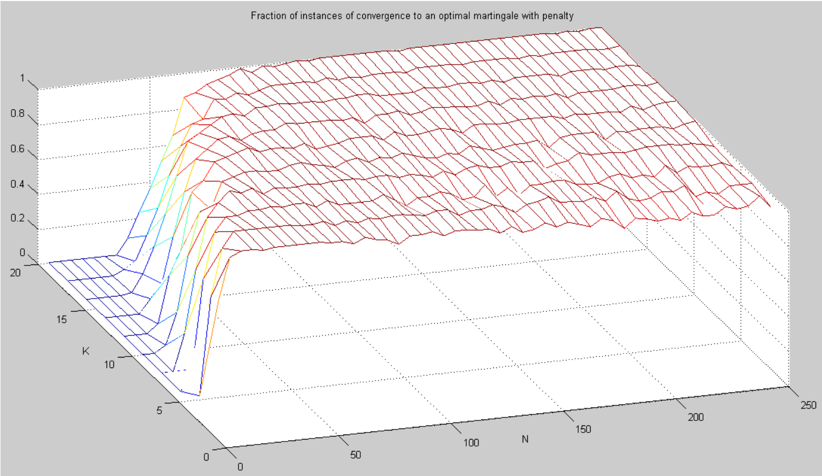
We solve the two optimization problems (3) and (6) for $K = 2, 4, \ldots, 20$ and with a number of samples $N = 5, 10, 15, \ldots, 250$. For the martingale $\hat{M}$ which gives the optimal solution of problems (3) and (6), respectively, we compute the expected value and the variance of the expression $\max_{t=0,1,2}(Z_t - \hat{M}_t)$. Note that both this expected value and the variance are random variables, because they depend on the random realization of the paths $(Z_t^{(n)}, M_t^{j,(n)})$. For each pair (K, N), we perform 100 independent runs.

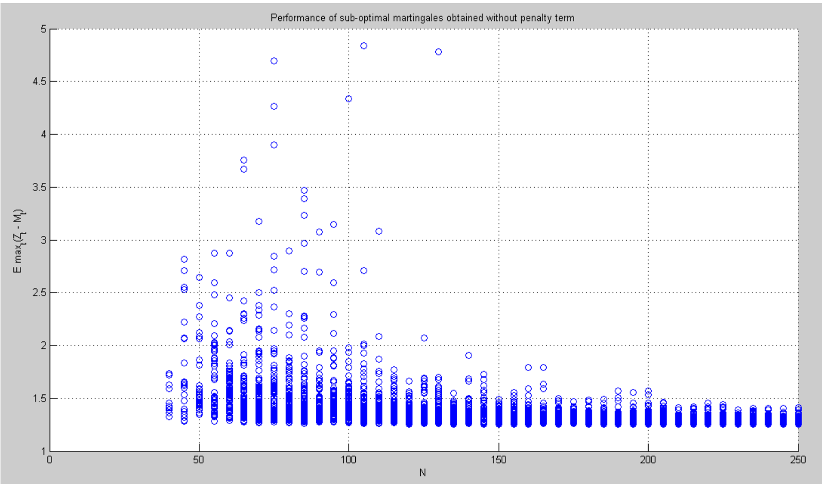
On Fig. 6 we depict the fraction of runs for which the solution of (6) yielded an *optimal* martingale. The failures for small N are due to runs where the paths were outside of the set $\mathcal{E}_{0,K,N}$. The exponential decay of $\mathbb{P}\left(\Omega \setminus \mathcal{E}_{0,K,N}\right)$ for growing N can be clearly seen. A tiny fraction of runs ($< 0.1\%$) yielded martingales which were not optimal, these were confined to the region in parameter space where $N \leq 5K$. The failures for large N were due to numerical difficulties experienced by the solver.

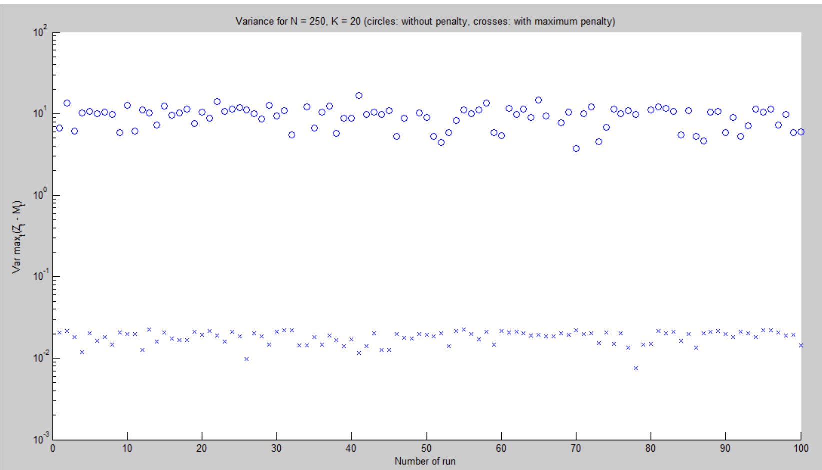
In contrast to problem (6) with maximum penalty term, whose solution in most cases yielded an optimal martingale, the martingale obtained from the solution of (3) without penalty was rarely optimal. In Fig. 6 we depict the obtained values of $\mathbb{E}\max_{t=0,1,2}(Z_t - \hat{M}_t)$ from solving problem (3), under the condition that the simulated paths were in the set $\mathcal{E}_{0,K,N}$ (paths outside this set do not yield a meaningful solution). It can be seen that the performance gets better with increasing N, but the obtained martingales nevertheless remain sub-optimal. Problem (6) is also a bit more robust numerically: a somewhat higher fraction of runs failed due to numerical difficulties when solving problem (3) without penalty as compared to (6).

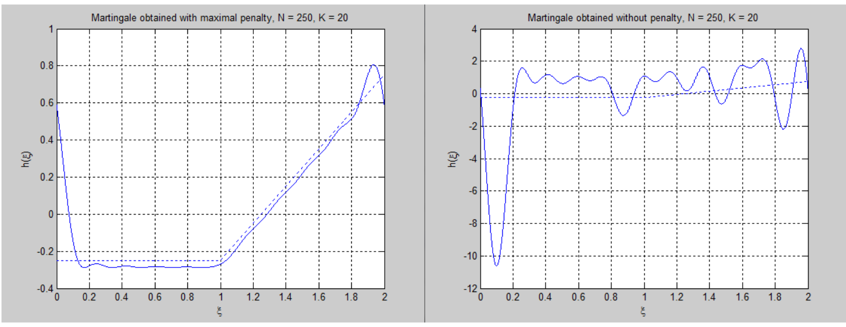
In Fig. 6 we compare the values of $Var\max_{t=0,1,2}(Z_t - \hat{M}_t)$ for the 100 runs with parameters $N = 250, K = 20$. It can be seen that the variance of $\max_{t=0,1,2}(Z_t - \hat{M}_t)$ drops dramatically if problem (6) is solved in place of problem (3). In Fig. 6 we depict martingales obtained by solving problems (3),(6) for $N = 250$, $K = 20$. The surely optimal function $h^*(\xi)$ is given for reference.

To summarize the results of the simulation, adding the maximum penalty term not only largely decreases the variance of $\max_{t=0,1,2}(Z_t - \hat{M}_t)$, but also its expectation. As a result, the obtained martingale is actually in most cases optimal. The presence of the penalty term in (6) also


Fraction of instances of convergence to an optimal martingale with penalty
K
N


Performance of sub-optimal martingales obtained without penalty term
E max$_t$(Z$_t$ - M$_t$)
N


Variance for N = 250, K = 20 (circles: without penalty, crosses: with maximum penalty)
Var max$_t$(Z$_t$ - M$_t$)
Number of run



leads to a slight numerical robustification against the uncertainty introduced by the sampling procedure.

Appendix

Proof of Theorem 19

We first need the following Lemma.

Lemma 24 *Let $K, N \in \mathbb{N}_+$ and $\beta \in \mathbb{R}^K$ be fixed. For a fixed set of N Monte Carlo realizations, let $t_\beta^{(n)}$, $n = 1, ..., N$, be such that*

$$\max_{t=0,\dots,T}\left(Z_t^{(n)} - \sum_{k=1}^{K}\beta_k M_t^{k,(n)}\right) = Z_{t_\beta^{(n)}}^{(n)} - \sum_{k=1}^{K}\beta_k M_{t_\beta^{(n)}}^{k,(n)}.$$

If

$$\max_{n=1,\dots,N}\sum_{k=1}^{K}\delta_k M_{t_\beta^{(n)}}^{k,(n)} \geq 0 \text{ for all } \delta \in \mathbb{R}^K \tag{19}$$

then it holds that

$$\begin{aligned}
&\min_{n=1,\dots,N}\left(Z_{t_\beta^{(n)}}^{(n)} - \sum_{k=1}^{K}\beta_k M_{t_\beta^{(n)}}^{k,(n)}\right)\\
&\leq \inf_{\alpha}\max_{n=1,\dots,N}\max_{t=0,\dots,T}\left(Z_t^{(n)} - \sum_{k=1}^{K}\alpha_k M_t^{k,(n)}\right)\\
&\leq \max_{n=1,\dots,N}\max_{t=0,\dots,T}\left(Z_t^{(n)} - \sum_{k=1}^{K}\beta_k M_t^{k,(n)}\right).
\end{aligned}$$

Proof. With $\alpha = \beta - \delta$ for $\delta \in \mathbb{R}^K$ we have on the one hand

$$\begin{aligned}
&\inf_{\alpha} \max_{n=1,...,N} \max_{t=0,...,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} \right) \\
&\quad = \inf_{\delta} \max_{n=1,...,N} \max_{t=0,...,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \beta_k M_t^{k,(n)} + \sum_{k=1}^{K} \delta_k M_t^{k,(n)} \right) \\
&\quad \leq \max_{n=1,...,N} \max_{t=0,...,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \beta_k M_t^{k,(n)} \right),
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&\inf_{\alpha} \max_{n=1,...,N} \max_{t=0,...,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} \right) \\
&\geq \inf_{\delta} \max_{n=1,...,N} \left(Z_{t_\beta^{(n)}}^{(n)} - \sum_{k=1}^{K} \beta_k M_{t_\beta^{(n)}}^{k,(n)} + \sum_{k=1}^{K} \delta_k M_{t_\beta^{(n)}}^{k,(n)} \right) \\
&\geq \inf_{\delta} \left(\max_{n=1,...,N} \left(\min_{n'=1,...,N} \left(Z_{t_\beta^{(n')}}^{(n')} - \sum_{k=1}^{K} \beta_k M_{t_\beta^{(n')}}^{k,(n')} \right) + \sum_{k=1}^{K} \delta_k M_{t_\beta^{(n)}}^{k,(n)} \right) \right) \\
&= \inf_{\delta} \left(\min_{n'=1,...,N} \left(Z_{t_\beta^{(n')}}^{(n')} - \sum_{k=1}^{K} \beta_k M_{t_\beta^{(n')}}^{k,(n')} \right) + \max_{n=1,...,N} \sum_{k=1}^{K} \delta_k M_{t_\beta^{(n)}}^{k,(n)} \right) \\
&\geq \min_{n=1,...,N} \left(Z_{t_\beta^{(n)}}^{(n)} - \sum_{k=1}^{K} \beta_k M_{t_\beta^{(n)}}^{k,(n)} \right),
\end{aligned}$$

by using (19). ■

Corollary 25 *Suppose that for a fixed $K \in \mathbb{N}_+$ there exists an $\alpha^* \in \mathbb{R}^K$ such that*

$$M^* := \sum_{k=1}^{K} \alpha_k^* M_t^k \tag{20}$$

is surely optimal in the sense of [10]. That is

$$Y^* = \max_{t=0,...,T} \left(Z_t - \sum_{k=1}^{K} \alpha_k^* M_t^k \right) \quad \textit{almost surely,}$$

and so we have

$$Y^* = \max_{t=0,...,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k^* M_t^{k,(n)} \right), \quad n = 1, ..., N.$$

Let $t_*^{(n)}$, $n = 1, ..., N$, be such that

$$Y^* = \max_{t=0,...,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k^* M_t^{k,(n)} \right) = Z_{t_*^{(n)}}^{(n)} - \sum_{k=1}^{K} \alpha_k^* M_{t_*^{(n)}}^{k,(n)}$$

for each n. By virtue of Lemma 24 we then obtain for $\beta = \alpha^*$

$$Y^* = \inf_{\alpha} \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} \right),$$

provided that (19) holds for $\beta = \alpha^*$.

Proposition 26 *Let us assume M^* as in* (20) *in Corollary 25 and that*

$$\max_{n=1,\ldots,N} \sum_{k=1}^{K} \delta_k M_{t_*^{(n)}}^{k,(n)} \geq c \, \|\delta\| \quad \textit{for all} \quad \delta \in \mathbb{R}^K \quad \textit{and some} \quad c > 0, \tag{21}$$

that is, a stronger version of (19) holds. If now

$$\alpha^\circ = \arg\inf_{\alpha} \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} \right),$$

then it follows that $\alpha^\circ = \alpha^*$.

Proof. Let us define

$$F(\alpha) = \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} \right).$$

So by Corollary 25, $F(\alpha^\circ) = F(\alpha^*) = Y^*$, and for any $\delta \in \mathbb{R}^K$ we have

$$\begin{aligned} F(\alpha^* - \delta) &= \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k^* M_t^{k,(n)} + \sum_{k=1}^{K} \delta_k M_t^{k,(n)} \right) \\ &\geq \max_{n=1,\ldots,N} \left(Z_{t_*^{(n)}}^{(n)} - \sum_{k=1}^{K} \alpha_k^* M_{t_*^{(n)}}^{k,(n)} + \sum_{k=1}^{K} \delta_k M_{t_*^{(n)}}^{k,(n)} \right) \\ &= Y^* + \max_{n=1,\ldots,N} \sum_{k=1}^{K} \delta_k M_{t_*^{(n)}}^{k,(n)} \geq c \, \|\delta\|, \end{aligned}$$

hence α^* is a strict local minimum of F. Since F is convex, α^* is also a unique strict global minimum. Thus, it must hold that $\alpha^\circ = \alpha^*$. ■

We next suppose that an almost surely optimal martingale M^* satisfies

$$M^* := \sum_{k=1}^{\infty} \alpha_k^* M_t^k$$

where the convergence is understood almost surely (and if it is needed to be in an L_p sense for some $p \geq 1$). Let us introduce two convex functions

$$G_{K,N}(\alpha) = \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} - \sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)} \right)$$

and

$$F_{K,N}(\alpha) = \max_{n=1,\dots,N} \max_{t=0,\dots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} \right).$$

It then holds that

$$\sup_{\alpha} |F_{K,N}(\alpha) - G_{K,N}(\alpha)| \leq \max_{n=1,\dots,N} \max_{t=0,\dots,T} \left| \sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)} \right|.$$

Indeed, for fixed α, n^* and $t_*^{n^*}$ such that

$$F_{K,N}(\alpha) = Z_{t_*^{n^*}}^{(n^*)} - \sum_{k=1}^{K} \alpha_k M_{t_*^{n^*}}^{k,(n^*)}$$

we have on the one hand

$$\begin{aligned}
&F_{K,N}(\alpha) - G_{K,N}(\alpha) \\
&\leq Z_{t_*^{n^*}}^{(n^*)} - \sum_{k=1}^{K} \alpha_k M_{t_*^{n^*}}^{k,(n^*)} - \left(Z_{t_*^{n^*}}^{(n^*)} - \sum_{k=1}^{K} \alpha_k M_{t_*^{n^*}}^{k,(n^*)} - \sum_{k=K+1}^{\infty} \alpha_k^* M_{t_*^{n^*}}^{k,(n^*)} \right) \\
&= \sum_{k=K+1}^{\infty} \alpha_k^* M_{t_*^{n^*}}^{k,(n^*)} \leq \max_{n=1,\dots,N} \max_{t=0,\dots,T} \left| \sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)} \right|,
\end{aligned}$$

and on the other hand, with n° and $t_\circ^{n^\circ}$ such that

$$G_{K,N}(\alpha) = Z_{t_\circ^{n^\circ}}^{(n^\circ)} - \sum_{k=1}^{K} \alpha_k M_{t_\circ^{n^\circ}}^{k,(n^\circ)} - \sum_{k=K+1}^{\infty} \alpha_k^* M_{t_\circ^{n^\circ}}^{k,(n^\circ)}$$

$$\begin{aligned}
&G_{K,N}(\alpha) - F_{K,N}(\alpha) \\
&\leq Z_{t_\circ^{n^\circ}}^{(n^\circ)} - \sum_{k=1}^{K} \alpha_k M_{t_\circ^{n^\circ}}^{k,(n^\circ)} - \sum_{k=K+1}^{\infty} \alpha_k^* M_{t_\circ^{n^\circ}}^{k,(n^\circ)} - \left(Z_{t_\circ^{n^\circ}}^{(n^\circ)} - \sum_{k=1}^{K} \alpha_k M_{t_\circ^{n^\circ}}^{k,(n^\circ)} \right) \\
&= - \sum_{k=K+1}^{\infty} \alpha_k^* M_{t_\circ^{K,n^\circ}}^{k,(n^\circ)} \leq \max_{n=1,\dots,N} \max_{t=0,\dots,T} \left| \sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)} \right|.
\end{aligned}$$

Now let $t_*^{(n)}$, $n = 1, \dots, N$, be defined such that for $\alpha^* := (\alpha_1^*, \dots, \alpha_K^*)$,

$$G_{K,N}(\alpha^*) = Z_{t_*^{(n)}}^{(n)} - \sum_{k=1}^{K} \alpha_k^* M_{t_*^{(n)}}^{k,(n)} - \sum_{k=K+1}^{\infty} \alpha_k^* M_{t_*^{(n)}}^{k,(n)} = Y^*$$

for each n, and assume that

$$\max_{n=1,\dots,N} \sum_{k=1}^{K} \delta_k M_{t_*^{(n)}}^{k,(n)} \geq c \,\|\delta\| \quad \text{for all} \quad \delta \in \mathbb{R}^K \quad \text{and some} \quad c > 0. \tag{22}$$

By applying Proposition 26 to the cash-flow

$$Z_t - \sum_{k=K+1}^{\infty} \alpha_k^* M_t^k$$

it thus follows that

$$\arg\inf_{\alpha \in \mathbb{R}^K} G_{K,N}(\alpha) = (\alpha_1^*, \ldots, \alpha_K^*)$$

on $\mathcal{E}_{c,K,N}$. Then, using the Markov and Doob inequalities, we get

$$\begin{aligned}
\mathbb{P}\left(\sup_{\alpha} |F_{K,N}(\alpha) - G_{K,N}(\alpha)| \geq \varepsilon\right) &\leq \mathbb{P}\left(\max_{n} \max_{t=0,\ldots,T} \left|\sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)}\right| \geq \varepsilon\right) \\
&= 1 - \mathbb{P}\left(\max_{n} \max_{t=0,\ldots,T} \left|\sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)}\right| < \varepsilon\right) \\
&= 1 - \left(\mathbb{P}\left(\max_{t=0,\ldots,T} \left|\sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)}\right| < \varepsilon\right)\right)^N \\
&\leq 1 - (1 - A_{p,\eta}\, \varepsilon^{-p}\, K^{-\rho})^N \leq A_{p,\eta} N \varepsilon^{-p}\, K^{-\rho}
\end{aligned} \tag{23}$$

for $K > K_0$ and some constant $A_{p,\eta}$ depending on p, η. Now consider K and N to be fixed and let

$$\alpha_{\inf}^F := (\alpha_{\inf,1}^F, ..., \alpha_{\inf,K}^F) := \arg\inf_{\alpha \in \mathbb{R}^K} F_{K,N}(\alpha).$$

Due to

$$\begin{aligned}
&G_{K,N}(\alpha_{\inf}^F) = G_{K,N}\left(\alpha^* - \left(\alpha^* - \alpha_{\inf}^F\right)\right) \\
&\quad = \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k^* M_t^{k,(n)} - \sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)} + \sum_{k=1}^{K} \left(\alpha_k^* - \alpha_{\inf,k}^F\right) M_t^{k,(n)}\right) \\
&\quad \geq \max_{n=1,\ldots,N} \left(Z_{t_*^{(n)}}^{(n)} - \sum_{k=1}^{K} \alpha_k^* M_{t_*^{(n)}}^{k,(n)} - \sum_{k=K+1}^{\infty} \alpha_k^* M_{t_*^{(n)}}^{k,(n)} + \sum_{k=1}^{K} \left(\alpha_k^* - \alpha_{\inf,k}^F\right) M_{t_*^{(n)}}^{k,(n)}\right) \\
&\quad = Y^* + \max_{n=1,\ldots,N} \sum_{k=1}^{K} \left(\alpha_k^* - \alpha_{\inf,k}^F\right) M_{t_*^{(n)}}^{k,(n)} \\
&\quad \geq Y^* + c \left\|\alpha^* - \alpha_{\inf}^F\right\|,
\end{aligned}$$

by virtue of (22), it holds that

$$\begin{aligned}
\left\|\alpha^* - \alpha_{\inf}^F\right\| &\leq \frac{1}{c}\left(G_{K,N}(\alpha_{\inf}^F) - G_{K,N}(\alpha^*)\right) \\
&\leq \frac{1}{c}\left|G_{K,N}(\alpha_{\inf}^F) - F_{K,N}(\alpha_{\inf}^F)\right| + \frac{1}{c}\left(F_{K,N}(\alpha_{\inf}^F) - F_{K,N}(\alpha^*)\right) \\
&\quad + \frac{1}{c}\left|F_{K,N}(\alpha^*) - G_{K,N}(\alpha^*)\right| \\
&\leq \frac{2}{c} \sup_{\alpha} \left|F_{K,N}(\alpha) - G_{K,N}(\alpha)\right|.
\end{aligned}$$

So we have

$$\begin{aligned}\mathbb{P}\left(\left\{\|\alpha^{*K}-\alpha^F_{\inf}\|\geq\varepsilon\right\}\cap\mathcal{E}_{c,K,N}\right) &\leq \mathbb{P}\left(\frac{2}{c}\sup_\alpha|F_{K,N}(\alpha)-G_{K,N}(\alpha)|\geq\varepsilon\right)\\ &\leq A_{p,\eta}2^p\,N(c\varepsilon)^{-p}\,K^{-\rho}\end{aligned}$$

by (23).

Exactness of the convex relaxation

The following consideration shows that if the condition in Lemma 16 is not satisfied, then no convex relaxation can be guaranteed to be exact, because problem (5) may not be convex at all.

Let $\alpha^*\in\mathbb{R}^K$ be an arbitrary vector, and define the vector c^* by $c^*_k=Z^{(k)}(M(\alpha^*))$. Assume that g is differentiable and g' is not nonnegative at c^*, i.e., there exists an index l such that $\nabla_l g(c^*)<0$. Suppose further that the maximum $\max_t(Z^{(l)}_t-\sum_{r=1}^K\alpha^*_rM^{r,(l)}_t)$ is attained at more than one index t, e.g., the indices i,j, and suppose that there exists a direction $\delta\in\mathbb{R}^K$ such that $\sum_{r=1}^K\delta_rM^{r,(k)}_t$ is zero for pairs (k,t) other than (l,i) and (l,j) such that $Z^{(k)}_t-\sum_{r=1}^K\alpha^*_rM^{r,(k)}_t=Z^{(k)}(M(\alpha^*))$, and $\sum_{r=1}^K\delta_rM^{r,(l)}_i\neq\sum_{r=1}^K\delta_rM^{r,(l)}_j$. Then problem (5) is not convex.

Indeed, for real ε define $\alpha_\varepsilon=\alpha^*+\varepsilon\delta$ and a vector $c(\varepsilon)$ by $c_k(\varepsilon)=Z^{(k)}(M(\alpha_\varepsilon))$. Let without loss of generality $\sum_{r=1}^K\delta_rM^{r,(l)}_i<\sum_{r=1}^K\delta_rM^{r,(l)}_j$. For $\varepsilon>0$ small enough we then have $c_k(\pm\varepsilon)=c^*_k$ for all $k\neq l$, $c_l(\varepsilon)=c^*_l-\varepsilon\sum_{r=1}^K\delta_rM^{r,(l)}_i$, and $c_l(-\varepsilon)=c^*_l+\varepsilon\sum_{r=1}^K\delta_rM^{r,(l)}_j$. The cost function of problem (5) is given by $g(c(\varepsilon))$ for $\alpha=\alpha_\varepsilon$. We have $\frac{d}{d\varepsilon}\frac{g(c(\varepsilon))+g(c(-\varepsilon))}{2}|_{\varepsilon=0}=\nabla_lg(c^*)\frac{\sum_{r=1}^K\delta_rM^{r,(l)}_j-\sum_{r=1}^K\delta_rM^{r,(l)}_i}{2}<0$, and the cost function is not convex.

If K is not too small, then the above conditions are in general verified for some value of α. Hence it is reasonable to demand the condition given in Lemma 16.

Proof of Theorem 21

Let $X\in\mathbb{R}^K$ be a random vector with a probability density. In this subsection we estimate the probability $p(K,N,c)$ of the event $\{\max_{k=1,\dots,N}\langle z,X_k\rangle\geq c|z|\ \forall z\in\mathbb{R}^K\}$, where $X_1,\dots,X_N$ are i.i.d. samples of the random vector X, and $c>0$ is a fixed constant.

First we assume that $X\sim\mathcal{N}(0,I)$. In this case the above-mentioned probability is explicitly known for the limiting case $c=0$ [11]:

$$p(K,N,0)=\mathbb{P}\left\{\max_{k=1,\dots,N}\langle z,X_k\rangle\geq 0\,\forall z\in\mathbb{R}^K\right\}=1-2^{-N+1}\sum_{i=0}^{K-1}\binom{N-1}{i}.$$

We now consider the case $c>0$. For every $x\in\mathbb{R}^K$, define the set $S_x=\{z\in S^{K-1}\,|\,\langle z,x\rangle\geq$

$c\}$. This set is a spherical cap with opening angle

$$2\arctan\frac{\sqrt{||x||^2-c^2}}{c},$$

centered on $\frac{x}{||x||}$. It is nonempty if and only if $||x||\geq c$. Clearly we have $\max_{k=1,\dots,N}\langle z,X_k\rangle\geq c|z|$ for all $z\in\mathbb{R}^K$ if and only if $\bigcup_{k=1}^N S_{X_k}=S^{K-1}$.

In order to estimate the probability of this event, we shall employ an idea from [8]. Fix $\delta\in(0,\frac{\pi}{2})$ and let $S_{x,\delta}$ be the cap centered on $\frac{x}{||x||}$ with angle $2\left(\arctan\frac{\sqrt{||x||^2-c^2}}{c}-\delta\right)$ if the inequality $\arctan\frac{\sqrt{||x||^2-c^2}}{c}\geq\delta$ holds and $S_{x,\delta}=\emptyset$ otherwise.

Let now $z\in S^{K-1}$ be a point such that $z\notin\bigcup_{k=1}^N S_{X_k}$. Then the spherical cap $B(z,2\delta)$ centered on z and with opening angle 2δ is contained in the complement of the union $\bigcup_{k=1}^N S_{X_k,\delta}$. In particular, the fraction $u_\delta(X_1,\dots,X_N)$ of points of the sphere S^{K-1} which is not covered by the union $\bigcup_{k=1}^N S_{X_k,\delta}$ is not smaller than

$$\frac{|B(z,2\delta)|}{|S^{K-1}|}=\frac{\frac{2\pi^{(K-1)/2}}{\Gamma((K-1)/2)}\int_0^\delta(\sin\varphi)^{K-2}d\varphi}{\frac{2\pi^{K/2}}{\Gamma(K/2)}}=\frac{\Gamma(K/2)\int_0^\delta(\sin\varphi)^{K-2}d\varphi}{\sqrt{\pi}\Gamma((K-1)/2)}.$$

Hence

$$\mathbb{E}u_\delta(X_1,\dots,X_N)\geq\mathbb{P}\left\{\bigcup_{k=1}^N S_{X_k}\neq S^{K-1}\right\}\cdot\frac{\Gamma(K/2)\int_0^\delta(\sin\varphi)^{K-2}d\varphi}{\sqrt{\pi}\Gamma((K-1)/2)}. \tag{24}$$

We shall now compute the expectation of $u_\delta(X_1,\dots,X_N)$. For a fixed point z, let p_δ be the probability that $z\in S_{X,\delta}$. Since X is isotropic, this quantity does not depend on z, and equals the expectation of the fraction of S^{K-1} covered by the spherical cap $S_{X,\delta}$. We have

$$\frac{|S_{X,\delta}|}{|S^{K-1}|}=\begin{cases}\frac{\Gamma(K/2)\int_0^{\arctan\frac{\sqrt{||X||^2-c^2}}{c}-\delta}(\sin\varphi)^{K-2}d\varphi}{\sqrt{\pi}\Gamma((K-1)/2)}, & ||X||^2>c^2(1+\tan^2\delta),\\ 0, & ||X||^2\leq c^2(1+\tan^2\delta).\end{cases}$$

Therefore

$$\begin{aligned}p_\delta&=\int_{c^2(1+\tan^2\delta)}^\infty\frac{\chi^{K/2-1}e^{-\chi/2}}{2^{K/2}\Gamma(K/2)}\frac{\Gamma(K/2)\int_0^{\arctan\frac{\sqrt{\chi-c^2}}{c}-\delta}(\sin\varphi)^{K-2}d\varphi}{\sqrt{\pi}\Gamma((K-1)/2)}\,d\chi\\&=\frac{1}{2^{K/2}\sqrt{\pi}\Gamma((K-1)/2)}\int_{c^2(1+\tan^2\delta)}^\infty\chi^{K/2-1}e^{-\chi/2}\int_0^{\arctan\frac{\sqrt{\chi-c^2}}{c}-\delta}(\sin\varphi)^{K-2}d\varphi\,d\chi.\end{aligned}$$

Note that at $\delta=0$ we get

$$p_0=\mathbb{P}\{\langle z,X\rangle\geq c\}=\frac{1}{\sqrt{2\pi}}\int_c^\infty e^{-t^2/2}\,dt,\qquad 1-p_0=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^c e^{-t^2/2}\,dt.$$

and

$$\left.\frac{dp_\delta}{d\delta}\right|_{\delta=0} = -\frac{1}{2^{K/2}\sqrt{\pi}\Gamma((K-1)/2)}\int_{c^2}^{\infty}\chi^{K/2-1}e^{-\chi/2}\sin^{K-2}\left(\arctan\frac{\sqrt{\chi-c^2}}{c}\right)\,d\chi$$
$$= -\frac{1}{2^{K/2}\sqrt{\pi}\Gamma((K-1)/2)}\int_{c^2}^{\infty}e^{-\chi/2}(\chi-c^2)^{K/2-1}\,d\chi = -\frac{e^{-c^2/2}\Gamma(K/2)}{\sqrt{\pi}\Gamma((K-1)/2)}.$$

Moreover, for $K \geq 3$ we have

$$\begin{aligned}\frac{d^2p_\delta}{d\delta^2} &= \frac{K-2}{2^{K/2}\sqrt{\pi}\Gamma((K-1)/2)}\int_{c^2(1+\tan^2\delta)}^{\infty}\chi^{K/2-1}e^{-\chi/2}\,\cdot \\ &\cdot(\sin(\arctan\frac{\sqrt{\chi-c^2}}{c}-\delta))^{K-3}\cos(\arctan\frac{\sqrt{\chi-c^2}}{c}-\delta)\,d\chi \geq 0,\end{aligned}$$

and the quantity p_δ is convex in δ. In particular, we have

$$1-p_\delta \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c}e^{-t^2/2}\,dt + \frac{e^{-c^2/2}\Gamma(K/2)}{\sqrt{\pi}\Gamma((K-1)/2)}\delta.$$

For fixed $z \in S^{K-1}$ we have $\mathbb{P}\{z \notin \bigcup_{k=1}^{N} S_{X_K,\delta}\} = (1-p_\delta)^N$. By Robbins theorem [6, pp.109–110] we then have

$$\mathbb{E}u_\delta(X_1,\ldots,X_N) = \frac{1}{|S^{K-1}|}\int_{S^{K-1}}\mathbb{P}\{z \notin \bigcup_{k=1}^{N} S_{X_K,\delta}\}dz = (1-p_\delta)^N. \tag{25}$$

Inserting this into (24), we obtain

$$\begin{aligned}1-p(K,N,c) = P\left\{\bigcup_{k=1}^{N} S_{X_k} \neq S^{K-1}\right\} &\leq \frac{\sqrt{\pi}\Gamma((K-1)/2)}{\Gamma(K/2)\int_0^\delta(\sin\varphi)^{K-2}d\varphi}(1-p_\delta)^N \\ \leq \frac{\sqrt{\pi}\Gamma((K-1)/2)}{\Gamma(K/2)\int_0^\delta(\sin\varphi)^{K-2}d\varphi}&\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c}e^{-t^2/2}\,dt + \frac{e^{-c^2/2}\Gamma(K/2)}{\sqrt{\pi}\Gamma((K-1)/2)}\delta\right)^N.\end{aligned}$$

Now we set $N = \beta(K-1)$ with $\beta > 1$ and $\delta = \frac{\Gamma((K-1)/2)\int_{-\infty}^{c}e^{-t^2/2}\,dt}{\sqrt{2}(\beta-1)e^{-c^2/2}\Gamma(K/2)}$. Then we get

$$\begin{aligned}1-p(K,N,c) &\leq \frac{\sqrt{\pi}\Gamma((K-1)/2)}{\Gamma(K/2)\int_0^\delta\varphi^{K-2}(1-\frac{\varphi^2}{6})^{K-2}d\varphi}\left(\frac{\beta}{\beta-1}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c}e^{-t^2/2}\,dt\right)^{\beta(K-1)} \\ &\leq \frac{(K-1)e^{-c^2(K-1)/2}\Gamma(K/2)^{K-2}\beta^{\beta(K-1)}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c}e^{-t^2/2}\,dt\right)^{(\beta-1)(K-1)}}{\pi^{K/2-1}\Gamma((K-1)/2)^{K-2}(1-\frac{\delta^2}{6})^{K-2}(\beta-1)^{(\beta-1)(K-1)}}.\end{aligned}$$

For $K \geq 3$ and $a \leq a^*$, where $a^* \approx 0.4915$ is given by the positive root of the equation $1-\frac{4a^*}{\pi} = e^{-2a^*}$, we have $\left(1-\frac{a\Gamma((K-1)/2)^2}{\Gamma(K/2)^2}\right)^{K-2} \geq e^{-2a} \geq e^{-2a^*} > e^{-1}$. It follows that for $\beta \geq 1 + \frac{\int_{-\infty}^{c} e^{-t^2/2}\,dt}{2\sqrt{3a^* e^{-c^2/2}}}$ we have $(1 - \frac{\delta^2}{6})^{K-2} > e^{-1}$. Moreover, for $K \geq 3$ we have $\frac{(K-1)\Gamma(K/2)^{K-2}}{\Gamma((K-1)/2)^{K-2}} \leq \sqrt{\pi}\left(\frac{K-1}{2}\right)^{K/2}$. We also have $(\frac{\beta}{\beta-1})^{\beta-1} < e$. Inserting all this, we get

$$1 - p(K,N,c) < \frac{e\sqrt{\pi}(K-1)^{K/2} e^{(1-c^2/2)(K-1)} \beta^{K-1}}{2^{K/2}\pi^{K/2-1}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-t^2/2}\,dt\right)^{(\beta-1)(K-1)}.$$

Now set $\beta = 1 + \gamma \log(K-1)$ with $\gamma > -\frac{1}{2\log(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-t^2/2}\,dt)}$. Then we get

$$\begin{aligned} 1 - p(K,N,c) \quad < \quad & \frac{e\sqrt{\pi}(K-1)^{K/2} e^{(1-c^2/2)(K-1)} (1+\gamma\log(K-1))^{K-1}}{2^{K/2}\pi^{K/2-1}} \cdot \\ & \cdot \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-t^2/2}\,dt\right)^{\gamma(K-1)\log(K-1)}, \end{aligned}$$

and

$$\begin{aligned} \frac{\log(1-p(K,N,c))}{K-1} &< 1 - \frac{c^2}{2} + \log(1+\gamma\log(K-1)) \\ &+ \frac{2 + \log(K-1) - K\log 2 - (K-3)\log\pi}{2(K-1)} \\ &+ \log(K-1)\left(\gamma\log\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-t^2/2}\,dt\right) + \frac{1}{2}\right) \\ &\leq \frac{3 - \log 2 - c^2}{2} + \log(1+\gamma\log(K-1)) \\ &+ \log(K-1)\left(\gamma\log\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-t^2/2}\,dt\right) + \frac{1}{2}\right). \end{aligned}$$

The main term on the right-hand side is the last one, which tends to $-\infty$ by the choice of γ if $K \to \infty$.
Defining $\tilde{\gamma} = -\left(\gamma\log\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-t^2/2}\,dt\right) + \frac{1}{2}\right) > 0$, we then get

$$1 - p(K,N,c) <\sim (K-1)^{-\tilde{\gamma}(K-1)} \to 0.$$

The number of Monte-Carlo simulations hence has to be chosen like

$$N = (K-1)(1+\gamma\log(K-1))$$

for some $\gamma > -\frac{1}{2\log(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c} e^{-t^2/2}\,dt)}$.

Let us now assume that the density of X majorizes the normal density, $\mu(X) \geq \frac{a}{(2\pi)^{K/2}} e^{-||X||^2/2}$, where $a \in (0,1]$ might also depend on K. Then we get $\mathbb{P}\{z \in S_{X,\delta}\} \geq ap_\delta$, and hence instead of (25) we obtain $\mathbb{E}u_\delta \leq (1-ap_\delta)^N$. The subsequent reasoning is analogous to the normal case considered above. Set again $N = \beta(K-1)$, but $\delta = \frac{(1-\frac{a}{\sqrt{2\pi}}\int_{c}^{\infty} e^{-t^2/2}\,dt)\sqrt{\pi}\Gamma((K-1)/2)}{(\beta-1)ae^{-c^2/2}\Gamma(K/2)}$.

For $\beta \geq 1 + \frac{(1-\frac{a}{\sqrt{2\pi}}\int_c^\infty e^{-t^2/2}\,dt)\sqrt{\pi}}{\sqrt{6a^*ae^{-c^2/2}}}$ and $K \geq 3$ we then get $(1-\frac{\delta^2}{6})^{K-2} > e^{-1}$. From (24) we then get

$$
\begin{aligned}
\mathbb{P}\left\{\bigcup_{k=1}^{N} S_{X_k} \neq S^{K-1}\right\} &\leq \frac{\sqrt{\pi}(K-1)\Gamma((K-1)/2)}{\Gamma(K/2)\delta^{K-1}(1-\frac{\delta^2}{6})^{K-2}}(1-ap_\delta)^N \\
&< \frac{(K-1)e\beta^{\beta(K-1)}a^{K-1}(\Gamma(K/2))^{K-2}}{e^{c^2(K-1)/2}\pi^{K/2-1}(\Gamma((K-1)/2))^{K-2}}\left(\frac{1-\frac{a}{\sqrt{2\pi}}\int_c^\infty e^{-t^2/2}\,dt}{\beta-1}\right)^{(\beta-1)(K-1)} \\
&\leq \frac{(K-1)^{K/2}\beta^{K-1}e^K a^{K-1}}{2^{K/2}e^{c^2(K-1)/2}\pi^{(K-3)/2}}\left(1-\frac{a}{\sqrt{2\pi}}\int_c^\infty e^{-t^2/2}\,dt\right)^{(\beta-1)(K-1)}.
\end{aligned}
$$

Now set $\beta = 1+\gamma\log(K-1)$ such that $\gamma = -\frac{1+\nu}{2\log\left(1-\frac{a}{\sqrt{2\pi}}\int_c^\infty e^{-t^2/2}\,dt\right)}$ for $\nu > 0$. We obtain

$$
\begin{aligned}
\frac{\log\mathbb{P}\left\{\bigcup_{k=1}^{N} S_{X_k} \neq S^{K-1}\right\}}{K-1} &< \frac{\log(K-1)+2K-K\log 2-(K-3)\log\pi}{2(K-1)} \\
&\quad +\log(1+\gamma\log(K-1))+\log a-\frac{c^2}{2} \\
&\quad +\left(\gamma\log\left(1-\frac{a}{\sqrt{2\pi}}\int_c^\infty e^{-t^2/2}\,dt\right)+\frac{1}{2}\right)\log(K-1) \\
&\leq \frac{3-\log 2}{2}-\frac{c^2}{2}+\log(a+\gamma a\log(K-1))-\frac{\nu}{2}\log(K-1) \\
&< \frac{3-\log 2}{2}-\frac{c^2}{2}+\log\left(a+\frac{1+\nu}{\sqrt{\frac{2}{\pi}}\int_c^\infty e^{-t^2/2}\,dt}\log(K-1)\right)-\frac{\nu}{2}\log(K-1).
\end{aligned}
$$

The main term is again the last summand, which tends to $-\infty$ for $K \to \infty$. The number of Monte-Carlo simulations has thus to be chosen not smaller than

$$
\begin{aligned}
N &= \left(1-\frac{(1+\nu)\log(K-1)}{2\log\left(1-\frac{a}{\sqrt{2\pi}}\int_c^\infty e^{-t^2/2}\,dt\right)}\right)(K-1) \\
&\leq \left(1+\frac{(1+\nu)\log(K-1)}{\frac{\sqrt{2}a}{\sqrt{\pi}}\int_c^\infty e^{-t^2/2}\,dt}\right)(K-1).
\end{aligned}
$$

This proves Theorem 21.

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