Weierstraß-Institut

für Angewandte Analysis und Stochastik

Leibniz-Institut im Forschungsverbund Berlin e. V.

Preprint ISSN 2198-5855

Spectrahedral cones generated by rank 1 matrices

Roland Hildebrand

submitted: September 25, 2014

Weierstrass Institute
Mohrenstrasse 39
10117 Berlin
Germany
E-Mail: roland.hildebrand@wias-berlin.de

No. 2014

Berlin 2014

2010 *Mathematics Subject Classification.* 15A48, 90C22.

Key words and phrases. Semi-definite relaxation, exactness, rank 1 extreme ray, quadratically constrained quadratic optimization problem.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Leibniz-Institut im Forschungsverbund Berlin e. V.
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 20372-303
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

Let $\mathcal{S}_+^n \subset \mathcal{S}^n$ be the cone of positive semi-definite matrices as a subset of the vector space of real symmetric $n \times n$ matrices. The intersection of $\mathcal{S}_+^n$ with a linear subspace of $\mathcal{S}^n$ is called a spectrahedral cone. We consider spectrahedral cones K such that every element of K can be represented as a sum of rank 1 matrices in K. We shall call such spectrahedral cones rank one generated (ROG). We show that ROG cones which are isomorphic as convex cones are also isomorphic as linear sections of the positive semi-definite matrix cone, which is not the case for general spectrahedral cones. We give many examples of ROG cones and show how to construct new ROG cones from given ones by different procedures. We provide classifications of some subclasses of ROG cones, in particular, we classify all ROG cones for matrix sizes not exceeding 4. Further we prove some results on the structure of ROG cones. ROG cones are in close relation with the exactness of semi-definite relaxations of quadratically constrained quadratic optimization problems or of relaxations approximating the cone of nonnegative functions in squared functional systems.

1 Introduction

Let $\mathcal{S}^n$ be the real vector space of $n \times n$ real symmetric matrices and $\mathcal{S}_+^n \subset \mathcal{S}^n$ the cone of positive semi-definite matrices. The intersection of the cone $\mathcal{S}_+^n$ with an affine subspace of $\mathcal{S}^n$ is called a *spectrahedron*. Spectrahedra appear as the feasible sets of semi-definite programs and are thus of importance for convex optimization. If the affine subspace happens to be a linear subspace $L \subset \mathcal{S}^n$, then the intersection $K = L \cap \mathcal{S}_+^n$ is a *spectrahedral cone*. The facial structure of spectrahedra and spectrahedral cones has been studied in [13].

The subject of this contribution are spectrahedral cones K satisfying the following property.

Property 1. *Every matrix in K can be represented as a sum of rank 1 matrices in K.*

We shall call such spectrahedral cones *rank 1 generated* (ROG). A convex cone in some real vector space will be called ROG cone if it is linearly isomorphic to a spectrahedral cone possessing Property 1. The corresponding isomorphism will define a representation of the ROG cone. We shall adopt the convention that the empty sum evaluates to zero, such that the zero matrix can be represented as an empty sum. Clearly the cone $\mathcal{S}_+^n$ itself is ROG.

The condition of being a ROG spectrahedral cone can equivalently be stated in terms of bounded spectrahedra. Namely, the conic hull K of a bounded spectrahedron C not containing the zero matrix is ROG if and only if C is the convex hull of the rank 1 matrices in C. Therefore, if C is a compact section of a ROG spectrahedral cone, then minimizing a linear function over the nonconvex set of rank 1 matrices in C is equivalent to minimizing this linear function over the bounded spectrahedron C.

Thus Property 1 is in close relation with the exactness of semi-definite relaxations of nonconvex problems in the case when the relaxation is obtained by dropping a rank constraint. Many nonconvex optimization problems which are arising in computational practice fall into this framework, i.e., they can be cast as semi-definite programs with an additional rank constraint. It is this rank constraint which makes the problem nonconvex and difficult to solve. At the same time, dropping the rank constraint provides a convenient way of relaxing the problem into an easily solvable semi-definite program.

A classical example is the MAXCUT problem [4], which can be formulated as the problem of maximizing a linear function over the set of positive semi-definite rank 1 matrices whose diagonal elements all equal 1. By dropping the rank 1 condition, one obtains a semi-definite program which yields an upper bound on the maximum cut.

We shall now consider two applications of ROG spectrahedral cones.

Quadratically constrained quadratic problems. The most general class of problems which can be formulated as semi-definite programs with an additional rank 1 constraint are the quadratically constrained quadratic problems [13],[10]. This class includes also problems with binary decision variables, as the condition $x \in \{a, b\}$ can be cast as the quadratic condition $(x - a)(x - b) = 0$.

A generic quadratically constrained quadratic problem can be written as

$$\min_{x \in \mathbb{R}^n} x^T S x \; : \quad x^T A_i x = 0, \; i = 1, \dots, k; \quad x^T B x = 1.$$

Here $A_1, \dots, A_k; B; S$ are real symmetric $n \times n$ matrices defining the homogeneous quadratic constraints, the inhomogeneous quadratic constraint, and the quadratic cost function, respectively. Introducing the matrix variable $X = xx^T \in \mathcal{S}_+^n$, we can write the problem as

$$\min_{X \in K} \langle S, X \rangle \; : \quad \langle B, X \rangle = 1, \quad \operatorname{rk} X = 1, \tag{1}$$

where $K = L \cap \mathcal{S}_+^n$, and $L \subset \mathcal{S}^n$ is the linear subspace given by $\{X \in \mathcal{S}^n \mid \langle A_i, X \rangle = 0 \; \forall \; i = 1, \dots, k\}$. The cone K is hence a linear section of the positive semi-definite matrix cone. Problem (1) can be relaxed to a semi-definite program by dropping the rank constraint,

$$\min_{X \in K} \langle S, X \rangle \; : \quad \langle B, X \rangle = 1. \tag{2}$$

Naturally, the question arises when the semi-definite relaxation (2) obtained from the nonconvex problem (1) is *exact*, i.e., yields the same optimal value as (1). In general, this question is NP-hard [13]. One NP-hard instance is, for example, the MAXCUT problem [3]. However, Property 1 of the spectrahedral cone K provides a simple sufficient condition.

Lemma 1. *Let the linear subspace $L \subset \mathcal{S}^n$ be such that the cone $K = L \cap \mathcal{S}_+^n$ is rank 1 generated. Then either problems* (1),(2) *are both infeasible, or problem* (2) *is unbounded, or problems* (1),(2) *have the same optimal value.*

Proof. Define the spectrahedron $C = \{X \in K \,|\, \langle B, X\rangle = 1\}$. Then the feasible set of problem (2) is C, while that of problem (1) is $C_1 = \{X \in C \mid \operatorname{rk} X = 1\}$. If $C = \emptyset$, then both problems are infeasible. Assume that $C \neq \emptyset$. Then $K \neq \{0\}$, and by Property 1 every extreme ray of the cone K is generated by a rank 1 matrix. If problem (2) is bounded, then its optimal value is achieved at an extreme point $X \in C$. Since X generates an extreme ray of K, we must have $\operatorname{rk} X = 1$. Thus X is feasible also for problem (1), and the optimal value of (1) is not greater than that of (2). But $C_1 \subset C$, and hence the optimal value of (1) is not smaller than that of (2). Therefore both optimal values must coincide. □

In particular, if the spectrahedron C is bounded, then problems (1) and (2) are equivalent.

Squared functional systems. Another motivation for the study of ROG spectrahedral cones comes from squared functional systems [11]. Let Δ be an arbitrary set and F an n-dimensional real vector space of real-valued functions on Δ. Choose basis functions $u_1, \dots, u_n \in F$. The *squared functional system* generated by these basis functions is the set $\{u_i u_j \,|\, i, j = 1, \dots, n\}$ of product functions. This system spans another real vector space V of real-valued functions on Δ. Clearly V does not depend on the choice of the basis functions u_i, since it is also the linear span of the squares f^2, $f \in F$.

Let us define a linear map $\Lambda : V^* \to \mathcal{S}^n$ and its adjoint $\Lambda^* : \mathcal{S}^n \to V$ by $\Lambda^*(A) = \sum_{i,j=1}^n A_{ij} u_i u_j$. Here the space $\mathcal{S}^n$ is identified with its dual by means of the Frobenius scalar product[1]. By definition of V the map Λ^* is surjective, and hence the map Λ is injective.

The *sum of squares* (SOS) cone $\Sigma \subset V$, given by the set of all functions of the form $\sum_{k=1}^N f_k^2$ for $f_1, \dots, f_N \in F$, can be represented as the image $\Lambda^*[\mathcal{S}_+^n]$ of the positive semi-definite matrix cone and has nonempty interior. The dual Σ^* of the SOS cone is given by the set of all dual vectors $w \in V^*$ such that $\Lambda(w) \succeq 0$ [11, Theorem 17.1]. By injectivity of Λ it follows that Σ^* is linearly isomorphic to its image $K = \Lambda[\Sigma^*] \subset \mathcal{S}^n$. This image equals the intersection of $\mathcal{S}_+^n$ with the linear subspace $L = Im\Lambda$. It follows that Σ^* is isomorphic to a spectrahedral cone.

Let $P \subset V$ be the cone of nonnegative functions in V. Since every sum of squares of real numbers is nonnegative, we have the inclusion $\Sigma \subset P$. It is then interesting to know when the cones P and Σ coincide. The following result shows that the cone K being ROG is a necessary condition.

Lemma 2. *Assume above notations. If $P = \Sigma$, then the spectrahedral cone $K = L \cap \mathcal{S}_+^n$ is rank 1 generated.*

Proof. For $x \in \Delta$, define the dual vector $w_x \in V^*$ by $\langle w_x, v\rangle = v(x)$ for all $v \in V$. We first show that for all $x \in \Delta$ we have that $\Lambda(w_x)$ is contained in the set $K_1 = \{X \in K \mid \operatorname{rk} X \leq 1\}$.

[1] The reason for defining the operator Λ by virtue of its adjoint Λ^* is to stay in line with the notations in [11]. This definition explicitly uses a basis of the space F. In a coordinate-free definition, the source space of Λ^* should be the space $\operatorname{Sym}^2(F)$ of contravariant symmetric 2-tensors over F, and the operator Λ^* itself should be defined by linear continuation of the map $f \otimes f \mapsto f^2$, $f \in F$.

Fix $x \in \Delta$ and define the vector $s \in \mathbb{R}^n$ element-wise by $s_i = u_i(x)$, $i = 1, \dots, n$. Then we have for all $A \in \mathcal{S}^n$ that

$$\langle \Lambda(w_x), A \rangle = \langle w_x, \Lambda^*(A) \rangle = \sum_{i,j=1}^{n} A_{ij} \langle w_x, u_i u_j \rangle = \sum_{i,j=1}^{n} A_{ij} u_i(x) u_j(x) = \langle ss^T, A \rangle.$$

It follows that $\Lambda(w_x) = ss^T$. it follows that the rank of $\Lambda(w_x)$ does not exceed 1. Moreover, we have $\Lambda(w_x) \succeq 0$ and $w_x \in V^*$, and hence $\Lambda(w_x) \in K$. This proves our claim.

For the sake of contradiction, assume now that $K = \Lambda[\Sigma^*]$ is not ROG. Then there exists a dual vector $y \in \Sigma^*$ such that the matrix $\Lambda(y)$ can be strictly separated from the convex hull of K_1. In other words, there exists $A \in \mathcal{S}^n$ such that $\langle A, \Lambda(y) \rangle < 0$, but $\langle A, X \rangle \geq 0$ for every $X \in K_1$.

Consider the function $q = \Lambda^*(A) \in V$. For every $x \in \Delta$ we have $q(x) = \langle w_x, \Lambda^*(A) \rangle = \langle \Lambda(w_x), A \rangle \geq 0$, because $\Lambda(w_x) \in K_1$. Hence we have $q \in P$. But $\langle q, y \rangle = \langle \Lambda^*(A), y \rangle = \langle A, \Lambda(y) \rangle < 0$, and therefore $y \notin P^*$.

It follows that $P^* \neq \Sigma^*$ and hence $P \neq \Sigma$. This completes the proof. $\square$

Thus in every squared functional system where the cone of nonnegative functions coincides with the SOS cone Σ, the dual SOS cone Σ^* is isomorphic to a ROG spectrahedral cone. This allows us to construct ROG cones from such squared functional systems. Let us consider two examples.

- The first example is taken from [11, Section 3.1]. Here $\Delta = \mathbb{R}$, and F is the space of all polynomials of degree not exceeding $n-1$, equipped with the basis of monomials $1, x, \dots, x^{n-1}$. It is well-known that a univariate polynomial is nonnegative if and only if it is a sum of squares of polynomials of lower degree. The corresponding ROG cone K is the cone of all Hankel matrices in $\mathcal{S}_+^n$ and has dimension $2n-1$. We shall denote this cone by Han_+^n.

- Let $\Delta = \mathbb{R}^3$ and let F be the 6-dimensional space of homogeneous quadratic polynomials on $\mathbb{R}^3$, equipped with the basis $x_1^2, x_2^2, x_3^2, x_2x_3, x_1x_3, x_1x_2$. The space V is then the 15-dimensional space of ternary quartics, and in this space the cone of nonnegative polynomials coincides with the SOS cone [8]. The corresponding ROG cone K is given by all matrices in $\mathcal{S}_+^6$ of the form

$$A = \begin{pmatrix} a_1 & a_6 & a_5 & a_7 & a_{11} & a_{10} \\ a_6 & a_2 & a_4 & a_{13} & a_8 & a_{12} \\ a_5 & a_4 & a_3 & a_{15} & a_{14} & a_9 \\ a_7 & a_{13} & a_{15} & a_4 & a_9 & a_8 \\ a_{11} & a_8 & a_{14} & a_9 & a_5 & a_7 \\ a_{10} & a_{12} & a_9 & a_8 & a_7 & a_6 \end{pmatrix}, \qquad a_1, \dots, a_{15} \in \mathbb{R}.$$

Besides the motivations coming from optimization, the ROG spectrahedral cones may represent an interesting subject of study in their own right.

The remainder of the paper is structured as follows. In Section 2 we study the most basic properties of ROG cones. In particular, we establish that the minimal polynomial of a ROG cone, when the latter is viewed as an algebraic interior, is determinantal, and the degree of the cone is given by the maximal rank of the matrices it contains (Subsection 2.1). In Subsection 2.2 we study the facial structure of ROG cones and establish the identity of the rank and the Carathéodory number of its elements. In particular, the rank is an invariant of the elements of a ROG cone under linear isomorphisms. In Subsection 2.3 we prove that the geometry of a ROG cone as a conic convex subset of a real vector space determines its representations as ROG spectrahedral cones uniquely up to a trivial notion of isomorphism, which is not true for spectrahedral cones in general. In Section 3 we describe different methods to construct ROG cones of higher degree from ROG cones of lower degree. The most simple way is taking direct sums, which is considered in Subsection 3.1. This leads to the notion of simple ROG cones, which are defined as those not representable as a nontrivial direct sum. In Subsections 3.2, 3.3 we consider two other ways of constructing ROG cones. The second one can be seen as a generalization of taking direct sums. In Section 4 we consider some examples of ROG cones. In Subsection 4.1 we investigate ROG cones defined by conditions of the type that a subset of entries in the representing matrices vanishes. This class of ROG cones is linked to chordal graphs and has been studied in [1],[12], see also [9] for a generalization to higher matrix ranks. We show that these cones can be constructed from full matrix cones $\mathcal{S}_+^k$ by the methods presented in Section 3. In Subsection 4.2 we construct an example of a continuous family of mutually non-isomorphic ROG cones. In Section 5 we consider ROG cones of low codimension (Subsections 5.1, 5.2) and simple ROG cones of low dimension (Subsection 5.3). In Section 6 we consider the variety of extreme rays of ROG cones. We show that the discrete part of this variety factors out and does not interfere with the part corresponding to the continuous components. Finally, we give a complete classification of ROG cones for degrees $n \leq 4$ up to isomorphism in Section 7. We conclude the paper with an outlook on future work.

For $n \in \mathbb{N}$, we define two operators $\mathcal{L}_n, \mathcal{F}_n$ from the set of linear subspaces of $\mathbb{R}^n$ into the set of linear subspaces of $\mathcal{S}^n$ and the set of faces of the cone $\mathcal{S}_+^n$, respectively. Let $H \subset \mathbb{R}^n$ be a linear subspace. Then $\mathcal{L}_n(H)$, $\mathcal{F}_n(H)$ will be defined as the linear span and the convex hull of the set $\{xx^T \in \mathcal{S}^n \,|\, x \in H\}$, respectively. Note that $\mathcal{F}_n(H)$ is linearly isomorphic to the cone $\mathcal{S}_+^{\dim H}$, and $\mathcal{L}_n(H)$ is isomorphic to the matrix space $\mathcal{S}^{\dim H}$, although the isomorphisms are not unique. For a matrix $X \in \mathcal{S}_+^n$, the smallest face of $\mathcal{S}_+^n$ containing X is then given by $\mathcal{F}_n(ImX)$, where $ImX \subset \mathbb{R}^n$ is the image of the matrix X.

In order to indicate the size n of the matrices making up a spectrahedral cone K, we shall write $K = L \cap \mathcal{S}_+^n$ or $K \subset \mathcal{S}_+^n$, where $L \subset \mathcal{S}^n$ is a linear subspace. Later in the paper we shall also work with ROG cones as abstract convex conic subsets of a real vector space. They may then have representations in matrix spaces of different sizes.

2 Basic properties

In this section we establish some basic properties of ROG cones.

2.1 Minimal defining polynomial

In this subsection we consider aspects of spectrahedral and ROG cones which emanate from real algebraic geometry.

Definition 1. *[7, Section 2.2] A closed set $C \subset \mathbb{R}^m$ is an algebraic interior if there exists a polynomial p on $\mathbb{R}^m$ such that C equals the closure of a connected component of the set $\{x \in \mathbb{R}^m \mid p(x) > 0\}$. Such a polynomial is called defining polynomial.*

Lemma 3. *[7, Lemma 2.1] Let C be an algebraic interior. Then the defining polynomial p of C with minimal degree is unique up to multiplication by a positive constant. Any other defining polynomial of C is divisible by p.*

Definition 2. *The defining polynomial with minimal degree of an algebraic interior C is called minimal defining polynomial. The degree of C is defined as the degree of the minimal defining polynomial.*

Lemma 4. *[7, Theorem 2.2] Every spectrahedron is a convex algebraic interior.*

From Lemma 3 it follows that the minimal defining polynomial of a spectrahedral cone is homogeneous. Indeed, under a homothety of the cone the minimal defining polynomial transforms to another minimal defining polynomial, which must differ from the original one by a multiplicative positive constant.

Definition 3. *We say that a spectrahedral cone $K = L \cap \mathcal{S}_+^n$ is non-degenerate if the interior of K consists of positive definite matrices.*

Note that non-degeneracy is not an invariant under linear isomorphisms of spectrahedral cones as conic convex subsets of real vector spaces. This property depends on the spectrahedral representation of the abstract cone as a linear section of a positive semi-definite matrix cone.

From a degenerate spectrahedral cone $K = L \cap \mathcal{S}_+^n$ we may always construct an isomorphic non-degenerate spectrahedral cone by replacing $\mathcal{S}_+^n$ by the minimal face of $\mathcal{S}_+^n$ that contains K [7, Lemma 2.3]. This replacement has no effect on Property 1. Hence for every ROG spectrahedral cone there exists an isomorphic non-degenerate ROG spectrahedral cone.

For a non-degenerate spectrahedral cone $K \subset \mathcal{S}_+^n$, a defining polynomial of K is given by the restriction of the determinant in $\mathcal{S}^n$ to $\operatorname{span} K$. We shall call this polynomial the *determinantal defining polynomial*. In contrast to the minimal defining polynomial, the determinantal defining polynomial explicitly uses the representation of K as a linear section of a positive semi-definite matrix cone. Linearly isomorphic spectrahedral cones may have determinantal defining polynomials of different degrees. Our main result in this subsection is that this cannot happen if K is a ROG cone, because the determinantal and minimal defining polynomials coincide.

Theorem 1. *Let $K = L \cap \mathcal{S}_+^n$ be a non-degenerate ROG spectrahedral cone. Then the determinantal defining polynomial d of K is a minimal defining polynomial.*

Proof. Let $X \in K$ be positive definite. Since K is ROG, there exist vectors $x_1, \dots, x_N \in \mathbb{R}^n$ such that $X = \sum_{i=1}^N x_i x_i^T$ and $x_i x_i^T \in K$ for all $i = 1, \dots, N$. Since $X \succ 0$, the linear span of $\{x_1, \dots, x_N\}$ equals $\mathbb{R}^n$. In particular, among the x_i there are n linearly independent vectors, let these be $x_1, \dots, x_n$.

Denote the linear span of the matrices $x_1x_1^T, \dots, x_nx_n^T \in K$ by D, and the intersection $D \cap K$ by K_D. We have $D \subset L$, and hence $D \cap \mathcal{S}_+^n = D \cap L \cap \mathcal{S}_+^n = K_D$. However, in the coordinates defined by the basis $\{x_1, \dots, x_n\}$ of $\mathbb{R}^n$ the subspace $D \subset \mathcal{S}^n$ is the subspace of diagonal matrices. Hence $K_D = D \cap \mathcal{S}_+^n$ equals the convex conic hull of $\{x_1x_1^T, \dots, x_nx_n^T\}$, which in turn is linearly isomorphic to the nonnegative orthant $\mathbb{R}_+^n$. Moreover, the relative interior of K_D consists of positive definite matrices and is hence contained in the relative interior of K. On the other hand, the boundary of K_D is contained in the boundary of K.

Let $p : L \to \mathbb{R}$ be a minimal defining polynomial of K. Since the determinantal defining polynomial d has degree n, the degree of p is at most n. By Lemma 3 p divides d. Since $d > 0$ on the relative interior of K, we also have $p > 0$ on the relative interior of K. Hence $p > 0$ on the relative interior of K_D. On the other hand, $p = 0$ on the boundary of K_D, because $p = 0$ on the boundary of K. Therefore the restriction of p on D is a defining polynomial for the cone $K_D \cong \mathbb{R}_+^n$.

However, the degree of the algebraic interior $\mathbb{R}_+^n$ is n, and hence p has degree at least n. It follows that $\deg p = n$, and d must be a minimal defining polynomial of K. $\square$

Note that Theorem 1 is applicable to any non-degenerate spectrahedral cone $K \subset \mathcal{S}_+^n$ such that there exist linearly independent vectors $x_1, \dots, x_n \in \mathbb{R}^n$ satisfying $x_ix_i^T \in K$, $i = 1, \dots, n$, because the proof uses only this condition.

Corollary 1. *Let $K = L \cap \mathcal{S}_+^n$ be a ROG spectrahedral cone. Then the degree of K is given by $\deg K = \max_{X \in K} \operatorname{rk} X$.*

Proof. Let $m = \max_{X \in K} \operatorname{rk} X$. Then the minimal face F of $\mathcal{S}_+^n$ which contains K is isomorphic to $\mathcal{S}_+^m$, and the linear span of F is isomorphic to $\mathcal{S}^m$. As outlined in [7, Lemma 2.3], we can then construct a determinantal defining polynomial of K by considering K as a subset of $\operatorname{span} F \cong \mathcal{S}^m$. This polynomial has degree m. The proof is concluded by application of Theorem 1. $\square$

Corollary 2. *Let $K \subset \mathcal{S}_+^n$ be a ROG cone of degree m. Then K is linearly isomorphic to a non-degenerate ROG cone $K' \subset \mathcal{S}_+^m$. Moreover, every non-degenerate ROG cone which is isomorphic to K is represented by matrices of size m.*

Proof. As outlined above, for every ROG cone K there exists a linearly isomorphic non-degenerate ROG cone.

Let K' be such a non-degenerate ROG cone. We have $\deg K = \deg K'$, and hence by Corollary 1 the interior of K' consists of matrices of rank m. Since these matrices are positive definite by the non-degeneracy condition, the size of the matrices in K' must be m. $\square$

In other words, a ROG cone of degree m possesses a non-degenerate ROG spectrahedral representation of size m, and every non-degenerate ROG spectrahedral representation has this size.

2.2 Facial structure

In this subsection we study the facial structure and the Carathéodory number of ROG cones. We shall call an element of a cone K *extreme* if it generates an extreme ray of K.

Lemma 5. *Let $K \subset \mathcal{S}_+^n$ be a ROG spectrahedral cone. Then the set of extreme elements of K is given by $\{X \in K \mid \operatorname{rk} X = 1\}$.*

Proof. Since K is ROG, every $X \in K$ with $\operatorname{rk} X > 1$ can be represented as sum of elements $X_i \in K$ of rank 1. Hence such X cannot be extreme. On the other hand, every $X \in K$ with $\operatorname{rk} X = 1$ generates an extreme ray of $\mathcal{S}_+^n$. Extremality in K for such X follows immediately. □

Let us recall the results of [13] on the facial structure of general spectrahedral cones. Let $K = L \cap \mathcal{S}_+^n$ be a spectrahedral cone. Then the faces of K are given by the intersections of L with the faces of $\mathcal{S}_+^n$ [13, Theorem 1], see also [14, Prop. 2.1]. In particular, the kernel of the matrices $X \in K$ is constant over the relative interior of each face of K, and every face of K is exposed [13, Corollary 1]. It follows that the faces of spectrahedral cones are also spectrahedral cones.

The smallest face of $K = L \cap \mathcal{S}_+^n$ containing a matrix $X \in K$ is given by the intersections $L \cap \mathcal{F}_n(ImX) = L \cap \mathcal{S}_+^n \cap \mathcal{L}_n(ImX) = K \cap \mathcal{L}_n(ImX)$, because $\mathcal{F}_n(ImX)$ is the smallest face of $\mathcal{S}_+^n$ containing X. The smallest face of $\mathcal{S}_+^n$ containing K is given by $\mathcal{F}_n(ImX)$, where X is an arbitrary matrix in the interior of K.

Lemma 6. *Every face of a ROG cone is a ROG cone.*

Proof. Let $K = L \cap \mathcal{S}_+^n$ be a ROG cone and $K' \subset K$ a face of K. Then there exists a face F of $\mathcal{S}_+^n$ such that $K' = L \cap F$. Let $X \in K'$ be an arbitrary nonzero matrix. Since $X \in K$ and K is ROG, there exist rank 1 matrices $X_1, \dots, X_l \in K$ such that $X = \sum_{i=1}^l X_i$. At the same time, $X \in F$. Since F is a face of $\mathcal{S}_+^n$, the rank 1 matrices $X_i \in \mathcal{S}_+^n$ must also be elements of this face. It follows that $X_i \in K'$, and X can be represented as sum of rank 1 matrices in K'. Thus K' is ROG. □

Definition 4. *[5, p.59] Let $K \subset \mathbb{R}^m$ be a closed pointed convex cone. The Carathéodory number $\kappa(x)$ of a point $x \in K$ is the minimal number k such that there exist extreme elements $x_1, \dots, x_k$ of K satisfying $x = \sum_{i=1}^k x_i$.*

The Carathéodory number $\kappa(K)$ of the cone K is the maximum of $\kappa(x)$ over $x \in K$.

Lemma 7. *Let $K \subset \mathcal{S}_+^n$ be a spectrahedral cone. The Carathéodory number of $X \in K$ satisfies $\kappa(X) \leq \operatorname{rk} X$.*

Proof. We proceed by induction. If $\operatorname{rk} X \leq 1$, then by virtue of Lemma 5 we trivially have $\kappa(X) = \operatorname{rk} X$. Suppose the relation $\kappa(X) \leq \operatorname{rk} X$ is proven for $\operatorname{rk} X \leq k-1$, and let $X \in K$ with $\operatorname{rk} X = k \geq 2$.

Without loss of generality we may assume $n = k$, otherwise we replace K by $K_X = L \cap \mathcal{F}_n(ImX)$, the minimal face of K which contains X. Neither the rank nor the Carathéodory number of X will change by this substitution of the ambient cone, but now $\mathcal{F}_n(ImX) \cong \mathcal{S}_+^k$ and K_X can be seen as a spectrahedral cone defined by $k \times k$ matrices.

Then the boundary of K consists of matrices Y with $\operatorname{rk} Y < k = n$, and hence $\kappa(Y) < k$ by the induction hypothesis. Let $E \in K$ an extreme element of K, normalized such that $\operatorname{tr} E = \operatorname{tr} X$. Consider the compact line segment l which is defined by the intersection of K with the affine line passing through X and E. Since X is in the interior of K, it is also in the interior of the segment l. One endpoint of l is given by E, while the other one is some matrix $Y \in \partial K$. Then there exists $\lambda \in (0,1)$ such that $X = \lambda E + (1-\lambda)Y$. Hence $\kappa(X) \leq \kappa(E) + \kappa(Y) \leq 1 + (k-1) = k$.

This completes the proof. □

Lemma 8. *Let $K \subset \mathcal{S}_+^n$ be a ROG spectrahedral cone. The Carathéodory number of $X \in K$ is given by $\kappa(X) = \operatorname{rk} X$.*

Proof. We have $\kappa(X) \geq \operatorname{rk} X$, because by virtue of Lemma 5 all generators of extreme rays of K have rank 1, and a matrix X cannot be the sum of less that $\operatorname{rk} X$ matrices of rank 1. On the other hand, $\kappa(X) \leq \operatorname{rk} X$ by Lemma 7. □

Corollary 3. *The Carathéodory number of a ROG cone equals its degree.*

Proof. The claim follows immediately from Lemma 8 and Corollary 1. □

Corollary 4. *Let $K \subset \mathcal{S}_+^n$ be a ROG spectrahedral cone, and let $X \in K$ be an element of rank k. Then there exist rank 1 matrices $X_i = x_i x_i^T \in K$, $i = 1, \dots, k$, such that $X = \sum_{i=1}^k X_i$ and the vectors $x_1, \dots, x_k$ are linearly independent. In particular, if $d = \deg K$, then there exist d linearly independent vectors $r_1, \dots, r_d \in \mathbb{R}^n$ such that $r_i r_i^T \in K$ for $i = 1, \dots, d$.*

Proof. The first claim of the Corollary is a consequence of Lemmas 5 and 8. The second claim follows from the first claim, Lemma 8, and Corollary 3. □

As a consequence, we have the following result on the diagonalization of matrices in a ROG cone.

Lemma 9. *Let $K \subset \mathcal{S}_+^n$ be a ROG spectrahedral cone, and let $X \in K$ be an element of rank k. Then there exists a basis of $\mathbb{R}^n$ such that in the corresponding coordinates we have $X = \operatorname{diag}(1, \dots, 1, 0, \dots, 0)$, and all diagonal matrices of the form $\operatorname{diag}(d_1, \dots, d_k, 0, \dots, 0)$, where $d_i \geq 0$, $i = 1, \dots, k$, are in K.*

Proof. By Corollary 4 there exist linearly independent vectors $x_1, \dots, x_k \in \mathbb{R}^n$ such that $X_i = x_i x_i^T \in K$, $i = 1, \dots, k$, and $X = \sum_{i=1}^k X_i$. Extend the set $\{x_1, \dots, x_k\}$ to a basis of $\mathbb{R}^n$, then in the coordinates defined by this basis we have $X = \operatorname{diag}(1, \dots, 1, 0, \dots, 0)$.

Moreover, for all $d_1, \dots, d_k \geq 0$ we have $\sum_{i=1}^k d_i x_i x_i^T \in K$, and in the coordinates defined above this matrix has the form $\operatorname{diag}(d_1, \dots, d_k, 0, \dots, 0)$. □

2.3 Isomorphisms and invariants

In this paper we consider spectrahedral cones as linear sections of cones of positive semi-definite matrices. Two such sections may be linearly isomorphic as subsets of their linear hull, while the matrices put into relation by the isomorphism may have very different properties. We must therefore distinguish between isomorphisms of the spectrahedral cones as convex conic subsets of a real vector space and isomorphisms between spectrahedral cones together with their defining representations. In order to formalize the latter notion, we introduce the following definition.

Definition 5. *Let $L \subset \mathcal{S}^n$, $L' \subset \mathcal{S}^{n'}$ be linear subspaces of matrix spaces, and suppose that $n \leq n'$. We call L, L' isomorphic if there exists an injective linear map $f : \mathbb{R}^n \to \mathbb{R}^{n'}$ with coefficient matrix $A \in \mathbb{R}^{n' \times n}$ such that the induced map $\tilde{f} : \mathcal{S}^n \to \mathcal{S}^{n'}$ given by $\tilde{f} : X \mapsto AXA^T$ takes L onto L'.*

Note that the linear map $\tilde{f}$ defines an isomorphism between the cone $\mathcal{S}_+^n$ and a face of $\mathcal{S}_+^{n'}$. Therefore, if $L \subset \mathcal{S}^n$, $L' \subset \mathcal{S}^{n'}$ are isomorphic subspaces, then the spectrahedral cones $K = L \cap \mathcal{S}_+^n$, $K' = L' \cap \mathcal{S}_+^{n'}$ are linearly isomorphic. For general spectrahedral cones, the isomorphism between L and L' in the sense of Definition 5 is a much stronger condition than the usual linear isomorphism between K and K'. For instance, a linear isomorphism between spectrahedral cones in general does not preserve the rank, while the map $\tilde{f}$ is rank-preserving.

In particular, the isomorphisms of linear subspaces of matrix spaces preserve Property 1 of the spectrahedral cones these subspaces define. For ROG cones, however, the rank is even an invariant of linear isomorphisms, as the following result shows.

Lemma 10. *Let $K \subset \mathcal{S}_+^n$, $K' \subset \mathcal{S}_+^{n'}$ be linearly isomorphic ROG cones, and let $I : L \to L'$ realize the linear isomorphism, where L, L' are the linear hulls of K, K', respectively. Then $\operatorname{rk} I(X) = \operatorname{rk} X$ for all $X \in K$.*

Proof. The claim follows from Lemma 8 and the fact that the Carathéodory number is an invariant under linear isomorphisms. □

We shall now show that for ROG cones, the two notions of isomorphism considered above define the same equivalence relation. The proof requires some auxiliary results on the image of the Plücker embedding of Grassmanians. We begin with results on the rank 1 completion of partially specified matrices.

Definition 6. *A real partially specified $n \times m$ matrix is defined by an index subset $\mathcal{P} \subset \{1, \dots, n\} \times \{1, \dots, m\}$, called a pattern, together with a collection of real numbers $(A_{ij})_{(i,j) \in \mathcal{P}}$. A completion of a partially specified matrix $(\mathcal{P}, (A_{ij})_{(i,j) \in \mathcal{P}})$ is a real $n \times m$ matrix C such that $C_{ij} = A_{ij}$ for all $(i, j) \in \mathcal{P}$.*

We shall be concerned with the question when a partially specified matrix possesses a completion of rank 1. This problem has been solved in [2], see also [6]. In order to formulate the result, we need to define a weighted bipartite graph G associated to the partially specified matrix. The two groups of vertices will be the row indices $1, \dots, n$ and the column indices $1, \dots, m$. The edges will be the elements of $\mathcal{P}$, with the weight of (i, j) equal to A_{ij}.

Lemma 11. *[6, Theorem 5] A partially specified matrix $(\mathcal{P}, (A_{ij})_{(i,j)\in\mathcal{P}})$ has a rank 1 completion if and only if the following conditions are satisfied. If for some $(i,j) \in \mathcal{P}$ we have $A_{ij} = 0$, then either $A_{ij'} = 0$ for all $(i,j') \in \mathcal{P}$, or $A_{i'j} = 0$ for all $(i',j) \in \mathcal{P}$. Further, for every cycle i_1-j_1-i_2-$\cdots$-i_k-j_k-i_1 of the bipartite graph G corresponding to the partially specified matrix, where i_1 in the representation of the cycle is a row index, we have $\prod_{l=1}^k A_{i_l j_l} = A_{i_k j_1} \cdot \prod_{l=1}^{k-1} A_{i_{l+1} j_l}$.*

Note that the relation in the second condition of the lemma depends only on the cycle itself, but not on its starting point or on the direction in which the edges are traversed. Since the products in the lemma are multiplicative under the concatenation of paths [6, p.2171], we may also restrict the condition to elementary cycles. Moreover, for an elementary cycle i_1-j_1-i_1 of length 2 the relation reduces to $A_{i_1 j_1} = A_{i_1 j_1}$ and is hence trivially satisfied.

Corollary 5. *Let $A = (\mathcal{P}, (A_{ij})_{(i,j)\in\mathcal{P}})$ be a partially specified matrix such that $A_{ij} = \pm 1$ for all $(i,j) \in \mathcal{P}$, and G the corresponding bipartite graph. Assume further that for every elementary cycle i_1-j_1-$\cdots$-j_k-i_1 of G with $k \geq 2$, where the representation of the cycle begins with a row index, we have $\prod_{l=1}^k A_{i_l j_l} = A_{i_k j_1} \cdot \prod_{l=1}^{k-1} A_{i_{l+1} j_l}$. Then there exists a rank 1 completion $C = ef^T$ of A such that $e \in \{-1,+1\}^n$, $f \in \{-1,+1\}^m$.*

Proof. By Lemma 11 there exists a rank 1 completion $\tilde{C} = \tilde{e}\tilde{f}^T$ of A, where $\tilde{e} \in \mathbb{R}^n$, $\tilde{f} \in \mathbb{R}^m$. Suppose there exists an index i such that $\tilde{e}_i = 0$. Then all elements of the i-th row of $\tilde{C}$ vanish, and all elements of this row are unspecified in A. We may then set $\tilde{e}_i = 1$ and $\tilde{e}\tilde{f}^T$ would still be a completion of A. Hence assume without loss of generality that all elements of $\tilde{e}$ are nonzero. In a similar manner, we may assume that the elements of $\tilde{f}$ are nonzero.

We then define the vectors $e \in \mathbb{R}^n$, $f \in \mathbb{R}^m$ element-wise by the signs of the elements of $\tilde{e}, \tilde{f}$, respectively. For every $(i,j) \in \mathcal{P}$ we then have $e_i f_j = \frac{\tilde{e}_i \tilde{f}_j}{|\tilde{e}_i \tilde{f}_j|} = \frac{A_{ij}}{|A_{ij}|} = A_{ij}$, because $A_{ij} = \pm 1$. It follows that $C = ef^T$ is also a completion of A. □

We now come to the Grassmanian $Gr(n, \mathbb{R}^m)$, i.e., the space of linear n-planes in $\mathbb{R}^m$. Fix a basis in $\mathbb{R}^m$. Then an n-plane Λ can be represented by an n-tuple of linear independent vectors in $\mathbb{R}^m$, namely those spanning Λ. Let us treat these vectors as row vectors and stack them into an $n \times m$ matrix M. The matrix M is determined only up to left multiplication by a nonsingular $n \times n$ matrix, reflecting the ambiguity in the choice of vectors spanning Λ. The *Plücker coordinate* $\Delta_{i_1 \dots i_n}$ of Λ, where $1 \leq i_1 < \cdots < i_n \leq m$, is given by the determinant of the $n \times n$ submatrix formed of the columns $i_1, \dots, i_n$ of M. The vector Δ of all Plücker coordinates is determined by the n-plane Λ up to multiplication by a nonzero constant and corresponds to a point in projective space.

Lemma 12. *Let $\Lambda, \Lambda' \subset \mathbb{R}^m$ be two n-planes with Plücker coordinate vectors Δ, Δ', respectively. Suppose there exists a positive constant c such that $|\Delta_{i_1 \dots i_n}| = c|\Delta'_{i_1 \dots i_n}|$ for all n-tuples $(i_1, \dots, i_n)$. Then there exists a linear automorphism of $\mathbb{R}^m$, given by a diagonal coefficient matrix $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_m)$, where $\sigma_i \in \{-1,+1\}$ for all $i = 1, \dots, m$, which takes the n-plane Λ to Λ'.*

Proof. Assume the conditions of the lemma. Let without restriction of generality $\Delta_{1\dots n} \neq 0$, then also $\Delta'_{1\dots n} \neq 0$. Otherwise we may permute the basis vectors of $\mathbb{R}^m$ to obtain these inequalities. Then we may choose the $n \times m$ matrix M representing Λ such that the first n columns of M form the identity matrix. Make a similar choice for the $n \times m$ matrix M' representing Λ'. Then we have $\Delta_{1\dots n} = \Delta'_{1\dots n} = 1$ and hence $c = 1$ for this choice of M, M'. If $m = n$, then we may take Σ as the identity matrix. Let $m > n$.

Let k, l be indices such that $1 \leq k \leq n$, $n < l \leq m$. The determinant $\Delta_{1,\dots,k-1,k+1,\dots,n,l}$ is then given by $(-1)^{n-k} M_{kl}$. Likewise, $\Delta'_{1,\dots,k-1,k+1,\dots,n,l} = (-1)^{n-k} M'_{kl}$, and hence $|M_{kl}| = |M'_{kl}|$ by the assumption on Δ, Δ'. We then get $|M_{kl}| = |M'_{kl}|$ also for all $k = 1, \dots, n$, $l = 1, \dots, m$.

Let now $\mathcal{P}$ be the set of index pairs (k, l) such that $M_{kl} \neq 0$, and set $A_{kl} = \frac{M'_{kl}}{M_{kl}} \in \{-1, +1\}$ for $(k, l) \in \mathcal{P}$. Then for every completion C of the partially specified matrix $A = (\mathcal{P}, (A_{kl})_{(k,l) \in \mathcal{P}})$ we have $M' = M \bullet C$, where $\bullet$ denotes the Hadamard matrix product.

We shall now show that the partially specified matrix A satisfies the condition of Corollary 5. Let i_1-j_1-$\cdots$-j_k-i_1 be an elementary cycle of the bipartite graph G corresponding to A, where $k \geq 2$, $i_1, \dots, i_k$ are row indices, and $j_1, \dots, j_k$ are column indices. Since the cycle is elementary, the row and column indices are mutually distinct. The $k \times k$ submatrix $\hat{M}$ of M consisting of elements with row indices $i_1, \dots, i_k$ and column indices $j_1, \dots, j_k$ does not have any nonzero elements except those specified by the edges of the cycle, because any such element would render the cycle non-elementary. In particular, every row and every column of $\hat{M}$ contains exactly two nonzero elements. The index set $\{j_1, \dots, j_k\}$ then has an empty intersection with $\{1, \dots, n\}$, because the first n columns of M contain strictly less than two nonzero elements each. Moreover, in the Leibniz formula for the determinant $\det \hat{M}$ only two products are nonzero, and the corresponding permutations are related by a cyclic permutation, which has sign $(-1)^{k-1}$. Therefore we have $|\det \hat{M}| = \left|\prod_{l=1}^{k} M_{i_l j_l} - (-1)^k M_{i_k j_1} \cdot \prod_{l=1}^{k-1} M_{i_{l+1} j_l}\right|$.

Consider the $n \times n$ submatrix of M consisting of columns with indices in $(\{1, \dots, n\} \setminus \{i_1, \dots, i_k\}) \cup \{j_1, \dots, j_k\}$. The determinant of this submatrix has absolute value $|\det \hat{M}|$ by construction. A similar formula holds for the absolute value of the determinant of the corresponding $n \times n$ submatrix of M'. By the assumption on Δ, Δ' we then have

$$\left|\prod_{l=1}^{k} M_{i_l j_l} - (-1)^k M_{i_k j_1} \cdot \prod_{l=1}^{k-1} M_{i_{l+1} j_l}\right| = \left|\prod_{l=1}^{k} M'_{i_l j_l} - (-1)^k M'_{i_k j_1} \cdot \prod_{l=1}^{k-1} M'_{i_{l+1} j_l}\right|.$$

It follows that either

$$\left(1 - \prod_{l=1}^{k} A_{i_l j_l}\right) \prod_{l=1}^{k} M_{i_l j_l} = \left(1 - A_{i_k j_1} \cdot \prod_{l=1}^{k-1} A_{i_{l+1} j_l}\right) (-1)^k M_{i_k j_1} \cdot \prod_{l=1}^{k-1} M_{i_{l+1} j_l}$$

or

$$\left(1 + \prod_{l=1}^{k} A_{i_l j_l}\right) \prod_{l=1}^{k} M_{i_l j_l} = \left(1 + A_{i_k j_1} \cdot \prod_{l=1}^{k-1} A_{i_{l+1} j_l}\right) (-1)^k M_{i_k j_1} \cdot \prod_{l=1}^{k-1} M_{i_{l+1} j_l}.$$

Note that all the involved elements of M are nonzero, while those of A equal ± 1. The relation $\prod_{l=1}^{k} A_{i_l j_l} = -A_{i_k j_1} \cdot \prod_{l=1}^{k-1} A_{i_{l+1} j_l}$ would then imply that in each of the two equations above, one side is zero while the other is not. Therefore we must have $\prod_{l=1}^{k} A_{i_l j_l} = A_{i_k j_1} \cdot \prod_{l=1}^{k-1} A_{i_{l+1} j_l}$, and the condition in Corollary 5 is fulfilled.

By this corollary there exists a rank 1 completion $C = ef^T$ of A such that $e \in \{-1,+1\}^n$, $f \in \{-1,+1\}^m$. We then have $M' = M \bullet (ef^T) = \mathrm{diag}(e) \cdot M \cdot \mathrm{diag}(f)$. Setting $\Sigma = \mathrm{diag}(f)$ completes the proof. □

We now are in a position to prove the main result of this subsection.

Lemma 13. *Let $K, K' \subset \mathcal{S}_+^n$ be linearly isomorphic non-degenerate ROG cones. Then their linear hulls $L, L' \subset \mathcal{S}^n$ are also isomorphic.*

Proof. Denote the determinantal polynomial on $\mathcal{S}^n$ by d. Then $p = d|_L$, $p' = d|_{L'}$ are the determinantal defining polynomials of K, K', respectively. By Theorem 1 both p, p' are minimal defining polynomials.

Let $\tilde{f} : L \to L'$ be an invertible linear map realizing the isomorphism between K and K'. By Lemma 3 there exists a positive constant $c > 0$ such that $p = c \cdot (p' \circ \tilde{f})$. Our goal is to extend $\tilde{f}$ to an automorphism of $\mathcal{S}^n$.

For every nonzero $x \in \mathbb{R}^n$ such that $xx^T \in K$, the image $\tilde{f}(xx^T) \in K'$ is a rank 1 matrix by Lemma 10. Hence there exists a vector $y \in \mathbb{R}^n$ such that $\tilde{f}(xx^T) = yy^T$. This vector is determined up to a sign. We shall now construct an automorphism f of $\mathbb{R}^n$ such that $\tilde{f}(xx^T) = f(x)f(x)^T$ for all such x.

Let $x_1, \dots, x_m \in \mathbb{R}^n$ be such that the set $\{x_i x_i^T \mid i = 1, \dots, m\}$ forms a basis of L. This is possible because K is a ROG cone. Let $y_1, \dots, y_m \in \mathbb{R}^n$ be such that $\tilde{f}(x_i x_i^T) = y_i y_i^T$ for all $i = 1, \dots, m$. By virtue of the relation $p = c \cdot (p' \circ \tilde{f})$ we then have $\det(\sum_{i=1}^m \delta_i x_i x_i^T) = c \det(\sum_{i=1}^m \delta_i y_i y_i^T)$ identically in the variables $\delta_1, \dots, \delta_m$. In particular, for every n-tuple of indices $(i_1, \dots, i_n)$ we have $\det(\sum_{k=1}^n x_{i_k} x_{i_k}^T) = c \det(\sum_{k=1}^n y_{i_k} y_{i_k}^T)$.

Assemble the column vectors x_i into an $n \times m$ matrix X and the column vectors y_i into an $n \times m$ matrix Y. The above relation is then equivalent to $\det(X_{i_1 \dots i_n} X_{i_1 \dots i_n}^T) = c \det(Y_{i_1 \dots i_n} Y_{i_1 \dots i_n}^T)$, where $X_{i_1 \dots i_n}, Y_{i_1 \dots i_n}$ are the $n \times n$ submatrices formed of the columns $i_1, \dots, i_n$ of X, Y, respectively. This finally yields $|\det X_{i_1 \dots i_n}| = \sqrt{c} |\det Y_{i_1 \dots i_n}|$ for all n-tuples $(i_1, \dots, i_n)$.

Thus the n-planes spanned in $\mathbb{R}^m$ by the row vectors of X, Y, respectively, fulfill the conditions of Lemma 12. By this lemma there exist a nonsingular $n \times n$ matrix S and a diagonal matrix $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_m)$ with $\sigma_i \in \{-1,+1\}$ such that $Y = SX\Sigma$, or equivalently $Y\Sigma = SX$.

Define a linear automorphism f of $\mathbb{R}^n$ by the coefficient matrix S. Then we have for every $i = 1, \dots, m$ that $f(x_i) = Sx_i = \sigma_i y_i$, and hence $f(x_i)f(x_i)^T = y_i y_i^T = \tilde{f}(x_i x_i^T)$. Thus the automorphism $A \mapsto SAS^T$ of $\mathcal{S}^n$ which is generated by f extends the map $\tilde{f}$ between the subspaces L, L'. In particular, it defines an isomorphism between L and L'. □

Theorem 2. *Let $K \subset \mathcal{S}_+^n, K' \subset \mathcal{S}_+^{n'}$ be linearly isomorphic ROG cones. Then their linear hulls L, L' are also isomorphic in the sense of Definition 5.*

Proof. Let without loss of generality $n \leq n'$ and denote by k the degree of K and K'. Let $H \subset \mathbb{R}^n$, $H' \subset \mathbb{R}^{n'}$ be the images of arbitrary matrices in the interiors of K, K', respectively. By Corollary 1 we have $\dim H = \dim H' = k$. The minimal faces of $\mathcal{S}_+^n, \mathcal{S}_+^{n'}$ containing

K, K', respectively, are given by $\mathcal{F}_n(H)$ and $\mathcal{F}_{n'}(H')$. The linear hulls of $\mathcal{F}_n(H), \mathcal{F}_{n'}(H')$ are given by $\mathcal{L}_n(H)$ and $\mathcal{L}_{n'}(H')$, respectively, which are both isomorphic to the space $\mathcal{S}^k$.

Let us view the cones K, K' as subsets of $\mathcal{S}^k$ by means of the corresponding isomorphisms. Then by Lemma 13 there exists an automorphism of $\mathcal{S}^k$ which takes L onto L'. In other words, there exists an invertible linear map $f : H \to H'$ such that $f(x)f(x)^T \in L'$ for all $x \in H$ such that $xx^T \in L$. Let us extend this linear map to an injective linear map $\varphi : \mathbb{R}^n \to \mathbb{R}^{n'}$, and define by Φ its coefficient matrix. By construction, we then have $\Phi xx^T\Phi^T = f(x)f(x)^T \in L'$ for all $x \in \mathbb{R}^n$ such that $xx^T \in L$. Thus the map $X \mapsto \Phi X\Phi^T$ is the sought isomorphism of L and L'. □

Theorem 2 states that the geometry of a ROG cone as a subset of real space determines its representations as linear sections of a positive semi-definite matrix cone uniquely up to isomorphisms as in Definition 5. Of course, this does not preclude the existence of nonisomorphic representations as a spectrahedral cone, but in these the cone will not be ROG. In the sequel, when we speak of a representation of a ROG cone, we will always mean a spectrahedral representation where the cone is ROG.

Corollary 6. *Let $K, K' \subset \mathcal{S}_+^n$ be linearly isomorphic ROG cones. Then the conic subsets $X_K = \{x \in \mathbb{R}^n \,|\, xx^T \in K\}$ and $X_{K'} = \{x \in \mathbb{R}^n \,|\, xx^T \in K'\}$ of $\mathbb{R}^n$ are linearly isomorphic and hence define projectively equivalent varieties in real projective space $\mathbb{R}P^{n-1}$.*

Proof. Let $L, L' \subset \mathcal{S}^n$ be the linear spans of K, K', respectively. By Theorem 2 there exists a linear automorphism f of $\mathbb{R}^n$ with coefficient matrix A, such that L' is the image of L under the automorphism $\tilde{f}$ of $\mathcal{S}^n$ given by $\tilde{f} : X \mapsto AXA^T$.

Let $x \in X_K$. Then $xx^T \in L$, and hence $\tilde{f}(xx^T) = (Ax)(Ax)^T = f(x)f(x)^T \in L'$. Since $f(x)f(x)^T$ is positive semi-definite, we also have $f(x)f(x)^T \in K'$ and hence $f(x) \in X_{K'}$. It follows that $f[X_K] \subset X_{K'}$. In the same way one proves that $f^{-1}[X_{K'}] \subset X_K$, which implies that f is the sought isomorphism between X_K and $X_{K'}$. □

3 Construction of new ROG cones from given ones

In this section we consider several ways to construct ROG spectrahedral cones of higher degree from given ones. By iterating these procedures, one may construct ROG cones of arbitrarily high complexity.

3.1 Direct sums

In this subsection we consider direct sums of ROG cones and introduce the notion of a simple ROG cone[2]. The following definition is standard.

[2]We propose to reserve the notion *irreducible* for ROG cones $K \subset \mathcal{S}_+^n$ such that the real projective variety defined by the set $\{x \in \mathbb{R}^n \,|\, xx^T \in K\}$ is irreducible.

Definition 7. *Let $K \subset \mathbb{R}^n$, $K' \subset \mathbb{R}^{n'}$ be convex cones. Their direct sum $K \oplus K'$ is defined as the set $\{(x, x') \in \mathbb{R}^n \oplus \mathbb{R}^{n'} \mid x \in K,\ x' \in K'\}$.*

Let $A \in \mathcal{S}^n$, $A' \in \mathcal{S}^{n'}$ be matrices. Their direct sum $A \oplus A' \in \mathcal{S}^{n+n'}$ is defined as the block-diagonal matrix $\operatorname{diag}(A, A')$.

These notions naturally extend to an arbitrary number of factors.

If $K = L \cap \mathcal{S}_+^n$, $K' = L' \cap \mathcal{S}_+^{n'}$ are spectrahedral cones, then their direct sum is linearly isomorphic to a spectrahedral cone. The isomorphism $K \oplus K' \cong (L \oplus L') \cap \mathcal{S}_+^{n+n'}$ assigns the block-diagonal matrix $A \oplus A' \in (L \oplus L') \cap \mathcal{S}_+^{n+n'}$ to the element $(A, A') \in K \oplus K'$. In other words, direct sums of spectrahedral cones are spectrahedral cones with corresponding block-diagonal matrix representations. We shall show that for ROG cones also the converse is true, i.e., if a ROG cone has a block-diagonal representation, then it is the direct sum of the ROG cones defined by the individual blocks.

First we show the forward implication for ROG cones.

Lemma 14. *Let $K_1, \dots, K_m$ be ROG cones of degrees $n_1, \dots, n_m$. Then their direct sum $K = \oplus_{k=1}^m K_k$ is also a ROG cone, which has degree $n = \sum_{k=1}^m n_k$. If $K_k = L_k \cap \mathcal{S}_+^{n_k'}$ are representations of K_k, then the block-diagonal representation of $K = \oplus_{k=1}^m K_k$ constructed from these representations of K_k is also ROG. In particular the cone K possesses a block-diagonal non-degenerate representation with block sizes $n_1, \dots, n_m$, such that block k defines a non-degenerate representation of K_k.*

Proof. Let $K_k = L_k \cap \mathcal{S}_+^{n_k'}$ be arbitrary representations of the cones K_k as ROG cones. Let $X = \oplus_{k=1}^m X_k = \operatorname{diag}(X_1, \dots, X_m)$ be an arbitrary element of K in the corresponding block-diagonal representation, where $X_k \in K_k$. Since the factor cones K_k are ROG, every X_k decomposes into a sum of rank 1 matrices $r_{k,j} \in K_k$, $j = 1, \dots, \eta_k$. For every such rank 1 matrix $r_{k,j}$, the matrix $R_{k,j} = \operatorname{diag}(0, \dots, 0, r_{k,j}, 0, \dots, 0)$ is a rank 1 matrix in K, where the non-zero block is located at position k. Then $X = \sum_{k=1}^m \sum_{j=1}^{\eta_k} R_{k,j}$, which proves that K is ROG.

By Corollary 2, for every k the cone K_k has a non-degenerate representation as a linear section of $\mathcal{S}_+^{n_k}$. The ROG spectrahedral representation of K given by the corresponding block-diagonal representation has the required block structure. It is also non-degenerate, because a block-diagonal matrix with all blocks being positive definite is itself positive definite. Hence K has degree $n = \sum_{k=1}^m n_k$ by Corollary 1. □

Corollary 7. *Let $K_1, \dots, K_m$ be regular convex cones and $K = \oplus_{k=1}^m K_k$ their direct sum. Then K is ROG if and only if all cones K_k are ROG.*

Proof. If the K_k are ROG, then K is ROG by Lemma 14. Let K be ROG. Each of the cones K_k is isomorphic to a face of K, and hence ROG by Lemma 6. □

Consider a ROG cone which is a direct sum of other cones. The next result shows that every ROG spectrahedral representation of such a cone can be brought to a corresponding block-diagonal form by an appropriate choice of the coordinate system.

Lemma 15. *Let the ROG cone $K = \oplus_{k=1}^m K_k$ be a direct sum of lower-dimensional cones. Then the factor cones K_k are ROG. Moreover, for every representation $K \subset \mathcal{S}_+^n$ there exists a direct sum decomposition $\mathbb{R}^n = \oplus_{k=1}^m H_k$ into subspaces of dimensions $\dim H_k \geq \deg K_k$ such that the intersection $F_k = \mathcal{L}_n(H_k) \cap K$ is linearly isomorphic to K_k for all $k = 1, \dots, m$, and $K = \sum_{k=1}^m F_k$. If $n = \deg K$, then $\dim H_k = \deg K_k$ for $k = 1, \dots, m$.*

Proof. By Corollary 7 each factor cone K_k is ROG, denote its degree by n_k.

By Lemma 14 we have $\deg K = \sum_{k=1}^m n_k$ and K possesses a block-diagonal representation as a linear section of $\mathcal{S}_+^{\deg K}$ with block sizes n_k, such that block k defines a representation of the factor cone K_k. By Theorem 2 an arbitrary representation of K as a linear section of $\mathcal{S}_+^n$ is isomorphic to this block-diagonal representation. Let $f : \mathbb{R}^{\deg K} \to \mathbb{R}^n$ be the injective linear map from Definition 5 which defines the isomorphism, and denote by $H \subset \mathbb{R}^n$ the image of f. The map f then puts the direct sum decomposition of $\mathbb{R}^{\deg K}$ defined by the block structure of the block-diagonal representation in correspondence to some direct sum decomposition $H = \oplus_{k=1}^m H_k'$, where $\dim H_k' = \deg K_k$. Let $\mathbb{R}^n = \oplus_{k=1}^m H_k$ be an arbitrary direct sum decomposition such that $H_k' \subset H_k$ for all $k = 1, \dots, m$. By construction this decomposition has the required properties.

If $n = \deg K$, then f is bijective, and $H_k = H_k'$ is the only possible choice for H_k. It follows that $\dim H_k = \deg K_k$ in this case. □

On the other hand, a ROG cone possessing a block-diagonal representation is isomorphic to the direct sum of the ROG cones defined by the individual blocks.

Lemma 16. *Let $K = L \cap \mathcal{S}_+^n$ be a ROG cone. Let $\mathbb{R}^n = H_1 \oplus \dots \oplus H_m$ be a direct sum decomposition of $\mathbb{R}^n$ and suppose that $L \subset \sum_{k=1}^m \mathcal{L}_n(H_k)$. Then K is the sum of the ROG cones $K_k = K \cap \mathcal{L}_n(H_k)$, $k = 1, \dots, m$, and is canonically isomorphic to their direct sum.*

Proof. First note that the cones K_k are faces of K and hence indeed ROG cones by Lemma 6. Moreover, the sum $\sum_{k=1}^m K_k$ is canonically isomorphic to the direct sum $\oplus_{k=1}^m K_k$, because we have $\dim(\sum_{k=1}^m \mathcal{L}_n(H_k)) = \sum_{k=1}^m \dim \mathcal{L}_n(H_k)$.

Clearly $\sum_{k=1}^m K_k \subset K$, because $K_k \subset K$ for all k and K is a convex cone.

Let now $X \in K$ be arbitrary. By Property 1 there exist rank 1 matrices $X_i \in K$, $i = 1, \dots, N$, such that $X = \sum_{i=1}^N X_i$. Now for every i we have $X_i \in L \subset \sum_{k=1}^m \mathcal{L}_n(H_k)$. Since X_i is rank 1, there must exist $k_i \in \{1, \dots, m\}$ such that $X_i \in \mathcal{L}_n(H_{k_i})$. It follows that $X_i \in \mathcal{L}_n(H_{k_i}) \cap K = K_{k_i}$. Therefore $X \in \sum_{k=1}^m K_k$, and hence $K \subset \sum_{k=1}^m K_k$.

Thus we get $K = \sum_{k=1}^m K_k$, which completes the proof. □

Definition 8. *We call a ROG cone K simple if it is not isomorphic to a nontrivial direct sum of lower-dimensional cones.*

The decomposition of K into simple factor cones is unique up to permutation of the factors.

Lemma 17. *A non-degenerate ROG cone $K \subset \mathcal{S}_+^n$ is simple if and only if there does not exist a nontrivial decomposition $\mathbb{R}^n = H_1 \oplus \dots \oplus H_m$ such that $\operatorname{span} K \subset \sum_{k=1}^m \mathcal{L}_n(H_k)$.*

Proof. If K is not simple, then Lemma 15 applies with a nontrivial direct sum decomposition of $\mathbb{R}^n$. The assertion of this lemma then implies $\operatorname{span} K \subset \sum_{k=1}^m \mathcal{L}_n(H_k)$.

On the other hand, if there exists a nontrivial decomposition $\mathbb{R}^n = H_1 \oplus \cdots \oplus H_m$ such that $\operatorname{span} K \subset \sum_{k=1}^m \mathcal{L}_n(H_k)$, then by Lemma 16 K is isomorphic to the direct sum of the ROG cones $K_k = K \cap \mathcal{L}_n(H_k)$, $k = 1, \dots, m$. Since $K \subset \mathcal{S}_+^n$ is non-degenerate, we have $\deg K_k = \dim H_k > 0$ for all k, and the direct sum is nontrivial. □

Lemma 18. *Let $K \subset \mathcal{S}_+^n$ be a non-degenerate ROG cone. Then there exists a unique (up to a permutation of factors) direct sum decomposition $\mathbb{R}^n = H_1 \oplus \cdots \oplus H_m$ such that K is the sum of the faces $K_k = \mathcal{L}_n(H_k) \cap K$, and such that the factor cones K_k are simple.*

Proof. The claim of the lemma follows from Lemma 15, applied to the unique decomposition of K into simple factor cones, and the fact that the subspaces H_k are uniquely determined by the faces K_k representing the factor cones. □

Thus there are two different criteria that allow to check whether a ROG cone K is composed, i.e., not simple. On the one hand, one may consider the geometric decomposition of K into factor cones. On the other hand, one has the algebraic criterion whether in a non-degenerate representation, K is contained in the sum of two complementary faces of the ambient matrix cone. This second criterion is not valid for general spectrahedral cones, as for example the 1-dimensional cone generated by the identity matrix shows.

3.2 Full extensions

In this subsection we consider a special class of ROG cones which are essentially determined by ROG cones of smaller degree.

Let $K = L \cap \mathcal{S}_+^n$ be a spectrahedral cone, and suppose that there exists a linear subspace $E \subset \mathbb{R}^n$ of dimension k such that all matrices of the form $xy^T + yx^T$ for $x \in \mathbb{R}^n$, $y \in E$ are contained in L. Denote the linear subspace spanned by these matrices by L_E. Let $H \subset \mathbb{R}^n$ be an $(n-k)$-dimensional linear subspace which is complementary to E. Then we have the decompositions $\mathcal{S}^n = \mathcal{L}_n(H) \oplus L_E$, $L = (L \cap \mathcal{L}_n(H)) \oplus L_E$.

The next result shows that the face $K_H = \mathcal{L}_n(H) \cap K = (L \cap \mathcal{L}_n(H)) \cap \mathcal{S}_+^n$ of K is essentially independent of the choice of the subspace H.

Lemma 19. *Assume above notations. Let $E \subset \mathbb{R}^n$ be a k-dimensional subspace. Let $H, H' \subset \mathbb{R}^n$ be $(n-k)$-dimensional subspaces which are complementary to E. Then the intersections $L \cap \mathcal{L}_n(H), L \cap \mathcal{L}_n(H') \subset \mathcal{S}^n$ are isomorphic.*

Proof. Let Π be the canonical projection on the quotient space $\mathbb{R}^n/E$. Then the restrictions $\Pi|_H, \Pi|_{H'}$ are isomorphisms between H, H' and $\mathbb{R}^n/E$. There exists a unique automorphism f of $\mathbb{R}^n$ such that $f|_H = (\Pi|_{H'})^{-1} \circ \Pi|_H$ and $f|_E = Id_E$. In other words, f is the unique automorphism that projects H onto H' along E and leaves E point-wise invariant.

The automorphism f induces an automorphism $\tilde{f}$ of $\mathcal{S}^n$ by $X \mapsto AXA^T$, where A is the coefficient matrix of f. Let us show that $\tilde{f}$ defines the sought isomorphism between $\mathcal{L}_n(H) \cap L$ and $\mathcal{L}_n(H') \cap L$.

Since $f[H] = H'$, we have $\tilde{f}[\mathcal{L}_n(H)] = \mathcal{L}_n(H')$. Let now $x \in \mathbb{R}^n$ be arbitrary, and denote by $d \in E$ the difference $f(x) - x$. Then we have $\tilde{f}(xx^T) = (x+d)(x+d)^T = xx^T + xd^T + dx^T + dd^T$. It follows that $\tilde{f}(xx^T) - xx^T \in L_E$. By linearity we get $\tilde{f}(X) - X \in L_E$ for every matrix $X \in \mathcal{S}^n$. By virtue of the inclusion $L_E \subset L$, for every $X \in L$ we then have $\tilde{f}(X) \in L$. It follows that $\tilde{f}[L] = L$.

Therefore $\tilde{f}[\mathcal{L}_n(H) \cap L] = \mathcal{L}_n(H') \cap L$, which proves our claim. □

It follows that the faces $K_H = \mathcal{L}_n(H) \cap K$, $K_{H'} = \mathcal{L}_n(H') \cap K$ defined above and their representations as linear sections of $\mathcal{S}_+^n$ are isomorphic. Moreover, since $L = (\mathcal{L}_n(H) \cap L) \oplus L_E$, the knowledge of the cone $K_H \subset \mathcal{L}_n(H)$ is sufficient to recover the cone $K \subset \mathcal{S}_+^n$.

Definition 9. *Let k, n be positive integers, $k < n$, and let $K' = L' \cap \mathcal{S}_+^{n-k}$, $K = L \cap \mathcal{S}_+^n$ be spectrahedral cones. We call K a full extension of K' if there exists a direct sum decomposition $\mathbb{R}^n = H \oplus E$ into subspaces of dimensions $n-k, k$, respectively, and a corresponding direct sum decomposition of $\mathcal{S}^n$ into subspaces $L_E = \operatorname{span}\{xy^T + yx^T \mid x \in \mathbb{R}^n,\ y \in E\}$, $\mathcal{L}_n(H)$, with the following properties. The inclusion $L_E \subset L$ holds, and the subspaces $L' \subset \mathcal{S}^{n-k}$, $\mathcal{L}_n(H) \cap L \subset \mathcal{S}^n$ are isomorphic.*

A spectrahedral cone $K \subset \mathcal{S}_+^n$ is hence a full extension of some other spectrahedral cone if and only if there exists a linear subspace $L \subset \mathcal{S}^n$ and a nonzero vector $y \in \mathbb{R}^n$ such that $K = L \cap \mathcal{S}_+^n$ and $xy^T + yx^T \in L$ for all $x \in \mathbb{R}^n$. It is also easy to see that $\dim K = \dim K' + (\max_{X \in K'} \operatorname{rk} X)k + \frac{k(k+1)}{2}$, and $K \subset \mathcal{S}_+^n$ is non-degenerate if and only if $K' \subset \mathcal{S}_+^{n-k}$ is non-degenerate.

Lemma 20. *Let $K = L \cap \mathcal{S}_+^n$ be a full extension of $K' = L' \cap \mathcal{S}_+^{n-k}$. Then K is ROG if and only if K' is ROG.*

Proof. Assume the notations of Definition 9.

If K is ROG, then the face $K_H = \mathcal{L}_n(H) \cap K = (\mathcal{L}_n(H) \cap L) \cap \mathcal{S}_+^n$ of K is also ROG by Lemma 6. The isomorphism between L' and $\mathcal{L}_n(H) \cap L$ induces an isomorphism between K' and K_H, and hence K' is also ROG. This proves one direction of the equivalence.

Adopt a basis of $\mathbb{R}^n$ such that H is spanned by the first $n-k$ basis vectors, and E by the last k basis vectors. Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix} \in K$ be an arbitrary matrix, with the partition adapted to the decomposition $\mathbb{R}^n = H \oplus E$. Define the matrix $Y = \begin{pmatrix} X_{11} & 0 \\ 0 & 0 \end{pmatrix}$. Then we have $Y \succeq 0$ by virtue of $X \succeq 0$. Moreover, $X - Y \in L_E \subset L$, and hence $Y = X - (X - Y) \in L$. It follows that $Y \in K_H$.

Assume now that K' is ROG. Since L' and $\mathcal{L}_n(H) \cap L$ are isomorphic, the face K_H is also ROG. Therefore Y is representable as a sum of rank 1 matrices in K_H. In other words, there exist vectors $v_1, \dots, v_N \in \mathbb{R}^{n-k}$ such that $X_{11} = \sum_{i=1}^N v_i v_i^T$ and $V_i = \begin{pmatrix} v_i v_i^T & 0 \\ 0 & 0 \end{pmatrix} \in L$ for all i. Let V be the $(n-k) \times N$ matrix formed of the column vectors v_i. The condition $X \succeq 0$ implies that the columns of X_{12} are in the image of $X_{11} = VV^T$. Therefore there exists a $k \times N$ matrix W such that $X_{12} = VW^T$. Let the columns of W be $w_1, \dots, w_N \in \mathbb{R}^k$. We then have the representation

$$X = \begin{pmatrix} V \\ W \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}^T + \begin{pmatrix} 0 & 0 \\ 0 & X_{22} - WW^T \end{pmatrix} = \sum_{i=1}^N \begin{pmatrix} v_i \\ w_i \end{pmatrix} \begin{pmatrix} v_i \\ w_i \end{pmatrix}^T + \begin{pmatrix} 0 & 0 \\ 0 & X_{22} - WW^T \end{pmatrix}.$$

Denote the rank 1 matrix $\begin{pmatrix} v_i \\ w_i \end{pmatrix} \begin{pmatrix} v_i \\ w_i \end{pmatrix}^T$ by U_i, $i = 1, \dots, N$. We have $U_i - V_i \in L_E \subset L$, and hence $U_i \in L$ for all i. The $k \times k$ matrix $X_{22} - WW^T$ is the Schur complement of X_{11} in X and is hence positive semi-definite. It can then we written as a sum $\sum_{j=1}^{N'} z_j z_j^T$ with $z_j \in \mathbb{R}^k$. The rank 1 matrices $Z_j = \begin{pmatrix} 0 & 0 \\ 0 & z_j z_j^T \end{pmatrix}$ are also in L_E, and hence $X = \sum_{i=1}^N U_i + \sum_{j=1}^{N'} Z_j$ is a sum of rank 1 matrices in K. This shows that K is also ROG and proves the other direction of the equivalence. $\square$

Note that the full extension of a ROG cone is simple.

3.3 Intertwinings

In this subsection we present a way to construct new ROG cones from pairs of given ROG cones of smaller degree.

Definition 10. *Let $K = L \cap \mathcal{S}_+^n$ be a spectrahedral cone. We call a face F of K full if it is also a face of $\mathcal{S}_+^n$. The number $k = \max_{X \in F} \operatorname{rk} X$ is called the rank of the face.*

A face F of a ROG cone is full if and only if $\dim F = \frac{\deg F(\deg F + 1)}{2}$. Indeed, F is a linear section of the minimal face S of $\mathcal{S}_+^n$ which contains F. Hence $F = S$ if and only if $\dim F = \dim S$. But $\dim S = \frac{\deg F(\deg F+1)}{2}$ by Corollary 1.

We shall need the following auxiliary result.

Lemma 21. *Let $M = \begin{pmatrix} A & B & 0 \\ B^T & C & D \\ 0 & D^T & E \end{pmatrix}$ be a block-partitioned positive semi-definite matrix. Then there exists a decomposition $C = C_1 + C_2$ such that the matrices $\begin{pmatrix} A & B \\ B^T & C_1 \end{pmatrix}$, $\begin{pmatrix} C_2 & D \\ D^T & E \end{pmatrix}$ are positive semi-definite.*

Proof. The Schur complement of A in M is given by $\begin{pmatrix} C - B^T A^\dagger B & D \\ D^T & E \end{pmatrix}$ and is positive semi-definite. Here $A^\dagger$ is the pseudo-inverse of A, which is also positive semi-definite. Setting $C_1 = B^T A^\dagger B$, $C_2 = C - B^T A^\dagger B$ yields the desired decomposition. $\square$

Lemma 22. *Let F_1, F_2 be faces of the positive semi-definite matrix cone $\mathcal{S}_+^n$ and L_1, L_2 their linear hulls. Let $L \subset \mathcal{S}^n$ be a linear subspace such that $L_1 \cap L_2 \subset L = (L \cap L_1) + (L \cap L_2)$. Then the spectrahedral cone $K = L \cap \mathcal{S}_+^n$ equals the sum of its faces $K_1 = L_1 \cap K$, $K_2 = L_2 \cap K$.*

Proof. Clearly $K_1 + K_2 \subset K$, because K is a convex cone.

There exist linear subspaces $H_1, H_2 \subset \mathbb{R}^n$ such that $L_i = \mathcal{L}_n(H_i)$, $i = 1, 2$. Set $H = H_1 \cap H_2$. Then $L_1 \cap L_2 = \mathcal{L}_n(H)$. Introduce a direct sum decomposition $\mathbb{R}^n = H_1' \oplus H \oplus H_2' \oplus H_0$ such that $H_1 = H_1' \oplus H$, $H_2 = H \oplus H_2'$. Adopt a coordinate system in $\mathbb{R}^n$ which is adapted to this decomposition and partition the matrices in $\mathcal{S}^n$ accordingly. Then every matrix in $L_1 + L_2$, and hence also in L, has the form $X = \begin{pmatrix} X_{11} & X_{12} & 0 & 0 \\ X_{12}^T & X_{22} & X_{23} & 0 \\ 0 & X_{23}^T & X_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Moreover, every matrix whose only nonzero block is X_{22} is in $L_1 \cap L_2$ and hence in L.

Let now $X \in K$ be an arbitrary matrix, partitioned as above. By Lemma 21 there exists a decomposition $X_{22} = X_{22,1} + X_{22,2}$ such that the matrices

$$X_1 = \begin{pmatrix} X_{11} & X_{12} & 0 & 0 \\ X_{12}^T & X_{22,1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & X_{22,2} & X_{23} & 0 \\ 0 & X_{23}^T & X_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are positive semi-definite. On the other hand, since $X \in L = (L \cap L_1) + (L \cap L_2)$, there exists a decomposition $X = X_3 + X_4$ such that

$$X_3 = \begin{pmatrix} X_{11} & X_{12} & 0 & 0 \\ X_{12}^T & X_{22,3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in L \cap L_1, \quad X_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & X_{22,4} & X_{23} & 0 \\ 0 & X_{23}^T & X_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in L \cap L_2.$$

We have $D_1 = X_1 - X_3 \in L_1 \cap L_2 \subset L$, $D_2 = X_2 - X_4 \in L_1 \cap L_2 \subset L$. Hence $X_1 = D_1 + X_3 \in L_1 \cap L$, $X_2 = D_2 + X_4 \in L_2 \cap L$. It follows that $X_1 \in K_1$, $X_2 \in K_2$. Therefore $X = X_1 + X_2 \in K_1 + K_2$. Thus $K \subset K_1 + K_2$, which completes the proof. □

Lemma 23. *Assume the conditions of Lemma 22. Then K is a ROG cone if and only if K_1, K_2 are ROG cones.*

Proof. If K is ROG, then K_1, K_2 are ROG by Lemma 6.

Assume that K_1, K_2 are ROG, and let $X \in K$ be arbitrary. Since $K = K_1 + K_2$, there exist $X_1 \in K_1$, $X_2 \in K_2$ such that $X = X_1 + X_2$. Since K_1, K_2 are ROG, both X_1 and X_2 can be represented as a sum of rank 1 matrices in K_1 and K_2, respectively. Hence X can be represented as a sum of rank 1 matrices in $K_1 \cup K_2 \subset K$. Thus K is ROG. □

Lemma 22 hence presents a way to construct new ROG cones from pairs of lower-dimensional ROG cones. Note that in this lemma both faces K_1, K_2 of K contain the intersection $F = F_1 \cap F_2$ of faces of $\mathcal{S}_+^n$. The face F is hence a full face of both K_1 and K_2.

Let us now start from two ROG cones K_1, K_2, each of which has a full face of the same rank k. Denote these faces by Φ_1, Φ_2. Both faces are isomorphic to $\mathcal{S}_+^k$ and hence they are also mutually isomorphic. Let $f : \operatorname{span}\Phi_1 \to \operatorname{span}\Phi_2$ be an arbitrary isomorphism between Φ_1, Φ_2. Set $n = \deg K_1 + \deg K_2 - k$ and choose linear subspaces $H_1, H_2 \subset \mathbb{R}^n$ such that $\dim H_1 = \deg K_1$, $\dim H_2 = \deg K_2$, $\dim H = k$, where $H = H_1 \cap H_2$. Then $\mathcal{F}_n(H_1) \cap \mathcal{F}_n(H_2) = \mathcal{F}_n(H)$ is a face of $\mathcal{S}_+^n$ of degree k, and $\mathcal{L}_n(H_1) \cap \mathcal{L}_n(H_2) = \mathcal{L}_n(H)$.

We may then embed K_1, K_2 as linear sections of the faces $\mathcal{F}_n(H_1), \mathcal{F}_n(H_2)$ in a way such that both full faces Φ_1, Φ_2 are represented by $\mathcal{F}_n(H)$, and that $X_1 \in \Phi_1, X_2 \in \Phi_2$ are represented by the same matrix in $\mathcal{F}_n(H)$ if and only if $f(X_1) = X_2$. Then we have $\operatorname{span} K_i \subset \mathcal{L}_n(H_i)$, $i = 1, 2$, and $\operatorname{span} K_1 \cap \operatorname{span} K_2 = \mathcal{L}_n(H_1) \cap \mathcal{L}_n(H_2)$. Set $L = \operatorname{span} K_1 + \operatorname{span} K_2$. Then the conditions of Lemma 22 are fulfilled and by Lemma 23 the sum $K = K_1 + K_2 = L \cap \mathcal{S}_+^n$ is also a ROG cone. The next result shows that the isomorphism class of the cone K depends only on f.

Lemma 24. *Assume above notations. Then the cone K is isomorphic to a linear projection of the direct sum $K_1 \oplus K_2$. The kernel of this projection is given by pairs $(X_1, X_2) \in \operatorname{span} K_1 \oplus \operatorname{span} K_2$ such that $X_1 \in \operatorname{span}\Phi_1$, $X_2 \in \operatorname{span}\Phi_2$, and $f(X_1) + X_2 = 0$.*

Proof. The first claim follows from Lemma 22, by defining the projection by $\Pi : (X_1, X_2) \to X_1 + X_2$. Now $\operatorname{span} K_1 \cap \operatorname{span} K_2 = \mathcal{L}_n(H)$, and hence for every $(X_1, X_2) \in \operatorname{span} K_1 \oplus \operatorname{span} K_2$ such that $X_1 + X_2 = 0$ we must have $X_1 \in \operatorname{span}\Phi_1$, $X_2 \in \operatorname{span}\Phi_2$. Further, $X_1 + X_2 = 0$ if and only if X_1 and $-X_2$ are represented by the same element of $\mathcal{L}_n(H)$. This is the case if and only if $f(X_1) = -X_2$. $\square$

Definition 11. *Let K_1, K_2 be ROG cones, let $\Phi_1 \subset K_1$, $\Phi_2 \subset K_2$ be full faces of rank k, and let $f : \operatorname{span}\Phi_1 \to \operatorname{span}\Phi_2$ be an isomorphism between Φ_1, Φ_2. Define the linear subspace $\Lambda = \{(X_1, X_2) \,|\, X_1 \in \operatorname{span}\Phi_1,\ X_2 \in \operatorname{span}\Phi_2,\ f(X_1) + X_2 = 0\}$ of the direct sum $\operatorname{span} K_1 \oplus \operatorname{span} K_2$. Then the projection K of the direct sum $K_1 \oplus K_2$ on the quotient space $(\operatorname{span} K_1 \oplus \operatorname{span} K_2)/\Lambda$ is called an intertwining of K_1, K_2.*

Since the intertwining of two ROG cones K_1, K_2 depends on the choice of the full faces $\Phi_1 \subset K_1$, $\Phi_2 \subset K_2$ as well as on the isomorphism f between these faces, there can be many non-isomorphic such intertwinings for given cones K_1, K_2. We shall give two examples in the next section.

Note that a 1-dimensional face of a ROG cone is generated by a rank 1 matrix and hence is always full. We obtain the following result.

Corollary 8. *Let K_1, K_2 be ROG cones and $X_1 \in K_1$, $X_2 \in K_2$ rank 1 matrices. Define the 1-dimensional subspace $\Lambda \subset \operatorname{span} K_1 \oplus \operatorname{span} K_2$ as the linear span of the element $(X_1, -X_2)$. Then the image of the sum $K_1 \oplus K_2$ under the natural projection $\Pi : \operatorname{span} K_1 \oplus \operatorname{span} K_2 \to (\operatorname{span} K_1 \oplus \operatorname{span} K_2)/\Lambda$ is a ROG cone.*

Proof. There exists a unique linear map $f : \operatorname{span} X_1 \to \operatorname{span} X_2$ such that $f(X_1) = X_2$, and this map defines an isomorphism between the 1-dimensional faces defined by X_1, X_2. The corollary now follows from Lemmas 23 and 24. $\square$

Note that a direct sum of ROG cones can also be seen as an intertwining, defined by virtue of the 0-dimensional full face of the factor cones.

4 Examples of ROG cones

In this section we consider two nontrivial examples of ROG cones. We show that the class of ROG cones defined by chordal graphs can be constructed from the full matrix cones $\mathcal{S}_+^k$ by applying the constructive procedures presented in the previous section. We also provide an example of a continuous family of mutually non-isomorphic ROG cones.

4.1 Cones defined by chordal graphs

In this subsection we consider spectrahedral cones $K_G = L_G \cap \mathcal{S}_+^n$ defined by linear subspaces of the form $L_G = \{X \in \mathcal{S}^n \mid X_{ij} = 0 \; \forall \, (i,j) \not\in E(G)\}$, where $E(G)$ is the edge set of a graph G on the vertices $1, \dots, n$. Note that the identity matrix is an element of K_G. Hence K_G has a nonempty intersection with the interior of $\mathcal{S}_+^n$, and the linear span of K_G equals L_G.

Lemma 25. *[1, Theorem 2.3], [12, Theorem 2.4] Assume above notations. Then the cone K_G is ROG if and only if the graph G is chordal.*

Chordal graphs are characterized by the condition that they admit a *perfect elimination ordering* of the vertices $1, \dots, n$. This is an ordering such that for every $k = 1, \dots, n$, the subset $N_k = \{l < k \mid (l,k) \in E(G)\} \cup \{k\}$ of vertices forms a *clique*, i.e., the subgraph of G defined by N_k is complete.

Lemma 26. *Let G be a chordal graph with vertex set $\{1, \dots, n\}$, and let K_G be the corresponding ROG cone. Then K_G can be constructed out of full matrix cones by iterated intertwinings or taking direct sums.*

Proof. Assume that the vertices are arranged in a perfect elimination ordering. For a subset $I \subset \{1, \dots, n\}$ of indices, define the linear subspace $H_I = \{x \in \mathbb{R}^n \mid x_i = 0 \; \forall \, i \not\in I\}$. For $k = 1, \dots, n$, set $K_k = K_G \cap \mathcal{F}_n(H_{\{1,\dots,k\}})$.

Note that K_1 is isomorphic to the full matrix cone $\mathcal{S}_+^1$. We shall now show for all $k = 2, \dots, n$ that the cone K_k is either an intertwining of K_{k-1} with a full matrix cone, or a direct sum $K_{k-1} \oplus \mathcal{S}_+^1$.

Since G is chordal, the set $N_k = \{l < k \mid (l,k) \in E(G)\} \cup \{k\}$ and its subset $N_k' = \{l < k \mid (l,k) \in E(G)\}$ define cliques of G. Therefore the faces $\mathcal{F}_n(H_{N_k}), \mathcal{F}_n(H_{N_k'})$ of $\mathcal{S}_+^n$ are contained in K and are full faces of this cone. In particular, $\mathcal{F}_n(H_{N_k'})$ is a full face of both $\mathcal{F}_n(H_{N_k})$ and K_{k-1}. On the other hand, $K_k = K_{k-1} + \mathcal{F}_n(H_{N_k})$ by definition of N_k. Hence K_k is an intertwining of K_{k-1} with the full matrix cone $\mathcal{F}_n(H_{N_k})$ in case that $N_k' \neq \emptyset$, and a direct sum $K_{k-1} \oplus \mathcal{F}_n(H_{\{k\}})$ in case that $N_k' = \emptyset$.

The proof is completed by the observation that $K_G = K_n$. □

Lemma 27. *Let G be a chordal graph with vertex set $\{1,\dots,n\}$, and let K_G be the corresponding ROG cone. Then $\deg K_G = n$, and K_G is simple if and only if G is connected.*

Proof. By construction $K_G \subset \mathcal{S}_+^n$ contains the identity matrix, and hence $\deg K_G = n$ by Corollary 1.

Suppose that K_G is not simple. Then there exists a nontrivial direct sum decomposition $\mathbb{R}^n = H \oplus H'$ such that for every rank 1 matrix $xx^T \in K_G$, either $x \in H$ or $x \in H'$. In particular, if $x = e_i$ is a canonical basis vector, then $e_i e_i^T \in K_G$ by construction of K_G and hence $e_i \in H \cup H'$ for all $i = 1,\dots,n$. Define the index sets $I = \{i \mid e_i \in H\}$ and $I' = \{i \mid e_i \in H'\}$. Then $I \cap I' = \emptyset$ and $I \cup I' = \{1,\dots,n\}$, because $\mathbb{R}^n = H \oplus H'$ is a direct sum decomposition. It follows that $H = \operatorname{span}\{e_i \mid i \in I\}$ and $H' = \operatorname{span}\{e_i \mid i \in I'\}$. Let now $x \in \mathbb{R}^n$ be a nonzero vector such that $X = xx^T \in K_G$. Then for every index pair $(i,j) \in I \times I'$ we have $x_i x_j = 0$ and hence $X_{ij} = 0$. From the fact that K_G is a ROG cone it follows that $X_{ij} = 0$ for all $X \in \operatorname{span} K_G = L_G$ in general for $(i,j) \in I \times I'$. But then $(i,j) \notin E(G)$, and there is no edge in G which connects the vertex subsets I, I'. Hence G is not connected.

Suppose, on the other hand, that G is not connected. Let I, I' be disjoint nonempty vertex sets such that $I \cup I' = \{1,\dots,n\}$ and there is no edge in G which connects I to I'. Then by definition for every $X \in L_G$ we have $X_{ij} = x_i x_j = 0$ for every index pair $(i,j) \in I \times I'$. Define subspaces $H = \operatorname{span}\{e_i \mid i \in I\}$, $H' = \operatorname{span}\{e_i \mid i \in I'\}$ of $\mathbb{R}^n$. Then $\mathbb{R}^n = H \oplus H'$ is by construction a nontrivial direct sum decomposition. It then follows that $L_G \subset \mathcal{L}_n(H) + \mathcal{L}_n(H')$, and the cone K_G is not simple. $\square$

4.2 A continuous family of non-isomorphic cones

In this subsection we construct a family of mutually non-isomorphic ROG cones in $\mathcal{S}^6$ which depends on a real parameter. Fix mutually distinct angles $\varphi_1,\dots,\varphi_4 \in [0,\pi)$. For $\varphi \in [0,\pi)$, let $l(\varphi) \subset \mathbb{R}^2$ be the line through the origin with incidence angle φ. Then the lines $l(\varphi_1),\dots,l(\varphi_4)$ define a quadruple of points in real projective space $\mathbb{R}P^1$.

Consider the 11-dimensional subspace $L_{\varphi_1,\varphi_2,\varphi_3,\varphi_4} \subset \mathcal{S}^6$ of matrices of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3\cos\varphi_1 & \alpha_4\cos\varphi_2 & \alpha_5\cos\varphi_3 & \alpha_6\cos\varphi_4 \\ \alpha_2 & \alpha_7 & \alpha_3\sin\varphi_1 & \alpha_4\sin\varphi_2 & \alpha_5\sin\varphi_3 & \alpha_6\sin\varphi_4 \\ \alpha_3\cos\varphi_1 & \alpha_3\sin\varphi_1 & \alpha_8 & 0 & 0 & 0 \\ \alpha_4\cos\varphi_2 & \alpha_4\sin\varphi_2 & 0 & \alpha_9 & 0 & 0 \\ \alpha_5\cos\varphi_3 & \alpha_5\sin\varphi_3 & 0 & 0 & \alpha_{10} & 0 \\ \alpha_6\cos\varphi_4 & \alpha_6\sin\varphi_4 & 0 & 0 & 0 & \alpha_{11} \end{pmatrix}, \quad \alpha_1,\dots,\alpha_{11} \in \mathbb{R}. \tag{3}$$

Let $H_0,\dots,H_4 \subset \mathbb{R}^6$ be the two-dimensional subspaces spanned by the columns of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \cos\varphi_1 & 0 \\ \sin\varphi_1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \cos\varphi_2 & 0 \\ \sin\varphi_2 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \cos\varphi_3 & 0 \\ \sin\varphi_3 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \cos\varphi_4 & 0 \\ \sin\varphi_4 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4}$$

respectively. Then $L_{\varphi_1,\varphi_2,\varphi_3,\varphi_4} = \sum_{i=0}^4 \mathcal{L}_6(H_i)$.

Let us define subspaces $L_j = \sum_{i=0}^j \mathcal{L}_6(H_i) \subset \mathcal{S}^6$ and spectrahedral cones $K_j = L_j \cap \mathcal{S}_+^6$, $j = 0, \dots, 4$. Then $L_{\varphi_1,\varphi_2,\varphi_3,\varphi_4} = L_4$, and $K_0 = \mathcal{F}_6(H_0)$ is a ROG cone.

Lemma 28. *The cone K_j, $j = 1, \dots, 4$, is a ROG cone given by the sum $\sum_{i=0}^j \mathcal{F}_6(H_i)$.*

Proof. We prove the lemma by induction over j. Assume that $K_{j-1} = \sum_{i=0}^{j-1} \mathcal{F}_6(H_i)$ is a ROG cone. We shall show that K_j is an intertwining of K_{j-1} with the face $\mathcal{F}_6(H_j)$ of $\mathcal{S}_+^6$, which would imply $K_j = \sum_{i=0}^j \mathcal{F}_6(H_i)$ by Lemma 22 and that K_j is ROG by Lemma 23.

To this end we have to show that the conditions of Lemma 22 apply to the faces $\mathcal{F}_6(\sum_{i=0}^{j-1} H_i)$ and $\mathcal{F}_6(H_j)$ of $\mathcal{S}_+^6$ and to the subspace $L_j \subset \mathcal{S}^6$, i.e., that

$$\mathcal{L}_6(\sum_{i=0}^{j-1} H_i) \cap \mathcal{L}_6(H_j) \subset L_j = (L_j \cap \mathcal{L}_6(\sum_{i=0}^{j-1} H_i)) + (L_j \cap \mathcal{L}_6(H_j)).$$

Indeed, the intersection $\Delta_j = (\sum_{i=0}^{j-1} H_i) \cap H_j$ is 1-dimensional and is contained in H_0. Namely, Δ_j is the linear span of the first column of the matrix generating H_j in (4). We obtain $\mathcal{L}_6(\sum_{i=0}^{j-1} H_i) \cap \mathcal{L}_6(H_j) = \mathcal{L}_6(\Delta_j) \subset \mathcal{L}_6(H_0) \subset L_j$, which proves the required inclusion.

Further, we have $(L_j \cap \mathcal{L}_6(\sum_{i=0}^{j-1} H_i)) = L_{j-1}$, $(L_j \cap \mathcal{L}_6(H_j)) = \mathcal{L}_6(H_j)$. But $L_j = L_{j-1} + \mathcal{L}_6(H_j)$ by definition, which proves the required equality. This completes the proof. □

Lemma 29. *The cone $K_{\varphi_1,\varphi_2,\varphi_3,\varphi_4} = L_{\varphi_1,\varphi_2,\varphi_3,\varphi_4} \cap \mathcal{S}_+^6$ is a ROG cone. Two cones $K_{\varphi_1,\varphi_2,\varphi_3,\varphi_4}$, $K_{\varphi_1',\varphi_2',\varphi_3',\varphi_4'}$ of this form are isomorphic if and only if the corresponding quadruples of lines $l(\varphi_1), \dots, l(\varphi_4) \subset \mathbb{R}^2$ and $l(\varphi_1'), \dots, l(\varphi_4') \subset \mathbb{R}^2$ define projectively equivalent quadruples of points in $\mathbb{R}P^1$.*

Proof. The first part of the lemma follows from Lemma 28 for $j = 4$.

Let us prove the second part. Consider cones $K_{\varphi_1,\varphi_2,\varphi_3,\varphi_4}$, $K_{\varphi_1',\varphi_2',\varphi_3',\varphi_4'}$ for quadruples $(\varphi_1, \dots, \varphi_4)$, $(\varphi_1', \dots, \varphi_4')$ of mutually distinct angles. Let $H_0, \dots, H_4$ and $H_0', \dots, H_4'$, respectively, be the corresponding 2-dimensional subspaces of $\mathbb{R}^6$ as defined by the column spaces of the matrices (4). Note that $H_0 = H_0'$. By Lemma 28 the set $\{x \in \mathbb{R}^6 \mid xx^T \in K_{\varphi_1,\varphi_2,\varphi_3,\varphi_4}\}$ is given by the union $\bigcup_{j=0}^4 H_j$, and the set $\{x \in \mathbb{R}^6 \mid xx^T \in K_{\varphi_1',\varphi_2',\varphi_3',\varphi_4'}\}$ by the union $\bigcup_{j=0}^4 H_j'$. Therefore, by virtue of Corollary 6, the cones $K_{\varphi_1,\varphi_2,\varphi_3,\varphi_4}$ and $K_{\varphi_1',\varphi_2',\varphi_3',\varphi_4'}$ are isomorphic if and only if there exists an invertible linear map $f : \mathbb{R}^6 \to \mathbb{R}^6$ which takes $\bigcup_{j=0}^4 H_j$ to $\bigcup_{j=0}^4 H_j'$.

Suppose that such a map f exists. The intersections $l(\varphi_i) = H_0 \cap H_i$, $l(\varphi_i') = H_0' \cap H_i'$, $i = 1, 2, 3, 4$, are 1-dimensional, while the intersections $H_i \cap H_j$, $H_i' \cap H_j'$, $i \neq j$, $i, j = 1, \dots, 4$, are 0-dimensional. Hence we must have $f[H_0] = H_0'$ and $f[H_i] = H'_{\sigma(i)}$, $i = 1, \dots, 4$, where $\sigma \in S_4$ is a permutation of the index set $\{1, \dots, 4\}$. Moreover, $f|_{H_0}[l_i] = l'_{\sigma(i)}$, $i = 1, \dots, 4$. It follows that $l(\varphi_1), \dots, l(\varphi_4) \subset H_0$ and $l(\varphi_1'), \dots, l(\varphi_4') \subset H_0$ define projectively equivalent quadruples of points in the projectivization of H_0.

Suppose now that the lines $l(\varphi_1), \dots, l(\varphi_4) \subset H_0$ and $l(\varphi_1'), \dots, l(\varphi_4') \subset H_0$ define projectively equivalent quadruples of points in the projectivization of H_0. Then there exists an invertible linear map $h : H_0 \to H_0$ and a permutation $\sigma \in S_4$ such that $h[l_i] = l'_{\sigma(i)}$, $i = 1, \dots, 4$. Let now $x_i \in H_i \setminus l_i$, $x_i' \in H_i' \setminus l_i'$, $i = 1, \dots, 4$, be arbitrary points. We then have $H_i = \mathrm{span}(l_i \cup \{x_i\})$, $H_i' = \mathrm{span}(l_i' \cup \{x_i'\})$, $i = 1, \dots, 4$. Moreover, $\mathrm{span}(H_0 \cup \{x_1, x_2, x_3, x_4\}) = \mathrm{span}(H_0 \cup \{x_1', x_2', x_3', x_4'\}) = \mathbb{R}^6$. We then can extend the map h to a linear map $f : \mathbb{R}^6 \to \mathbb{R}^6$ such that $f(x_i) = x'_{\sigma(i)}$, $i = 1, \dots, 4$. This map is invertible by construction and $f[H_i] = H'_{\sigma(i)}$, $i = 1, \dots, 4$. It follows that $f[\bigcup_{j=0}^4 H_j] = \bigcup_{j=0}^4 H_j'$, which completes the proof. $\square$

It is well-known that there exist infinitely many projectively non-equivalent quadruples of points in $\mathbb{R}P^1$. The equivalence classes are parameterized by the orbits of the *cross-ratio*

$$\lambda(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (l_1, l_2; l_3, l_4) = \frac{(\cot\varphi_1 - \cot\varphi_3)(\cot\varphi_2 - \cot\varphi_4)}{(\cot\varphi_2 - \cot\varphi_3)(\cot\varphi_1 - \cot\varphi_4)}$$

with respect to the action of the symmetric group S_4 on the arguments $\varphi_1, \dots, \varphi_4$. Thus there exists a continuum of mutually non-isomorphic ROG cones defined by subspaces $L \subset \mathcal{S}^6$ of type (3).

The cone $K_{\varphi_1,\varphi_2,\varphi_3,\varphi_4}$ is obtained from the face $\mathcal{F}_6(H_0) \cong \mathcal{S}_+^2$ by consecutive intertwining with the faces $\mathcal{F}_6(H_i) \cong \mathcal{S}_+^2$, $i = 1, \dots, 4$. It is hence an intertwining of 5 positive semi-definite matrix cones $\mathcal{S}_+^2$. More complicated ROG cones can be obtained by starting with a matrix cone $\mathcal{S}_+^n$ and consecutively intertwining it with matrix cones $\mathcal{S}_+^{k_1}, \dots, \mathcal{S}_+^{k_m}$ along full faces of ranks $d_1, \dots, d_m$, where $d_i < \min(n, k_i)$, $i = 1, \dots, m$. In this way, families of mutually non-isomorphic ROG cones can be obtained which are parameterized by an arbitrary number of real parameters.

5 Dimension and degree of ROG cones

In this section we consider the relation between the dimension and the degree of a ROG cone K. Evidently we have the inequality chain $\deg K \leq \dim K \leq \frac{\deg K(\deg K+1)}{2}$, with equality on the left if and only if K is isomorphic to the nonnegative orthant, and equality on the right if and only if K is isomorphic to the full cone of positive semi-definite matrices. We shall say that a ROG cone K has *codimension* k if $\dim K = \frac{\deg K(\deg K+1)}{2} - k$. In this case k can be interpreted as the number of linearly independent linear constraints on the matrices $X \in K$.

Lemma 30. *Let K be a ROG cone of degree n and dimension $\frac{n(n+1)}{2} - k$. Then K has a representation $K = \{X \in \mathcal{S}_+^n \mid \langle X, Q_i \rangle = 0 \ \forall\ i = 1, \dots, k\}$, where $Q_1, \dots, Q_k$ are* *linearly independent quadratic forms on $\mathbb{R}^n$ such that every nonzero form in the linear span $\mathrm{span}\{Q_1, \dots, Q_k\}$ is indefinite.*

Proof. Consider a representation of K as a linear section of $\mathcal{S}^n_+$. We have $\dim \mathcal{S}^n - \dim K = k$, and the orthogonal complement of $\mathrm{span}\, K$ in the space of quadratic forms on $\mathbb{R}^n$ has dimension k. Let $\{Q_1, \dots, Q_k\}$ be a basis of this complement. Then by construction we have $K = \mathrm{span}\, K \cap \mathcal{S}^n_+ = \{X \in \mathcal{S}^n_+ \mid \langle X, Q_i \rangle = 0 \,\forall\, i = 1, \dots, k\}$.

Since $\deg K = n$, by Corollary 1 there exists a positive definite matrix $X \in K$. Now suppose for the sake of contradiction that there exists a nonzero linear combination Q of $Q_1, \dots, Q_k$ which is semi-definite. By possibly replacing Q by $-Q$, we may assume that Q is positive semi-definite. Then $\langle Q, X \rangle > 0$, leading to a contradiction. □

In the next subsections we consider ROG cones of codimensions 1 and 2, and give a lower bound on the dimension of simple cones K of fixed degree.

5.1 ROG cones of codimension 1

Lemma 31. *Let $L \subset \mathcal{S}^n$ be a linear subspace of dimension $\frac{n(n+1)}{2} - d$. Then the spectrahedral cone $K = L \cap \mathcal{S}^n_+$ has no extreme elements of rank $k > -\frac{1}{2} + \sqrt{\frac{1}{4} + 2(d+1)}$.*

Proof. Let X lie on an extreme ray of K, and let $k = \mathrm{rk}\, X$. Then the minimal face of $\mathcal{S}^n_+$ which contains X has dimension $\frac{k(k+1)}{2}$. Denote this face by F. The minimal face of K which contains X is given by the intersection $F \cap L$ and has dimension 1. But since L has codimension d, we have $1 = \dim(F \cap L) \geq \dim F - d = \frac{k(k+1)}{2} - d$. This yields $k(k+1) \leq 2(d+1)$, which implies $k \leq -\frac{1}{2} + \sqrt{\frac{1}{4} + 2(d+1)}$. □

Corollary 9. *Let $L \subset \mathcal{S}^n$ be a linear subspace of dimension $\frac{n(n+1)}{2} - 1$. Then the cone $K = L \cap \mathcal{S}^n_+$ is ROG.*

Proof. By Lemma 31 the cone K has no extreme elements of rank $k \geq 2 > \frac{-1+\sqrt{17}}{2}$. Thus K is ROG. □

Corollary 10. *Every ROG cone of degree n and codimension 1 has a representation of the form $K = \{X \in \mathcal{S}^n_+ \mid \langle X, Q \rangle = 0\}$ for some indefinite quadratic form Q, and every cone of this form is ROG of degree n and codimension 1. Two such cones K, K', defined by indefinite quadratic forms Q, Q', respectively, are isomorphic if and only if either Q, Q' or $Q, -Q'$ have the same signature.*

Proof. The first claim follows from Lemma 30.

Let now Q be an indefinite quadratic form. Then the cone $K = \{X \in \mathcal{S}^n_+ \mid \langle X, Q \rangle = 0\}$ is ROG by Corollary 9. Since Q is indefinite, there exists a positive definite matrix X such that $\langle X, Q \rangle = 0$. Hence K intersects the interior of $\mathcal{S}^n_+$, and therefore $\dim K = \dim \mathcal{S}^n - 1$. Moreover, by Corollary 1 K is of degree n.

Let now the cones K, K' be defined by indefinite quadratic forms Q, Q', respectively. By Theorem 2 the cones K, K' are isomorphic if and only if their linear hulls $L = \{X \in \mathcal{S}^n \mid \langle X, Q\rangle = 0\}$, $L' = \{X \in \mathcal{S}^n \mid \langle X, Q'\rangle = 0\}$ are isomorphic. This holds if and only if the orthogonal complements of L, L', namely the 1-dimensional subspaces generated by Q and Q', are isomorphic. The last claim now easily follows. $\square$

It is not hard to establish that there are $[\frac{n^2}{4}]$ isomorphism classes of ROG cones of degree n and codimension 1. For $n \geq 3$ all of them are simple.

5.2 ROG cones of codimension 2

In this subsection we classify the ROG cones of degree n and dimension $\frac{n(n+1)}{2} - 2$.

By Lemma 31, a spectrahedral cone $K \subset \mathcal{S}_+^n$ of dimension $\frac{n(n+1)}{2} - d$, where $d \leq 4$, can only have extreme elements of rank $k \leq 2$. Therefore such a cone is ROG if and only if there exist no extreme elements of rank 2.

Let $L \subset \mathcal{S}^n$ be a subspace of codimension d, $2 \leq d \leq 4$. We can represent L as the set $\{X \in \mathcal{S}^n \mid \langle X, Q_i\rangle = 0,\ i = 1, \dots, d\}$, where $Q_1, \dots, Q_d$ are linearly independent quadratic forms on $\mathbb{R}^n$. The next result establishes in which cases the spectrahedral cone $K = L \cap \mathcal{S}_+^n$ has an extreme element of rank 2.

Lemma 32. *Assume above notations and conditions. The cone K is ROG if and only if for all pairs of vectors $x, y \in \mathbb{R}^n$ the $d \times 3$ matrix*

$$M(x,y) = \begin{pmatrix} x^TQ_1x & 2x^TQ_1y & y^TQ_1y \\ \vdots & \vdots & \vdots \\ x^TQ_dx & 2x^TQ_dy & y^TQ_dy \end{pmatrix}$$

either has rank < 2 or its kernel does not contain a vector $(a,b,c)^T$ such that the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is definite.

Proof. A vector $(a,b,c)^T$ is contained in the kernel of $M(x,y)$ if and only if

$$\left\langle \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} x^TQ_ix & x^TQ_iy \\ x^TQ_iy & y^TQ_iy \end{pmatrix} \right\rangle = \langle axx^T + b(xy^T + yx^T) + cyy^T, Q_i\rangle = 0 \,\forall\, i = 1, \dots, d, \tag{5}$$

or equivalently, if $axx^T + b(xy^T + yx^T) + cyy^T \in L$. Therefore, if x, y are linearly independent, then the dimension of the intersection $\operatorname{span}\{xx^T, xy^T + yx^T, yy^T\} \cap L = \mathcal{L}_n(\operatorname{span}\{x,y\}) \cap L$ equals the dimension of $\ker M(x,y)$.

Suppose that there exist $x, y \in \mathbb{R}^n$ and $a, b, c \in \mathbb{R}$ such that $\operatorname{rk} M(x,y) \geq 2$, $(a,b,c)^T$ is in the kernel of $M(x,y)$, and $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \succ 0$. Since $\operatorname{rk} M(x,y) \geq 2$, the vectors x, y are linearly independent. Set $H = \operatorname{span}\{x,y\}$. Then $\mathcal{F}_n(H)$ is a face of $\mathcal{S}_+^n$ of rank 2, and $F = \mathcal{F}_n(H) \cap L$ is a face of K. Define the matrix $X = axx^T + b(xy^T + yx^T) + cyy^T \in \mathcal{L}_n(H)$.

We have the representation $X = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix}^T$. Hence X is positive semi-definite of rank 2 and is contained in the relative interior of $\mathcal{F}_n(H)$. Further, by (5) we have $X \in L$, and hence $X \in F$. Since $\operatorname{rk} M(x,y) \geq 2$, the kernel of $M(x,y)$ has dimension 1. Therefore $\dim(\mathcal{L}_n(H) \cap L) = 1$, and every matrix in this intersection is proportional to X. It follows that the face F of K is 1-dimensional and the rank 2 matrix X generates an extreme ray of K. Thus K is not ROG.

Suppose now that K is not ROG. By Lemma 31 there exists a matrix $X \in K$ of rank 2 which generates an extreme ray of K. Let $H \subset \mathbb{R}^n$ be the image of X, and let $\{x, y\}$ be a basis of H. Then there exist $a, b, c \in \mathbb{R}$ such that $X = axx^T + b(xy^T + yx^T) + cyy^T$ and $\begin{pmatrix} a & b \\ b & c \end{pmatrix} \succ 0$. The inclusion $X \in L$ then implies (5) and hence $(a,b,c)^T \in \ker M(x,y)$. The matrix X is contained in the relative interior of $\mathcal{F}_n(H)$, the minimal face of $\mathcal{S}_+^n$ which contains X. Hence the dimension of the intersection $\mathcal{F}_n(H) \cap L$, which contains X, equals the dimension of $\mathcal{L}_n(H) \cap L$. But $\mathcal{F}_n(H) \cap L$ is the minimal face of K which contains X, and has dimension 1 by the extremality of X. By the above, the kernel of $M(x,y)$ then also has dimension 1. It follows that $\operatorname{rk} M(x,y) = 2$.

This completes the proof. $\square$

Corollary 11. *Let $L = \{X \in \mathcal{S}^n \mid \langle X, Q_1 \rangle = \langle X, Q_2 \rangle = 0\}$ be a linear subspace, where Q_1, Q_2 are linearly independent quadratic forms. Then the cone $K = L \cap \mathcal{S}_+^n$ is ROG if and only if the bi-quartic polynomial given by*

$$\begin{aligned} p(x,y) \;=\; & (y^TQ_1y \cdot x^TQ_2x - x^TQ_1x \cdot y^TQ_2y)^2 \\ & -4(x^TQ_1y \cdot y^TQ_2y - x^TQ_2y \cdot y^TQ_1y)(x^TQ_1x \cdot x^TQ_2y - x^TQ_1y \cdot x^TQ_2x) \end{aligned} \tag{6}$$

is nonnegative for all $x, y \in \mathbb{R}^n$.

Proof. By Lemma 32 the cone K is not ROG if and only if there exist $x, y \in \mathbb{R}^n$ such that the 2×3 matrix

$$M(x,y) = \begin{pmatrix} x^TQ_1x & 2x^TQ_1y & y^TQ_1y \\ x^TQ_2x & 2x^TQ_2y & y^TQ_2y \end{pmatrix}$$

has full rank and its kernel contains a vector $(a,b,c)^T$ such that $ac - b^2 > 0$. Now note that $M(x,y)$ has full rank if and only if the cross product

$$\begin{pmatrix} x^TQ_1x \\ 2x^TQ_1y \\ y^TQ_1y \end{pmatrix} \times \begin{pmatrix} x^TQ_2x \\ 2x^TQ_2y \\ y^TQ_2y \end{pmatrix} = \begin{pmatrix} 2(x^TQ_1y \cdot y^TQ_2y - x^TQ_2y \cdot y^TQ_1y) \\ y^TQ_1y \cdot x^TQ_2x - x^TQ_1x \cdot y^TQ_2y \\ 2(x^TQ_1x \cdot x^TQ_2y - x^TQ_1y \cdot x^TQ_2x) \end{pmatrix}$$

is nonzero, and in this case the kernel of $M(x,y)$ is generated by this cross product. The claim of the corollary now easily follows. $\square$

We now prove an auxiliary result on real symmetric matrix pencils. Recall that y is called an eigenvector of the pencil $Q_1 + \lambda Q_2$ if the linear forms Q_1y, Q_2y are linearly dependent.

Lemma 33. *Let Q_1, Q_2 be quadratic forms on $\mathbb{R}^n$ such that the pencil $Q_1 + \lambda Q_2$ possesses n linearly independent real eigenvectors. Then there exists a direct sum decomposition $\mathbb{R}^n = H_0 \oplus H_1 \oplus \cdots \oplus H_m$, non-degenerate quadratic forms Φ_k on H_k, $k = 1, \dots, m$, and mutually distinct angles $\varphi_1, \dots, \varphi_m \in [0, \pi)$ with the following properties. For every vector $x = \sum_{k=0}^m x_k$, where $x_k \in H_k$, we have $Q_1(x) = \sum_{k=1}^m \cos\varphi_k \Phi_k(x_k)$, $Q_2(x) = \sum_{k=1}^m \sin\varphi_k \Phi_k(x_k)$. Moreover, the set of real eigenvectors of the pencil $Q_1 + \lambda Q_2$ is given by the union $\bigcup_{k=1}^m (H_0 + H_k)$.*

Proof. We define the subspace H_0 as the intersection $\ker Q_1 \cap \ker Q_2$. For every real eigenvector $y \not\in H_0$ of the pencil $Q_1 + \lambda Q_2$, the linear span of the set $\{Q_1 y, Q_2 y\}$ of linear forms has then dimension 1. Hence there exists a unique angle $\varphi(y) \in [0, \pi)$ such that $\sin\varphi(y) Q_1 y - \cos\varphi(y) Q_2 y = 0$.

By assumption we find linearly independent real eigenvectors $y_1, \dots, y_{n-\dim H_0}$ of the pencil $Q_1 + \lambda Q_2$ such that $\mathrm{span}(H_0 \cup \{y_1, \dots, y_{n-\dim H_0}\}) = \mathbb{R}^n$. Regroup these vectors into subsets $\{y_{11}, \dots, y_{1d_1}\}, \dots, \{y_{m1}, \dots, y_{md_m}\}$ such that $\varphi(y_{kl}) = \varphi_k$, $k = 1, \dots, m$, $l = 1, \dots, d_k$, where $\varphi_1, \dots, \varphi_m \in [0, \pi)$ are mutually distinct angles, and d_k is the number of eigenvectors corresponding to angle φ_k. Define the subspace H_k as the linear span of $y_{k1}, \dots, y_{kd_k}$, $k = 1, \dots, m$. Then by construction we have that $H_0 \oplus H_1 \oplus \cdots \oplus H_m$ is a direct sum decomposition of $\mathbb{R}^n$. Moreover, every vector $y \in H_k$ is an eigenvector and we have $\sin\varphi_k Q_1 y - \cos\varphi_k Q_2 y = 0$ for all $y \in H_k$, $k = 1, \dots, m$. It follows that there exist quadratic forms Φ_k on H_k, $k = 1, \dots, m$, such that $Q_1|_{H_k} = \cos\varphi_k \Phi_k$, $Q_2|_{H_k} = \sin\varphi_k \Phi_k$.

Let now $k, k' \in \{1, \dots, m\}$ be distinct indices and $y \in H_k$, $y' \in H_{k'}$ be arbitrary vectors. By construction we have $\sin\varphi_k Q_1 y - \cos\varphi_k Q_2 y = \sin\varphi_{k'} Q_1 y' - \cos\varphi_{k'} Q_2 y' = 0$. Therefore $\sin\varphi_k y^T Q_1 y' - \cos\varphi_k y^T Q_2 y' = \sin\varphi_{k'} y^T Q_1 y' - \cos\varphi_{k'} y^T Q_2 y' = 0$. But $\varphi_k, \varphi_{k'}$ are distinct, and thus this linear system on $y^T Q_i y'$ has only the trivial solution $y^T Q_1 y' = y^T Q_2 y' = 0$. The decomposition formulas $Q_1(x) = \sum_{k=1}^m \cos\varphi_k \Phi_k(x_k)$, $Q_2(x) = \sum_{k=1}^m \sin\varphi_k \Phi_k(x_k)$ now readily follow.

Let $1 \leq k \leq m$. Suppose there exists a vector $y \in H_k$ such that $\Phi_k y = 0$. Then we have $Q_1 y = Q_2 y = 0$, and $y \in H_0$. Thus $y = 0$, and the form Φ_k must be non-degenerate.

Let now $x = \sum_{k=0}^m x_k$ be a real eigenvector of the pencil $Q_1 + \lambda Q_2$, where $x_k \in H_k$. Then we have $\sin\varphi Q_1 x - \cos\varphi Q_2 x = 0$ for some angle $\varphi \in [0, \pi)$. Let $z_k \in H_k$, $k = 0, \dots, m$ be arbitrary vectors, and set $z = \sum_{k=0}^m z_k$. Then we have $x^T Q_1 z = \sum_{k=1}^m \cos\varphi_k \Phi_k(x_k, z_k)$, $x^T Q_2 z = \sum_{k=1}^m \sin\varphi_k \Phi_k(x_k, z_k)$. It follows that

$$\begin{aligned} 0 &= \sin\varphi x^T Q_1 z - \cos\varphi x^T Q_2 z = \sum_{k=1}^m (\sin\varphi\cos\varphi_k \Phi_k(x_k, z_k) - \cos\varphi\sin\varphi_k \Phi_k(x_k, z_k)) \\ &= \sum_{k=1}^m \sin(\varphi - \varphi_k)\Phi_k(x_k, z_k). \end{aligned}$$

Here $\Phi_k(x_k, z_k) = \frac{1}{4}(\Phi_k(x_k + z_k) - \Phi_k(x_k - z_k))$ is as usual the bilinear form defined by the quadratic form Φ_k. Since this holds identically for all $z_k \in H_k$ and the forms Φ_k are non-degenerate, we must have either $\varphi = \varphi_k$ or $x_k = 0$ for each $k = 1, \dots, m$. Therefore

$x \in H_0 + H_k$ for some k. On the other hand, every vector $x \in H_0 + H_k$ is an eigenvector of the pencil $Q_1 + \lambda Q_2$, since it satisfies $\sin \varphi_k Q_1 x - \cos \varphi_k Q_2 x = 0$. $\square$

Let K be a ROG cone of degree 2. Then the dimension of K is either 2 or 3, and K cannot be of codimension 2. We shall henceforth consider ROG cones of degree $n \geq 3$.

Lemma 34. *Let $K = \{X \in \mathcal{S}^n_+ \,|\, \langle X, Q_1 \rangle = \langle X, Q_2 \rangle = 0\}$ be a ROG cone of degree $n \geq 3$, where Q_1, Q_2 are linearly independent quadratic forms. Then there exists $z \in \mathbb{R}^n$ such that $z^T Q_1 z = z^T Q_2 z = 0$ and the linear forms $Q_1 z, Q_2 z$ are linearly independent.*

Proof. By Corollary 1 the cone K has a nonempty intersection with the interior of $\mathcal{S}^n_+$. Hence the linear span of K is given by $L = \operatorname{span} K = \{X \in \mathcal{S}^n \,|\, \langle X, Q_1 \rangle = \langle X, Q_2 \rangle = 0\}$.

For the sake of contradiction, assume that for every $z \in \mathbb{R}^n$ such that $z^T Q_1 z = z^T Q_2 z = 0$ the linear forms $Q_1 z, Q_2 z$ are linearly dependent. Let $Z = zz^T$ be an arbitrary rank 1 matrix in K. Then $z^T Q_1 z = z^T Q_2 z = 0$, and by our assumption z is an eigenvector of the pencil $Q_1 + \lambda Q_2$.

Since the degree of K is n, by Corollary 4 there exist n linearly independent vectors $z_1, \dots, z_n \in \mathbb{R}^n$ such that the rank 1 matrices $z_k z_k^T$ are in K for $k = 1, \dots, n$. This implies that the pencil $Q_1 + \lambda Q_2$ has n linearly independent real eigenvectors. Therefore the conditions of Lemma 33 are satisfied. Let $\mathbb{R}^n = H_0 \oplus H_1 \oplus \cdots \oplus H_m$ be the direct sum decomposition from this lemma. If $m \leq 1$, then the forms Q_1, Q_2 are linearly dependent, which contradicts our assumptions. Hence $m \geq 2$.

Let $x_1 \in H_1$, $x_2 \in H_2$ be nonzero vectors. Consider the matrix $X = x_1 x_2^T + x_2 x_1^T$. We have $\langle Q_i, X \rangle = 2x_1^T Q_i x_2 = 0$ for $i = 1, 2$, and hence $X \in L$. On the other hand, L is spanned by all rank 1 matrices in K because K is ROG. However, if $z \in \mathbb{R}^n$ is such that $zz^T \in K$, then by Lemma 33 we have $z \in \bigcup_{k=1}^m (H_0 + H_k)$. It follows that $L \subset \sum_{k=1}^m \mathcal{L}_n(H_0 + H_k)$. But $X \notin \sum_{k=1}^m \mathcal{L}_n(H_0 + H_k)$, leading to a contradiction. $\square$

Lemma 35. *Let $K = \{X \in \mathcal{S}^n_+ \,|\, \langle X, Q_1 \rangle = \langle X, Q_2 \rangle = 0\}$ be a ROG cone of degree $n \geq 3$, where Q_1, Q_2 are linearly independent quadratic forms. Let $z \in \mathbb{R}^n$ be such that $z^T Q_1 z = z^T Q_2 z = 0$. Suppose that the linear forms $q_1 = Q_1 z$, $q_2 = Q_2 z$ are linearly independent. Then there exists a linear form u which is linearly independent form q_1, q_2 and such that $Q_1 = u \otimes q_1 + q_1 \otimes u$, $Q_2 = u \otimes q_2 + q_2 \otimes u$.*

Proof. By virtue of the condition $z^T Q_1 z = z^T Q_2 z = 0$ the polynomial $p(x, y)$ defined by (6) vanishes for $x = z$ and all $y \in \mathbb{R}^n$. By Corollary 11 this polynomial is nonnegative. Therefore $\left.\frac{\partial p(x,y)}{\partial x}\right|_{x=z} = 0$ for all $y \in \mathbb{R}^n$. By virtue of $z^T Q_1 z = z^T Q_2 z = 0$, at $x = z$ this gradient is given by

$$\left.\frac{\partial p(x,y)}{\partial x}\right|_{x=z} = -8(q_1^T y \cdot y^T Q_2 y - q_2^T y \cdot y^T Q_1 y)(q_2^T y \cdot q_1 - q_1^T y \cdot q_2) = 0.$$

Since q_1, q_2 are linearly independent, the linear form $q_2^T y \cdot q_1 - q_1^T y \cdot q_2$ is nonzero if $q_1^T y \neq 0$ or $q_2^T y \neq 0$. Therefore $q_1^T y \cdot y^T Q_2 y = q_2^T y \cdot y^T Q_1 y$ for all such y, i.e., for a dense subset of $\mathbb{R}^n$. It follows that $q_1^T y \cdot y^T Q_2 y = q_2^T y \cdot y^T Q_1 y$ identically for all $y \in \mathbb{R}^n$.

In particular, for every $y \in \mathbb{R}^n$ such that $q_1^T y = 0$, $q_2^T y \neq 0$ we have $y^T Q_1 y = 0$. This subset of vectors y is dense in the kernel of q_1, and hence Q_1 is zero on this kernel. It follows that there exists a linear form u_1 such that $Q_1 = q_1 \otimes u_1 + u_1 \otimes q_1$. In a similar manner, there exists a linear form u_2 such that $Q_2 = q_2 \otimes u_2 + u_2 \otimes q_2$. It follows that $q_1^T y \cdot q_2^T y \cdot u_2^T y = q_2^T y \cdot q_1^T y \cdot u_1^T y$ identically for all $y \in \mathbb{R}^n$. For all $y \in \mathbb{R}^n$ such that $q_1^T y \neq 0$ and $q_2^T y \neq 0$ it follows that $u_2^T y = u_1^T y$. Since the set of such vectors y is dense in $\mathbb{R}^n$, we get that u_1, u_2 are equal to the same linear form u.

Note that $q_1^T z = q_2^T z = 0$ by assumption. We obtain $q_1 = Q_1 z = q_1^T z \cdot u + u^T z \cdot q_1 = u^T z \cdot q_1$, and hence $u^T z = 1$. Therefore u must be linearly independent of q_1, q_2. □

Theorem 3. *Let $K = \{X \in \mathcal{S}_+^n \mid \langle X, Q_1 \rangle = \langle X, Q_2 \rangle = 0\}$ be a ROG cone of degree $n \geq 3$, where Q_1, Q_2 are linearly independent quadratic forms. Then K is isomorphic to the direct sum $\mathcal{S}_+^1 \oplus \mathcal{S}_+^2$ if $n = 3$ and to a full extension of this sum if $n > 3$.*

Proof. By Lemma 34 Lemma 35 is applicable. By choosing an appropriate basis of $\mathbb{R}^n$, we can assume that the linear forms u, q_1, q_2 from Lemma 35 are the first elements of the dual basis. Then the cone K is given by the set $\{X \in \mathcal{S}_+^n \mid X_{12} = X_{13} = 0\}$. The claim of the theorem now easily follows. □

5.3 Lower bound on the dimension of simple ROG cones

In this section we show that for simple ROG cones K the dimension is bounded from below by $2 \cdot \deg K - 1$. We shall need the following auxiliary result.

Lemma 36. *Let $x_1, \dots, x_m \in \mathbb{R}^n$ be linearly independent vectors, and let $S \subset \mathcal{S}^n$ be the m-dimensional subspace spanned by the rank 1 matrices $x_1 x_1^T, \dots, x_m x_m^T$. Let further $H \subset \mathbb{R}^n$ be a linear subspace. Then the dimension of the intersection $S \cap \mathcal{L}_n(H)$ is given by the number of indices i such that $x_i \in H$. In particular, $\dim(S \cap \mathcal{L}_n(H)) \leq \dim H$.*

Proof. Define the index set $I = \{i \mid x_i \in H\}$. Let $A = \sum_{i=1}^m \alpha_i x_i x_i^T$ be an arbitrary element of S, where α_i are scalar coefficients. Suppose there exists an index $j \notin I$ such that $\alpha_j \neq 0$. Let $y \in \mathbb{R}^n$ be a vector such that $y^T x_j = 1$, and $y^T x_i = 0$ for all $i \neq j$. Such a vector y exists by the linear independence of $x_1, \dots, x_m$. We then get $Ay = \sum_{i=1}^m \alpha_i (y^T x_i) x_i = \alpha_j x_j \neq H$. Hence $A \notin \mathcal{L}_n(H)$.

It follows that every matrix in the intersection $S \cap \mathcal{L}_n(H)$ is of the form $A = \sum_{i \in I} \alpha_i x_i x_i^T$ for some scalars α_i. On the other hand, for every such matrix A and every vector $y \in \mathbb{R}^n$ we have $Ay = \sum_{i \in I} (y^T x_i) x_i \in H$, and $A \in \mathcal{L}_n(H)$. Therefore the intersection $S \cap \mathcal{L}_n(H)$ equals the linear span of the set $\{x_i x_i^T \mid i \in I\}$. The claims of the lemma now easily follow. □

Theorem 4. *Let K be a simple ROG cone of degree n. Then $\dim K \geq 2n - 1$.*

Proof. Represent K as a linear section of $\mathcal{S}_+^n$. Recall that by Lemma 6 every face of K is a ROG cone, and that K itself is the face of K of largest degree n. Denote by $\mathbf{F}$ the set of faces F of K such that $\dim F \geq 2 \deg F - 1$. The set $\mathbf{F}$ is not empty, because every extreme ray of K is an element of $\mathbf{F}$. Set $k = \max_{F \in \mathbf{F}} \deg F$. Assume for the sake of contradiction that $K \not\in \mathbf{F}$, and hence $k < n$. Let $F_k \in \mathbf{F}$ be a face of K which achieves the maximal degree k. Denote the linear span of K by L, and the linear span of F_k by L_k. By construction we have $\dim L_k \geq 2k - 1$.

By Corollary 1 the maximal rank of matrices in F_k equals k. Let $Y \in F_k$ be a matrix of maximal rank k, and let the k-dimensional subspace $H \subset \mathbb{R}^n$ be its image. Then we have $L_k = L \cap \mathcal{L}_n(H)$ and $F_k = L \cap \mathcal{F}_n(H)$. By Corollary 4 there exists a basis $\{r_1, \dots, r_k\}$ of H such that $r_i r_i^T \in K$ for all $i = 1, \dots, k$, and $Y = \sum_{i=1}^k r_i r_i^T$. By virtue of $\deg K = n$ and the last part of Corollary 4 we may complete this basis of H to a basis $\{r_1, \dots, r_n\}$ of $\mathbb{R}^n$ such that $r_i r_i^T \in K$ for all $i = 1, \dots, n$. Adopt the coordinate system defined by this basis. Then all diagonal matrices are in L, and the subspace L_k consists of the matrices in L all whose non-zero elements are located in the upper left $k \times k$ block.

Since K is simple, there exists a rank 1 matrix $zz^T \in K$ such that the vector $z = (z_1, \dots, z_n)^T$ is neither in H nor in $\mathrm{span}\{r_{k+1}, \dots, r_n\}$. In other words, the subvector $z_H = (z_1, \dots, z_k)^T$ is not zero, and not all of the elements $z_{k+1}, \dots, z_n$ are zero. Without loss of generality, let the nonzero elements in the second group be $z_{k+1}, \dots, z_{k+m}$. By scaling the vector z, we may also assume that $z^T z = 1$.

Denote by F_{k+m} the face of K which consists of all matrices in K whose non-zero elements are located in the upper left $(k+m) \times (k+m)$ block. Denote the linear span of F_{k+m} by L_{k+m}. Since all diagonal matrices are in L, the maximal rank of the matrices in F_{k+m} equals $k + m$. By Corollary 1 we get $\deg F_{k+m} = k + m > k$. By our definition of k we then have $F_{k+m} \not\in \mathbf{F}$, and hence $\dim L_{k+m} < 2(k+m) - 1$. Let S be the $(\dim L_k + m)$-dimensional subspace of L_{k+m} spanned by L_k and the rank 1 matrices $r_{k+1} r_{k+1}^T, \dots, r_{k+m} r_{k+m}^T$.

We have $zz^T \in F_{k+m}$. Consider the matrix $X = \mathrm{diag}(I_{k+m}, 0, \dots, 0) - zz^T \in L_{k+m}$, where I_{k+m} is the $(k+m) \times (k+m)$ identity matrix. By $z^T z = 1$ the matrix X is positive semi-definite of rank $k + m - 1$, with z as kernel vector. It follows that $X \in F_{k+m}$, and by Corollary 4 there exist $k + m - 1$ linearly independent vectors $x_1, \dots, x_{k+m-1} \in \mathbb{R}^n$ such that $x_i x_i^T \in F_{k+m}$ for all $i = 1, \dots, k+m-1$, and $X = \sum_{i=1}^{k+m-1} x_i x_i^T$. Since $z^T X z = \sum_{i=1}^{k+m-1} (z^T x_i)^2 = 0$, it follows that $z^T x_i = 0$ for all $i = 1, \dots, k + m - 1$.

Consider the $(k + m - 1)$-dimensional subspace $S' \subset L_{k+m}$ spanned by the rank 1 matrices $x_i x_i^T$, $i = 1, \dots, k + m - 1$. Let us bound the dimension of the intersection $S \cap S'$. Let $A \in S \cap S'$ be arbitrary. Since $A \in S$, the matrix A has a block-diagonal structure $A = \mathrm{diag}(A_H, a_{k+1}, \dots, a_{k+m}, 0, \dots, 0)$, with A_H a block of size $k \times k$. On the other hand, $A \in S'$ implies $Az = 0$. It follows that $a_{k+1} z_{k+1} = \dots = a_{k+m} z_{k+m} = 0$ and $a_{k+1} = \dots = a_{k+m} = 0$, because the corresponding elements of z are non-zero. The image of A is hence contained in the intersection of the subspace H with the orthogonal complement of z. By virtue of $z_H \neq 0$ this intersection has dimension $k - 1$. By Lemma 36 we then get that $\dim(S \cap S') \leq k - 1$.

Thus $\dim(S + S') = \dim S + \dim S' - \dim(S \cap S') \geq (\dim L_k + m) + (k + m - 1) - (k - 1) \geq 2k - 1 + 2m$, leading to a contradiction with the bound $\dim L_{k+m} < 2(k+m) - 1$. This completes the proof. $\square$

6 Isolated extreme rays

The extreme rays of a ROG cone are generated by its rank 1 matrices. In this section we study the situation when an extreme ray of a ROG cone K is isolated. We shall show that in this case K is a direct sum of $\mathcal{S}_+^1$ with a lower-dimensional ROG cone, and the isolated extreme ray is the face of K corresponding to the factor $\mathcal{S}_+^1$. We will need the following concept.

Definition 12. *The vectors $x_1, \dots, x_{k+1} \in \mathbb{R}^n$ are called minimally linearly dependent if they are linearly dependent, but every k of them are linearly independent.*

Lemma 37. *A set of vectors $x_1, \dots, x_{k+1} \in \mathbb{R}^n$ is minimally linearly dependent if and only if their span has dimension k and there exist nonzero real numbers $c_1, \dots, c_{k+1}$ such that $\sum_{i=1}^{k+1} c_i x_i = 0$.*

Proof. Denote by L the linear span of $\{x_1, \dots, x_{k+1}\}$, and let X be the $n \times (k+1)$ matrix formed of the column vectors x_i.

Let $x_1, \dots, x_{k+1} \in \mathbb{R}^n$ be minimally linearly dependent. Then the dimension of L equals k, because there exist k linearly independent vectors in L. The matrix X then has rank k and its kernel has dimension 1. Let $(c_1, \dots, c_{k+1})^T \in \mathbb{R}^{k+1}$ be a generator of $\ker X$. Then $\sum_{i=1}^{k+1} c_i x_i = 0$ and not all c_i are zero. Let $I \subset \{1, \dots, k+1\}$ be the set of indices i such that $c_i \neq 0$. Then the vectors in the set $\{c_i \mid i \in I\}$ are linearly dependent. By assumption, no k vectors are linearly dependent, and therefore I has not less than $k+1$ elements. It follows that $c_i \neq 0$ for all i.

Let now $c_1, \dots, c_{k+1}$ be nonzero real numbers such that $\sum_{i=1}^{k+1} c_i x_i = 0$, and suppose $\dim L = k$. Then $x_1, \dots, x_{k+1}$ are linearly dependent. Moreover, $\operatorname{rk} X = k$, and hence the vector $(c_1, \dots, c_{k+1})^T$ generates the kernel of X. In particular, there is no nonzero kernel vector with a zero element. It follows that every subset of k vectors is linearly independent. Thus $x_1, \dots, x_{k+1}$ are minimally linearly dependent. $\square$

Lemma 38. *Let $S \subset \mathbb{R}^n$ be a subset and $x \in S$ a nonzero vector. Then either*

1) there exists a subspace $H \subset \mathbb{R}^n$ of dimension $n-1$ which does not contain x, such that for every $y \in S$ either $y \in H$ or y is a multiple of x,

or 2) there exists a minimally linearly dependent subset $T \subset S$ of size at least 3 such that $x \in T$.

Proof. Let $L \subset \mathbb{R}^n$ be the linear span of S, and let k be its dimension. Let us complete $x_1 = x$ to a basis $\{x_1, \dots, x_k\} \subset S$ of L. Then every vector $y \in S$ can be in a unique way represented as a sum $y = \sum_{i=1}^k c_i x_i$. We have two possibilities.

1) For every vector $y = \sum_{i=1}^k c_i x_i \in S$, either $c_1 = 0$, or $c_2 = \dots = c_k = 0$. Then we can take H as any hyperplane which contains the span of $\{x_2, \dots, x_k\}$ but not x_1, and are in the situation 1) of the lemma.

2) There exists $y = \sum_{i=1}^k c_i x_i \in S$ such that $c_1 \neq 0$ and at least one of the coefficients $c_2, \dots, c_k$ is not zero. Let without loss of generality the nonzero coefficients among the

$c_2, \ldots, c_k$ be the coefficients $c_2, \ldots, c_l$, $l \geq 2$. Then we obtain $y - \sum_{i=1}^{l} c_i x_i = 0$, and the set $\{x_1, \ldots, x_l, y\} \subset S$ is minimally linearly dependent by Lemma 37. Thus we are in the situation 2) of the lemma. □

Lemma 39. *Let K be a ROG cone and let $R_1, \ldots, R_{k+1} \in K$ be extreme rays of K. Let the rank 1 matrices $X_i = x_i x_i^T$ be generators of these extreme rays, respectively, in some representation of K as a linear section of a positive semi-definite matrix cone $\mathcal{S}_+^n$. Whether the set $\{x_1, \ldots, x_{k+1}\} \subset \mathbb{R}^n$ is minimally linearly dependent then depends only on the extreme rays $R_1, \ldots, R_{k+1}$ of K, but not on the representation of K, its size, or the generators X_i.*

Proof. Let $c_1, \ldots, c_{k+1}$ be non-zero real numbers. Then a subset $\{x_1, \ldots, x_{k+1}\} \subset \mathbb{R}^n$ is minimally linearly dependent if and only if the subset $\{c_1 x_1, \ldots, c_{k+1} x_{k+1}\}$ is minimally linearly dependent. This follows directly from Definition 12. Hence the property does not depend on the generators X_i of the extreme rays for a given representation of K. Let now $X_i = x_i x_i^T$, $Y_i = y_i y_i^T$ be generators of the rays R_i in different representations of sizes n, m, respectively. Let $n \leq m$ without loss of generality. By Theorem 2 there exists an injective linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(x_i) = \sigma_i y_i$, where $\sigma_i \in \{-1, +1\}$, for all $i = 1, \ldots, k+1$. By the injectivity of f, we have for every index subset $I \subset \{1, \ldots, k+1\}$ that the set $\{x_i\}_{i \in I}$ is linearly dependent if and only if the set $\{\sigma_i y_i\}_{i \in I}$ is linearly dependent. Hence $\{x_1, \ldots, x_{k+1}\}$ is minimally linearly dependent if and only if the set $\{y_1, \ldots, y_{k+1}\}$ is. This completes the proof. □

Lemma 39 allows to make the following definition.

Definition 13. *Let K be a ROG cone. We call a subset $\{R_1, \ldots, R_{k+1}\}$ of extreme rays of K, generated by rank 1 matrices $X_i = x_i x_i^T$, respectively, an MLD set, if the set $\{x_1, \ldots, x_{k+1}\}$ is minimally linearly dependent.*

Lemma 40. *Let K be a ROG cone of degree $n \geq 2$, possessing an MLD set $\{R_1, \ldots, R_{n+1}\}$ of extreme rays. Then the following holds:*

i) the cone K is simple;

ii) the extreme rays $R_1, \ldots, R_{n+1}$ of K are not isolated.

Proof. Represent K as a linear section of $\mathcal{S}_+^n$, and let the rank 1 matrices $X_i = x_i x_i^T$ be generators of the extreme rays R_i, $i = 1, \ldots, n+1$. Then the set $\{x_1, \ldots, x_{n+1}\} \subset \mathbb{R}^n$ is minimally linearly dependent. Denote the linear span of K by L.

Suppose for the sake of contradiction that K is not simple. Then there exists a nontrivial direct sum decomposition $\mathbb{R}^n = H_1 \oplus H_2$ such that $K \subset \mathcal{L}_n(H_1) + \mathcal{L}_n(H_2)$ and hence $x_i \in H_1 \cup H_2$ for all $i = 1, \ldots, n+1$. Let n_1, n_2 be the dimensions of H_1, H_2, respectively, and n_1', n_2' the number of indices i such that $x_i \in H_1$ or $x_i \in H_2$, respectively. Then $n_1', n_2' > 0$, because the vectors $x_1, \ldots, x_{n+1}$ span the whole space $\mathbb{R}^n$ and the decomposition $\mathbb{R}^n = H_1 \oplus H_2$ is nontrivial. On the other hand, we have $n_1 + n_2 = n$ and $n_1' + n_2' = n + 1$. Hence either $n_1' > n_1$, or $n_2' > n_2$, and there exists a strict subset of the set $\{x_1, \ldots, x_{n+1}\}$ which is linearly dependent, leading to a contradiction. This proves i).

We shall now prove ii). For $n = 2$ we have $K = \mathcal{S}^2_+$, and the assertion is evident. Suppose $n \geq 3$.

By the definition of minimal linear dependence the vectors $x_1, \dots, x_n$ form a basis of $\mathbb{R}^n$. Choose a coordinate system in which this is the canonical basis. By Lemma 37 there exist nonzero scalars $c_1, \dots, c_{n+1}$ such that $\sum_{i=1}^{n+1} c_i x_i = 0$. We may normalize these scalars by a common factor to achieve $c_{n+1} = -1$. Then we have $x_{n+1} = (c_1, \dots, c_n)^T$.

The subspace $L \subset \mathcal{S}^n$ contains the $(n+1)$-dimensional linear span $\tilde{L}$ of the rank 1 matrices $x_i x_i^T$, $i = 1, \dots, n+1$. Let $d_1, \dots, d_n > 0$ be positive scalars, and set $d_{n+1} = -\left(\sum_{i=1}^n d_i^{-1} c_i^2\right)^{-1}$. Then the matrix $M = \sum_{i=1}^{n+1} d_i x_i x_i^T$ is an element of $\tilde{L}$. Moreover, for every vector $r = (r_1, \dots, r_n)^T$ we have

$$\begin{aligned} r^T M r &= \sum_{i=1}^{n+1} d_i (r^T x_i)^2 = \sum_{i=1}^{n} d_i r_i^2 - \frac{\left(\sum_{i=1}^n c_i r_i\right)^2}{\sum_{i=1}^n d_i^{-1} c_i^2} \\ &= \sum_{i=1}^{n} \left(\sqrt{d_i} r_i - \frac{c_i \sum_{j=1}^n c_j r_j}{\sqrt{d_i} \sum_{j=1}^n d_j^{-1} c_j^2} \right)^2 \geq 0. \end{aligned}$$

It follows that $M \succeq 0$ and hence $M \in K$.

Moreover, we have $r^T M r = 0$ if and only if $\sqrt{d_i} r_i = \frac{c_i \sum_{j=1}^n c_j r_j}{\sqrt{d_i} \sum_{j=1}^n d_j^{-1} c_j^2}$ for all $i = 1, \dots, n$. An equivalent condition is that $r = \alpha s$ for some scalar α, where $s = (s_1, \dots, s_n)^T$ is a vector given by $s_i = d_i^{-1} c_i$ for all $i = 1, \dots, n$. Hence M is of rank $n-1$, in particular, it is not rank 1.

Let H be the $(n-1)$-dimensional subspace of vectors $v \in \mathbb{R}^n$ such that $v^T s = 0$. Then the minimal face of $\mathcal{S}^n_+$ which contains M is given by $\mathcal{F}_n(H)$. It consists of all matrices $X \in \mathcal{S}^n_+$ such that $Xs = 0$. The linear span $\mathcal{L}_n(H)$ of this face is given by all $X \in \mathcal{S}^n$ such that $Xs = 0$. We shall now compute the intersection $\mathcal{L}_n(H) \cap \tilde{L}$. Let $X = \sum_{i=1}^{n+1} \alpha_i x_i x_i^T \in \mathcal{L}_n(H) \cap \tilde{L}$. Then we have

$$\begin{aligned} Xs &= \sum_{i=1}^{n+1} \alpha_i (x_i^T s) x_i = \sum_{i=1}^{n} \alpha_i s_i x_i + \alpha_{n+1} \cdot \sum_{j=1}^{n} c_j s_j \cdot \sum_{i=1}^{n} c_i x_i \\ &= \sum_{i=1}^{n} \left(\alpha_i d_i^{-1} + \alpha_{n+1} \sum_{j=1}^{n} d_j^{-1} c_j^2 \right) c_i x_i = 0. \end{aligned}$$

It follows that $\alpha_i d_{n+1} = \alpha_{n+1} d_i$ for all $i = 1, \dots, n$. An equivalent condition is that the vectors $\alpha = (\alpha_1, \dots, \alpha_{n+1})^T$ and $d = (d_1, \dots, d_{n+1})^T$ are proportional, and hence X is proportional to M. It follows that $\mathcal{L}_n(H) \cap \tilde{L}$ is the 1-dimensional subspace generated by M.

By Lemma 8 there exist $n-1$ linearly independent vectors $y_1, \dots, y_{n-1} \in \mathbb{R}^n$ such that $y_i y_i^T \in K$ for all i and $M = \sum_{i=1}^{n-1} y_i y_i^T$. Note that $y_i y_i^T \in \mathcal{L}_n(H)$ for all i, and hence $y_i y_i^T \in \mathcal{L}_n(H) \cap L$.

Assume for the sake of contradiction that the extreme ray R_1 generated by the rank 1 matrix $x_1 x_1^T$ is an isolated extreme ray of K. Then there exists $\beta > 0$ such that for every vector $z \in \mathbb{R}^n$, not proportional to x_1 and such that $zz^T \in K$, we have $\sum_{i=2}^n (z^T x_i)^2 > \beta (z^T x_1)^2$.

Since the intersection $\mathcal{L}_n(H) \cap \tilde{L}$ does not contain a rank 1 matrix, $x_1x_1^T \in \tilde{L}$, and $y_iy_i^T \in \mathcal{L}_n(H)$, we have that y_i is not proportional to x_1 for every $i = 1, \dots, n-1$. It follows that $\sum_{j=2}^n (y_i^T x_j)^2 > \beta(y_i^T x_1)^2$ for all $i = 1, \dots, n-1$. Therefore

$$\begin{aligned}\sum_{j=2}^n \left(d_j + d_{n+1}c_j^2\right) &= \sum_{j=2}^n x_j^T M x_j = \sum_{i=1}^{n-1}\sum_{j=2}^n (x_j^T y_i)^2 > \beta \sum_{i=1}^{n-1} (y_i^T x_1)^2 = \beta x_1^T M x_1 \\ &= \beta\left(d_1 + d_{n+1}c_1^2\right). \end{aligned} \tag{7}$$

Fix now $d_2, \dots, d_n$ and let $d_1 \to +\infty$. Then $d_{n+1} \to -\left(\sum_{i=2}^n d_i^{-1}c_i^2\right)^{-1}$, and the leftmost term in (7) tends to a finite value. On the other hand, the rightmost term in (7) tends to $+\infty$, leading to a contradiction.

For the other extreme rays of K the reasoning is similar after an appropriate permutation of the MLD set $\{R_1, \dots, R_{n+1}\}$. □

Corollary 12. *Let $k \geq 2$ and let K be a ROG cone possessing an MLD set $\{R_1, \dots, R_{k+1}\}$ of extreme rays. Then the following holds:*

i) the dimension and degree of K satisfy $\dim K \geq 2k-1$, $\deg K \geq k$;

ii) the extreme rays $R_1, \dots, R_{k+1}$ of K are not isolated.

Proof. Represent K as a linear section of $\mathcal{S}_+^n$ for some n, and let the rank 1 matrices $X_i = x_ix_i^T$ be generators of the extreme rays R_i, $i = 1, \dots, k+1$. Then the set $\{x_1, \dots, x_{k+1}\} \subset \mathbb{R}^n$ is minimally linearly dependent. By Lemma 37 the linear span H of the vectors $x_1, \dots, x_{k+1}$ is a subspace of dimension k. Then $K_H = \mathcal{L}_n(H) \cap K$ is a face of K and hence a ROG cone by Lemma 6. Moreover, K_H contains the rank 1 matrices $x_1x_1^T, \dots, x_{k+1}x_{k+1}^T$ and is of degree k. In particular, the set $\{R_1, \dots, R_{k+1}\}$ is also an MLD set of extreme rays for K_H.

Applying Lemma 40 to the cone K_H, we see that K_H is simple and the extreme rays $R_1, \dots, R_{k+1}$ of K_H are not isolated for all $i = 1, \dots, k+1$. By Theorem 4 we have $\dim K_H \geq 2k-1$. But $\dim K \geq \dim K_H$, $\deg K \geq \deg K_H$, and every extreme ray of K_H is also an extreme ray of K. The claim of the corollary now easily follows. □

Corollary 13. *Let K be a ROG cone of degree n, and let R be an isolated extreme ray of K. Then K can be represented as a direct sum $K' \oplus \mathcal{S}_+^1$, where K' is a ROG cone of degree $n-1$, such that the extreme ray R is given by the set $\{0\} \oplus \mathcal{S}_+^1$.*

Proof. Represent K as a linear section of the cone $\mathcal{S}_+^n$, and let $x \in \mathbb{R}^n$ be such that $X = xx^T$ generates the isolated extreme ray R of K.

Define the set $S = \{y \in \mathbb{R}^n \,|\, yy^T \in K\}$ and note that $x \in S$. By virtue of Corollary 12 the vector x cannot be contained in a minimally linearly dependent subset of S of cardinality at least 3. By Lemma 38 there exists a subspace $H \subset \mathbb{R}^n$ of dimension $n-1$ such that $x \notin H$ and $S \subset H \cup \operatorname{span}\{x\}$.

Hence $\operatorname{span} K = \operatorname{span}\{yy^T \,|\, y \in S\} \subset \mathcal{L}_n(H) + \operatorname{span} R$, and by Lemma 16 we have $K = K' + R$, where $K' = K \cap \mathcal{L}_n(H)$ is the face of K generated by H, and the sum is isomorphic to the direct sum of the summands. By Lemma 14 the cone K' has degree $n-1$. This completes the proof. □

Theorem 5. *Let K be a ROG cone of degree n. Then the number of its isolated extreme rays does not exceed n. Let $R_1,\dots,R_k$ be the isolated extreme rays of K. Then K is isomorphic to a direct sum $K' \oplus \mathbb{R}_+^k$, where K' is a ROG cone of degree $n-k$ without isolated extreme rays, and the extreme rays $R_1,\dots,R_k$ correspond to the extreme rays of the summand $\mathbb{R}_+^k$.*

Proof. We prove the theorem by induction over n. If $n=1$, then $K=\mathbb{R}_+$, and the assertion is evident. Suppose now that $n \geq 2$ and the assertion is proven for cones of degrees not exceeding $n-1$.

If K has no isolated extreme ray, then the assertion of the theorem holds with $K'=K$.

Assume now that R is an isolated extreme ray of K. By Corollary 13 K can be represented as a direct sum $K_1 \oplus \mathbb{R}_+$, where K_1 is a ROG cone of degree $n-1$. By the assumption of the induction, the number of isolated extreme rays of K_1 is finite and does not exceed $n-1$, let these be $\rho_2,\dots,\rho_{k'}$, $1 \leq k' \leq n$. Moreover, K_1 is isomorphic to a direct sum $K' \oplus \mathbb{R}_+^{k'-1}$, where K' is a ROG cone of degree $n-k'$ without isolated extreme rays. It follows that $K \cong K' \oplus \mathbb{R}_+^{k'}$.

Now every extreme ray of the direct sum $K' \oplus \mathbb{R}_+^{k'}$ is either an extreme ray of the factor K' or an extreme ray of the factor $\mathbb{R}_+^{k'}$, and it is isolated in the direct sum if and only if it is isolated in the factor. The extreme rays of K' are not isolated in K', and hence they are not isolated in K. The factor $\mathbb{R}_+^{k'}$ has k' extreme rays, and all of them are isolated. These k' rays hence exhaust the isolated extreme rays of $K' \oplus \mathbb{R}_+^{k'}$. It follows that $k'=k$ and the assertion of the theorem readily follows. $\square$

The discrete and the continuous part of the set of extreme rays of K thus generate separate factors of the cone K. The factor generated by the discrete part is isomorphic to the nonnegative orthant, with the discrete extreme rays of K being its generators.

Corollary 14. *Let K be a simple ROG cone of degree $\deg K \geq 2$. Then K has no isolated extreme rays.*

Proof. The corollary is an immediate consequence of Theorem 5. $\square$

Lemma 41. *Let $K \subset \mathcal{S}_+^n$ be a ROG cone, and let $x \in \mathbb{R}^n$ be such that the rank 1 matrix xx^T generates an extreme ray of K which is not isolated. Then there exists a vector $y \in \mathbb{R}^n$, linearly independent of x, such that $xy^T + yx^T \in \operatorname{span} K$.*

Proof. Assume the conditions of the lemma. Then there exists a sequence $v_1, v_2, \dots$ of nonzero vectors in $\mathbb{R}^n$ such that $x^T v_k = 0$, $(x+v_k)(x+v_k)^T \in K$ for all k, and $\lim_{k\to\infty} v_k = 0$. Set $y_k = \frac{v_k}{||v_k||}$. Then we have $\frac{(x+v_k)(x+v_k)^T - xx^T}{||v_k||} = xy_k^T + y_k x^T + \frac{v_k v_k^T}{||v_k||} \in \operatorname{span} K$. Since $\lim_{k\to\infty} \frac{v_k v_k^T}{||v_k||} = 0$ and $\operatorname{span} K$ is closed, we have $xy^T + yx^T \in \operatorname{span} K$ for every accumulation point of the sequence $y_1, y_2, \dots$. But such accumulation points exist due to the compactness of the unit sphere, and every such accumulation point is orthogonal to x. This completes the proof. $\square$

Corollary 15. *Let $K \subset \mathcal{S}_+^n$ be a simple ROG cone of degree $\deg K \geq 2$. Then for every nonzero vector $x \in \mathbb{R}^n$ such that $xx^T \in K$ there exists a vector $y \in \mathbb{R}^n$, linearly independent of x, such that $xy^T + yx^T \in \operatorname{span} K$.*

Proof. The corollary is an immediate consequence of Lemma 41 and Corollary 14. □

Lemma 42. *Let K be a simple ROG cone of degree $\deg K \geq 2$. If K has a face $F \subset K$ such that $\dim K - \dim F = 2$, then K is isomorphic to an intertwining of F and $\mathcal{S}_+^2$.*

Proof. Assume the conditions of the lemma, and set $n = \deg K$, $k = \deg F$. Represent K as a linear section of the cone $\mathcal{S}_+^n$, and let $X \in F$ be a matrix of maximal rank k. Denote the image of X by H. Then $F = \mathcal{L}_n(H) \cap K$. By the last part of Corollary 4 there exist linearly independent vectors $r_{k+1}, \dots, r_n \in \mathbb{R}^n$ such that $\mathbb{R}^n = \operatorname{span}(H \cup \{r_{k+1}, \dots, r_n\})$ and $r_j r_j^T \in K$, $j = k+1, \dots, n$. We obtain $\dim K \geq \dim F + \dim \operatorname{span}\{r_{k+1}r_{k+1}^T, \dots, r_n r_n^T\} = (\dim K - 2) + (n - k)$. It follows that $k \geq n - 2$. If $k = n - 2$, then $\operatorname{span} K = \operatorname{span} F + \operatorname{span}\{r_{n-1}r_{n-1}^T, r_n r_n^T\}$ and K is isomorphic to the direct sum $F \oplus \mathbb{R}_+^2$, contradicting the simplicity of K.

Hence $k = n - 1$. Set $x = r_n$ for simplicity of notation. By Corollary 15 there exists a nonzero vector $y \in H$ such that $xy^T + yx^T \in \operatorname{span} K$.

Since the codimension of F in K is two, we have $\operatorname{span} K = \operatorname{span} F \oplus \operatorname{span} xx^T \oplus \operatorname{span}(xy^T + yx^T)$. Since K is simple, there exists a vector $z \in \mathbb{R}^n$ such that $z \notin H \cup \operatorname{span}\{x\}$ and $zz^T \in K$. Let $z = z_H + \beta x$ be the decomposition of z corresponding to the direct sum decomposition $\mathbb{R}^n = H \oplus \operatorname{span}\{x\}$. Then $z_H \neq 0, \beta \neq 0, zz^T = z_H z_H^T + \beta(z_H x^T + x z_H^T) + \beta^2 xx^T$. On the other hand, we have the decomposition $zz^T = Z_F + \alpha_1 xx^T + \alpha_2(xy^T + yx^T)$, where $Z_F \in \operatorname{span} F$.

Let l be a linear form which is zero on H, but $l(x) = 1$. Contracting both decompositions of the rank 1 matrix zz^T with l, we obtain $\beta z_H + \beta^2 x = \alpha_1 x + \alpha_2 y$, and hence $\alpha_1 = \beta^2$, $\beta z_H = \alpha_2 y$, $\alpha_2 \neq 0$, $Z_F = z_H z_H^T = (\beta^{-1}\alpha_2)^2 yy^T \in \operatorname{span} F$.

Hence $yy^T \in F \subset K$. Thus $\mathcal{L}_n(\operatorname{span}\{x, y\}) \subset \operatorname{span} K$ and

$$\mathcal{F}_n(\operatorname{span}\{x, y\}) = \mathcal{L}_n(\operatorname{span}\{x, y\}) \cap K$$

is a face of K which is isomorphic to $\mathcal{S}_+^2$. By construction K is an intertwining of the faces F and $\mathcal{F}_n(\operatorname{span}\{x, y\})$, with the intersection $F \cap \mathcal{F}_n(\operatorname{span}\{x, y\})$ generated by yy^T. This yields the assertion of the lemma. □

7 Classification for small degrees

In this section we classify all simple ROG cones K of degree $n = \deg K \leq 4$ up to isomorphism. Denote by Tri_+^n the cone of all tridiagonal matrices in $\mathcal{S}_+^n$.

7.1 Cones of degree $n \leq 3$

For $n = 1$ the only ROG cone is $\mathcal{S}^1_+$.

For $n = 2$ we have the ROG cones $\mathcal{S}^1_+ \oplus \mathcal{S}^1_+$ and $\mathcal{S}^2_+$, of which only the latter is simple.

For $n = 3$ the only ROG cone of dimension 6 is $\mathcal{S}^3_+$, which is simple. By Theorem 4 any other simple ROG cone must have dimension 5, i.e., is given by $K = \{X \in \mathcal{S}^3_+ \mid \langle X, Q\rangle = 0\}$ for some indefinite quadratic form Q. The isomorphism class of K depends only on the signature of Q, and the forms $\pm Q$ define the same cone K. Moreover, every cone K of this form is ROG by Corollary 9. The possible isomorphism classes are hence given by the signatures $(++-)$ and $(+-0)$ of Q. It is easily seen that the corresponding ROG cones are isomorphic to Han^3_+ and the full extension of $\mathcal{S}^1_+ \oplus \mathcal{S}^1_+$, respectively. The latter cone is isomorphic to Tri^3_+.

7.2 Auxiliary results

For the classification of all simple ROG cones of degree $n = 4$ we shall need a couple of auxiliary results.

Lemma 43. *Let $e_1, \dots, e_4$ be the canonical basis vectors of $\mathbb{R}^4$. Let $P \subset \mathbb{R}^4$ be a 2-dimensional subspace which is transversal to all coordinate planes spanned by pairs of basis vectors. Define the set of vectors*

$$\begin{aligned}\mathcal{R} \;=\; & \{\alpha e_i \mid \alpha \in \mathbb{R},\ i = 1,2,3,4\} \quad \cup \\ & \{z = (z_1, z_2, z_3, z_4)^T \in \mathbb{R}^4 \mid z_i \neq 0\ \forall\, i = 1,\dots,4;\ (z_1^{-1}, z_2^{-1}, z_3^{-1}, z_4^{-1})^T \in P\}.\end{aligned}$$

Let $L \subset \mathcal{S}^4$ be the linear span of the set $\{zz^T \mid z \in \mathcal{R}\}$. Then $\dim L = 7$, and the spectrahedral cone $K = L \cap \mathcal{S}^4_+$ is isomorphic to the cone Han^4_+ of positive semi-definite 4×4 Hankel matrices.

Proof. Let $(r_1 \cos\varphi_1, \dots, r_4 \cos\varphi_4)^T, (r_1 \sin\varphi_1, \dots, r_4 \sin\varphi_4)^T \in \mathbb{R}^4$ be two linearly independent vectors spanning P. By the transversality property of P all 2×2 minors of the 4×2 matrix composed of these vectors are nonzero. Hence the angles $\varphi_1, \dots, \varphi_4$ are mutually distinct modulo π, and the scalars $r_1, \dots, r_4$ are nonzero. We may also assume without loss of generality that none of the angles φ_i is a multiple of π, otherwise we choose slightly different basis vectors in P.

For all $\xi \in [0, \pi)$ we then have that the vector $(r_1 \sin(\varphi_1 + \xi), \dots, r_4 \sin(\varphi_4 + \xi))^T$ is an element of P. More precisely, we get

$$\begin{aligned}\mathcal{R} \;=\; & \{\alpha e_i \mid \alpha \in \mathbb{R},\ i = 1,2,3,4\} \;\cup \\ & \cup \left\{\alpha \left(\frac{1}{r_1 \sin(\varphi_1 + \xi)}, \dots, \frac{1}{r_4 \sin(\varphi_4 + \xi)}\right)^T \mid \alpha \in \mathbb{R},\ \xi \neq \varphi_i \mod \pi\right\}.\end{aligned}$$

Now set $\cos\xi = \frac{1-t^2}{1+t^2}$, $\sin\xi = \frac{2t}{1+t^2}$, and $s = t - \frac{1}{t}$. Then $\frac{1}{r_i \sin(\varphi_i + \xi)} = \frac{1+t^2}{r_i t(2\cos\varphi_i - s\sin\varphi_i)}$. Define the vector $\mu(s) = \left(\frac{1}{r_1(2\cos\varphi_1 - s\sin\varphi_1)}, \dots, \frac{1}{r_4(2\cos\varphi_4 - s\sin\varphi_4)}\right)^T$ for all $s \in \mathbb{R}$ except the values $s = 2\cot\varphi_i$, $i = 1, \dots, 4$. We then get

$$\begin{aligned} \mathcal{R} &= \{\alpha e_i \,|\, \alpha \in \mathbb{R},\ i = 1,2,3,4\} \cup \{\alpha\mu(s) \,|\, \alpha \in \mathbb{R},\ s \in \mathbb{R},\ s \neq 2\cot\varphi_i\} \cup \\ & \cup \left\{ \alpha \left(\frac{1}{r_1 \sin\varphi_1}, \dots, \frac{1}{r_4 \sin\varphi_4} \right)^T \,|\, \alpha \in \mathbb{R} \right\}. \end{aligned}$$

Multiplying the vector $\mu(s)$ by the common denominator of its elements, we obtain the vector

$$\begin{aligned} \nu(s) &= \mu(s) \cdot \prod_{i=1}^{4} (2\cos\varphi_i - s\sin\varphi_i) \\ &= \operatorname{diag}(r_1^{-1}, r_2^{-1}, r_3^{-1}, r_4^{-1}) \cdot M \cdot \operatorname{diag}(8, -4, 2, -1) \cdot \eta(s), \end{aligned} \tag{8}$$

where $\eta(s) = (1, s, s^2, s^3)^T$ and the matrix M is given by

$$\begin{pmatrix} \cos\varphi_2\cos\varphi_3\cos\varphi_4 & \sin\varphi_2\sin\varphi_3\sin\varphi_4 + \sin(\varphi_2+\varphi_3+\varphi_4) & \cos\varphi_2\cos\varphi_3\cos\varphi_4 - \cos(\varphi_2+\varphi_3+\varphi_4) & \sin\varphi_2\sin\varphi_3\sin\varphi_4 \\ \cos\varphi_1\cos\varphi_3\cos\varphi_4 & \sin\varphi_1\sin\varphi_3\sin\varphi_4 + \sin(\varphi_1+\varphi_3+\varphi_4) & \cos\varphi_1\cos\varphi_3\cos\varphi_4 - \cos(\varphi_1+\varphi_3+\varphi_4) & \sin\varphi_1\sin\varphi_3\sin\varphi_4 \\ \cos\varphi_1\cos\varphi_2\cos\varphi_4 & \sin\varphi_1\sin\varphi_2\sin\varphi_4 + \sin(\varphi_1+\varphi_2+\varphi_4) & \cos\varphi_1\cos\varphi_2\cos\varphi_4 - \cos(\varphi_1+\varphi_2+\varphi_4) & \sin\varphi_1\sin\varphi_2\sin\varphi_4 \\ \cos\varphi_1\cos\varphi_2\cos\varphi_3 & \sin\varphi_1\sin\varphi_2\sin\varphi_3 + \sin(\varphi_1+\varphi_2+\varphi_3) & \cos\varphi_1\cos\varphi_2\cos\varphi_3 - \cos(\varphi_1+\varphi_2+\varphi_3) & \sin\varphi_1\sin\varphi_2\sin\varphi_3 \end{pmatrix}.$$

Here for the calculus of M we used the formulas

$$\begin{aligned} \sin\varphi_i\cos\varphi_j\cos\varphi_k + \sin\varphi_j\cos\varphi_i\cos\varphi_k + \sin\varphi_k\cos\varphi_i\cos\varphi_j &= \sin\varphi_i\sin\varphi_j\sin\varphi_k + \sin(\varphi_i+\varphi_j+\varphi_k), \\ \sin\varphi_2\sin\varphi_3\cos\varphi_4 + \sin\varphi_2\sin\varphi_4\cos\varphi_3 + \sin\varphi_3\sin\varphi_4\cos\varphi_2 &= \cos\varphi_i\cos\varphi_j\cos\varphi_k - \cos(\varphi_i+\varphi_j+\varphi_k). \end{aligned}$$

Note that the vector $\nu(s)$ can also be defined by the right-hand side of (8) for $s = 2\cot\varphi_i$, and for this value of s it is proportional to e_i. Defining $\eta(\infty) = e_4$ and

$$\nu(\infty) = \operatorname{diag}(r_1^{-1}, r_2^{-1}, r_3^{-1}, r_4^{-1}) \cdot M \cdot \operatorname{diag}(8, -4, 2, -1) \cdot \eta(\infty),$$

we finally get

$$\mathcal{R} = \{\alpha\nu(s) \,|\, \alpha \in \mathbb{R},\ s \in \mathbb{R} \cup \{\infty\}\}.$$

A symbolic computation with a computer algebra system yields

$$\det M = \sin(\varphi_1-\varphi_2)\sin(\varphi_1-\varphi_3)\sin(\varphi_1-\varphi_4)\sin(\varphi_2-\varphi_3)\sin(\varphi_2-\varphi_4)\sin(\varphi_3-\varphi_4) \neq 0.$$

Hence the matrix product $\operatorname{diag}(r_1^{-1}, r_2^{-1}, r_3^{-1}, r_4^{-1}) \cdot M \cdot \operatorname{diag}(8, -4, 2, -1)$ is non-degenerate, and the subspace $L = \operatorname{span}\{zz^T \,|\, z = \nu(s),\ s \in \mathbb{R} \cup \{\infty\}\}$ is isomorphic to the subspace $L' = \operatorname{span}\{zz^T \,|\, z = \eta(s),\ s \in \mathbb{R} \cup \{\infty\}\}$.

The subspace L', however, is the subspace of Hankel matrices in $\mathcal{S}^4$. The claim of the lemma now easily follows. $\square$

Lemma 44. *Let $K \subset \mathcal{S}_+^4$ be a simple ROG cone of dimension 7 and degree 4. Suppose that the subspace of block-diagonal matrices consisting of two blocks of size 2×2 each is contained in $\operatorname{span} K$. Then K is isomorphic to the cone Tri_+^4 of positive semi-definite tridiagonal matrices.*

Proof. The subspace of block-diagonal matrices as defined in the formulation of the lemma is 6-dimensional. Hence there exist scalars $a_{13}, a_{14}, a_{23}, a_{24}$, not all equal zero, such that the linear span of K is given by all matrices of the form

$$A = \sum_{i=1}^{7} \alpha_i A_i = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_7 a_{13} & \alpha_7 a_{14} \\ \alpha_2 & \alpha_3 & \alpha_7 a_{23} & \alpha_7 a_{24} \\ \alpha_7 a_{13} & \alpha_7 a_{23} & \alpha_4 & \alpha_5 \\ \alpha_7 a_{14} & \alpha_7 a_{24} & \alpha_5 & \alpha_6 \end{pmatrix}, \quad \alpha_1, \dots, \alpha_7 \in \mathbb{R},$$

where the matrices $A_1, \dots, A_7$ are defined by the above identity. Since K is ROG, there exists a rank 1 matrix $Z = zz^T = \sum_{i=1}^7 \zeta_i A_i$ with $\zeta_7 \neq 0$. Its upper right 2×2 block is also rank 1. Hence $a_{13}a_{24} = a_{14}a_{23}$ and there exist angles φ_1, φ_2 and a positive scalar r such that $a_{13} = r\cos\varphi_1\cos\varphi_2$, $a_{14} = r\cos\varphi_1\sin\varphi_2$, $a_{23} = r\sin\varphi_1\cos\varphi_2$, $a_{24} = r\sin\varphi_1\sin\varphi_2$. Define a basis of $\mathbb{R}^4$ by the vectors $x_1 = (-\sin\varphi_1, \cos\varphi_1, 0, 0)^T$, $x_2 = (\cos\varphi_1, \sin\varphi_1, 0, 0)^T$, $x_3 = (0, 0, \cos\varphi_2, \sin\varphi_2)^T$, $x_4 = (0, 0, -\sin\varphi_2, \cos\varphi_2)^T$. In the coordinates given by this basis K equals the cone Tri_+^4, which proves our claim. □

Lemma 45. *Let $K \subset \mathcal{S}_+^n$ be a ROG cone, let $e_1, \dots, e_n$ be the canonical basis vectors of $\mathbb{R}^n$, and let $y = (0, y_2, \dots, y_n)^T \in \mathbb{R}^n$ be a vector such that $y_2, \dots, y_n \neq 0$. If $e_1e_1^T, \dots, e_ne_n^T$, $e_1y^T + ye_1^T \in \operatorname{span} K$, then K is simple and $\dim K \geq 2n - 1$.*

Proof. Suppose for the sake of contradiction that K is not simple. Then there exists a nontrivial direct sum decomposition $\mathbb{R}^n = H_1 \oplus H_2$ such that for every rank 1 matrix $xx^T \in K$ we have either $x \in H_1$ or $x \in H_2$. Hence $e_i \in H_1 \cup H_2$ for $i = 1, \dots, n$. It follows that H_1, H_2 are spanned by complementary subsets of the canonical basis of $\mathbb{R}^n$. Hence there exists a permutation of the basis vectors such that in the corresponding coordinate system every matrix in K, and hence also in $\operatorname{span} K$, becomes block-diagonal with a nontrivial block structure. But this is in contradiction with the assumption $e_1y^T + ye_1^T \in \operatorname{span} K$. Hence K must be simple.

Since the identity matrix is an element of K, we have $\deg K = n$. The bound on the dimension now follows from Theorem 4. □

Corollary 16. *Let $K \subset \mathcal{S}_+^4$ be a simple ROG cone of dimension 7 and degree 4. Suppose there exist linearly independent vectors $z_1, z_2, z_3 \in \mathbb{R}^4$ and nonzero scalars α, β such that $z_1z_1^T$, $z_2z_2^T$, $z_3z_3^T$, $\alpha(z_1z_2^T + z_2z_1^T) + \beta(z_1z_3^T + z_3z_1^T) \in \operatorname{span} K$. Then K is isomorphic to either Tri_+^4, or the full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1 \oplus \mathcal{S}_+^1$, or an intertwining of Han_+^3 and $\mathcal{S}_+^2$.*

Proof. Denote by $H \subset \mathbb{R}^4$ the hyperplane spanned by z_1, z_2, z_3. The face $F = \mathcal{L}_4(H) \cap K$ of K is a ROG cone by Lemma 6. Applying Lemma 45 to F, we obtain $\dim F \geq 5$.

However, $\dim F \neq 6$, because a ROG cone K with a face F of codimension 1 in K is isomorphic to $F \oplus \mathcal{S}_+^1$ and hence not simple. Therefore F has codimension 2 in K. By Lemma 42 the cone K is isomorphic to an intertwining of F with $\mathcal{S}_+^2$.

In the previous subsection we established that a simple ROG cone of dimension 5 and degree 3 is isomorphic to either Han_+^3 or Tri_+^3. If $F \cong \mathrm{Han}_+^3$, then K is isomorphic to an intertwining of Han_+^3 and $\mathcal{S}_+^2$. If $F \cong \mathrm{Tri}_+^3$, then there exist two possibilities for K, because Tri_+^3 has two non-isomorphic types of extreme rays. It is not hard to see that an intertwining of Tri_+^3 with $\mathcal{S}_+^2$ along these two types of extreme rays leads to cones which are isomorphic to Tri_+^4 or the full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1 \oplus \mathcal{S}_+^1$, respectively. □

Lemma 46. *Let $K \subset \mathcal{S}_+^4$ be a simple ROG cone of dimension 7 and degree 4. Suppose that K has a face which is isomorphic to $\mathcal{S}_+^2$. Then K fulfills the conditions of Corollary 16.*

Proof. By assumption there exist linearly independent vectors $x_1, x_2 \in \mathbb{R}^4$ such that $x_1x_1^T$, $x_2x_2^T$, $x_1x_2^T + x_2x_1^T \in \operatorname{span} K$. By Corollary 4 we may complete x_1, x_2 with vectors x_3, x_4 to a basis of $\mathbb{R}^4$ such that $x_3x_3^T, x_4x_4^T \in K$. Pass to the coordinate system defined by this basis.

By Corollary 15 there exists a nonzero vector $y = (y_1, y_2, 0, y_4)^T$ such that $x_3y^T + yx_3^T \in \operatorname{span} K$.

If $y_1 = y_2 = 0$, then $x_3x_4^T + x_4x_3^T \in \operatorname{span} K$ and K fulfills the conditions of Lemma 44. Hence K is isomorphic to Tri_+^4. The claim of the lemma then immediately follows in this case.

Suppose now that y_1, y_2 are not simultaneously zero.

Let us first consider the case $y_4 = 0$. Let $F = \mathcal{L}_4(\operatorname{span}\{x_1, x_2, x_3\}) \cap K$ be the face of K which consists of matrices $X \in K$ whose last column vanishes. Then $x_1x_1^T, x_2x_2^T, x_1x_2^T + x_2x_1^T, x_3x_3^T, x_3y^T + yx_3^T \in \operatorname{span} F$, and $\dim F \geq 5$. Since $\dim F = 6$ is not possible by the simplicity of K, we then must have $\operatorname{span} F = \operatorname{span}\{x_1x_1^T, x_2x_2^T, x_1x_2^T + x_2x_1^T, x_3x_3^T, x_3y^T + yx_3^T\}$. It follows that F is isomorphic to Tri_+^3. From Lemma 42 it follows that K is isomorphic to an intertwining of Tri_+^3 and $\mathcal{S}_+^2$, which proves the claim of the lemma in this case.

Suppose now that $y_4 \neq 0$. Define the nonzero vector $z_3 = (y_1, y_2, 0, 0)^T$. Then $z_3z_3^T \in K$ and $y = \alpha x_4 + \beta z_3$ with $\alpha = y_4$ and $\beta = 1$. The linearly independent vectors $z_1 = x_3$, $z_2 = x_4$, z_3, and scalars α, β then satisfy the conditions of Corollary 16. $\square$

We are now in a position to classify the simple ROG cones of degree 4.

7.3 Cones of degree 4

Let K be a ROG cone of degree $\deg K = 4$.

If $\dim K = 10$, then $K = \mathcal{S}_+^4$.

If $\dim K = 9$, then K is of the form $\{X \in \mathcal{S}_+^3 \mid \langle X, Q \rangle = 0\}$ for some indefinite quadratic form Q. As in the case $n = 3$, the isomorphism class of K is defined by the signature of Q, where $\pm Q$ yields the same cone K. The possible isomorphism classes of K are then defined by the signatures $(+-00)$, $(++-0)$, $(++--)$, and $(+++-)$ of Q. In the first two cases K is a full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1$ and Han_+^3, respectively. In the third case K is isomorphic to the cone of positive semi-definite 4×4 matrices consisting of four 2×2 symmetric blocks. It can be interpreted as the moment cone of the homogeneous biquadratic forms on $\mathbb{R}^2 \times \mathbb{R}^2$. It is not hard to see that all four isomorphism classes consist of simple cones.

Let $\dim K = 8$. If K is simple, then by Theorem 3 it is isomorphic to a full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^2$.

By Theorem 4 any other simple ROG cone must have dimension 7.

Theorem 6. *Let K be a simple ROG cone of degree $\deg K = 4$ and dimension $\dim K = 7$. Then K is isomorphic to either Tri_+^4, or the full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1 \oplus \mathcal{S}_+^1$, or an intertwining of Han_+^3 and $\mathcal{S}_+^2$, or Han_+^4.*

Proof. Let $K \subset \mathcal{S}_+^4$ be a simple ROG cone of degree 4 and dimension 7. By Corollary 4 there exist linearly independent vectors x_1, x_2, x_3, x_4 such that $x_ix_i^T \in K$, $i = 1, \dots, 4$. Pass to the coordinate system defined by the basis $\{x_1, x_2, x_3, x_4\}$. Then all diagonal matrices are in the linear span of K. Moreover, by Corollary 15 there exist nonzero vectors $y_i = (y_{i1}, y_{i2}, y_{i3}, y_{i4})^T$ such that $x_i^Ty_i = y_{ii} = 0$ and $x_iy_i^T + y_ix_i^T \in \operatorname{span} K$, $i = 1, 2, 3, 4$. Therefore $\operatorname{span} K$

contains all matrices of the form

$$\begin{pmatrix} \alpha_1 & \alpha_5 y_{12} + \alpha_6 y_{21} & \alpha_5 y_{13} + \alpha_7 y_{31} & \alpha_5 y_{14} + \alpha_8 y_{41} \\ \alpha_5 y_{12} + \alpha_6 y_{21} & \alpha_2 & \alpha_6 y_{23} + \alpha_7 y_{32} & \alpha_6 y_{24} + \alpha_8 y_{42} \\ \alpha_5 y_{13} + \alpha_7 y_{31} & \alpha_6 y_{23} + \alpha_7 y_{32} & \alpha_3 & \alpha_7 y_{34} + \alpha_8 y_{43} \\ \alpha_5 y_{14} + \alpha_8 y_{41} & \alpha_6 y_{24} + \alpha_8 y_{42} & \alpha_7 y_{34} + \alpha_8 y_{43} & \alpha_4 \end{pmatrix}, \qquad \alpha_1, \dots, \alpha_8 \in \mathbb{R}. \tag{9}$$

Since the dimension of $\operatorname{span} K$ is 7, the matrices at the coefficients $\alpha_1, \dots, \alpha_8$ must be linearly dependent. This is equivalent to the condition that the matrix

$$Y = \begin{pmatrix} y_{12} & y_{21} & 0 & 0 \\ y_{13} & 0 & y_{31} & 0 \\ y_{14} & 0 & 0 & y_{41} \\ 0 & y_{23} & y_{32} & 0 \\ 0 & y_{24} & 0 & y_{42} \\ 0 & 0 & y_{34} & y_{43} \end{pmatrix} \tag{10}$$

is rank-deficient, $\operatorname{rk} Y \leq 3$. Here the rows of Y correspond to the elements $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$, $(3,4)$ of (9), respectively, and the columns to the expressions at the coefficients $\alpha_5, \dots, \alpha_8$, respectively. By construction every column of Y is nonzero.

If there exists a column of Y with exactly one nonzero element, let it be y_{ij}, then $x_i x_i^T$, $x_j x_j^T$, $x_i x_j^T + x_j x_i^T \in \operatorname{span} K$ and K has a face which is isomorphic to $\mathcal{S}_+^2$. By Lemma 46 the cone K is then isomorphic to either Tri_+^4, or the full extension of $\mathcal{S}_+^1 \oplus \mathcal{S}_+^1 \oplus \mathcal{S}_+^1$, or an intertwining of Han_+^3 and $\mathcal{S}_+^2$.

If there exists a column of Y with exactly two nonzero elements, let them be y_{ij}, y_{ik}, then the linearly independent vectors $z_1 = x_i$, $z_2 = x_j$, $z_3 = x_k$ and scalars $\alpha = y_{ij}$, $\beta = y_{ik}$ satisfy the conditions of Corollary 16, and K is again isomorphic to one of the aforementioned cones.

Let us now assume that all elements y_{ij} for $i \neq j$ are nonzero. Then $\operatorname{rk} Y = 3$, and the subspace spanned by the set $\{x_i x_i^T, x_i y_i^T + y_i x_i^T \mid i = 1,2,3,4\}$ has dimension 7. Since this subspace is contained in $\operatorname{span} K$, it must actually equal $\operatorname{span} K$. There exists a nonzero vector $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)^T$ such that $Y\beta = 0$. It is easy to see that no three columns of Y can be linearly dependent, and hence all elements β_i are nonzero. By possibly multiplying y_i by the nonzero constant β_i, we may assume without loss of generality that $\beta = (1,1,1,1)^T$. Then $y_{ij} = -y_{ji}$ for all $i, j = 1, \dots, 4$, $i \neq j$.

It is not hard to check that $\operatorname{span} K$ can then alternatively be written as the set $\{X \in \mathcal{S}^4 \mid \langle X, Q_i \rangle = 0,\ i = 1,2,3\}$, where the linearly independent quadratic forms Q_1, Q_2, Q_3 are given by

$$\begin{pmatrix} 0 & y_{13}y_{23} & -y_{12}y_{23} & 0 \\ y_{13}y_{23} & 0 & y_{12}y_{13} & 0 \\ -y_{12}y_{23} & y_{12}y_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & y_{14}y_{24} & 0 & -y_{12}y_{24} \\ y_{14}y_{24} & 0 & 0 & y_{12}y_{14} \\ 0 & 0 & 0 & 0 \\ -y_{12}y_{24} & y_{12}y_{14} & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & y_{14}y_{34} & -y_{13}y_{34} \\ 0 & 0 & 0 & 0 \\ y_{14}y_{34} & 0 & 0 & y_{13}y_{14} \\ -y_{13}y_{34} & 0 & y_{13}y_{14} & 0 \end{pmatrix},$$

respectively.

The rank 1 matrices in K are then given by zz^T such that $z \neq 0$ and $z^T Q_i z = 0$ for $i = 1, 2, 3$. Let us determine the set of vectors $z = (z_1, z_2, z_3, z_4)^T$ which satisfy this quadratic system of equations. It is not hard to see that if a solution z is not equal to a canonical basis vector, then all elements of z are nonzero. For such z the quadratic system can be written as

$$\begin{aligned} y_{23}^{-1} z_1^{-1} - y_{13}^{-1} z_2^{-1} + y_{12}^{-1} z_3^{-1} &= 0, \\ y_{24}^{-1} z_1^{-1} - y_{14}^{-1} z_2^{-1} + y_{12}^{-1} z_4^{-1} &= 0, \\ y_{34}^{-1} z_1^{-1} - y_{14}^{-1} z_3^{-1} + y_{13}^{-1} z_4^{-1} &= 0. \end{aligned} \tag{11}$$

This is a linear system in the unknowns z_i^{-1}. If the coefficient matrix of this system is full rank, then the solution $(z_1^{-1}, \dots, z_4^{-1})$ is proportional to $(0, y_{12}^{-1}, y_{13}^{-1}, y_{14}^{-1})$ and does not correspond to a real vector z. In this case the only rank 1 matrices in the subspace $\operatorname{span} K$ are the matrices $x_i x_i^T$, $i = 1, \dots, 4$, and K is not ROG.

Thus the coefficient matrix of system (11) is rank deficient. This implies that all 3×3 minors of this matrix vanish, which leads to the condition $y_{14}^{-1} y_{23}^{-1} - y_{13}^{-1} y_{24}^{-1} + y_{12}^{-1} y_{34}^{-1} = 0$. The general solution of system (11) is then given by

$$\begin{pmatrix} z_1^{-1} \\ z_2^{-1} \\ z_3^{-1} \\ z_4^{-1} \end{pmatrix} = \gamma_1 \begin{pmatrix} y_{12}^{-2} + y_{13}^{-2} + y_{14}^{-2} \\ y_{13}^{-1} y_{23}^{-1} + y_{14}^{-1} y_{24}^{-1} \\ y_{14}^{-1} y_{34}^{-1} - y_{12}^{-1} y_{23}^{-1} \\ -y_{12}^{-1} y_{24}^{-1} - y_{13}^{-1} y_{34}^{-1} \end{pmatrix} + \gamma_2 \begin{pmatrix} 0 \\ y_{12}^{-1} \\ y_{13}^{-1} \\ y_{14}^{-1} \end{pmatrix}, \qquad \gamma_1, \gamma_2 \in \mathbb{R}.$$

It can be checked by direct calculation that none of the 2×2 minors of the 4×2 matrix composed of the two vectors at γ_1, γ_2, respectively, vanishes. Hence the 2-dimensional subspace of solutions of system (11) is transversal to all coordinate planes spanned by pairs of canonical basis vectors. By Lemma 43 the cone K is then isomorphic to Han_+^4. □

8 Conclusions and open questions

In this contribution we have defined and considered a special class of spectrahedral cones, the rank 1 generated cones. These cones are characterized by Property 1. They have applications in optimization, namely for the approximation of difficult optimization problems by semi-definite programs, in the common case where the semi-definite program is obtained by dropping a rank 1 constraint on the matrix-valued decision variable. They are closely linked to the property of such a semi-definite relaxation being exact.

We provided many examples of ROG cones and several structural results. One of the main results has been that the geometry of a ROG cone as a convex conic subset of a real vector space uniquely determines its representation as a linear section of the positive semi-definite matrix cone, if this representation is required to satisfy Property 1, up to isomorphism (Theorem 2). In particular, every point of the cone has the same rank in every such representation. The rank also equals its Carathéodory number (Lemma 8). The Carathéodory number of the cone itself equals its degree as an algebraic interior (Corollary 3).

There exist surprisingly many ROG cones. This is due to the fact that there are several non-trivial ways to construct ROG cones of higher degree out of ROG cones of lower degree, which we have called full extensions (Subsection 3.2) and intertwinings (Subsection 3.3). Besides, there is the obvious way of taking direct sums (Subsection 3.1). Iterating these procedures, one may obtain families of mutually non-isomorphic ROG cones with arbitrarily many real parameters. One may call ROG cones that are neither direct sums nor intertwinings nor full extensions of other ROG cones *elementary*. Examples of elementary ROG cones are the cones of positive semi-definite Hankel matrices and the cones $K = \{X \in \mathcal{S}_+^n \mid \langle X, Q \rangle = 0\}$ of codimension 1 (Subsection 5.1), where Q is an indefinite non-degenerate quadratic form. Besides these infinite series of elementary ROG cones, there exists the exceptional moment cone of the ternary quartics of dimension 15 and degree 6. It is unknown whether there exist other elementary cones.

We classified the isomorphism classes of simple ROG cones, i.e., those not representable as non-trivial direct sums, up to degree 4. There are 1,1,3,10 equivalence classes of such cones for degrees 1,2,3,4, respectively.

The set of extreme rays of a ROG cone defines a real projective variety (Corollary 6). The varieties defined by direct sums or intertwinings are finite unions of other projective varieties. The classification of the irreducible varieties defined by ROG cones is an open question. It would follow from a classification of the elementary ROG cones.

In this contribution we dealt with real symmetric matrices. The concept of ROG cones can equally well be defined for complex hermitian or quaternionic hermitian matrices.

Acknowledgements

The author kindly acknowledges support from the ANR GéoLMI of the French National Research Agency. A short version of the paper was presented at the Oberwolfach Workshop 1415 "Real Algebraic Geometry with a view toward Systems Control and Free Positivity".

References

[1] Jim Agler, J. William Helton, Scott McCullough, and Leiba Rodman. Positive semidefinite matrices with a given sparsity pattern. *Linear Algebra Appl.*, 107:101–149, 1988.

[2] Nir Cohen, Charles R. Johnson, Leiba Rodman, and Hugo J. Woerdeman. Ranks of completions of partial matrices. *Oper. Th. Adv. Appl.*, 40:165–185, 1989.

[3] M.R. Garey and D.S. Johnson. *Computers and Intractability: A guide to the Theory of NP-Completeness*. Freeman, San Francisco, 1979.

[4] Michel X. Goemans and David P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. Assoc. Comput. Mach.*, 42(6):1115–1145, Nov 1995.

[5] Osman Güler and Levent Tunçel. Characterization of the barrier parameter of homogeneous convex cones. *Math. Program.*, 81(1):55–76, 1998.

[6] Don Hadwin, K.J. Harrison, and J.A. Ward. Rank-one completions of partial matrices and completely rank-nonincreasing linear functionals. *P. Am. Math. Soc.*, 134(8):2169–2178, 2006.

[7] J. William Helton and Victor Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Applied Math.*, 60(5):654–674, 2007.

[8] David Hilbert. Über die Darstellung definiter Formen als Summe von Formenquadraten. *Mathematische Annalen*, 32:342–350, 1888.

[9] Monique Laurent. On the sparsity order of a graph and its deficiency in chordality. *Combinatorica*, 21(4):543–570, 2001.

[10] Zhi-Quan Luo, Wing-Kin Ma, Anthony Man-Cho So, Yinyu Ye, and Shuzhong Zhang. Semidefinite relaxation of quadratic optimization problems. *IEEE Signal Proc. Mag.*, 27(3):20–34, 2010.

[11] Yuri Nesterov. Squared functional systems and optimization problems. In H. Frenk, K. Roos, T. Terlaky, and S. Zhang, editors, *High Performance Optimization*, chapter 17, pages 405–440. Kluwer Academic Press, Dordrecht, 2000.

[12] Vern I. Paulsen, Stephen C. Power, and Roger R. Smith. Schur products and matrix completions. *J. Funct. Anal.*, 85:151–178, 1987.

[13] Motakuri Ramana and A.J. Goldman. Some geometric results in semidefinite programming. *J. Global Optim.*, 7:33–50, 1995.

[14] Bernard Ycart. Extrémales du cône des matrices de type non négatif, à coefficients positifs ou nuls. *Linear Algebra Appl.*, 48:317–330, 1982.