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Optimal Sobolev regularity for linear second-order divergence elliptic operators occurring in real-world problems

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Abstract. On bounded domains $\Omega \subset \mathbb{R}^3$, we consider divergence-type operators $-\nabla \cdot \mu\nabla$, including mixed homogeneous Dirichlet and Neumann boundary conditions on $\partial\Omega \setminus \Gamma$ and $\Gamma \subset \partial\Omega$, respectively, and discontinuous coefficient functions μ. We develop a general geometric framework for Ω, Γ and μ in which it is possible to prove that $-\nabla \cdot \mu\nabla + 1$ provides an isomorphism from $W^{1,q}_\Gamma(\Omega)$ to $W^{-1,q}_\Gamma(\Omega)$ for some $q > 3$. We indicate relevant examples from real-world applications.

1. Introduction. In the modelling of real-world problems, one is often confronted with elliptic and parabolic differential equations which act on nonsmooth domains Ω, possess discontinuous coefficients and/or are complemented by mixed boundary conditions. For instance, in simulations of electron transport in semiconductor devices the latter are unavoidable — or the model is meaningless [74]. When treating — mostly nonlinear — models from different application areas which manifest such phenomena (see e.g. [55], [18] [45], [62], [14], [22], [27], [24], [46], [42]), an isomorphism theorem of Gröger ([31], see also [32]) turned out to be of great use. It states that any divergence-type operator $-\nabla \cdot \mu\nabla + 1$ provides a topological isomorphism between $W^{1,q}_\Gamma(\Omega)$ and $W^{-1,q}_\Gamma(\Omega)$ for some $q > 2$, provided that μ is bounded and elliptic, and Ω and the Neumann boundary part $\Gamma \subset \partial\Omega$ satisfy some minimal regularity properties. It is well-known that the upper bound for possible q's depends on the domain Ω (see [43]), on the coefficient function μ (see [59]), and on Γ (see [63]). In general, it exceeds 2 by an arbitrarily small margin only, see [21, Ch. 4] for a striking example. This is exactly what in fact restricts the applicability of Gröger's theorem — in this generality — more or less to two-dimensional problems. In the meanwhile, the necessity grows to consider not only two-dimensional problems (mostly as cuts of the original three-dimensional ones) but the three-dimensional models themselves. In particular, this is true in device modelling (see [26], [56]) and in many of the above mentioned applications. An analogue of Gröger's theorem is thus desirable; namely to find a class of three-dimensional domains Ω, coefficient functions μ and Dirichlet boundary parts $D \overset{\text{def}}{=} \partial\Omega \setminus \Gamma$, so that the operator

$$-\nabla \cdot \mu\nabla + 1 : W^{1,q}_\Gamma(\Omega) \to W^{-1,q}_\Gamma(\Omega) \tag{1.1}$$

provides a topological isomorphism for some $q > 3$. The number 3 here has the meaning of the underlying space dimension — which is crucial in many aspects. For instance, Gajewski and Gröger observed already in [23] that the additional knowledge concerning the gradient of the electrostatic potential to lie in $L^{3+\epsilon}$ would lead to a satisfactory analysis of the 3-dimensional van-Roosbroeck system, which models electron transport in semiconductors.

In order to cover real-world situations, the special features of such a setting should be as mentioned at the beginning; but the essential point is that several of these nonsmooth phenomena should be allowed to meet in one point. As far as we know, in this complexity, the isomorphism property in (1.1) has never been treated before in the literature (see e.g. [75], [17], [19], [7], [10], [63], [72], [43], [81] and compare also [11], [15], [50], [52], [57], [60], [66], [67] and [71]). In this article, one of our aims is to provide the reader with a wide variety of explicit geometric configurations for domains, including mixed Dirichlet and Neumann boundary parts and heterostructures, for which (1.1) is a topological isomorphism. This makes it possible to decide 'by appearance' for many settings, whether they fall into this class — in this sense, our approach can serve as a black box for future applications. In the following subsection, let us discuss in some more detail the motivation for our work coming from real-world applications. In Subsection 1.2, we then discuss our approach from a mathematical point of view.

1.1. Geometric material constellations from science and technology. In principle, our setting is oriented towards the requirements of modern technology simulations, in particular, for semiconductors. Thus, we primarily have in mind polyhedral domains and domains which result by (local) C^1-deformations of polyhedra. A simple structure of this type is the three-dimensional L-shape, which may be composed of two different materials, compare [36]. It is often regarded as a benchmark problem for numerical simulations, cf. [15, Fig. 2], [67, Fig. 1], and it appears naturally in device modelling, such as the three-dimensional thermistor or the quantum well laser, see Figure 1 (left and centre). Similarly, the *Fichera cube* may be included in our setting. In general, the constellations we consider are not required to be *strong* Lipschitz domains. For a striking example, we refer to the wood-pile structure of photonic crystals shown in Figure 1 (right). For details on this specific topic, see [44, p. 100 ff.]; and for a modelling of photonic crystals which includes elliptic operators, see [70]. On the other hand, our setting possibly excludes applications from biology where the domain itself may vary with time and forms geometries such as cusps, see [3], [2] for an alternative approach and compare also [49].

The material discontinuities (heterostructures) we allow for are of a *layered* type, where this term is to be understood in a very broad sense, cf. Assumption 4.2. In particular, the meeting of three (or more) materials is not allowed to happen within the domain, but may occur on the boundary in special cases, cf. Figure 4. Moreover, the domains of continuity of the coefficient function may not admit vertices or edges within the domain. On the other hand, we allow e.g. for smoothly bounded material inclusions in a different host material. An example is given in Figure 2 by spherical Ag nanoparticles embedded in an ZnO:Al environment in Thin-Film Solar Cells, see [68, Figure 5.2].

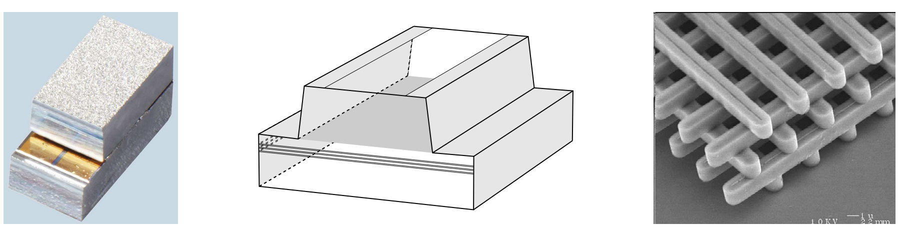


Fig. 1. *Left: Micropelt Thinfilm Peltier Cooler MPC-D403, Courtesy Micropelt. — Centre: Scheme of a ridge waveguide quantum well laser (detail $3.2\mu m \times 1.5\mu m \times 4\mu m$). A material interface (darkly shaded) and a boundary part (lightly shaded) carrying Neumann boundary conditions meet at an edge of the device domain. At the bottom and the top of the structure are contacts giving rise to Dirichlet boundary conditions for the electrostatic potential in the electronic simulation of the laser, while other parts of the device are insulated (Neumann boundary conditions). A triple quantum well structure is indicated where the light beam forms in the symmetry plane of the domain. — Right: 3D photonic crystal. Courtesy Sandia National Laboratories.*

1.2. A mathematical point of view on our approach. Our treatment of settings which include discontinuous coefficient functions heavily rests on previous results on model constellations, provided in [20], [21] [35], [36], [33], [47]. All these insights are based on [59] and, in essence, a detailed investigation of the occurring edge singularities. However, our intention was to avoid the explicit study of elliptic singularities in this work. Our program is rather ambitious nevertheless: while for the proof of the isomorphy property (1.1) for q close to 2, intelligent perturbation arguments for the case $q = 2$ are sufficient (see [31], compare also [9] and [34]), one is

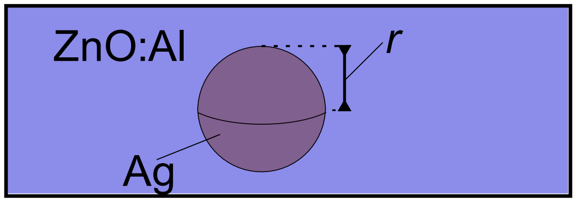


Fig. 2. *Spherical Ag nanoparticles embedded in ZnO:Al in Thin-Film Solar Cells, [68, Figure 5.2],* `http://darwin.bth.rwth-aachen.de/opus3/volltexte/2013/4602/pdf/4602.pdf`*. Courtesy Ulrich Wilhelm Paetzold and Forschungszentrum Jülich GmbH.*

confronted with difficulties of a different quality when aiming at $q > 3$. The overall strategy will be to collect a set of suitable model constellations for which one can show the isomorphism property (1.1) for some $q > 3$ and then generalize by adequate permanence principles. The first and perhaps most essential one of these principles is localization, already used in [31].

In short, the study of our model constellations is based on the following previous insights:

(i) Maz'ya's pioneering theorem [59, Thm. 2.3], which states that for polyhedral domains and coefficient functions which are constant on polyhedral subdomains, the isomorphism property

$$-\nabla \cdot \mu \nabla : W_0^{1,q}(\Omega) \to W^{-1,q}(\Omega)$$

is implied for some $q > 3$, if all occurring elliptic edge singularity exponents are larger than $\frac{1}{3}$,

(ii) the (nontrivial) fact that bimaterial outer edges satisfy this condition, if the surrounding boundary carries a pure Dirichlet or a pure Neumann condition, see [20, Thm. 2.1] or [35, Appendix],

(iii) a highly nontrivial deformation argument [33, Ch. 3], deduced by means of piecewise linear geometric topology,

(iv) the fact that edge singularities are invariant under continuous, piecewise linear transformations, see [33, Thm. 4.15],

(v) interpolation properties (see [28]) of the scales

$$\{W_\Gamma^{1,p}(\Omega)\}_{p \in]1,\infty[} \quad \text{and} \quad \{W_\Gamma^{-1,p}(\Omega)\}_{p \in]1,\infty[},$$

(vi) some subtle reflection arguments [35, Prop. 17 and Lemma 22],

(vii) Elschner's theorem on the isomorphism property

$$-\nabla \cdot \mu \nabla + 1 : W^{1,q}(\mathbb{R}^d) \to W^{-1,q}(\mathbb{R}^d), \quad q \in]1, \infty[,$$

if the coefficient function μ is constant on one half space and also on its complement, see [21, Thm 3.11] and compare also [5, Ch. 4.5].

Let us next discuss the question why spaces of $W^{-1,q}$-type are adequate for elliptic and parabolic problems. Of course, if the right hand side of the equation has a Lebesgue density in the volume of the underlying domain, and if the boundary condition is either homogeneous or purely Dirichlet, then L^q-spaces are adequate. However, if one has to treat inhomogeneous Neumann conditions and/or one is confronted with distributions on the right hand side, L^q-spaces are no longer suitable, compare [54, Ch. 3.2]. It is our belief that then spaces of $W^{-1,q}$-type are adequate, since they indeed contain distributional objects such as surface charge densities (concerning their relevance in the theory of electricity consult [78, Ch. 1]). In particular, for some $1 < q' < 3/2$, our result yields q'-integrability of the gradient of solutions even if the right hand side is given by an arbitrary Radon measure, cf. Remark 4.9. However, we note that in this work, we aim at the abstract isomorphism property (1.1) only and do not find *explicit* or *quantitative* estimates. We refer to e.g. [13] for recent results on this topic. In addition, we note that the isomorphism property (1.1) for $q > 3$ trivially implies Hölder continuity of the solution by embedding. However, Hölder regularity for elliptic mixed boundary value problems is a much weaker property than (1.1) and can thus be obtained in more general geometric settings, cf. e.g. [29], [37].

In addition to concerning the elliptic equations themselves, the isomorphism property (1.1) provides a good perspective on the investigation of — even quasilinear — *parabolic* equations, based on the following observations:

(i) Second order divergence operators satisfy maximal parabolic regularity in the spaces $W^{-1,q}_\Gamma$; in particular, they generate analytic semigroups, see [38] or [39].

(ii) The following multiplier law holds: if (1.1) is a topological isomorphism for some $q \in [2, 6]$, then this property is maintained if μ is replaced by a coefficient function $\vartheta\mu$, where ϑ is a strictly positive, uniformly continuous function on Ω. Moreover, the operators $\nabla \cdot \vartheta\mu\nabla$ behave well concerning their dependence on such ϑ, see Section 6 below for details.

(iii) For $q > 3$, $W^{1,q}$ is a well-suited multiplier space for a great variety of function/distribution spaces, see [38] for applications of this fact.

(iv) if q is larger than the space dimension, then $L^{q/2} \hookrightarrow W^{-1,q}$. Thus, the isomorphism (1.1) allows for the treatment of equations with quadratic gradient terms, see [40], [38] and compare also the investigation of the thermistor problem in [4] or [42].

(v) the possibility of including surface densities makes $W^{-1,q}$ a particularly useful space for treating both dynamic boundary conditions and for solving parabolic optimal control problems, see e.g. [41].

The outline of this paper is as follows: First we fix some notation and state general assumptions. In Section 3 we introduce our local model constellations. In Section 4, detailed assumptions on our setting are presented, specifying how the local constituents establish the global framework. Afterwards, the main result is stated as Theorem 4.8. The proof is given in Section 5, along with some auxiliary results. The multiplier law described above is established in Section 6. A discussion of the limitations of our concept is provided in Section 7. Finally, Section 8 contains some concluding remarks.

2. Notations and a general assumption. If P_1, P_2 are points in one of the Euclidean spaces $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\overline{P_1P_2}$ denotes the *open* segment between P_1 and P_2. If $\Lambda \subset \mathbb{R}^3$ is a bounded domain, then we denote by $W^{1,q}(\Lambda)$ the (complex) Sobolev space on Λ. If Υ is an open subset of $\partial\Lambda$, we use the symbol $W^{1,q}_\Upsilon(\Lambda)$ for the closure of

$$\{v|_\Lambda : v \in C_0^\infty(\mathbb{R}^3), \text{supp } v \cap (\partial\Lambda \setminus \Upsilon) = \emptyset\}$$

in $W^{1,q}(\Lambda)$. We write $W^{1,q}_0(\Lambda)$ instead of $W^{1,q}_\emptyset(\Lambda)$ and $W^{1,q}(\Lambda)$ instead of $W^{1,q}_{\partial\Lambda}(\Lambda)$. This notation is justified, since we will assume that Λ is a Lipschitz domain. It follows that the set of restrictions to Λ of functions in $C_0^\infty(\mathbb{R}^3)$ is dense in $W^{1,q}(\Lambda)$, cp. [30, Thm. 1.4.2.1]. $W^{-1,q'}(\Lambda)$ denotes the space of continuous antilinear forms on $W_0^{1,q}(\Lambda)$, and $W_\Upsilon^{-1,q'}(\Lambda)$ denotes the anti-dual to $W_\Upsilon^{1,q}(\Lambda)$, if $\frac{1}{q}+\frac{1}{q'}=1$ holds. Finally, $W^{1,\infty}(\Lambda)$ denotes the space of Lipschitz functions on Λ and $W_0^{1,\infty}(\Lambda)$ denotes the subspace of functions which vanish on $\partial\Lambda$.

Definition 2.1. *A function μ, defined on some open subset $\mathcal{O}$ of $\mathbb{R}^3$, which is bounded, measurable and takes its values in the set of* real, symmetric 3×3 *matrices will be called a* coefficient function *on $\mathcal{O}$. If μ additionally satisfies the condition*

$$\operatorname*{ess\,inf}_{\mathbf{x}\in\mathcal{O}} \inf_{\|\xi\|_{\mathbb{R}^3}=1} \mu(\mathbf{x})\xi\cdot\xi > 0, \tag{2.1}$$

then it will be called elliptic. As usual, we define

$$-\nabla\cdot\mu\nabla : W_\Upsilon^{1,2}(\Lambda) \to W_\Upsilon^{-1,2}(\Lambda)$$

by

$$\langle -\nabla\cdot\mu\nabla v, w\rangle := \int_\Lambda \mu\nabla v\cdot\nabla\overline{w}\,d\mathrm{x} \quad v,w\in W_\Upsilon^{1,2}(\Lambda), \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the corresponding (anti-) dual pairing.
The maximal restriction of $-\nabla\cdot\mu\nabla$ to any of the spaces $W_\Upsilon^{-1,q}(\Lambda)$ $(q>2)$ will be denoted by the same symbol.

Remark 2.2. *The maximal restriction of $-\nabla\cdot\mu\nabla$ to $L^2(\Lambda)$ leads to a homogeneous Dirichlet condition on $\partial\Lambda\setminus\Upsilon$ and a (generalized) homogeneous Neumann condition for the elements of its domain on Υ (cf. [12, Ch. 1.2] or [25, Ch. II.2], compare also [16]). For simplicity, we will call Υ the* Neumann boundary part *of Λ in any case.*

For two Banach spaces X and Y, we denote the space of linear, bounded operators from X into Y by $\mathcal{L}(X;Y)$. If $X=Y$, then we abbreviate by $\mathcal{L}(X)$. The symbol $\mathcal{LH}(X;Y)$ denotes the set of linear homeomorphisms between the Banach spaces X and Y. The norm in a Banach space X will be denoted by $\|\cdot\|_X$.

Let us introduce the following general assumption on our three-dimensional domain Ω and the corresponding Neumann boundary part Γ:

Assumption 2.3. *We suppose that all domains under consideration are bounded, in particular, Ω is a bounded Lipschitz domain (cf. [30, Def.1.2.1.2] or [58, Ch. 1.1.9 Def. 3]) and Γ is an (relatively) open part of the boundary $\partial\Omega$. Moreover, we assume that the boundary of Γ within $\partial\Omega$ is locally bi-Lipschitz diffeomorphic to the unit interval $]0,1[$, and $\partial\Omega\setminus\Gamma$ is the closure of its interior (in $\partial\Omega$).*

In general, if a three-dimensional domain Λ and an open part of its boundary Υ satisfy these conditions, we will say that *the pair (Λ,Υ) satisfies Assumption 2.3.*

Remark 2.4. *If the pair (Λ,Υ) satisfies Assumption 2.3, it follows that the interpolation results proved in [28] apply to the spaces $W_\Upsilon^{1,p}(\Lambda)$, cf. Proposition 5.2 below. In particular, by [37, Ch. 5], in space dimension* 3*, Assumption 2.3 guarantees that $\Lambda\cup\Upsilon$ is regular in the sense of Gröger [31, Defn. 2], which is the requirement of [28].*

Remark 2.5. *Note that we do not assume Ω to be a* strong *or, equivalently, a* graph *Lipschitz domain, cf. [58, Ch. 1.1.9].*

Finally, we use the term *polyhedral Lipschitz domain* for a Lipschitz domain the closure of which is a *polyhedron* (see [64, Introduction]).

3. Model constellations. Before arriving at the details of the main result in Section 4, our intention is to give the reader an impression of which geometric settings can be expected in view of the isomorphism property (1.1) as soon as possible. Thus, already in this section, we collect a list of local constituents for which this is true. The examples presented are polyhedral Lipschitz domains which may, in particular, *not* be strong Lipschitz domains, see Figure 3 (right). Moreover, the coefficients are allowed to jump across surfaces, and the boundary conditions may be mixed. The essential point for us is that only these constituents cover a sufficiently rich class of three-dimensional real-world applications, including the examples from Subsection 1.1. In order to provide clear impressions of the model configurations, we include a series of sketches.

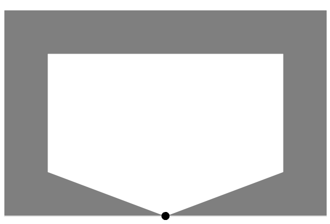
DEFINITION 3.1. *We call a bounded subset M of $\mathbb{R}^3$ a* polyhedral 3-manifold with boundary *if*

(i) *M is a 3-manifold with boundary*

(ii) *M is a polyhedron, cf. [64, Introduction].*

REMARK 3.2. *The interior of a polyhedral 3-manifold with boundary is a Lipschitz domain, cf. [33, Thm. 3.10].*

REMARK 3.3. *A pair of pincers (Figure 3 left) or domains with cracks such as an open ball minus half of its equatorial plane are not Lipschitz domains and are thus excluded by Assumption 2.3.*



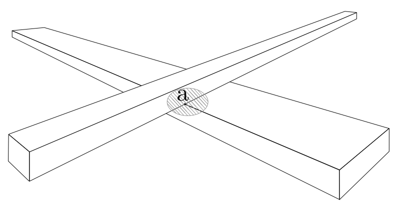


FIG. 3. *Left: A pair of pincers is* not *a Lipschitz domain. — Right: The double beam, see also Figure 1 (right), can serve as a prototypical example of the situation in Proposition 3.4 i) or ii), if we assume that ρ is constant on each beam. One of the crucial points* a *is highlighted here.*

The first model constellations we consider are those of pure homogeneous Dirichlet or Neumann boundary conditions.

PROPOSITION 3.4. *Let M be a polyhedral 3-manifold with boundary and let Λ be its interior. Suppose that there is a plane $\mathcal{H}$, intersecting Λ, such that the elliptic coefficient function ρ is constant on each of the (finitely many) connected components of $\Lambda \setminus \mathcal{H}$. Moreover, every edge on the boundary of M which is induced by the interface $\mathcal{H}$ has to be adjacent to exactly two connected components, and the angles between $\mathcal{H}$ and adjacent boundary faces, measured on the inside of M, shall not exceed π.*

(i) *(Homogeneous Dirichlet boundary conditions) Then there is a $p_D > 3$ such that the operator*

$$-\nabla \cdot \rho \nabla : W_0^{1,q}(\Lambda) \to W^{-1,q}(\Lambda) \tag{3.1}$$

is a topological isomorphism for all $q \in [2, p_D[$.

(ii) *(Homogenous Neumann boundary conditions) Let* $\mathrm{a} \in \partial M \cap \mathcal{H}$ *be a vertex of M. Then, for every ball around* a*, there is an open neighbourhood $\mathcal{W}$ of* a*, contained in this ball, with the following property: Setting $\Pi := \Lambda \cap \mathcal{W}$ and $\Sigma := \partial \Lambda \cap \mathcal{W}$, the* *pair* (Π, Σ) *satisfies Assumption 2.3, and there exists a* $p_N > 3$ *such that the operator*

$$-\nabla \cdot \mu|_\Pi \nabla : W^{1,q}_\Sigma(\Pi) \to W^{-1,q}_\Sigma(\Pi) \tag{3.2}$$

is a topological isomorphism for all $q \in [2, p_N[$.

Proof. By Remark 3.2, Λ is a Lipschitz domain. Moreover, the edges induced by $\mathcal{H}$ are bi-material ones, and the angles between $\mathcal{H}$ and the neighbouring boundary faces do not exceed π. Thus, (i) follows from [20, Thm. 2.1].

ii) The isomorphism property is proved for some $q > 3$ in [33, Thm. 5.1], see also Example 5.4.1 there. In particular, $\mathcal{W}$ is constructed such that Π is the bi-Lipschitz image of a cube, and Γ is the bi-Lipschitz image of one of the sides of this cube. Hence, Π is a Lipschitz domain and the pair (Π, Σ) satisfies Assumption 2.3. Thus, Corollary 5.4 and Lemma 5.5 below apply by Remark 2.4. ∎

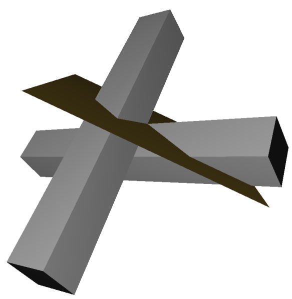


Fig. 4. *An example of a domain which is divided into* three *components by the plane* $\mathcal{H}$*. If the coefficient function is constant on each of these components, this configuration is admissible in both Proposition 3.4 i) and ii).*

Remark 3.5. *Geometries as in Figure 3 (right) and Figure 4 — not constituting a* strong *Lipschitz domain, but only a Lipschitz domain — are quite common: they not only appear in photonic crystals (Figure 1 right), but e.g. in the combination of a railroad track and the underlying railroad tie. This can be of relevance in view of a corresponding heat conduction problem. In view of future applications, we note that the geometry around vertices in Proposition 3.4 may be nearly arbitrarily wild, as far as polyhedra go, and refer to [8, Ch. XIV] for a number of examples.*

Remark 3.6. *The condition on* $\mathcal{H}$ *in Proposition 3.4 is clearly satisfied if* Λ *is convex.*

The following four Propositions concern different model constellations with mixed Neumann and Dirichlet boundary conditions.

Proposition 3.7. *[47, Thm. 1.1] Let* $\triangle_1, \triangle_2 \subset \mathbb{R}^2$ *be two open triangles which share one side with endpoints* P, Q*. Let* P_1, Q_1 *be the other vertices of* $\triangle_1, \triangle_2 \subset \mathbb{R}^2$ *which share a side with* P*. Define* $\Lambda_0 := \big(\triangle_1 \cup \triangle_2 \cup \overline{PQ}\big) \times \,]-1, 0[$ *and the boundary part* Υ_0 *as* $\big(\overline{Q_1P} \cup \overline{P_1P} \cup \{P\}\big) \times]-1, 0[$*. Let further* $\mathcal{H}$ *be a plane which intersects* Λ_0 *but does not touch its ground plate or its cover plate and let* ρ *be any elliptic coefficient function which is constant on both components of* $\Lambda_0 \setminus \mathcal{H}$*. Then there is a* $p > 3$ *such that*

$$-\nabla \cdot \rho \nabla : W^{1,q}_{\Upsilon_0}(\Lambda_0) \to W^{-1,q}_{\Upsilon_0}(\Lambda_0) \tag{3.3}$$

is a topological isomorphism for all $q \in [2, p[$.

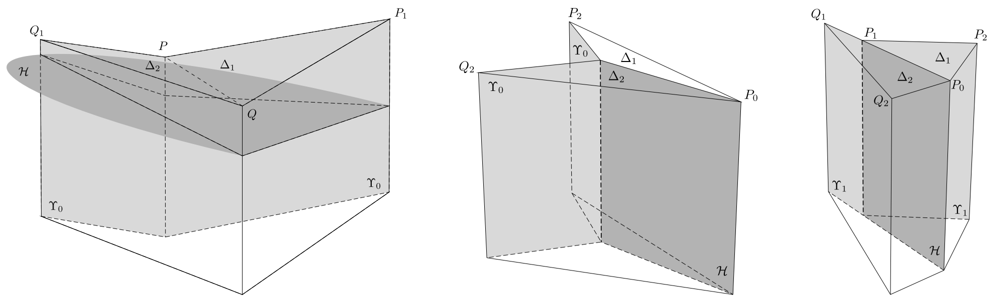


FIG. 5. *Model constellation of Proposition 3.7 (left) Proposition 3.8 (center) and Proposition 3.9 (right).*

PROPOSITION 3.8. *[47, Thm. 1.2] Let $\triangle_1, \triangle_2, \Lambda_0$ and the boundary part Υ_0 as in Proposition 3.7. If the coefficient function ρ is elliptic and constant on both prisms $\triangle_1 \times]-1,0[$ and $\triangle_2 \times]-1,0[$, then there is a $p > 3$ such that*

$$-\nabla \cdot \rho \nabla : W^{1,q}_{\Upsilon_0}(\Lambda_0) \to W^{-1,q}_{\Upsilon_0}(\Lambda_0) \tag{3.4}$$

is a topological isomorphism for all $q \in [2, p[$.

PROPOSITION 3.9. *[47, Thm. 1.2] Let $\triangle_1 \subset \mathbb{R}^2$ be an open triangle with vertices P_0, P_1, P_2 and $\triangle_2$ another open triangle, disjoint to $\triangle_1$, with vertices P_0, Q_1, Q_2, such that $P_1 \neq Q_1$ is contained in the segment joining P_0 and Q_1. Furthermore, let $\Lambda_1 \subseteq \mathbb{R}^3$ be the open right prism $(\triangle_1 \cup \triangle_2 \cup \overline{P_0 P_1}) \times]-1,0[$, and let $\Upsilon_1 := \big(\overline{P_1 P_2} \cup \{P_1\} \cup \overline{P_1 Q_1}\big) \times]-1,0[$. Let ϱ be any elliptic coefficient function on Λ_1, which is constant on the prisms $\triangle_1 \times]-1,0[$ and $\triangle_2 \times]-1,0[$. Then there is a $p > 3$, such that*

$$-\nabla \cdot \varrho \nabla : W^{1,q}_{\Upsilon_1}(\Lambda_1) \to W^{-1,q}_{\Upsilon_1}(\Lambda_1)$$

provides a topological isomorphism for all $q \in [2, p[$.

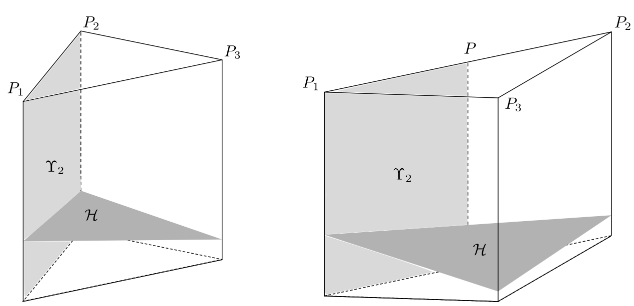


FIG. 6. *Model constellations of Proposition 3.10.*

PROPOSITION 3.10. *Let $\triangle \subset \mathbb{R}^2$ be an open triangle with vertices P_1, P_2, P_3. Define $\Lambda := \triangle \times]{-1}, 0[$.*

(i) *Let the boundary part Υ_2 be $\overline{P_1 P_2} \times]-1,0[$. Suppose $\mathcal{H}$ to be a plane within $\mathbb{R}^3$ that intersects Λ, but avoids the top and bottom sides of Λ, see Figure 6 (left). Let ρ be any elliptic coefficient function which is constant on both components of $\Lambda \setminus \mathcal{H}$. Then there is a $p > 3$ such that*

$$-\nabla \cdot \rho \nabla : W^{1,q}_{\Upsilon_2}(\Lambda) \to W^{-1,q}_{\Upsilon_2}(\Lambda) \tag{3.5}$$

is a topological isomorphism for all $q \in [2,p[$.

(ii) *Let further P denote the midpoint of $\overline{P_1P_2}$, and let the boundary part Υ_2 this time be $\overline{P_1P}\times]-1,0[$, see Figure 6 (right). Suppose $\mathcal{H}$ to be a plane within $\mathbb{R}^3$ that intersects Λ, but avoids the top and bottom sides and let ρ be any elliptic coefficient function which is constant on both components of $\Lambda \setminus \mathcal{H}$. Then there is a $p > 3$ such that*

$$-\nabla \cdot \rho\nabla : W^{1,q}_{\Upsilon_2}(\Lambda) \to W^{-1,q}_{\Upsilon_2}(\Lambda) \tag{3.6}$$

is a topological isomorphism for all $q \in [2,p[$.

(iii) *Let, in the notation from above, this time Υ_2 be given by*

$$\Upsilon_2 := (\overline{P_1P}\times]-1,0]) \cup (\triangle \times \{0\}) \tag{3.7}$$

and let ρ be constant and elliptic. Then there is a $p > 3$ such that

$$-\nabla \cdot \rho\nabla : W^{1,q}_{\Upsilon_2}(\Lambda) \to W^{-1,q}_{\Upsilon_2}(\Lambda) \tag{3.8}$$

is a topological isomorphism for all $[2,p[$.

(iv) *Let now Υ_2 be defined by*

$$\Upsilon_2 := (\overline{P_1P}\times]-1,0]) \cup (\widetilde{\triangle} \times \{0\}), \tag{3.9}$$

where $\widetilde{\triangle}$ is the triangle with vertices P_1, P, P_3. Let $\mathcal{H} \subset \mathbb{R}^3$ be a plane which contains the line segment $\overline{P_1P_2}$ and a point $Q := (P_3, \kappa)$ with $\kappa \in]-1,0[$ and let ρ be any elliptic coefficient function which is constant on both components of $\Lambda \setminus \mathcal{H}$. Then there is an open, arbitrarily small neighbourhood $\mathcal{V} \ni (P,0)$ and a $p > 3$ such that

$$-\nabla \cdot \rho\nabla : W^{1,q}_{\Upsilon_\blacktriangle}(\Lambda_\blacktriangle) \to W^{-1,q}_{\Upsilon_\blacktriangle}(\Lambda_\blacktriangle) \tag{3.10}$$

is a topological isomorphism for all $q \in [2,p[$, provided one defines $\Lambda_\blacktriangle := \Lambda \cap \mathcal{V}$ and $\Upsilon_\blacktriangle := \Upsilon_2 \cap \mathcal{V}$.

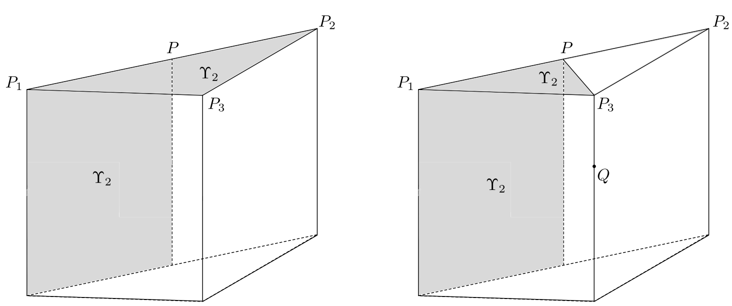


FIG. 7. *Model constellations of Proposition 3.10. The shaded part of the boundary again carries the Neumann condition. The plane $\mathcal{H}$ contains the points P_1, P_2, Q.*

Proof. The assertions (i) and (ii) have been proved in [35, Thm. 1 and 2]. Assertion (iii) in case of (3.7) is proved by a classical reflection argument: when reflecting the problem symmetrically across the plane $\mathcal{P}$ which is determined by P, P_1, P_2, P_3, one ends up again with a problem of type ii), where the plane of discontinuity of the coefficient function is exactly $\mathcal{P}$. The proof of iv) is postponed to the beginning of Subsection 5.2. ∎

REMARK 3.11. *In the model constellations of Proposition 3.10, 2, 3 and 4, Dirichlet and Neumann parts of the boundary meet in the plane at an angle of π. This is known to be the most critical case, cf. [63] and [71].*

4. The main result. Let us first explain the organization of this chapter. At the beginning, we introduce the relevant geometric assumptions and suppositions on the admissable coefficient functions μ. Primarly, this affects the question in which way hetero structures are admissable in our context. Then we introduce geometric suppositions on the pair (Ω, Γ) of the three-dimensional domain Ω and its Neumann boundary part $\Gamma \subset \partial\Omega$. We divide points on the boundary of Ω into three classes, namely

- points on the Dirichlet part $\partial\Omega \setminus \overline{\Gamma}$ (cf. Assumption 4.4),
- points on the Neumann part Γ (cf. Assumption 4.5), of three different types,
  NS points on a **N**eumann **S**urface;
  NE points on a **N**eumann **E**dge NE1, NE2 or NE3;
  NV **N**eumann **V**ertices.
- Finally, we consider points on the joint boundary of Dirichlet and Neumann parts $B := (\partial\Omega \setminus \Gamma) \cap \overline{\Gamma}$ (cf. Assumption 4.6). Again, we distinguish between
  BS points in B on a **S**urface;
  BE points in B on an **E**dge BE1–BE4;
  BV **V**ertices in B.

We then state the main result of this paper.

REMARK 4.1. *Up to now, we have not given a precise definition of the notions of surfaces, edges and vertices of the Lipschitz domain Ω. In the assumptions below, these disctinctions come into play only in the case of points on the closure of the Neumann boundary part $\overline{\Gamma}$. Locally around these points, we assume $\overline{\Omega}$ to be C^1-diffeomorphic to one of the nine polyhedral model constellations given in the previous section. Thus, a surface, edge or vertex of $\partial\Omega$ is defined locally as being a part of the boundary which is transformed onto a surface, an edge or a vertex of one of these prisms, respectively. Note that if Ω itself is a polyhedral Lipschitz domain, C^1-diffeomorphisms require surfaces to be mapped to surfaces, edges to be mapped to edges and vertices to vertices.*

ASSUMPTION 4.2. *Let the pair (Ω, Γ) satisfy Assumption 2.3 and let μ be a coefficient function according to Definition 2.1 such that, additionally:*

(i) *For every $\mathrm{x} \in \overline{\Omega}$, there is an open neighbourhood $\mathcal{U}_\mathrm{x}$ of x and a two-dimensional C^1-surface $\mathcal{S}_\mathrm{x}$, containing x, such that $(\mathcal{U}_\mathrm{x} \cap \Omega) \setminus \mathcal{S}_\mathrm{x}$ has finitely many components, and x is an accumulation point for each of them. The coefficient function μ is uniformly continuous on each of these connected components.*

(ii) *For every $\mathrm{x} \in \partial\Omega$, there is a C^1-diffeomorphism ϕ_x from a neighbourhood of $\overline{\mathcal{U}_\mathrm{x}}$ into $\mathbb{R}^3$ such that $\phi_\mathrm{x}(\Omega \cap \mathcal{U}_\mathrm{x})$ equals the interior of a polyhedral 3-manifold M with boundary, which we denote by Λ. We assume additionally that $\phi_\mathrm{x}(\Omega \cap \mathcal{U}_\mathrm{x} \cap \mathcal{S}_\mathrm{x}) = \Lambda \cap \mathcal{H}_\mathrm{x}$ holds for some plane $\mathcal{H}_\mathrm{x} \ni \phi_\mathrm{x}(\mathrm{x})$ in $\mathbb{R}^3$.*

REMARK 4.3. *Of course, the particular case of μ being uniformly continuous on the whole of $\mathcal{U}_\mathrm{x} \cap \Omega$ is also allowed. Moreover:*

(i) *Observe that $\phi_\mathrm{x}(\mathcal{U}_\mathrm{x} \cap \Omega)$ always equals a polyhedral Lipschitz domain Λ, cf. Remark 3.2.*

(ii) *If $\mathrm{x} \in \Omega$ is included in a C^1-surface $\mathcal{S}$, then there is always a neighbourhood $\mathcal{U}_\mathrm{x}$ such that $\mathcal{U}_\mathrm{x} \setminus \mathcal{S}$ has exactly two components, cf. [80, Ch. I.2]. This is* not true *for boundary points, see the example in Fig. 8.*

ASSUMPTION 4.4. *For Dirichlet points $\mathrm{x} \in \partial\Omega \setminus \overline{\Gamma}$, we demand the following: Let $\mathcal{U}_\mathrm{x}$ be the neighbourhood from Assumption 4.2, and ϕ_x and $\mathcal{S} = \mathcal{S}_\mathrm{x}$ the corresponding C^1-diffeomorphism and C^1-surface, respectively. We ask that $\mathcal{U}_\mathrm{x}$ can be chosen sufficiently small to guarantee $\mathcal{U}_\mathrm{x} \cap \overline{\Gamma} = \emptyset$.*

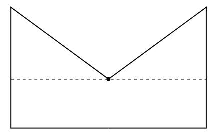


FIG. 8. *A 2d cross-section of a domain which is constant in the third direction. The dashed line indicates the position of the 2d interface $\mathcal{S}$, giving an example of a Lipschitz configuration excluded by Assumption 4.4.*

Then we assume that every edge on the boundary of Λ, induced by the hetero interface $\mathcal{H}$, is adjacent to exactly two connected components of $\Lambda \setminus \mathcal{H}$, and the angles between $\mathcal{H}$ and the two adjacent boundary faces, measured on the inside of Λ, shall not exceed π.

The next assumption affects points on the Neumann boundary part Γ.

ASSUMPTION 4.5. *Let $\mathrm{x} \in \Gamma$ and let $\mathcal{U}_\mathrm{x}$ be its neighbourhood given in Assumption 4.2, ϕ_x the corresponding C^1-diffeomorphism and $\mathcal{S}_\mathrm{x}$ the C^1-surface and let again $\mathcal{U}_\mathrm{x}$ be sufficiently small to yield $\mathcal{U}_\mathrm{x} \cap \partial\Omega = \mathcal{U}_\mathrm{x} \cap \Gamma =: \Gamma_\bullet$. We assume the following.*
If x *is a point on a geometrical* surface,
NS) then $\phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Omega)$ meets the geometrical requirements of Proposition 3.10 i), i.e. there is an open triangle Δ, S being one of its sides such that

$$\phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Omega) = \Delta \times]-1,0[=: \Lambda,$$

and

$$\phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \partial\Omega) = \phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Gamma) = S \times]-1,0[.$$

Moreover, $\mathcal{H} \ni \phi_\mathrm{x}(\mathrm{x})$ is a plane in $\mathbb{R}^3$ which touches neither the top nor the bottom side of Λ.
If x *is a point on a geometrical* edge, *then either*
NE1) $\phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Omega)$ meets the geometrical requirements of Proposition 3.7, i.e.

$$\phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Omega) = \Lambda_0 =: \Lambda$$

and

$$\phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \partial\Omega) = \phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Gamma) = \Upsilon_0$$

and furthermore, $\mathcal{H}$ is a plane which touches neither the top nor the bottom side of Λ_0, or
NE2) $\phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Omega)$ meets the geometrical requirements of Proposition 3.8, i.e.

$$\phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Omega) = \Lambda_0 =: \Lambda$$

and

$$\phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \partial\Omega) = \phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Gamma) = \Upsilon_0$$

and $\mathcal{H}$ is the plane which contains $\overline{PQ} \times]-1,0[$, or
NE3) $\phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Omega)$ meets the geometrical requirements of Proposition 3.9, i.e.

$$\phi_\mathbf{x}(\mathcal{U}_\mathbf{x} \cap \Omega) = \Lambda_1 =: \Lambda$$

and

$$\phi_{\mathbf{x}}(\mathcal{U}_{\mathbf{x}} \cap \partial\Omega) = \phi_{\mathbf{x}}(\mathcal{U}_{\mathbf{x}} \cap \Gamma) = \Upsilon_1$$

and $\mathcal{H}$ *is the plane which contains* $\overline{P_0P_1} \times]-1,0[$.
If x *is a* vertex,
NV) then $\phi_{\mathrm{x}}(\mathrm{x})$ *must be a vertex of* Λ *and in addition to Assumption 4.2, we require that every edge on the boundary of* Λ*, induced by the plane* $\mathcal{H}$*, is adjacent to exactly two connected components of* $\Lambda \setminus \mathcal{H}$*, and the angles between the hetero interface* $\mathcal{H}$ *and the two adjacent boundary faces do not exceed* π.

The next assumption covers the points on the joint boundary $B = \overline{\Gamma} \cap (\partial\Omega \setminus \Gamma)$ of Dirichlet and Neumann boundary parts.

Assumption 4.6. *For* $\mathrm{x} \in B$*, let* $\mathcal{U}_{\mathrm{x}}$ *be the neighbourhood from Assumption 4.2,* ϕ_{x} *the corresponding* C^1*-diffeomorphism and* $\mathcal{S}_{\mathrm{x}}$ *the corresponding* C^1*-surface. Then, there is a triangle* Δ *with vertices* P_1, P_2, P_3 *such that* $\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \Omega) = \Delta \times]-1,0[=: \Lambda$. *Denoting the midpoint of* $\overline{P_1P_2}$ *by* P *and the triangle with vertices* P_1, P, P_3 *by* $\widetilde{\triangle}$*, we suppose that one of the following is satisfied:*
If x *is a point on a geometrical* surface,
BS) then

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \partial\Omega) = \overline{P_1P_2} \times]-1,0[$$

and

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \Gamma) = \overline{P_1P} \times]-1,0[$$

and $\mathcal{H}$ *is a plane in* $\mathbb{R}^3$ *which touches neither the top nor the bottom side of* Λ*, cf. Proposition 3.10 ii).*
If x *is a point on a geometrical* edge*, then*
BE1) either

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \partial\Omega) = (\overline{P_1P_2} \cup \{P_2\} \cup \overline{P_2P_3}) \times]-1,0[,$$

and

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \Gamma) = \overline{P_1P_2} \times]-1,0[$$

and $\mathcal{H}$ *is a plane in* $\mathbb{R}^3$ *which touches neither the top nor the bottom side of* Λ*, cf. Proposition 3.10 i),*
BE2) or

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \partial\Omega) = \left[\overline{P_1P_2} \times]-1,0]\right] \cup \left[\triangle \times \{0\}\right],$$

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \Gamma) = \overline{P_1P} \times]-1,0[,$$

$\phi_{\mathrm{x}}(\mathrm{x}) = (P,0)$ *and* μ *is uniformly continuous on* $\mathcal{U}_{\mathrm{x}} \cap \Omega$*, cf. Proposition 3.10 ii),*
BE3) or

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \partial\Omega) = \left[\overline{P_1P_2} \times]-1,0]\right] \cup \left[\triangle \times \{0\}\right],$$

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \Gamma) = \left[\overline{P_1P} \times]-1,0[\right] \cup \left[\triangle \times \{0\}\right],$$

$\phi_{\mathrm{x}}(\mathrm{x}) = (P, 0)$ *and* μ *is uniformly continuous on* $\mathcal{U}_{\mathrm{x}} \cap \Omega$*, cf. Proposition 3.10 iii),*
BE4) or

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \partial\Omega) = \left[\overline{P_1P_2} \times]-1,0]\right] \cup \left[\triangle \times \{0\}\right],$$

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \Gamma) = \left[\overline{P_1P} \times]-1,0[\right] \cup \left[\widetilde{\triangle} \times \{0\}\right],$$

$\phi_{\mathrm{x}}(\mathrm{x}) = (P, 0)$ *and* $\mathcal{H}$ *is a plane which contains the segment* $\overline{P_1P_2} \times \{0\}$ *and a point* (P_3, κ) *with* $\kappa \in]-1,0[$*, cf. Proposition 3.10 iv).*
If x *is a* vertex*, then*
MV1) either

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \partial\Omega) = \left[(\overline{P_1P_2} \cup \{P_2\} \cup \overline{P_2P_3}) \times]-1,0]\right] \cup \left[\triangle \times \{0\}\right],$$

and

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \Gamma) = \overline{P_1P_2} \times]-1,0[,$$

and $\phi_{\mathrm{x}}(\mathrm{x}) = (P_2, 0)$ *and* μ *is uniformly continuous on* $\mathcal{U}_{\mathrm{x}} \cap \Omega$*, cf. Proposition 3.10 i),*
MV2) or

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \partial\Omega) = \left[(\overline{P_1P_2} \cup \{P_2\} \cup \overline{P_2P_3}) \times]-1,0]\right] \cup \left[\triangle \times \{0\}\right],$$

and

$$\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \Gamma) = \left[\overline{P_1P_2} \times]-1,0]\right] \cup \left[\triangle \times \{0\}\right],$$

and $\phi_{\mathrm{x}}(\mathrm{x}) = (P_2, 0)$ *and* μ *is uniformly continuous on* $\mathcal{U}_{\mathrm{x}} \cap \Omega$*, cf. Proposition 3.10 iii).*

Remark 4.7. *For clarification of these statements, we remark that*

(i) *in BS) it is implicitly contained that* $\phi_{\mathrm{x}}(\mathrm{x}) \in \{P\} \times]-1,0[$*. This follows from the property* $\mathrm{x} \in \overline{\Gamma} \cap (\partial\Omega \setminus \Gamma)$ *and the requirements* $\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \partial\Omega) = \overline{P_1P_2} \times]-1,0[$*,* $\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \Gamma) = \overline{P_1P} \times]-1,0[$*.*

(ii) *In BE1) it is implicitly contained that* $\phi_{\mathrm{x}}(\mathrm{x}) \in \{P_2\} \times]-1,0[$*. This follows from the property* $\mathrm{x} \in \overline{\Gamma} \cap (\partial\Omega \setminus \Gamma)$ *and the demands* $\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \partial\Omega) = (\overline{P_1P_2} \cup \{P_2\} \cup \overline{P_2P_3}) \times]-1,0[$*,* $\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \Gamma) = \overline{P_1P_2} \times]-1,0[$*.*

(iii) *In fact, Assumptions 4.2 and 4.6 imply that the set* $\Omega \cup \Gamma$ *is regular in the sense of Gröger. Assumption 2.3 is thus redundant and was stated in the beginning in order to simplify the exposition.*

Let us now state the main result.

Theorem 4.8. *Let* Ω*,* Γ*,* μ *be as introduced in Section 2. Then, under the Assumptions 4.2, 4.4, 4.5, 4.6, there is a* $p > 3$ *such that* (1.1) *is a topological isomorphism for all* $q \in [2, p[$*.*

The proof of this theorem will be given in the next chapter.

Remark 4.9. *In fact, using the argument given in the proof of Lemma 5.5 below, under the assumptions of Theorem 4.8, it follows that* (1.1) *is a topological isomorphism for all* $q \in]p', p[$*, where* $\frac{1}{p'} + \frac{1}{p} = 1$*. In particular, in three space dimensions, the space of Radon measures embeds into* $W^{-1,q'}$ *for all* $q' < \frac{3}{2}$ *and* $p > 3$ *provides* $p' < \frac{3}{2}$*. Thus, arbitrary Radon meausure right-hand sides may be treated by Theorem 4.8.*

Remark 4.10. *Probably, the reader is interested in the optimal magnitude of p in Theorem 4.8.*
If the domain Ω is C^1, then the assertion is true for any $p \in]1, \infty[$ in case of $\Gamma = \emptyset$ or $\Gamma = \partial\Omega$, as long as the coefficient function is uniformly continuous.
Moreover, it is shown in [43] that in case of the Dirichlet Laplacian (on a strong Lipschitz domain), p only depends on the Lipschitz constant of the domain and exceeds 3 by only an arbitrarily small margin in general. The same is true in case of the Neumann Laplacian [81].
In case of the Laplacian with either Neumann or mixed boundary conditions, Dauge proved in [17] that lower bounds for p can be obtained from the edge and vertex singularities. Unfortunately, this is really restricted to the Laplacian (or to be substantially and nontrivially modified in case of other operators), and these singularityies are hard to control if the geometry is (a bit) complicated.
Furthermore, Shamir's famous counterexample [75] shows that in case of mixed boundary conditions, $p \geq 4$ can generically not be expexted, even if the domain and the coefficients are smooth.
If the coefficient function is discontinuous, our results rest heavily on the deep insight of Maz'ya [59] that, in case of a polyhedral Lipschitz domain and $\Gamma = \emptyset$, it suffices to control the edge singularities. The proof of this, however, is essentially based on Hölder estimates, where the Hölder exponent is not known explicitly in general, compare [53]. Thus, one cannot expect any detailed information on p in these cases.

5. Proof of Theorem 4.8.

In this section, we give a proof of Theorem 4.8.

5.1. Auxiliaries.

We first establish some technical tools needed in the proof. We start out with a lemma which shows that the L^∞-norm on the set of coefficient functions is an adequate choice for a suitable perturbation theory for the operators (1.1). Next, we quote an interpolation theorem which, together with the pioneering result of Sneiberg [76], shows that the set $\mathcal{I}$ of indices q for which (1.1) is a topological isomorphism is an *open* interval. This enables us later to conclude Theorem 4.8 from the fact that (1.1) provides a topological isomorphism for $q = 3$. Afterwards, we present a result which shows that the isomorphy (1.1) is in some sense invariant under bi-Lipschitzian deformations of the domain. We conclude with a technique which allows us to localize the problem (1.1) and thus reduce the assertion to the regularity statements for the local model constellations given in Section 3.

Lemma 5.1. *Let Λ be a Lipschitz domain, and Υ be an open part of its boundary. Let ρ be a coefficient function.*

(i) *For all $q \in]1, \infty[$ we have the estimate*

$$\| - \nabla \cdot \rho \nabla \|_{\mathcal{L}(W^{1,q}_\Upsilon(\Lambda); W^{-1,q}_\Upsilon(\Lambda))} \leq \|\rho\|_{L^\infty(\Lambda;\mathcal{L}(\mathbb{R}^3))}. \tag{5.1}$$

(ii) *If $(\nabla \cdot \rho \nabla)^{-1} \in \mathcal{L}(W^{-1,q}_\Upsilon(\Lambda); W^{1,q}_\Upsilon(\Lambda))$ and $\hat{\rho}$ is another coefficient function on Λ which satisfies*

$$\|\rho - \hat{\rho}\|_{L^\infty(\Lambda;\mathcal{L}(\mathbb{R}^3))} \|(\nabla \cdot \rho \nabla)^{-1}\|_{\mathcal{L}(W^{-1,q}_\Upsilon(\Lambda); W^{1,q}_\Upsilon(\Lambda))} < 1,$$

then also $(\nabla \cdot \hat{\rho} \nabla)^{-1} \in \mathcal{L}(W^{-1,q}_\Upsilon(\Lambda); W^{1,q}_\Upsilon(\Lambda))$.

Proof. i) follows directly from the definition of $-\nabla \cdot \rho \nabla$ and Hölder's inequality. ii) is implied by i) and a classical perturbation theorem, cf. [48, Ch. IV.1.4, Thm. 1.16]. ∎

Proposition 5.2. *(see [28]) Let the pair (Λ, Υ) satisfy Assumption 2.3 and assume $q_0, q_1 \in]1, \infty[$. For $\theta \in]0,1]$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, one has the following identities concerning complex interpolation:*

$$[W^{1,q_0}_\Upsilon(\Lambda); W^{1,q_1}_\Upsilon(\Lambda)]_\theta = W^{1,q}_\Upsilon(\Lambda) \tag{5.2}$$

and

$$[W^{-1,q_0}_\Upsilon(\Lambda); W^{-1,q_1}_\Upsilon(\Lambda)]_\theta = W^{-1,q}_\Upsilon(\Lambda). \tag{5.3}$$

Remark 5.3. *In fact, (5.3) is proved in [28] in the case where $W^{-1,q}_\Upsilon(\Lambda)$ is the space of continuous linear forms on $W^{1,q'}_\Upsilon(\Lambda)$, nota bene not the space of antilinear forms. But from this the above assertion may be easily concluded by assigning to each linear form F the antilinear form $f \mapsto \langle F, \bar{f} \rangle$ and then applying the retraction/coretraction theorem, c.f. [79, Ch. 1.2.4].*

Corollary 5.4. *Assume $q_0, q_1 \in]1, \infty[$ and*

$$A \in \mathcal{L}(W^{-1,q_0}_\Gamma(\Omega); W^{1,q_0}_\Upsilon(\Lambda)) \cap \mathcal{L}(W^{-1,q_1}_\Upsilon(\Lambda); W^{1,q_1}_\Upsilon(\Lambda)).$$

Then $A \in \mathcal{L}(W^{-1,q}_\Upsilon(\Lambda); W^{1,q}_\Upsilon(\Lambda))$ for every $q \in]q_0, q_1[$.

Lemma 5.5. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded Lipschitz domain and Υ be an open subset of its boundary such that the pair (Λ, Υ) satisfies Assumption 2.3 and let ρ be an elliptic coefficient function (cf. Definition 2.1).*

(i) *Then the set $\mathcal{I}$ of indices q for which*

$$-\nabla \cdot \rho \nabla + 1 : W^{1,q}_\Upsilon(\Lambda) \to W^{-1,q}_\Upsilon(\Lambda) \tag{5.4}$$

is a topological isomorphism, is an interval which contains 2.

(ii) *This interval is open.*

Proof. i) The mapping (5.4) is continuous for all $q \in]1, \infty[$, due to (5.1). Due to the ellipticity of ρ, 2 belongs to the set $\mathcal{I}$ by Lax-Milgram. Secondly, since the coefficient function ρ is symmetric, it is not hard to see that for $q > 2$, the restriction of the adjoint of (5.4) to $W^{1,q}_\Upsilon(\Lambda)$ is again the operator (5.4). This shows that $q' \in \mathcal{I}$ iff $q \in \mathcal{I}$. That the set $\mathcal{I}$ forms an interval follows from Corollary 5.4. However, up to now, this interval could be degenerate with $\mathcal{I} = [2,2]$.

ii) If we put $X_{1-\frac{1}{p}} := W^{1,p}_\Upsilon(\Lambda)$ and assume $]\alpha, \beta[\subset]0,1[$, then the family $\{X_\tau\}_{\tau \in [\alpha,\beta]}$ forms a complex interpolation scale (see [51, Ch.1]), due to (5.2). The same is true for the family $\{Y_\tau\}_{\tau \in [\alpha,\beta]}$, with $Y_\tau = W^{-1,\frac{1}{1-\tau}}_\Upsilon(\Lambda)$, according to (5.3). But then a deep theorem of Sneiberg ([76], see also [77, Proposition 4.1]) yields that the set of parameters τ for which

$$-\nabla \cdot \rho \nabla + 1 : X_\tau \to Y_\tau$$

is a topological isomorphism, must be open. $\square$

Proposition 5.6. *[35, Prop. 16] Let $\Lambda \subset \mathbb{R}^d$ be a bounded Lipschitz domain and Υ be an open subset of its boundary. Assume that ϕ is a mapping from a neighbourhood of $\bar{\Lambda}$ into $\mathbb{R}^d$ which is bi-Lipschitz. We write $\phi(\Lambda) =: \Lambda_\star$ and $\phi(\Upsilon) =: \Upsilon_\star$. Then:*

(i) *For any $q \in]1, \infty[$, ϕ induces a linear, topological isomorphism*

$$\Psi_q : W^{1,q}_{\Upsilon_\star}(\Lambda_\star) \to W^{1,q}_\Upsilon(\Lambda)$$

which is given by $(\Psi_q f)(x) = f(\phi(x)) = (f \circ \phi)(x)$. These isomorphisms are consistent for different indices q.

(ii) $\Psi^*_{q'}$ *is a linear, topological isomorphism between* $W^{-1,q}_{\Upsilon}(\Lambda)$ *and* $W^{-1,q}_{\Upsilon_\star}(\Lambda_\star)$.

(iii) *If* ρ *is a coefficient function on* Λ, *then*

$$\Psi^*_{q'}\nabla\cdot\rho\nabla\Psi_q = \nabla\cdot\rho_\star\nabla \tag{5.5}$$

with

$$\rho_\star(\mathrm{y}) = \frac{1}{\big|\det(D\phi)(\phi^{-1}\mathrm{y})\big|}(D\phi)(\phi^{-1}(\mathrm{y}))\rho(\phi^{-1}(\mathrm{y}))\big(D\phi\big)^T(\phi^{-1}(\mathrm{y})), \tag{5.6}$$

where $D\phi$ *denotes the Jacobian of* ϕ *and* $\det(D\phi)$ *denotes the determinant of* $D\phi$. *If, in particular,* $-\nabla\cdot\rho\nabla : W^{1,q}_{\Upsilon}(\Lambda)\to W^{-1,q}_{\Upsilon}(\Lambda)$ *is a topological isomorphism, then the same is true for* $-\nabla\cdot\rho_\star\nabla : W^{1,q}_{\Upsilon_\star}(\Lambda_\star)\to W^{-1,q}_{\Upsilon_\star}(\Lambda_\star)$ *and vice versa.*

Remark 5.7. *The formula* (5.6) *shows that* $\rho_\star$ *is again a coefficient function in the sense of Definition 2.1 since boundedness, measurability and symmetry of the values are preserved. Moreover,* $\rho_\star$ *is elliptic, iff* ρ *is.*

Lemma 5.8. *Let* $\Lambda\subset\mathbb{R}^d$ *be a bounded Lipschitz domain and* Υ *be an open subset of its boundary. Let* $\mathcal{U}\subset\mathbb{R}^3$ *be open such that* $\Lambda_\bullet := \Lambda\cap\mathcal{U}$ *is again a domain. Furthermore, we put* $\Upsilon_\bullet := \Upsilon\cap\mathcal{U}$ *and fix an arbitrary function* $\eta\in W^{1,\infty}_0(\mathcal{U})$.

(i) *If* $v\in W^{1,q}_{\Upsilon}(\Lambda)$ *then* $\eta v|_{\Lambda_\bullet}\in W^{1,q}_{\Upsilon_\bullet}(\Lambda_\bullet)$.

(ii) *For any* $w\in W^{1,1}(\Lambda_\bullet)$, *let the symbol* $\widetilde{w}$ *indicate the extension of* w *to* Λ *by zero.*

a) For every $q\in[1,\infty[$ *the mapping*

$$W^{1,q}_{\Upsilon_\bullet}(\Lambda_\bullet)\ni v\longmapsto\widetilde{\eta v}$$

has its image in $W^{1,q}_{\Upsilon}(\Lambda)$ *and is continuous.*

b) If $\mathcal{U}\cap(\partial\Lambda\setminus\Upsilon)=\emptyset$ *then, for any* $q\in[1,\infty[$, *the mapping*

$$W^{1,q}(\Lambda_\bullet)\ni v\longmapsto\widetilde{\eta v}$$

has its image in $W^{1,q}_{\Upsilon}(\Lambda)$ *and is continuous.*

Proof. i) and ii a) are proved in [38], c.f. Lemma 5.8, or see [37] Lemma 4.6. We provide the proof of iib): obviously, supp η has a positive distance to $\overline{\Lambda}\setminus\mathcal{U}$. Therefore, the continuation by zero preserves the $W^{1,q}$-property and, additionally, the corresponding operation is continuous. Thus, in order to show the property $\widetilde{\eta v}\in W^{1,q}_{\Upsilon}(\Lambda)$ it suffices to show that this indeed holds true for every element $v\in W^{1,q}(\Lambda_\bullet)$ which equals the restriction of a function $\hat v$ which is C^∞ on $\mathbb{R}^3$. But it is clear that in this case, supp $\eta\hat v$ does not intersect $\partial\Lambda\setminus\Upsilon$ and, additionally, $\widetilde{\eta v}=\eta\hat v|_\Lambda$. □

Lemma 5.9. *Let* $\Lambda,\Upsilon,\mathcal{U},\eta,\Lambda_\bullet$ *be as in the previous lemma. If* $\mathcal{U}\cap(\partial\Lambda\setminus\Upsilon)\neq\emptyset$, *we set* $\Upsilon_\bullet := \Upsilon\cap\mathcal{U}$. *If* $\mathcal{U}\cap(\partial\Lambda\setminus\Upsilon)=\emptyset$, *we put either* $\Upsilon_\bullet=\partial\Lambda_\bullet$ *or* $\Upsilon_\bullet := \Upsilon\cap\mathcal{U}$. *Let* $\mu_\bullet$ *denote the restriction of the coefficient function* μ *to* $\Lambda_\bullet$ *and let the operator*

$$-\nabla\cdot\mu_\bullet\nabla : W^{1,2}_{\Upsilon_\bullet}(\Lambda_\bullet)\to W^{-1,2}_{\Upsilon_\bullet}(\Lambda_\bullet)$$

be defined as in (2.2). *Assume* $u\in W^{1,2}_{\Upsilon}(\Lambda)$ *to be the solution of*

$$-\nabla\cdot\mu\nabla u = f\in W^{-1,2}_{\Upsilon}(\Lambda). \tag{5.7}$$

Then for all of the above choices of $\Upsilon_\bullet$, *the following holds.*

(i) *The function* $v := \eta u|_{\Lambda_\bullet}$ *is in* $W^{1,2}_{\Upsilon_\bullet}(\Lambda_\bullet)$.

(ii) *The anti-linear form*

$$f_\eta : w \mapsto \langle f, \widetilde{\eta w}\rangle$$

($\widetilde{\eta w}$ again denotes the extension of ηw to Λ by zero) is well defined and continuous on $W^{1,q'}_{\Upsilon_\bullet}(\Lambda_\bullet)$ whenever $f \in W^{-1,q}_{\Upsilon}(\Lambda)$.

(iii) *Denoting the anti-linear form*

$$w \longmapsto \int_{\Lambda_\bullet} u\mu_\bullet \nabla\eta \cdot \nabla\overline{w}\, dx$$

by T_u, the function $v = \eta u|_{\Lambda_\bullet}$ satisfies

$$-\nabla\cdot\mu_\bullet\nabla v = -\mu_\bullet\nabla u|_{\Lambda_\bullet}\cdot\nabla\eta|_{\Lambda_\bullet} + T_u + f_\eta =: f_\bullet. \tag{5.8}$$

(iv) *Assume that $\Lambda_\bullet := \Lambda\cap\mathcal{U}$ is also a Lipschitz domain and suppose $q\in[2,6]$. Then the right hand side $f_\bullet$ of (5.8) is in $W^{-1,q}_{\Upsilon_\bullet}(\Lambda_\bullet)$, provided that $f\in W^{-1,q}_{\Upsilon}(\Lambda)$.*

Proof. i) We have to distinguish two cases: if $\Upsilon_\bullet = \partial\Lambda_\bullet$, the assertion is trivial. Otherwise, it is implied by Lemma 5.8 i). ii) — iv) are proved in [35, Ch. 4.2] for linear forms, but for anti-linear forms the proof is analogous. ∎

REMARK 5.10. *In contrast to Gröger's localization principle (see [31] Lemma 2), here, if $\mathcal{U}\cap(\partial\Lambda\setminus\Upsilon)=\emptyset$, then there are two possible choices for $\Upsilon_\bullet$, providing the localized problem with either pure Neumann or mixed boundary conditions.*

REMARK 5.11. *The requirement in iv) that $\Lambda_\bullet$ is again a Lipschitz domain guarantees the usual Sobolev embedding theorems. In particular, the term $\mu_\bullet\nabla u|_{\Lambda_\bullet}\cdot\nabla\eta|_{\Lambda_\bullet}$ is generically in L^2, but can be interpreted as an element of $W^{-1,q}_{\Upsilon_\bullet}(\Lambda_\bullet)$ for $q\in[2,6]$. This is the reason for the restriction to parameters $q\in[2,6]$ at this point.*

LEMMA 5.12. *Let Λ be a bounded Lipschitz domain and Υ be an open part of its boundary. Assume that Λ is the disjoint union $\Lambda = \cup_{j=1}^n \Lambda_j \cup (\Lambda\cap\mathcal{N})$, where every $\Lambda_j\subset\Lambda$ is open, and $\mathcal{N}\subset\mathbb{R}^3$ is closed and Lebesgue negligible. Assume that a prescribed $\mathrm{x}\in\mathcal{N}$ is an accumulation point for each of the sets $\Lambda_1,\dots,\Lambda_n$, and, additionally, that the limits $\lim_{\mathbf{y}\in\Lambda_j,\mathbf{y}\to\mathbf{x}}\rho(\mathbf{y}) =: \rho_j$, $j\in\{1,\dots,n\}$ exist. Define the coefficient function $\hat\rho_{\mathrm{x}}$ by*

$$\hat\rho_{\mathrm{x}}(\mathbf{y}) = \begin{cases}\rho_j, & \textit{if } \mathbf{y}\in\Lambda_j, j\in\{1,\dots n\},\\ 1, & \textit{if } \mathbf{y}\in\Lambda\cap\mathcal{N}.\end{cases}$$

and suppose that $\nabla\cdot\hat\rho\nabla\in\mathcal{LH}(W^{1,q}_{\Upsilon}(\Lambda);W^{-1,q}_{\Upsilon}(\Lambda))$ for some $q\in[2,6]$. Then, for every sufficiently small neighbourhood $\mathcal{W}$ of $\mathbf{x}$ and any function $\eta\in W^{1,\infty}_0(\mathcal{W})$, the function ηu belongs to $W^{1,q}_{\Upsilon}(\Lambda)$, provided that u satisfies $-\nabla\cdot\rho\nabla u = f\in W^{-1,q}_{\Upsilon}(\Lambda)$.

Proof. Taking $\mathcal{U}$ as a neighbourhood of $\overline{\Lambda}$, Lemma 5.9 shows that ηu satisfies an equation of the form

$$-\nabla\cdot\rho\nabla(\eta u) = f_\bullet \in W^{-1,q}_{\Upsilon}(\Lambda).$$

If $\mathcal{W}$ is any neighbourhood of x and $\operatorname{supp}\eta\subset\mathcal{W}$, then ηu also satisfies $-\nabla\cdot\check\rho\nabla(\eta u) = f_\bullet$, where $\check\rho$ is given by

$$\check\rho(\mathbf{y}) = \begin{cases}\rho(\mathbf{y}), & \text{if } \mathbf{y}\in\mathcal{W}\cap\Lambda,\\ \rho_j, & \text{if } \mathbf{y}\in\Lambda_j\setminus\mathcal{W},\\ 1, & \text{otherwise.}\end{cases}$$

One easily calculates

$$\begin{aligned} &\|\check\rho-\hat\rho_{\mathrm x}\|_{L^\infty(\Lambda;\mathcal L(\mathbb R^3))}\big\|\big(\nabla\cdot\hat\rho\nabla\big)^{-1}\big\|_{\mathcal L(W^{-1,q}_\Upsilon(\Lambda);W^{1,q}_\Upsilon(\Lambda))}\\ &\quad=\|\check\rho-\hat\rho_{\mathrm x}\|_{L^\infty(\Lambda\cap\mathcal W;\mathcal L(\mathbb R^3))}\big\|\big(\nabla\cdot\hat\rho\nabla\big)^{-1}\big\|_{\mathcal L(W^{-1,q}_\Upsilon(\Lambda);W^{1,q}_\Upsilon(\Lambda))}, \end{aligned} \tag{5.9}$$

because the coefficient functions $\hat\rho_{\mathrm x}$ and $\check\rho$ coincide on $\Lambda\setminus\mathcal W$. By the definition of $\hat\rho_{\mathrm x}$ and $\check\rho$, the factor $\|\check\rho-\hat\rho_{\mathrm x}\|_{L^\infty(\Lambda\cap\mathcal W;\mathcal L(\mathbb R^3))}$ can be made arbitrarily small by shrinking the neighbourhood $\mathcal W$. We chose $\mathcal W$, e.g. as a ball, so small that (5.9) becomes smaller 1. Then Lemma 5.1 tells us that the property $-\nabla\cdot\hat\rho_{\mathrm x}\nabla\in\mathcal{LH}(W^{1,q}_\Upsilon(\Lambda);W^{-1,q}_\Upsilon(\Lambda)$, carries over to $-\nabla\cdot\check\rho\nabla$. Finally, the equation $-\nabla\cdot\rho\nabla(\eta u)=-\nabla\cdot\check\rho\nabla(\eta u)=f_\bullet\in W^{-1,q}_\Upsilon(\Lambda)$, together with the isomorphism property of $-\nabla\cdot\check\rho\nabla$, gives the assertion. ∎

Lemma 5.13. *Suppose that Assumption 4.2 holds and that* $\mathrm x\in\Omega$.

(i) *Then there is a neighbourhood* $\mathcal V_{\mathbf x}\subseteq\Omega$ *such that*

$$-\nabla\cdot\mu\nabla:W^{1,q}_0(\mathcal V_{\mathbf x})\to W^{-1,q}(\mathcal V_{\mathbf x}) \tag{5.10}$$

is a topological isomorphism for any $q\in]1,\infty[$.

(ii) *If, in particular,* u *satisfies*

$$-\nabla\cdot\mu\nabla u=f\in W^{-1,q}_\Gamma(\Omega)\quad\textit{with}\quad q\in[2,6] \tag{5.11}$$

and $\eta\in W^{1,\infty}(\mathbb R^3)$ *has its support in* $\mathcal V_{\mathbf x}$*, then* $\eta u\in W^{1,q}_\Gamma(\Omega)$.

Proof. The first assertion i) is proved in [21, p. 244]. If, in particular, μ is continuous in a neighbourhood of x, then the assertion has been proved already in [65, p. 156/157] and [1, Ch. 15]). Assertion ii) follows from i) by means of Lemma 5.9. ∎

5.2. Proof of the main result. In this subsection we give the proof of our main result, starting with the proof of Proposition 3.10 iv):
Modulo an affine transformation, we may assume that $\overline{P_1P_2}$ is part of the e_1-axis, and that $P=0\in\mathbb R^2$. Let us define a bi-Lipschitzian mapping χ from $\mathbb R^3$ onto $\mathbb R^3$ as follows: on the halfspace below the plane $\mathcal H$ we define χ as the identity. On the (closed) half space above the plane we define χ as the linear mapping which leaves $\mathcal H$ invariant and transforms $(P_3,0)$ into $(P,1)$. Let $\square$ be the quadrilateral with vertices $P_1,P_2,P_3,-P_3$ and Π be the prism $\square\times]-1,1[$. Then it is not hard to see that, for small $\lambda\in]0,1[$,

$$\lambda\Pi\cap\chi(\Lambda)=\lambda(\triangle\times]-1,1[),\ \text{and}\ \lambda\Pi\cap\chi(\Upsilon_2)=\lambda(\overline{P_1P}\times]-1,1[). \tag{5.12}$$

If one defines, for sufficiently small, positive λ, $\mathcal V:=\chi^{-1}(\lambda\Pi)$, we get from (5.12) that

$$\chi(\mathcal V\cap\Lambda)=\lambda\Pi\cap\chi(\Lambda)=\lambda(\triangle\times]-1,1[), \tag{5.13}$$

and, analogously,

$$\chi(\mathcal V\cap\Upsilon_2)=\lambda\Pi\cap\chi(\Upsilon_2)=\lambda(\overline{P_1P}\times]-1,1[). \tag{5.14}$$

Furthermore, one easily observes that (cf. Formula (5.6)) the induced coefficient function on $\lambda(\triangle\times]-1,1[)$ is constant above $\mathcal H$ and constant below $\mathcal H$. This, together with (5.13) and (5.14) shows that one is essentially again in the situation of Proposition 3.10 ii). Since χ is bi-Lipschitzian, Proposition 5.6 gives the result.
Proposition 3.10 iv) has the following useful implication:

COROLLARY 5.14. *Suppose that Assumption 4.2 and Assumption 4.6/BE4 hold and that $\phi := \phi_{\mathrm{x}}$ is the corresponding isomorphism. Then there is an open neighbourhood $\widetilde{\mathcal{U}_{\mathrm{x}}}$ of* x *such that*

$$\phi(\Omega \cap \widetilde{\mathcal{U}_{\mathrm{x}}}) = \Lambda_{\blacktriangle} \quad \textit{and} \quad \phi(\Gamma \cap \widetilde{\mathcal{U}_{\mathrm{x}}}) = \Upsilon_{\blacktriangle}, \tag{5.15}$$

cf. Proposition 3.10 iv).

Proof. We put $\widetilde{\mathcal{U}_{\mathrm{x}}} := \phi^{-1}(\phi(\mathcal{U}_{\mathrm{x}}) \cap \mathcal{V})$, where $\mathcal{V}$ is the neighbourhood from Proposition 3.10 iv). Observing $\widetilde{\mathcal{U}_{\mathrm{x}}} = \mathcal{U}_{\mathrm{x}} \cap \widetilde{\mathcal{U}_{\mathrm{x}}}$ one obtains

$$\begin{aligned}\phi(\Omega \cap \widetilde{\mathcal{U}_{\mathrm{x}}}) &= \phi(\Omega \cap \mathcal{U}_{\mathrm{x}} \cap \phi^{-1}(\phi(\mathcal{U}_{\mathrm{x}}) \cap \mathcal{V})) = \phi(\Omega \cap \mathcal{U}_{\mathrm{x}}) \cap \phi(\mathcal{U}_{\mathrm{x}}) \cap \mathcal{V} \\ &= \phi(\Omega \cap \mathcal{U}_{\mathrm{x}}) \cap \mathcal{V} = \Lambda \cap \mathcal{V} = \Lambda_{\blacktriangle},\end{aligned}$$

cf. Assumption 4.6/BE4 and Proposition 3.10 iv). Analogously, one gets

$$\begin{aligned}\phi(\Gamma \cap \widetilde{\mathcal{U}_{\mathrm{x}}}) &= \phi(\Gamma \cap \mathcal{U}_{\mathrm{x}} \cap \phi^{-1}(\phi(\mathcal{U}_{\mathrm{x}}) \cap \mathcal{V})) = \phi(\Gamma \cap \mathcal{U}_{\mathrm{x}}) \cap \phi(\mathcal{U}_{\mathrm{x}}) \cap \mathcal{V} \\ &= \phi(\Gamma \cap \mathcal{U}_{\mathrm{x}}) \cap \mathcal{V} = \Upsilon_2 \cap \mathcal{V} = \Upsilon_{\blacktriangle},\end{aligned}$$

cf. Assumption 4.6/BE4 and Proposition 3.10 iv). □

REMARK 5.15. *Proposition 3.10 iv) and Corollary 5.14 show that the constellations BE4 and BE2 are 'equivalent' when aiming only at the regularity behavior around the point P, if one admits bi-Lipschitzian charts, instead of restricting to C^1-charts. These considerations illustrate that in general, given two different configurations, it may be difficult to judge whether they are Lipschitz-diffeomorphic. For this reason, we decided to introduce nine model constellations which are partly redundant with respect to Lipschitz charts, but* look *different. Thus, given a particular model problem, they may more easily be locally identified.*

LEMMA 5.16. *Suppose Assumption 4.2,* $\mathrm{x} \in \partial\Omega \setminus \overline{\Gamma}$ *and Assumption 4.4. Then there is an open neighbourhood $\mathcal{V}_{\mathrm{x}}$ of* x *with the following property: if u satisfies*

$$-\nabla \cdot \mu \nabla u = f \in W_{\Gamma}^{-1,3}(\Omega) \tag{5.16}$$

and $\eta \in W_0^{1,\infty}(\mathcal{V}_{\mathrm{x}})$, then $\eta u \in W_{\Gamma}^{1,3}(\Omega)$.

Proof. If $\eta_0 \in C_0^\infty(\mathbb{R}^3)$ is any function with support in $\mathcal{U}_{\mathrm{x}}$, then (5.16) leads, according to Lemma 5.9, to an equation

$$-\nabla \cdot \mu \nabla \eta_0 u = g \in W^{-1,3}(\Omega \cap \mathcal{U}_{\mathrm{x}}). \tag{5.17}$$

We denote the domain $\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \Omega)$ by Λ. Transforming (5.17) under the mapping ϕ_{x}, we find that the function $v := \eta_0 u$ satisfies an equation of the form

$$-\nabla \cdot \check{\mu} \nabla v = h \in W^{-1,3}(\Lambda). \tag{5.18}$$

Observe that the resulting coefficient function $\check{\mu}$ is uniformly continuous on all components $\Lambda_1, \ldots, \Lambda_n$ of $\Lambda \setminus \mathcal{H}$, thanks to Proposition 5.6. When replacing the coefficient function $\check{\mu}$ by another one, $\tilde{\mu}$, which is even constant on each of these components, the result in Proposition 3.4 i) tells us that

$$-\nabla \cdot \tilde{\mu} \nabla : W_0^{1,3}(\Lambda) \to W^{-1,3}(\Lambda) \tag{5.19}$$

is a toplogical isomorphism. In particular, one may take the matrices on each Λ_j as

$$\lim_{\mathrm{y} \in \Lambda_j,\, \mathrm{y} \mapsto \phi_{\mathrm{x}}(\mathrm{x})} \check{\mu}(\mathrm{y}),$$

respecticely. Thanks to Lemma 5.12, there is an open neighbourhood $\mathcal{W}_{\mathrm{x}} \subseteq \phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}})$ of $\phi_{\mathrm{x}}(\mathrm{x})$ such that $\eta_1 v \in W_0^{1,3}(\Lambda)$ for every function $W^{1,\infty}$-function η_1 with support in $\mathcal{W}_{\mathrm{x}}$. Define now $\mathcal{V}_{\mathrm{x}} := \phi_{\mathrm{x}}^{-1}(\mathcal{W}_{\mathrm{x}})$. If $\eta \in W^{1,\infty}(\mathbb{R}^3)$ with support in $\mathcal{V}_{\mathrm{x}}$ is given, then take η_0 as any $C_0^\infty(\mathcal{V}_{\mathrm{x}})$-function which is identical to 1 on $\operatorname{supp} \eta$. Then, putting $\eta_1 = \eta \circ \phi_{\mathrm{x}}^{-1}$, one obtains the asserted properties for $\eta u = (\eta_1 v) \circ \phi_{\mathrm{x}}$. □

LEMMA 5.17. *Suppose Assumption 4.2,* $\mathrm{x} \in \Gamma$ *and Assumption 4.5. Then there is an open neighbourhood* $\mathcal{V}_{\mathrm{x}}$ *with the following property: if* u *satisfies*

$$-\nabla \cdot \mu \nabla u = f \in W_\Gamma^{-1,3}(\Omega) \tag{5.20}$$

and $\eta \in W_0^{1,\infty}(\mathcal{V}_{\mathrm{x}})$*, then* $\eta u \in W_\Gamma^{1,3}(\Omega)$.

Proof. The proofs for the cases NS, NE1, NE2, NE3 run along the same lines as the proof of Lemma 5.16, resting on the results in Proposition 3.7, Proposition 3.8 and Proposition 3.9, respectively. The proof in the case NV is a bit more involved and we provide it in some more detail. According to the supposition, there is a polyhedral 3-manifold M with boundary, a neighbourhood $\mathcal{U}_{\mathrm{x}}$ and a C^1-diffeomorphism ϕ_{x}, defined on a neighbourhood of $\overline{\mathcal{U}_{\mathrm{x}}}$ which maps $\Omega \cap \mathcal{U}_{\mathrm{x}}$ onto $\Lambda := \mathrm{Interior}(M)$ such that $\phi(\mathrm{x})$ is a vertex of M. Moreover, $\phi_{\mathrm{x}}(\mathcal{U}_{\mathrm{x}} \cap \mathcal{S}) = \Lambda \cap \mathcal{H}$, where $\mathcal{H}$ is a plane satisfying the suppositions in Proposition 3.4. Localizing the problem according to Lemma 5.9 (with $\mathcal{U} := \mathcal{U}_{\mathrm{x}}$), one gets an equation for ηu with Neumann boundary conditions, compare Remark 5.10. Now one transforms the resulting problem using the C^1-diffeomorphism ϕ_{x}, again obtaining a Neumann problem. Let $\check{\mu}$ be the transformed coefficient function, which is then uniformly continuous on all the components $\Lambda_1, \ldots, \Lambda_n$ of $\Lambda \setminus \mathcal{H}$. On Λ, we define the modified coefficient function $\tilde{\mu}$ by

$$\tilde{\mu}(\mathrm{y}) = \begin{cases} \lim_{\mathrm{z} \in \Lambda_j, \mathrm{z} \mapsto \mathrm{x}} \check{\mu}(\mathrm{z}), & \text{if} \quad \mathrm{y} \in \Lambda_j; \quad j = 1, \ldots, n \\ 1 \quad \text{on} \quad \mathcal{H}. \end{cases}$$

According to Proposition 3.4, there is an open neighbourhood $\widetilde{\mathcal{U}_{\mathrm{x}}} \subset \phi_{\mathrm{x}}(\mathcal{U})$ of $\phi_{\mathrm{x}}(\mathrm{x})$ with the following property: Setting $\Pi := \Lambda \cap \widetilde{\mathcal{U}_{\mathrm{x}}}$ and $\Gamma := \partial\Lambda \cap \widetilde{\mathcal{U}_{\mathrm{x}}}$, we obtain

$$-\nabla \cdot \check{\mu}|_\Pi \nabla \in \mathcal{LH}(W_\Gamma^{1,3}(\Pi); W_\Gamma^{-1,3}(\Pi)).$$

Now one can apply Lemma 5.12 and argue as in the proof of Lemma 5.17 above. □

LEMMA 5.18. *Assume* $\mathrm{x} \in \overline{\Gamma} \cap (\partial\Omega \setminus \Gamma)$*. Then, under Assumptions 4.2/4.6, there is an open neighbourhood* $\mathcal{V}_{\mathrm{x}} \ni \mathrm{x}$ *with the following property: if* u *satisfies*

$$-\nabla \cdot \mu \nabla u = f \in W_\Gamma^{-1,3}(\Omega) \tag{5.21}$$

and $\eta \in W_0^{1,\infty}(\mathcal{V}_{\mathrm{x}})$*, then* $\eta u \in W_\Gamma^{1,3}(\Omega)$.

Proof. The proof runs essentially along the same lines as that of Lemma 5.16. The corresponding regularity results are as follows: BS$\leftrightharpoons$ Proposition 3.10 ii), BBE1$\leftrightharpoons$ Proposition 3.10 i), BE2$\leftrightharpoons$ Proposition 3.10 ii), BE3$\leftrightharpoons$ Proposition 3.10 iii), BE4$\leftrightharpoons$ Proposition 3.10 iv)/Corollary 5.14, MV1$\leftrightharpoons$ Proposition 3.10 i), MV2$\leftrightharpoons$ Proposition 3.10 iii). □

We now finish the proof of Theorem 4.8: For any $f \in W_\Gamma^{-1,3}(\Omega) \hookrightarrow W_\Gamma^{-1,2}(\Omega)$, the equation

$$-\nabla \cdot \mu \nabla u + u = f \tag{5.22}$$

admits exactly one solution $u \in W^{1,2}_\Gamma(\Omega)$. In three space dimensions, $W^{1,2}_\Gamma(\Omega)$ continuously embeds into $W^{-1,3}_\Gamma(\Omega)$, so that one can rewrite (5.22) as

$$-\nabla \cdot \mu \nabla u = f - u \in W^{-1,3}_\Gamma(\Omega). \tag{5.23}$$

Since we already know the solution u of this equation to exist in any case, the localization principle in Lemma 5.9 is applicable. Thus, according to Lemma 5.13, Lemma 5.16, Lemma 5.17 and Lemma 5.18, for every $\mathrm{x} \in \overline{\Omega}$, there is an open neighbourhood $\mathcal{U}_\mathrm{x} \ni \mathrm{x}$ such that $\eta u \in W^{1,3}_\Gamma(\Omega)$, if $\eta \in C_0^\infty(\mathcal{U}_\mathrm{x})$. Let $\mathcal{U}_{\mathbf{x}_1}, \ldots, \mathcal{U}_{\mathbf{x}_n}$ be a finite subcovering of $\overline{\Omega}$ and $\eta_1, \ldots, \eta_n$ be a partition of unity over $\overline{\Omega}$ which is subordinate to this subcovering. Clearly, then

$$u = \sum_{j=1}^n \eta_j u \in W^{1,3}_\Gamma(\Omega).$$

Thus, $-\nabla \cdot \mu \nabla + 1 : W^{1,3}_\Gamma(\Omega) \to W^{-1,3}_\Gamma(\Omega)$ is a continuous bijection. By the Open Mapping Theorem, its inverse is also continuous. Finally, Lemma 5.5 gives the result.

6. Scalar multipliers for the coefficient function. In this section, we show how uniformly continuous positive scalar multipliers preserve the isomorphism property (1.1). This abstract result is particularly useful for handling quasilinear elliptic or parabolic problems.

DEFINITION 6.1. *Let $\underline{C}(\Omega)$ denote the set of positive functions on Ω which are uniformly continuous and admit a positive lower bound.*

LEMMA 6.2. *Assume that*

$$-\nabla \cdot \mu \nabla : W^{1,q}_\Gamma(\Omega) \to W^{-1,q}_\Gamma(\Omega) \tag{6.1}$$

is a topological isomorphism for some number $q \in [2, 6]$. If $\vartheta \in \underline{C}(\Omega)$, then also

$$-\nabla \cdot \vartheta \mu \nabla : W^{1,q}_\Gamma(\Omega) \to W^{-1,q}_\Gamma(\Omega) \tag{6.2}$$

is a topological isomorphism.

Proof. Let us identify ϑ with its unique uniformly continuous extension to $\overline{\Omega}$, and let us denote the positive lower bound of ϑ by $\underline{\vartheta}$. Let, for any $\mathrm{x} \in \overline{\Omega}$, $\mathcal{W}_\mathrm{x}$ be a neighbourhood of x such that for any $\mathrm{y} \in \mathcal{W}_\mathrm{x} \cap \Omega$ we have

$$|(\vartheta(\mathrm{x}) - \vartheta(\mathrm{y})| \, \|\mu\|_{L^\infty(\Omega)} \, \frac{1}{\underline{\vartheta}} \, \|(\nabla \cdot \mu \nabla)^{-1}\|_{\mathcal{L}(W^{-1,q}_\Gamma(\Omega); W^{1,q}_\Gamma(\Omega))} < 1. \tag{6.3}$$

Let $\mathcal{W}_{x_1}, \ldots, \mathcal{W}_{x_n}$ be a finite covering of $\overline{\Omega}$ and let $\eta_1, \ldots, \eta_n$ be a subordinate partition of unity on $\overline{\Omega}$. Let now $f \in W^{-1,q}_\Gamma(\Omega)$ be arbitrary. Then the equation $-\nabla \cdot \vartheta \mu \nabla \varphi = f$ leads to the equations

$$-\nabla \vartheta \mu \nabla \eta_j \varphi = f_j, \tag{6.4}$$

where the right hand side f_j of (6.4) again belongs to $W^{-1,q}_\Gamma(\Omega)$ by Lemma 5.9. Let from now on $j \in \{1, \ldots, n\}$ be arbitrary but fixed and let us define the function ϑ_j on Ω by

$$\vartheta_j = \begin{cases} \vartheta \text{ on } \mathcal{W}_{\mathrm{x}_j} \cap \Omega, \\ \vartheta(\mathrm{x}_j) \text{ otherwise.} \end{cases}$$

Since η_j has its support in $\mathcal{W}_{\mathrm{x}_j}$, the function $\eta_j\varphi$ satisfies (6.4) and the equation

$$-\nabla\cdot\vartheta_j\mu\nabla\eta_j\varphi = f_j \in W^{-1,q}_\Gamma(\Omega). \tag{6.5}$$

Obviously, thanks to our supposition on $\nabla\cdot\mu\nabla$, the operator

$$\nabla\cdot\vartheta(\mathrm{x}_j)\mu\nabla = \vartheta(\mathrm{x}_j)\nabla\cdot\mu\nabla : W^{1,q}_\Gamma(\Omega) \to W^{-1,q}_\Gamma(\Omega)$$

provides a topological isomorphism which satisfies the estimate

$$\begin{aligned}\|\big(\nabla\cdot\vartheta(\mathrm{x}_j)\mu\nabla\big)^{-1}\|_{\mathcal{L}(W^{-1,q}_\Gamma(\Omega);W^{1,q}_\Gamma(\Omega))} &= \frac{1}{\vartheta(\mathrm{x}_j)}\|\big(\nabla\cdot\mu\nabla\big)^{-1}\|_{\mathcal{L}(W^{-1,q}_\Gamma(\Omega);W^{1,q}_\Gamma(\Omega))}\\ &\le \frac{1}{\underline{\vartheta}}\|\big(\nabla\cdot\mu\nabla\big)^{-1}\|_{\mathcal{L}(W^{-1,q}_\Gamma(\Omega);W^{1,q}_\Gamma(\Omega))}.\end{aligned} \tag{6.6}$$

We write

$$-\nabla\cdot\vartheta_j\mu\nabla = -\nabla\cdot\vartheta(\mathrm{x}_j)\mu\nabla - \nabla\cdot(\vartheta_j-\vartheta(\mathrm{x}_j))\mu\nabla. \tag{6.7}$$

The definition of the function ϑ_j yields, in combination with (6.6),

$$\begin{aligned}&\|\vartheta_j\mu-\vartheta(\mathrm{x}_j)\mu\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{R}^3))}\|(\nabla\cdot\vartheta(\mathrm{x}_j)\mu\nabla)^{-1}\|_{\mathcal{L}(W^{-1,q}_\Gamma(\Omega);W^{1,q}_\Gamma(\Omega))}\\ &\le \|\vartheta-\vartheta(\mathrm{x}_j)\|_{L^\infty(\mathcal{W}_{\mathrm{x}_j})}\|\mu\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{R}^3))}\frac{1}{\underline{\vartheta}}\|(\nabla\cdot\mu\nabla)^{-1}\|_{\mathcal{L}(W^{-1,q}_\Gamma(\Omega);W^{1,q}_\Gamma(\Omega))} < 1,\end{aligned} \tag{6.8}$$

thanks to condition (6.3). Now (6.7), (6.8) and Lemma 5.1 tell us that the isomorphism property $\nabla\cdot\vartheta(\mathrm{x}_j)\mu\nabla \in \mathcal{LH}(W^{1,q}_\Gamma(\Omega);W^{-1,q}_\Gamma(\Omega))$ carries over to the operator $-\nabla\cdot\vartheta_j\mu\nabla$. Hence, for each j, the solution $\eta_j v$ of (6.4) belongs to the space $W^{1,q}_\Gamma(\Omega)$, which is then also true for v. Thus, the operator in (6.2) is a continuous bijection and by the Open Mapping Theorem, its inverse is continuous as well. □

Theorem 6.3. *Assume that* (1.1) *is a topological isomorphism for some number* $q\in[2,6]$. *Then the mapping*

$$\underline{C}(\Omega) \ni \vartheta \mapsto (-\nabla\cdot\vartheta\mu\nabla)^{-1} \in \mathcal{L}\big(W^{-1,q}_\Gamma(\Omega);W^{1,q}_\Gamma(\Omega)\big) \tag{6.9}$$

is well-defined and even continuous.

Proof. The first assertion results from the previous lemma. The second assertion is implied by the first, Lemma 5.1 and the continuity of the mapping $\mathcal{LH}(X;Y)\ni B\mapsto B^{-1}\in\mathcal{LH}(Y;X)$, see [73, Ch. III.8]. □

Corollary 6.4. *Assume that* (6.1) *is a topological isomorphism for some number* $q\in[2,6]$. *Let* $\mathcal{M}$ *be a compact set in* $C(\overline{\Omega})$ *which admits a uniform lower positive bound. Then the function*

$$\mathcal{M}\ni\vartheta\mapsto\big(-\nabla\cdot\vartheta\mu\nabla\big)^{-1}\in\mathcal{L}\big(W^{-1,q}_\Gamma(\Omega);W^{1,q}_\Gamma(\Omega)\big)$$

is bounded and even Lipschitzian.

Proof. The first assertion follows from Theorem 6.3 and the compactness of $\mathcal{M}$ in $\underline{C}(\Omega)$. The second assertion follows from the first, the resolvent equation,

$$(-\nabla\cdot\vartheta_1\mu\nabla)^{-1}-(-\nabla\cdot\vartheta_2\mu\nabla)^{-1} = (-\nabla\cdot\vartheta_1\mu\nabla)^{-1}(-\nabla\cdot(\vartheta_1-\vartheta_2)\mu\nabla)(-\nabla\cdot\vartheta_2\mu\nabla)^{-1}, \tag{6.10}$$

and Lemma 5.1. □

REMARK 6.5. *Assume that* (6.1) *is a topological isomorphism for a* $q > 3$*, i.e.*

$$\mathcal{D} := \mathrm{dom}_{W_\Gamma^{-1,q}(\Omega)}(\nabla \cdot \mu \nabla) = W_\Gamma^{1,q}(\Omega).$$

Then, for $\theta \in]0, \frac{1}{2}[$*, one has by re-iteration (cf. [79, Ch. 1.9.3])*

$$\begin{aligned}
[\mathcal{D}, W_\Gamma^{-1,q}(\Omega)]_\theta &= [W_\Gamma^{1,q}(\Omega), W_\Gamma^{-1,q}(\Omega)]_\theta = \big[W_\Gamma^{1,q}(\Omega), [W_\Gamma^{1,q}(\Omega), W_\Gamma^{-1,q}(\Omega)]_{\frac{1}{2}}\big]_{2\theta} \\
&\hookrightarrow \big[W_\Gamma^{1,q}(\Omega), [W_\Gamma^{1,2}(\Omega), W_\Gamma^{-1,2}(\Omega)]_{\frac{1}{2}}\big]_{2\theta} = [W_\Gamma^{1,q}(\Omega), L^2(\Omega)]_{2\theta} \\
&\hookrightarrow C(\overline{\Omega}), \qquad (6.11)
\end{aligned}$$

if θ *is chosen sufficiently small, cf. [28].*
Let $F : \mathbb{R} \to]0, \infty[$ *be a locally Lipschitzian function. If one defines, for* $u \in C(\overline{\Omega})$*,*

$$A(u) := -\nabla \cdot F(u)\mu\nabla,$$

then the mapping $u \mapsto A(u)$*, considered between the spaces*

$$[\mathcal{D}, W_\Gamma^{-1,q}(\Omega)]_\theta \hookrightarrow C(\overline{\Omega}) \to \mathcal{LH}(W_\Gamma^{1,q}(\Omega); W_\Gamma^{-1,q}(\Omega))$$

is well-behaved according to Corollary 6.4 and (6.11)*. Thus, the quasilinear operators*

$$W_\Gamma^{1,q}(\Omega) \ni u \mapsto A(u)u$$

fit into Pruess' sheme for the treatment of quasilinear parabolic equations, see [69] for details.

7. Limitations/Obstructions. Let us, in this section, discuss the limitations of our concept. It has been known for a long time that a fundamental obstruction against higher integrability for the gradient are edge and vertex singularities. Aiming at a $W^{1,q} \Leftrightarrow W^{-1,q}$ concept for $q > 3$, it was shown in [59] that it suffices to delimitate the edge singularities suitably. Thus, in a first step, one has to exclude all constellations in which the edge singularities are too strong. This leads to the requirement that hetero-interfaces have to be C^1, if matrix-valued coefficient functions are involved (for the scalar case, compare [71]). However, this is in the nature of things: the presence of just one edge in an interface is a potential obstruction against the L^q-integrability of the gradient if $q > 2$, see Elschner's counterexample ([20, Rem. 5.1] or [21, Ch. 4]).

Concerning multi-material boundary edges, the only setting in which the singularity exponent is larger than $\frac{1}{3}$ in general (in fact: even larger than $\frac{1}{2}$) is if at most two material sectors are involved – each having an opening angle not larger than π — and pure Dirichlet or pure Neumann boundary conditions are imposed (see [20, Lemma 2.3] or the Appendix of [35]). In the case of mixed boundary conditions, these singularities are already too bad in general if only two materials are involved, or if one material is involved, but the corresponding angle is larger than π, see [61]. In particular, this excludes the crossing beams (Figure 3 right) if one of them is carrying a Dirichlet boundary condition and the other one a Neumann condition — even if the material is homogeneous. Regrettably, edge singularities are also too strong in general in the cases of pure Dirichlet or Neumann boundary conditions, if the edge is adjacent to at least three materials, see [61] and see also [36] for a detailed discussion of singularities caused by multimaterial edges. Unfortunately, we did not succeed in [36] in finding sufficiently rich classes of coefficient configurations, correponding to three or four materials, which admit a singularity exponent larger that $\frac{1}{3}$ as required in [59]. In particular, this means that interfaces which mark the heterogeneity of the material are not allowed to intersect; as this would lead to multimaterial inner edges. In this — very broad — sense, our material constellations are 'layered' ones. If one wants to include the case of intersecting interfaces in a similar concept, spaces with weights promise to yield an adequate framework, compare [60], [2], [3] and [6].

In view of Proposition 5.6, the question arises whether one could not admit bi-Lipschitzian mappings ϕ_{x} in Assumption 4.2 instead of only C^1 ones. In principle, this is possible, but: deforming e.g. a suitable model constellation (say, NV from above) by a bi-Lipschitzian mapping, one can obtain edges which are adjacent to arbitrarily many materials — not to be identified on the image side as a possibly 'harmless' constellation (see [33, Thm. 3.10/Thm. 4.15] and cf. Remark 5.15). But the intention of this paper was the following: Given a non-smooth model problem derived from a real-world application, it should be possible to decide 'by appearance', whether our setting is applicable. For example, we did not want to introduce implicit conditions on edge singularities (compare [59]) which are extremely difficult to control in palpable examples, see [36] for the case of only three or four materials. This was the reason for restricting the class of admissable transformations to C^1, in which surfaces are forced to be mapped into surfaces, edges into edges and vertices into vertices.

Strong Lipschitz domains provide another admissible class of model constellations. For such domains, the isomorphism property (1.1) also holds for some $q > 3$, if $\Gamma = \emptyset$ or $\Gamma = \partial\Omega$ and the coefficient function is uniformly continuous. This is based on the deep results of [43] in the Dirichlet case and of [81] in the Neumann case — there obtained for the Laplacian — and carried over to general elliptic uniformly continuous coefficient functions in [21]. As it is difficult to come up with examples of applications with domains which may be strongly Lipschitz but are not locally C_1-diffeomorphic to polyhedra, for simplicity, we finally decided not to include them here.

Taking all of this into account, we feel that our concept is not far from optimal in its generality, if one aims at an integrability exponent larger than 3 for the gradient of solutions. Of course, there are special constellations — also admitting elliptic regularity — which include for example three materials adjacent to one edge, see [59], [36] and also [50].

Finally, let us remark that in this generality, we cannot expect an analogous concept in dimensions four or higher. Since one aims at an integrability exponent q which is larger than the space dimension, the classical counterexample of Shamir [75] is an obstruction in the case of mixed boundary conditions.

8. Concluding remarks. As a conclusion, we remark on three immediate extensions of Theorem 4.8.

Remark 8.1. *Theorem 4.8 remains true for real spaces: if one takes $W^{1,q}_{\mathbb{R},\Gamma}$ as the real part of the complex Sobolev space and $W^{-1,q}_{\mathbb{R},\Gamma}$ as the dual of $W^{1,q'}_{\mathbb{R},\Gamma}$, then $-\nabla\cdot\mu\nabla + 1 : W^{1,q}_{\mathbb{R},\Gamma} \to W^{-1,q}_{\mathbb{R},\Gamma}$ is a topological isomorphism for the same range of q's since the operator commutes with complex conjugation.*

Remark 8.2. *Let p be the number from Theorem 4.8, and let $q \in]3,p[$. Assume $\mathfrak{a}, \mathfrak{b} \in L^\infty(\Omega;\mathbb{C}^3)$ and $c \in L^\infty(\Omega)$. Then the first order operator*

$$B : W^{1,q}_{\Gamma}(\Omega) \ni u \mapsto \nabla\big(\mathfrak{a}\, u\big) + \mathfrak{b}\cdot\nabla u + c\, u \in W^{-1,q}_{\Gamma}(\Omega) \tag{8.1}$$

is relatively compact with respect to $-\nabla\cdot\mu\nabla$. Hence, if $-\nabla\cdot\mu\nabla$ is perturbed by (8.1), it also has $W^{1,q}_{\Gamma}(\Omega)$ as its domain of definition. Moreover, (8.1) is relatively bounded with respect to $-\nabla\cdot\mu\nabla$ with an arbitrarily small relative bound, due to Ehrling's *lemma. Since $-\nabla \cdot \mu\nabla$ generates an analytic semigroup on $W_\Gamma^{-1,q}(\Omega)$ (see [38]/[39]) this is then true also for $-\nabla\cdot\mu\nabla+B$ — thanks to a well-known perturbation argument. In particular, this shows that $-\nabla \cdot \mu\nabla + B$ also has no spectrum in a suitable left half plane, and, additionally, its spectrum is purely discrete.*

REMARK 8.3. *In the proof of Theorem 4.8, Snejberg's abstract argument for $p > 3$ may be avoided by choosing the minimal $p > 3$ of all local constituents. However, this would in general not provide new insights on the size of p and would considerably blow up technicalities.*

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