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Hardy's inequality for functions vanishing on a part of the boundary

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Abstract

We develop a geometric framework for Hardy's inequality on a bounded domain when the functions do vanish only on a closed portion of the boundary.

1. Introduction

Hardy's inequality is one of the classical items in analysis [26, 40]. Two milestones among many others in the development of the theory seem to be the result of Necas [39] that Hardy's inequality holds on strongly Lipschitz domains and the insight of Maz'ya [36], [37, Ch. 2.3] that its validity depends on measure theoretic conditions on the domain. Rather recently, the geometric framework in which Hardy's inequality remains valid was enlarged up to the frontiers of what is possible – as long as the boundary condition is purely Dirichlet, see [24, 27], compare also [4, 30, 46]. Moreover, over the last years it became manifest that Hardy's inequality plays an eminent role in modern PDE theory, see e.g. [8, 44, 41, 2, 13, 10, 16, 22, 31, 32].

What has not been treated systematically is the case where *only a part* D of the boundary of the underlying domain Ω is involved, reflecting the Dirichlet condition of the equation on this part – while on $\partial\Omega \setminus D$ other boundary conditions may be imposed, compare [25, 2, 23, 9] including references therein. The aim of this paper is to set up a geometric framework for the domain Ω and the Dirichlet boundary part D that allow to deduce the corresponding Hardy inequality

$$\int_\Omega \left| \frac{u}{\operatorname{dist}_D} \right|^p \,\mathrm{dx} \leq c \int_\Omega |\nabla u|^p \,\mathrm{dx}.$$

As in the well established case $D = \partial\Omega$ we in essence only require that D is $(d-1)$-thick in the sense of [27]. In our context this condition can be understood as an extremely weak compatibility condition between D and $\partial\Omega \setminus D$.

Our strategy of proof then is to first reduce in Section 4 to the case $D = \partial\Omega$ by purely topological means, provided two major tools are applicable: An extension operator $\mathfrak{E} : W^{1,p}_D(\Omega) \to W^{1,p}_D(\mathbb{R}^d)$, the subscript D indicating the subspace of those Sobolev functions which vanish on D in an appropriate sense, and a Poincaré inequality on $W^{1,p}_D(\Omega)$. In a second step in Sections 5 and 6 these partly implicit conditions are substantiated by more geometric assumptions that can be checked – more or less – by appearance. In particular, we prove under fairly general assumptions that every linear continuous extension operator $W^{1,p}_D(\Omega) \to W^{1,p}(\mathbb{R}^d)$ preserves the Dirichlet condition on D. This result even carries over to higher-order Sobolev spaces and sheds new light on some of the deep results on Sobolev extension operators obtained in [5].

It is of course natural to ask, whether Hardy's inequality also characterizes the space $W^{1,p}_D(\Omega)$, i.e. whether the latter is precisely the space of those functions $u \in W^{1,p}(\Omega)$ for which u/dist_D belongs to $L^p(\Omega)$. Under very mild geometric assumptions we answer this question to the affirmative in Section 7.

Finally, in Section 8 we attend to the naive intuition that the part of $\partial\Omega$ that is far away from D should only be circumstantial for the validity of Hardy's inequality and in fact we succeed to weaken the previously discussed geometric assumptions considerably.

2. Notation

Throughout we work in Euclidean space $\mathbb{R}^d$, $d \geq 1$. We use x, y, etc. for vectors in $\mathbb{R}^d$ and denote the open ball in $\mathbb{R}^d$ around x with radius r by $B(\mathrm{x}, r)$. The letter c is reserved for generic constants that may change their value from occurrence to occurrence. Given $F \subset \mathbb{R}^d$ we write dist_F for the function that measures the distance to F and $\operatorname{diam}(F)$ for the diameter of F. For

$l \in]0, \infty[$ the l-dimensional Hausdorff measure of F is

$$\mathcal{H}_l(F) := \liminf_{\delta \to 0} \Big\{ \sum_{j=1}^{\infty} \mathrm{diam}(F_j)^l : F_j \subset \mathbb{R}^d, \, \mathrm{diam}(F_j) \leq \delta, \, F \subset \bigcup_{j=1}^{\infty} F_j \Big\}$$

and its centered Hausdorff content is defined by

$$\mathcal{H}_l^\infty(F) := \inf \Big\{ \sum_{j=1}^{\infty} r_j^l : x_j \in F, \, r_j > 0, \, A \subset \bigcup_{j=1}^{\infty} B(x_j, r_j) \Big\}.$$

We also recall the following notions from geometric measure theory.

Definition 2.1. Let $l \in]0, \infty[$. A non-empty compact set $F \subset \mathbb{R}^d$ is called *l-thick* if there exist $R > 0$ and $\gamma > 0$ such that

$$\mathcal{H}_l^\infty(F \cap B(\mathrm{x}, r)) \geq \gamma \, r^l, \tag{2.1}$$

holds for all $\mathrm{x} \in F$ and all $r \in]0, R]$. It is called *l-set* if there are two constants c_0, c_1 such that

$$c_0 r^l \leq \mathcal{H}_l(F \cap B(\mathrm{x}, r)) \leq c_1 r^l \tag{2.2}$$

holds for all $r \in]0, 1]$ and all $\mathrm{x} \in F$.

Remark 2.2.
(i) If (2.1) holds for constants R, γ, then for all $S \geq R$ it also holds with R and γ replaced by S and $\gamma R^l S^{-l}$, respectively. For more information on this notion of l-thick sets see e.g. [27].
(ii) The notion of l-sets is due to [21, Sec. II.1]. It can be extended literally to arbitrary Borel sets F, c.f. [21, Sec. VII.1.1].
(iii) In Section 4 we will see that every l-set is also l-thick.

Next, let us introduce the common first-order Sobolev spaces of functions 'vanishing' on a part of the closure of the underlying domain that are most essential for the formulation of Hardy's inequality. If Λ is an open subset of $\mathbb{R}^d$ and E is a closed subset of $\overline{\Lambda}$, then for $p \in [1, \infty[$ the space $W_E^{1,p}(\Lambda)$ is defined as the completion of

$$C_E^\infty(\Lambda) := \{ v|_\Lambda : v \in C_0^\infty(\mathbb{R}^d), \, \mathrm{supp}(v) \cap E = \emptyset \}$$

with respect to the norm $v \mapsto \big(\int_\Lambda |\nabla v|^p + |v|^p \, \mathrm{dx} \big)^{1/p}$. More generally, for $k \in \mathbb{N}$ we define $W_E^{k,p}(\Lambda)$ as the closure of $C_E^\infty(\Lambda)$ with respect to the norm $v \mapsto \big(\int_\Lambda \sum_{j=0}^k |D^j v|^p \, \mathrm{dx} \big)^{1/p}$. The situation we have in mind is of course when $\Lambda = \Omega$ and $E = D$ is the Dirichlet part D of $\partial\Omega$.

The Sobolev spaces $W^{k,p}(\Lambda)$ are defined as usual as the space of those $L^p(\Lambda)$ functions whose distributional derivatives up to order k are in $L^p(\Lambda)$ equipped with the natural norm. Note that by definition $W_0^{k,p}(\Lambda) = W_{\partial\Lambda}^{k,p}(\Lambda)$ but in general $W_\emptyset^{k,p}(\Lambda) \subsetneq W^{k,p}(\Lambda)$, cf. [37, Sec. 1.1.6]

3. Main results

The following version of Hardy's inequality for functions vanishing on a part of the boundary is our main result.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $D \subset \partial\Omega$ be a closed part of the boundary and $p \in]1, \infty[$. Suppose the following three conditions are satisfied.*
(i) *The set D is $(d-1)$-thick.*
(ii) *The space $W_D^{1,p}(\Omega)$ can be equivalently normed by $\|\nabla \cdot \|_{L^p(\Omega)}$.*
(iii) *There is a linear continuous extension operator $\mathfrak{E} : W_D^{1,p}(\Omega) \to W_D^{1,p}(\mathbb{R}^d)$.*

Then there is a constant $c > 0$ such that Hardy's inequality

$$\int_\Omega \left| \frac{u}{\operatorname{dist}_D} \right|^p \mathrm{dx} \le c \int_\Omega |\nabla u|^p \mathrm{dx} \tag{3.1}$$

holds for all $u \in W^{1,p}_D(\Omega)$.

Of course the conditions (ii) and (iii) in Theorem 3.1 are rather abstract and should be supported by more geometrical ones. This will be the content of Sections 5 and 6 where we shall give an extensive kit of such conditions. In particular, we will obtain the following version of Hardy's inequality. For the precise definitions of Lipschitz- and (ε, δ)-domains we refer to Subsection 5.4.

Theorem 3.2 (A special Hardy inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, let $D \subset \partial\Omega$ be a $(d-1)$-set and assume that for every $\mathrm{x} \in \overline{\partial\Omega \setminus D}$ there is an open neighborhood U_x of x such that $\Omega \cap U_\mathrm{x}$ is a Lipschitz, or more generally, an (ε, δ)-domain. Then for every $p \in]1, \infty[$ there is a constant $c_p > 0$ such that*

$$\int_\Omega \left| \frac{u}{\operatorname{dist}_D} \right|^p \mathrm{dx} \le c_p \int_\Omega |\nabla u|^p \mathrm{dx}, \quad u \in W^{1,p}_D(\Omega).$$

Still, as we believe, the abstract framework traced out by the second and the third condition of Theorem 3.1 has the advantage that other sufficient geometric conditions for Hardy's inequality – tailor-suited for future applications – can be found much more easily. In fact the second condition is equivalent to the validity of Poincaré's inequality

$$\|u\|_{L^p} \le c\|\nabla u\|_{L^p(\Omega)}, \quad u \in W^{1,p}_D(\Omega),$$

that is clearly *necessary* for Hardy's inequality (3.1). We give a detailed discussion of Poincaré's inequality within the present context in Section 6. For further reference the reader may consult [48, Ch. 4]. Concerning the third condition note carefully that we require the extension operator to preserve the Dirichlet boundary condition on D. Whereas extension of Sobolev functions is a well-established business, the preservation of traces is much more delicate and we devote Subsection 5.3 to this problem.

It is interesting to remark that in the setup of Theorem 3.2 the space $W^{1,p}_D(\Omega)$ is the largest subspace of $W^{1,p}(\Omega)$ one which Hardy's inequality can hold. This is made precise by our third main result.

Theorem 3.3. *Suppose the geometric setup of Theorem 3.2. If $p \in]1, \infty[$ and $u \in W^{1,p}(\Omega)$ are such that $u/\operatorname{dist}_D \in L^p(\Omega)$, then already $u \in W^{1,p}_D(\Omega)$.*

Remark 3.4. In the case $D = \partial\Omega$ the conclusion of Theorem 3.3 is classical [11, Thm. V.3.4] and remains true without any assumptions on $\partial\Omega$.

In the following section we give the proof of the general Hardy inequality from Theorem 3.1. The proofs of Theorem 3.2 and 3.3 are postponed to the end of Sections 5 and 7, respectively.

4. Proof of Theorem 3.1

We will deduce Theorem 3.1 from the following proposition that states the assertion in the case $D = \partial\Omega$.

Proposition 4.1 ([27], see also [24])**.** *Let $\Omega_\bullet \subseteq \mathbb{R}^d$ be a bounded domain with $(d-1)$-thick boundary $\partial\Omega_\bullet$. Then Hardy's inequality is satisfied for all $u \in W^{1,p}_0(\Omega_\bullet)$, i.e.* (3.1) *holds with Ω replaced by $\Omega_\bullet$ and D by $\partial\Omega_\bullet$.*

Below we will reduce to the case $D = \partial\Omega$ by purely topological means, so that we can apply Proposition 4.1 afterwards. We will repeatedly use the following topological fact.

(■) Let $\{M_\lambda\}_\lambda$ be a family of connected subsets of a topological space. If $\bigcap_\lambda M_\lambda \neq \emptyset$, then $\bigcup_\lambda M_\lambda$ is again connected.

As required in Theorem 3.1 let now $\Omega \subseteq \mathbb{R}^d$ be a bounded domain and let D be a closed part of $\partial\Omega$. Then choose an open ball $B \supseteq \overline{\Omega}$ that, in what follows, will be considered as the relevant topological space. Consider

$$\mathcal{C} := \{M \subset B \setminus D : M \text{ open, connected and } \Omega \subset M\}$$

and for the rest of the proof put

$$\Omega_\bullet := \bigcup_{M \in \mathcal{C}} M.$$

In the subsequent lemma we collect some properties of $\Omega_\bullet$. Our proof here is not the shortest possible, cf. [6, Lem. 6.4] but it has, however, the advantage to give a description of $\Omega_\bullet$ as the union of Ω, the boundary part $\partial\Omega \setminus D$ and those connected components of $B \setminus \overline{\Omega}$ whose boundary does not consist only of points from D. This completely reflects the naive geometric intuition.

Lemma 4.2. *It holds $\Omega \subseteq \Omega_\bullet \subseteq B$. Moreover, $\Omega_\bullet$ is open and connected and $\partial\Omega_\bullet = D$ in B.*

Proof. The first assertion is obvious. By construction $\Omega_\bullet$ is open. Since all elements from $\mathcal{C}$ contain Ω the connectedness of $\Omega_\bullet$ follows by (■). It remains to show $\partial\Omega_\bullet = D$.

Let $\mathrm{x} \in D$. Then x is an accumulation point of Ω and, since $\Omega \subseteq \Omega_\bullet$, also of $\Omega_\bullet$. On the other hand, $\mathrm{x} \notin \Omega_\bullet$ by construction. This implies $\mathrm{x} \in \partial\Omega_\bullet$ and so $D \subseteq \partial\Omega_\bullet$.

In order to show the inverse inclusion, we first show that points from $\partial\Omega \setminus D$ cannot belong to $\partial\Omega_\bullet$. Indeed, since D is closed, for $\mathrm{x} \in \partial\Omega \setminus D$ there is a ball $B_\mathrm{x} \subseteq B$ around x that does not intersect D. Since x is a boundary point of Ω, we have $B_\mathrm{x} \cap \Omega \neq \emptyset$. Both, Ω and B_x are connected, so (■) yields that $\Omega \cup B_\mathrm{x}$ is connected. Moreover, this set is open, contains Ω and avoids D, so it belongs to $\mathcal{C}$ and we obtain $\Omega \cup B_\mathrm{x} \subseteq \Omega_\bullet$. This in particular yields $\mathrm{x} \in \Omega_\bullet$, so $\mathrm{x} \notin \partial\Omega_\bullet$ since $\Omega_\bullet$ is open.

Summing up, we already know that $\mathrm{x} \in \overline{\Omega}$ belongs to $\partial\Omega_\bullet$ if and only if $\mathrm{x} \in D$. So, it remains to make sure that no point from $B \setminus \overline{\Omega}$ belongs to $\partial\Omega_\bullet$.

As $B \setminus \overline{\Omega}$ is open, it splits up into its open connected components $Z_0, Z_1, Z_2, \ldots$. There are possibly only finitely many such components but at least one. We will show in a first step that for all these components it holds $\partial Z_j \subseteq \partial\Omega$. This allows to distinguish the two cases $\partial Z_j \subseteq D$ and $\partial Z_j \cap (\partial\Omega \setminus D) \neq \emptyset$. In Steps 2 and 3 we will then complete the proof by showing that in both cases Z_j does not intersect $\partial\Omega_\bullet$.

Step 1: $\partial Z_j \subseteq \partial\Omega$ for all j. First note that $\partial Z_j \cap \Omega = \emptyset$ for all j. Indeed, assuming this set to be non-empty and investing that Ω is open, we find that the set $Z_j \cap \Omega$ cannot be empty either and this contradicts the definition of Z_j.

Now, to prove the claim of Step 1, assume by contradiction that, for some j, there is a point $\mathrm{x} \in \partial Z_j$ that does not belong to $\partial\Omega$. By the observation above we then have $\mathrm{x} \notin \overline{\Omega}$ and consequently there is a ball B_x around x that does not intersect $\overline{\Omega}$. Now, the set $B_\mathrm{x} \cup Z_j$ is connected thanks to (■), avoids $\overline{\Omega}$ and includes Z_j properly. But this contradicts the property of Z_j to be a connected component of $B \setminus \overline{\Omega}$.

Step 2: If $\partial Z_j \subseteq D$, then $\overline{\Omega}_\bullet \cap Z_j = \emptyset$. We first note that it suffices to show $\Omega_\bullet \cap Z_j = \emptyset$. In fact, due to $\overline{\Omega}_\bullet = \partial\Omega_\bullet \cup \Omega_\bullet$ we then get $\overline{\Omega}_\bullet \cap Z_j = \emptyset$ since Z_j is open.

So, let us assume there is some $\mathrm{x} \in \Omega_\bullet \cap Z_j$. Then $\Omega_\bullet \cup Z_j$ is connected due to (■). By assumption we have $\partial Z_j \subseteq D$ and by construction the sets Z_j and $\Omega_\bullet$ are both disjoint to D. So we can infer that $\partial Z_j \cap (\Omega_\bullet \cup Z_j) = \emptyset$ and this allows us to write

$$\Omega_\bullet \cup Z_j = \big(\Omega_\bullet \cup Z_j\big) \cap \big(Z_j \cup (B \setminus \overline{Z}_j)\big) = Z_j \cup \big(\Omega_\bullet \cap (B \setminus \overline{Z}_j)\big).$$

This is a decomposition of $\Omega_\bullet \cup Z_j$ into two open and mutually disjoint sets, so if we can show that both are nonempty then this yields a contradiction to the connectedness of $\Omega_\bullet \cup Z_j$ and the claim of Step 2 follows. Indeed, we even find

$$\Omega_\bullet \cap (B \setminus \overline{Z}_j) = \Omega_\bullet \setminus \overline{Z}_j = \Omega_\bullet \setminus (\partial Z_j \cup Z_j) \supset \Omega \setminus (D \cup Z_j) = \Omega \neq \emptyset,$$

since both D and Z_j do not intersect Ω.

Step 3: If $\partial Z_j \cap (\partial\Omega \setminus D) \neq \emptyset$, then $Z_j \subseteq \Omega_\bullet$. Let $\mathrm{x} \in \partial Z_j \cap (\partial\Omega \setminus D)$, and let B_{x} be a ball around x that does not intersect D. The point x is a boundary point of Z_j, so $B_{\mathrm{x}} \cap Z_j \neq \emptyset$ and we obtain that $B_{\mathrm{x}} \cup Z_j$ is connected by (■). By the same argument, also the set $B_{\mathrm{x}} \cup \Omega$ is connected and putting these two together a third reiteration of the argument yields that $(B_{\mathrm{x}} \cup \Omega) \cup (B_{\mathrm{x}} \cup Z_j) = \Omega \cup B_{\mathrm{x}} \cup Z_j$ is again connected. This last set now is open and does not intersect D, so it belongs to $\mathcal{C}$ and we end up with $\Omega \cup B_{\mathrm{x}} \cup Z_j \subseteq \Omega_\bullet$. In particular we have $Z_j \subseteq \Omega_\bullet$. □

Remark 4.3. Conversely, it can be shown that the asserted properties characterize $\Omega_\bullet$ uniquely in the sense that if an open, connected subset $\Xi \supset \Omega$ of B additionally satisfies $\partial\Xi = D$, then necessarily $\Xi = \Omega_\bullet$. In fact, since $\Xi \cap D = \emptyset$ one has $\Xi \subset \Omega_\bullet$, due to the definition of $\Omega_\bullet$. In order to obtain the inverse inclusion we write

$$\Omega_\bullet = \big(\Omega_\bullet \cap \Xi\big) \cup \big(\Omega_\bullet \cap \partial\Xi\big) \cup \big(\Omega_\bullet \cap (B \setminus \overline{\Xi})\big) = \Xi \cup \big(\Omega_\bullet \cap (B \setminus \overline{\Xi})\big), \tag{4.1}$$

since $\Omega_\bullet \cap \partial\Xi = \Omega_\bullet \cap D = \emptyset$. Both $\Xi = \Xi \cap \Omega_\bullet$ and $\Omega_\bullet \cap (B \setminus \overline{\Xi})$ are open in $\Omega_\bullet$, and $\Xi \supset \Omega$ is non-empty. Since $\Omega_\bullet$ is connected and $\Xi = \Xi \cap \Omega_\bullet$ is clearly disjoint to $\Omega_\bullet \cap (B \setminus \overline{\Xi})$, this latter set must be empty. Thus, (4.1) gives $\Xi = \Omega_\bullet$.

Corollary 4.4. *Consider $\Omega_\bullet$ as a subset of $\mathbb{R}^d$. Then $\Omega_\bullet$ is open and connected. Moreover, either $\partial\Omega_\bullet = D$ or $\partial\Omega_\bullet = D \cup \partial B$.*

Proof. It is clear that $\Omega_\bullet$ remains open. Assume that $\Omega_\bullet$ is not connected. Then there are disjoint open sets $U, V \subseteq \mathbb{R}^d$ such that $\Omega_\bullet = U \cup V$. But the property $\Omega_\bullet \subseteq B$ then gives $\Omega_\bullet = \Omega_\bullet \cap B = (U \cap B) \cup (V \cap B)$, where $U \cap B$ and $V \cap B$ are open in B and disjoint to each other. This contradicts Lemma 4.2.

For the last assertion consider an annulus $A \subseteq B$ that is adjacent to ∂B and does not intersect $\overline{\Omega}$. Let Z_j be the connected component of $B \setminus \overline{\Omega}$ that contains A. We distinguish again the two cases of Step 2 and Step 3 in the proof of Lemma 4.2: If $\partial Z_j \subseteq D$, we have shown in Step 2 that Z_j is disjoint to $\Omega_\bullet$ and this implies $\partial\Omega_\bullet = \partial\Omega_\bullet \cap B = D$. In the second case, we infer from Step 3 in the above proof that $A \subseteq Z_j \subseteq \Omega_\bullet$ and this implies $\partial\Omega_\bullet = D \cup \partial B$. □

Let us now prove Theorem 3.1. We first observe that in both cases appearing in Corollary 4.4 the set $\partial\Omega_\bullet$ is $(d-1)$-thick. In fact, for the boundary part D this was an assumption and the $(d-1)$-thickness of ∂B can easily be checked using its local representation as the graph of a Lipschitz function. Hence, Proposition 4.1 applies to our special choice of $\Omega_\bullet$.

Now, let $\mathfrak{E}$ be the extension operator provided by Assumption (iii) of Theorem 3.1. In view of Corollary 4.4 we can define an extension operator $\mathfrak{E}_\bullet : W^{1,p}_D(\Omega) \to W^{1,p}_0(\Omega_\bullet)$ as follows: If $\partial\Omega_\bullet = D$, then we put $\mathfrak{E}_\bullet v := \mathfrak{E}v|_{\Omega_\bullet}$ and if $\partial\Omega_\bullet = D \cup \partial B$, then we choose $\eta \in C^\infty_0(B)$ with the property $\eta \equiv 1$ on $\overline{\Omega}$ and put $\mathfrak{E}_\bullet v := (\eta\mathfrak{E}v)|_{\Omega_\bullet}$. This allows us to apply Proposition 4.1 to the functions $\mathfrak{E}_\bullet u \in W^{1,p}_0(\Omega_\bullet)$, where u is taken from $W^{1,p}_D(\Omega)$. With a final help of Assumption (ii) in Theorem 3.1 this gives

$$\int_\Omega \left|\frac{u}{\mathrm{d}_D}\right|^p \mathrm{dx} \le \int_\Omega \left|\frac{u}{\mathrm{d}_{\partial\Omega_\bullet}}\right|^p \mathrm{dx} \le \int_{\Omega_\bullet} \left|\frac{\mathfrak{E}_\bullet u}{\mathrm{d}_{\partial\Omega_\bullet}}\right|^p \mathrm{dx} \le c\int_{\Omega_\bullet} |\nabla(\mathfrak{E}_\bullet u)|^p \,\mathrm{dx}$$

$$\le c\|\mathfrak{E}_\bullet u\|^p_{W^{1,p}_0(\Omega_\bullet)} \le c\|u\|^p_{W^{1,p}_D(\Omega)} \le c\int_\Omega |\nabla u|^p \,\mathrm{dx} \tag{4.2}$$

for all $u \in W^{1,p}_D(\Omega)$ and the proof is complete.

Remark 4.5. (i) At the first glance one might think that $\Omega_\bullet$ could always be taken as $B \setminus D$. The point is that this set need not be connected, as the following example shows. Take $\Omega = \{\mathrm{x} : 1 < |\mathrm{x}| < 2\}$ and $D = \{\mathrm{x} : |\mathrm{x}| = 1\} \cup \{\mathrm{x} : |\mathrm{x}| = 2, x_1 \ge 0\}$. Obviously, if a ball B contains $\overline{\Omega}$, then $B \setminus D$ cannot be connected. In the spirit of Lemma 4.2, the set $\Omega_\bullet$ has here to be taken as $B \setminus (D \cup \{\mathrm{x} : |\mathrm{x}| < 1\})$. Thus, the somewhat subtle, topological considerations in Lemma 4.2 cannot be avoided in general.

(ii) One might suggest that the procedure of this work is not limited to the proof of Hardy's inequality in the non-Dirichlet case. Possibly the combination of an application of the extension operator $\mathfrak{E}/\mathfrak{E}_\bullet$ and the construction of $\Omega_\bullet$ may serve for the reduction of other problems on function spaces related to mixed boundary conditions to the pure Dirichlet case.

Finally, instead of its $(d-1)$-thickness we can also require the more common and – as it will turn out – more restrictive condition that D is a $(d-1)$-set.

Theorem 4.6. *The assertion of Theorem 3.1 remains valid if instead of its $(d-1)$-thickness we require that D is a $(d-1)$-set.*

Remark 4.7. Having already Theorem 3.1 then additionally establishing Theorem 4.6 is not purely academic. Indeed at many occasions the $(d-1)$-property of D will give much better access to assumptions (ii) and (iii) and thus may be viewed as a tailor-suited condition for Hardy's inequality (3.1) in case of only partly vanishing trace overall.

One access to Theorem 4.6 is to prove that the $(d-1)$-property of $\partial\Omega$ implies the p-fatness of $\mathbb{R}^d \setminus \Omega$ – a result which was first obtained by Maz'ya [38]. Knowing this, Hardy's inequality may then be deduced from the results in [30] or [46]. Our approach is quite different and rests on Proposition 4.1 and the following lemma being implicit in [7, Lem. 2]. For the reader's convenience we include the elementary proof.

Lemma 4.8. *Let $l \in]0, \infty[$. If $F \subset \mathbb{R}^d$ is an l-set, then there are constants $c_0, c_1 > 0$ such that*

$$c_0 r^l \le \mathcal{H}^\infty_l(F \cap B(\mathrm{x}, r)) \le c_1 r^l$$

holds for all $r \in]0, 1[$ and all $\mathrm{x} \in F$. In particular, F is l-thick.

Proof. Recall that by definition F is non-empty and compact. We shall prove the estimates $\mathcal{H}^\infty_l(A) \le \mathcal{H}_l(A) \le c\mathcal{H}^\infty_l(A)$ for all non-empty Borel subsets $A \subset F$.

First, fix $\varepsilon > 0$ and let $\{A_j\}_{j\in\mathbb{N}}$ be a covering of A by sets with diameter at most ε. If $A_j \cap F \ne \emptyset$, then A_j is contained in an open ball B_j centered in A and radius such that $r_j^l = \mathrm{diam}(A_j)^l + \varepsilon 2^{-j}$. If $A_j \cap A = \emptyset$ let B_j be some ball centered in A and radius r_j as before. Then $\{B_j\}_{j\in\mathbb{N}}$ is a covering of A and

$$\sum_{j=1}^\infty \mathrm{diam}(A_j)^l \ge \sum_{j=1}^\infty (r_j^l - \varepsilon 2^{-j}) \ge \mathcal{H}^\infty_l(A) - \varepsilon.$$

Taking the infimum over all such coverings $\{A_j\}_{j\in\mathbb{N}}$ and passing to the limit $\varepsilon \to 0$ afterwards, $\mathcal{H}^\infty_l(A) \le \mathcal{H}_l(A)$ follows.

Conversely, let $\{B(x_j, r_j)\}_{j\in\mathbb{N}}$ be a covering of A by open balls centered in A. If $r_j \le 1$, then $\mathcal{H}_l(F \cap B(x_j, r_j)) \le c r_j^l$ since by assumption F is an l-set, and if $r_j > 1$, then certainly $\mathcal{H}_l(F \cap B(x_j, r_j)) \le \mathcal{H}_l(F) r_j^l$. Note carefully that $0 < \mathcal{H}_l(F) < \infty$ holds for F can be covered by finitely many balls with radius 1 centered in F. Altogether,

$$\sum_{j=1}^{\infty} r_j^l \ge c \sum_{j=1}^{\infty} \mathcal{H}_l(F \cap B(x_j, r_j)) \ge c\mathcal{H}_l\Big(F \cap \bigcup_{j=1}^{\infty} B(x_j, r_j)\Big) \ge c\mathcal{H}_l(A).$$

Passing to the infimum, $\mathcal{H}_l^\infty(A) \ge c\mathcal{H}_l(A)$ follows. □

5. The extension operator

In this section we discuss the second condition in our main result Theorem 3.1, that is the extendability for $W^{1,p}_D(\Omega)$ within the same class of Sobolev functions. We develop three abstract principles concerning Sobolev extension.

- Dirichlet cracks can be removed: We open the possibility of passing from Ω to another domain $\Omega_\star$ with a reduced Dirichlet boundary part, while $\Gamma = \partial\Omega \setminus D$ remains part of $\partial\Omega_\star$. In most cases this improves the boundary geometry in the sense of Sobolev extendability, see the example in the following Figure.

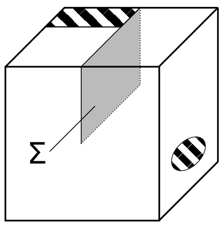

Figure 1. The set Σ does not belong to Ω, and carries – together with the striped parts – the Dirichlet condition.

- Sobolev extendability is a local property: We show that only the local geometry of the domain around the boundary part Γ plays a role for the existence of an extension operator.
- Preservation of traces: We prove under very general geometric assumptions that the extended functions do have the adequate trace behavior on D for *every* extension operator.

We believe that these result are of independent interest and therefore decided to directly present them for higher-order Sobolev spaces $W^{k,p}_E$. In the end we review some feasible commonly used geometric conditions which together with our abstract principles really imply the corresponding extendability.

5.1. **Dirichlet cracks can be removed.** As in Figure 1 there may be boundary parts which carry a Dirichlet condition and belong to the inner of the closure of the domain under consideration. Then one can extend the functions on Λ by 0 to such a boundary part, thereby enlarging the domain and simplifying the boundary geometry. In the following we make this precise.

Lemma 5.1. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain and let $E \subset \partial\Lambda$ be closed. Define $\Lambda_\bigstar$ as the interior of the set $\Lambda \cup E$. Then the following hold true.*

(i) *The set $\Lambda_\bigstar$ is again a domain, $\Xi := \partial\Lambda \setminus E$ is a (relatively) open subset of $\partial\Lambda_\bigstar$ and $\partial\Lambda_\bigstar = \Xi \cup (E \cap \partial\Lambda_\bigstar)$.*

(ii) *Let $k \in \mathbb{N}$ and $p \in [1,\infty[$. Extending functions from $W^{k,p}_E(\Lambda)$ by 0 to $\Lambda_\bigstar$, one obtains an isometric extension operator* $\mathrm{Ext}(\Lambda,\Lambda_\bigstar)$ *from $W^{k,p}_E(\Lambda)$ onto $W^{k,p}_E(\Lambda_\bigstar)$.*

Proof. (i) Due to the connectedness of Λ and the set inclusion $\Lambda \subset \Lambda_\bigstar \subset \overline{\Lambda}$, the set $\Lambda_\bigstar$ is also connected, and, hence a domain. Obviously, one has $\overline{\Lambda_\bigstar} = \overline{\Lambda}$. This, together with the inclusion $\Lambda \subset \Lambda_\bigstar$ leads to $\partial\Lambda_\bigstar \subset \partial\Lambda$. Since $\Xi \cap \Lambda_\bigstar = \emptyset$, one gets $\Xi \subset \partial\Lambda_\bigstar$. Furthermore, Ξ was relatively open in $\partial\Lambda$, the more it is relatively open in $\partial\Lambda_\bigstar$.

The last asserted equality follows from $\partial\Lambda_\bigstar = (\Xi \cap \partial\Lambda_\bigstar) \cup (E \cap \partial\Lambda_\bigstar)$ and $\Xi \subset \partial\Lambda_\bigstar$.

(ii) Consider any $\psi \in C^\infty_E(\mathbb{R}^d)$ and its restriction $\psi|_\Lambda$ to Λ. Since the support of ψ has a positive distance to E, one may extend $\psi|_\Lambda$ by 0 to the whole of $\Lambda_\bigstar$ without destroying the C^∞-property. Thus, this extension operator provides a linear isometry from $C^\infty_E(\Lambda)$ *onto* $C^\infty_E(\Lambda_\bigstar)$ (if both are equipped with the $W^{k,p}$-norm). This extends to a linear extension operator $\mathrm{Ext}(\Lambda,\Lambda_\bigstar)$ from $W^{k,p}_E(\Lambda)$ onto $W^{k,p}_E(\Lambda_\bigstar)$, see the two following commutative diagrams:

$$\begin{tikzcd} C^\infty_E(\mathbb{R}^d) \arrow[r, "\mathrm{restrict}_{\mathbb{R}^d\to\Lambda}"] \arrow[d, "\mathrm{restrict}_{\mathbb{R}^d\to\Lambda_\bigstar}"'] & C^\infty_E(\Lambda) \arrow[dl, "\mathrm{extend}_{\Lambda\to\Lambda_\bigstar}"] \\ C^\infty_E(\Lambda_\bigstar) & \end{tikzcd} \qquad \begin{tikzcd} W^{k,p}_E(\mathbb{R}^d) \arrow[r, "\mathrm{restrict}_{\mathbb{R}^d\to\Lambda}"] \arrow[d, "\mathrm{restrict}_{\mathbb{R}^d\to\Lambda_\bigstar}"'] & W^{k,p}_E(\Lambda) \arrow[dl, "\mathrm{extend}_{\Lambda\to\Lambda_\bigstar}"] \\ W^{k,p}_E(\Lambda_\bigstar) & \end{tikzcd}$$

□

Remark 5.2. (i) Note that no assumptions on E beside closedness are necessary.

(ii) Having extended the functions from Λ to $\Lambda_\bigstar$, the 'Dirichlet crack' Σ in Figure 1 has vanished, and one ends up with the whole cube. Here the problem of extending Sobolev functions is almost trivial. We suppose that this is the generic case – at least for problems arising in applications.

The above considerations suggest the following procedure: extend the functions from $W^{k,p}_E(\Lambda)$ first to $\Lambda_\bigstar$, and afterwards to the whole of $\mathbb{R}^d$. The next lemma shows that this approach is universal.

Lemma 5.3. *Let $k \in \mathbb{N}$ and $p \in [1,\infty[$. Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain, let $E \subset \partial\Lambda$ be closed and as before define $\Lambda_\bigstar$ as the interior of the set $\Lambda \cup E$. Every linear, continuous extension operator $\mathfrak{F} : W^{k,p}_E(\Lambda) \to W^{k,p}_E(\mathbb{R}^d)$ factorizes as $\mathfrak{F} = \mathfrak{F}_\bigstar \mathrm{Ext}(\Lambda,\Lambda_\bigstar)$ through a linear, continuous extension operator $\mathfrak{F}_\bigstar : W^{k,p}_E(\Lambda_\bigstar) \to W^{k,p}_E(\mathbb{R}^d)$.*

Proof. Let $\mathfrak{S}$ be the restriction operator from $W^{k,p}_E(\Lambda_\bigstar)$ to $W^{k,p}_E(\Lambda)$. Then we define, for every $f \in W^{k,p}_E(\Lambda_\bigstar)$, $\mathfrak{F}_\bigstar f := \mathfrak{F}\mathfrak{S}f$. We obtain $\mathfrak{F}_\bigstar \mathrm{Ext}(\Lambda,\Lambda_\bigstar) = \mathfrak{F}\mathfrak{S}\mathrm{Ext}(\Lambda,\Lambda_\bigstar) = \mathfrak{F}$. This shows that the factorization holds algebraically. But one also has

$$\begin{aligned} \|\mathfrak{F}_\bigstar \mathrm{Ext}(\Lambda,\Lambda_\bigstar) f\|_{W^{k,p}_E(\mathbb{R}^d)} &= \|\mathfrak{F} f\|_{W^{k,p}_E(\mathbb{R}^d)} \le \|\mathfrak{F}\|_{\mathcal{L}(W^{k,p}_E(\Lambda);W^{k,p}_E(\mathbb{R}^d))} \|f\|_{W^{k,p}_E(\Lambda)} \\ &= \|\mathfrak{F}\|_{\mathcal{L}(W^{k,p}_E(\Lambda);W^{k,p}_E(\mathbb{R}^d))} \|\mathrm{Ext}(\Lambda,\Lambda_\bigstar) f\|_{W^{k,p}_E(\Lambda_\bigstar)}. \end{aligned}$$

□

Having extended the functions already to $\Lambda_\bigstar$ one may proceed as follows: Since E is closed, so is $E_\bigstar := E \cap \partial\Lambda_\bigstar$. So, one can now consider the space $W^{1,p}_{E_\bigstar}(\Lambda_\bigstar)$ and has the task to establish an extension operator for this space – while afterwards one has to take into account that the original functions were 0 also on the set $E \cap \Lambda_\bigstar$ and have not been altered by the extension operator thereon. However, note carefully that $E_\bigstar := E \cap \partial\Lambda_\bigstar$ may have a worse geometry then E. For example, take Figure 2 and suppose that this time only Σ forms the whole Dirichlet part of the boundary. Then E is a $(d-1)$-set whereas even $\mathcal{H}_{d-1}(E_\bigstar) = 0$ holds.

To sum up, if one aims at an extension operator $\mathfrak{E} : W^{k,p}_D(\Lambda) \to W^{k,p}_D(\mathbb{R}^d)$, one is free to modify the domain Λ to $\Lambda_\bigstar$. In most cases this improves the local geometry concerning Sobolev extensions and we do not have examples where the situation gets worse. Though we do not claim that this is, in a whatever precise sense, the generic case.

5.2. **Sobolev extendability is a local property.** Below, we make precise in which sense Sobolev extendability is a local property. To set up notation, we say that a domain $\Lambda \subset \mathbb{R}^d$ is a $W^{k,p}$*-extension domain* for given $k \in \mathbb{N}$ and $p \in [1, \infty[$ if there exists a continuous extension operator $\mathfrak{E}_{k,p} : W^{k,p}(\Lambda) \to W^{k,p}(\mathbb{R}^d)$. If Λ is a $W^{k,p}$-extension domain for all $k \in \mathbb{N}$ and all $p \in [1, \infty[$ in virtue of the *same* extension operator, then we say that Λ is a *universal Sobolev extension domain.*

Proposition 5.4. *Fix $k \in \mathbb{N}$ and $p \in [1, \infty[$. Let Λ be a bounded domain and let E be a closed part of its boundary. Assume that for every $\mathrm{x} \in \overline{\partial\Lambda \setminus E}$ there is an open neighborhood U_x of x such that $\Lambda \cap U_\mathrm{x}$ is a $W^{k,p}$-extension domain. Then there is a continuous extension operator $\mathfrak{E}_{k,p} : W^{k,p}_E(\Lambda) \to W^{k,p}(\mathbb{R}^d)$.*

Proof. For every $\mathrm{x} \in \overline{\partial\Lambda \setminus E}$, let U_x be the open neighborhood of x from the assumption. Let $U_{\mathrm{x}_1}, \dots, U_{\mathrm{x}_n}$ be a finite subcovering of $\overline{\partial\Lambda \setminus E}$. Since the compact set $\overline{\partial\Lambda \setminus E}$ is contained in the open set $\bigcup_j U_{\mathrm{x}_j}$, there is an $\varepsilon > 0$, such that the sets $U_{\mathrm{x}_1}, \dots, U_{\mathrm{x}_n}$, together with the open set $U := \{\mathrm{y} \in \Lambda : \operatorname{dist}(\mathrm{y}, \overline{\partial\Lambda \setminus E}) > \varepsilon\}$, form an open covering of $\overline{\Lambda}$. Hence, on $\overline{\Lambda}$ there is a C^∞-partition of unity $\eta, \eta_1, \dots, \eta_n$, with the properties $\operatorname{supp}(\eta) \subset U$, $\operatorname{supp}(\eta_j) \subset U_{\mathrm{x}_j}$.

Assume $\psi \in C^\infty_E(\Lambda)$. Then $\eta\psi \in C^\infty_0(\Lambda)$. If one extends this function by 0 outside of Λ, then one obtains a function $\varphi \in C^\infty_{\partial\Lambda}(\mathbb{R}^d) \subset C^\infty_E(\mathbb{R}^d) \subset W^{k,p}_E(\mathbb{R}^d)$ with the property $\|\varphi\|_{W^{k,p}(\mathbb{R}^d)} = \|\eta\psi\|_{W^{k,p}(\Lambda)}$.

Now, for every fixed $j \in \{1, \dots, n\}$, consider the function $\psi_j := \eta_j\psi \in W^{k,p}(\Lambda \cap U_{\mathrm{x}_j})$. Since $\Lambda \cap U_{\mathrm{x}_j}$ is a $W^{k,p}$-extension domain by supposition, there is an extension of ψ_j to a $W^{k,p}(\mathbb{R}^d)$-function φ_j together with an estimate $\|\varphi_j\|_{W^{k,p}(\mathbb{R}^d)} \le c\|\psi_j\|_{W^{k,p}(\Lambda \cap U_{\mathrm{x}_j})}$, where c is independent from ψ. Clearly, one has a priori no control on the behavior of φ_j on the set $\Lambda \setminus U_{\mathrm{x}_j}$. In particular φ_j may there be nonzero and, hence, cannot be expected to coincide with $\eta_j\psi$ on the whole of Λ. In order to correct this, let ζ_j be a $C^\infty_0(\mathbb{R}^d)$-function which is identically 1 on $\operatorname{supp}(\eta_j)$ and has its support in U_{x_j}. Then $\eta_j\psi$ equals $\zeta_j\varphi_j$ on all of Λ. Consequently, $\zeta_j\varphi_j$ really is an extension of $\eta_j\psi$ to the whole of $\mathbb{R}^d$ which, additionally, satisfies the estimate

$$\|\zeta_j\varphi_j\|_{W^{k,p}(\mathbb{R}^d)} \le c\|\varphi_j\|_{W^{k,p}(\mathbb{R}^d)} \le c\|\eta_j\psi\|_{W^{k,p}(\Lambda \cap U_{\mathrm{x}_j})} \le c\|\psi\|_{W^{k,p}(\Lambda)},$$

where c is independent from ψ. Thus, defining $\mathfrak{E}_{k,p}(\psi) = \varphi + \sum_j \zeta_j\varphi_j$ one gets a linear, continuous extension operator from $C^\infty_E(\Lambda)$ into $W^{k,p}(\mathbb{R}^d)$. By density, $\mathfrak{E}_{k,p}$ uniquely extends to a linear, continuous operator

$$\mathfrak{E}_{k,p} : W^{k,p}_E(\Lambda) \to W^{k,p}(\mathbb{R}^d). \tag{5.1}$$

□

Remark 5.5. By construction one gets uniformity for $\mathfrak{E}$ with respect to p and k if one invests the respective uniformity concerning the extension property for the local domains $\Lambda \cap U_\mathrm{x}$. In particular, one obtains an extension operator that is bounded from $W^{k,p}_E(\Lambda)$ into $W^{k,p}(\mathbb{R}^d)$ for each $k \in \mathbb{N}$ and each $p \in [1, \infty[$ if the local domains are universal Sobolev extension domains.

5.3. **Preservation of traces.** Proposition 5.4 allows to construct Sobolev extension operators from $W^{k,p}_D(\Omega)$ into $W^{k,p}(\mathbb{R}^d)$. In this section we prove under fairly general assumptions on Ω and D that *every* such extension operator preserves the boundary behavior, i.e. maps into $W^{k,p}_D(\mathbb{R}^d)$. Recall that this is the crux of the matter in Assumption (iii) of Theorem 3.1.

The results in this section rely on deep insights from potential theory and we shall recall the necessary notions beforehand. For further background we refer e.g. to [1].

Definition 5.6. Let $k \in \mathbb{N}$, $p \in]1,\infty[$ and let $F \subset \mathbb{R}^d$. Denote by $G_k := \mathcal{F}^{-1}((1+|\xi|^2)^{-k/2})$ the Bessel kernel of order k. Then

$$C_{k,p}(F) := \inf\Big\{\int_{\mathbb{R}^d} |f|^p : f \geq 0 \text{ on } \mathbb{R}^d \text{ and } G_k * f \geq 1 \text{ on } F\Big\}$$

is called (k,p)-*capacity* of F.

The capacities $C_{k,p}$ are outer measures on $\mathbb{R}^d$ [1, Sec. 2.3]. A property that holds true for all x in some set $E \subset \mathbb{R}^d$ but those belonging to an exceptional set $F \subset E$ with $C_{k,p}(F) = 0$ is said to be true (k,p)-*quasieverywhere* on E, abbreviated (k,p)-*q.e.* A property that holds true (k,p)-q.e. also holds true (l,p)-q.e. if $l < k$. This is an easy consequence of [1, Prop. 2.3.13].

There is also a close connection between capacities and Hausdorff measures, cf. [1, Ch. 5.] for an exhaustive discussion. Most important for us is the following comparison theorem showing that already the $(1,p)$-capacities are more sensitive than the $(d-1)$-dimensional Hausdorff measure. In the case $p \in]1,d]$ this is [1, Thm. 5.1.13] and if $p \in]d,\infty[$ the result follows directly from [1, Prop. 2.6.1].

Theorem 5.7 (Comparison Theorem)**.** *Let $p \in]1,\infty[$ and let $E \subset \mathbb{R}^d$ be compact. If $C_{1,p}(E) = 0$, then $\mathcal{H}_{d-1}(E) = 0$.*

Bessel capacities naturally occur when studying convergence of average integrals for Sobolev functions. In fact, if $k \in \mathbb{N}$, $p \in]1,\frac{d}{k}]$ and $u \in W^{k,p}(\mathbb{R}^d)$, then (k,p)-quasievery $\mathrm{y} \in \mathbb{R}^d$ is a Lebesgue point for u in the L^p-sense, i.e.

$$\lim_{r\to 0} \frac{1}{|B(\mathrm{y},r)|}\int_{B(\mathrm{y},r)} u(\mathrm{x})\,\mathrm{dx} =: \mathfrak{u}(\mathrm{y}) \tag{5.2}$$

and

$$\lim_{r\to 0} \frac{1}{|B(\mathrm{y},r)|}\int_{B(\mathrm{y},r)} |u(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^p\,\mathrm{dx} = 0 \tag{5.3}$$

hold [1, Thm. 6.2.1]. The (k,p)-quasieverywhere defined function $\mathfrak{u}$ reproduces u within its Sobolev class. It gives rise to a meaningful (k,p)-quasieverywhere defined restriction $u|_E := \mathfrak{u}|_E$ of u to E whenever E has non-vanishing (k,p)-capacity. For convenience we agree upon that $u|_E = 0$ is true for all $u \in W^{k,p}(\mathbb{R}^d)$ if E has zero (k,p)-capacity. Note also that these results remain true if $p \in]\frac{d}{k},\infty[$, since in this case u has a Hölder continuous representative $\mathfrak{u}$ which then satisfies (5.2) and (5.3) for every $\mathrm{y} \in \mathbb{R}^d$.

We naturally obtain an alternate definition for Sobolev spaces with partially vanishing traces.

Definition 5.8. Let $k \in \mathbb{N}$, $p \in]1,\infty[$ and $E \subseteq \mathbb{R}^d$ be closed. Define

$$\begin{aligned}\mathcal{W}_E^{k,p}(\mathbb{R}^d) := \big\{u \in W^{k,p}(\mathbb{R}^d) : D^\beta u|_E = 0 \text{ holds } (k-|\beta|,p)\text{-q.e. on } E \\ \text{for all multiindices } \beta,\ 0 \leq |\beta| \leq k-1\big\}\end{aligned}$$

and equip it with the $W^{k,p}(\mathbb{R}^d)$-norm.

The following theorem of L.I. Hedberg and T.H. Wolff became known as the (k,p)-*synthesis*.

Theorem 5.9 ([1, Thm. 9.1.3])**.** *The spaces $W_E^{k,p}(\mathbb{R}^d)$ and $\mathcal{W}_E^{k,p}(\mathbb{R}^d)$ coincide whenever $p \in]1,\infty[$ and $E \subset \mathbb{R}^d$ is closed.*

Hedberg and Wolff's theorem manifests the use of capacities in the study of traces of Sobolev functions. However, if one invests more on the geometry of E, e.g. if one assumes that it is a $(d-1)$-set, then by the subsequent recent result of Mitrea, Mitrea, Mitrea and Brewster capacities can be replaced by the $(d-1)$-dimensional Hausdorff measure at each occurrence.

Theorem 5.10 ([5, Thm. 4.4, Cor. 4.5]). *Let $k \in \mathbb{N}$, $p \in]1,\infty[$ and let $E \subset \mathbb{R}^d$ be closed and additionally a $(d-1)$-set. Then*

$$W_E^{k,p}(\mathbb{R}^d) = \mathcal{W}_E^{k,p}(\mathbb{R}^d) = \big\{u \in W^{k,p}(\mathbb{R}^d) : D^\beta u|_E = 0 \text{ holds } \mathcal{H}_{d-1}\text{-a.e. on } E \\ \text{for all multiindices } \beta,\ 0 \le |\beta| \le k-1\big\},$$

where on the right-hand side $D^\beta u|_E = 0$ as before means that for $\mathcal{H}_{d-1}$-almost every $\mathrm{y} \in E$ the average integrals $\frac{1}{|B(\mathrm{y},r)|}\int_{B(\mathrm{y},r)} D^\beta u(\mathrm{x})\,\mathrm{dx}$ vanish in the limit $r \to 0$.

The next lemma prepares for the main goal of this section.

Lemma 5.11. *Let $k \in \mathbb{N}$ and $p \in]1,\infty[$. Let $\Lambda \subset \mathbb{R}^d$ be a domain and let $E \subset \partial\Lambda$ be closed. Suppose that for (k,p)-quasievery $\mathrm{y} \in E$ balls around y in Λ have* asymptotically nonvanishing relative volume, *i.e.*

$$\liminf_{r\to 0} \frac{|B(\mathrm{y},r) \cap \Lambda)|}{r^d} > 0. \tag{5.4}$$

Then given $\psi \in C_E^\infty(\Lambda)$, any Sobolev extension $u \in W^{k,p}(\mathbb{R}^d)$ of ψ in fact belongs to $W_E^{k,p}(\mathbb{R}^d)$. If in addition E is a $(d-1)$ set, then it suffices that (5.4) holds for $\mathcal{H}_{d-1}$-a.e. $\mathrm{y} \in E$.

Proof. Fix an arbitrary multiindex β with $|\beta| \le k-1$. Let $D^\beta \mathfrak{u}$ be the representative of the distributional derivative $D^\beta u$ of u defined $(k-|\beta|,p)$-q.e. on $\mathbb{R}^d$ via

$$D^\beta \mathfrak{u}(\mathrm{y}) := \lim_{r\to 0} \frac{1}{|B(\mathrm{y},r)|}\int_{B(\mathrm{y},r)} D^\beta u(\mathrm{x})\,\mathrm{dx}.$$

Recall from (5.3) that then

$$\begin{aligned} &\lim_{r\to 0} \frac{1}{|B(\mathrm{y},r)|}\int_{B(\mathrm{y},r)} |D^\beta \mathfrak{u}(\mathrm{x}) - D^\beta \mathfrak{u}(\mathrm{y})|\,\mathrm{dx} \\ &\le \lim_{r\to 0}\left(\frac{1}{|B(\mathrm{y},r)|}\int_{B(\mathrm{y},r)} |D^\beta \mathfrak{u}(\mathrm{x}) - D^\beta \mathfrak{u}(\mathrm{y})|^p\,\mathrm{dx}\right)^{1/p} = 0. \end{aligned} \tag{5.5}$$

holds for $(k-|\beta|,p)$-q.e. $\mathrm{y} \in \mathbb{R}^d$. Since (5.4) holds for (k,p)-quasievery $\mathrm{y} \in E$, it a fortiori holds for $(k-|\beta|,p)$-quasievery such y. Let now $N \subset \mathbb{R}^d$ be the exceptional set such that on $\mathbb{R}^d \setminus N$ the function $D^\beta\mathfrak{u}$ is defined and satisfies (5.5) and such that (5.4) holds for every $\mathrm{y} \in E \setminus N$. Owing to Theorem 5.9 the claim follows once we have shown $D^\beta \mathfrak{u}(\mathrm{y}) = 0$ for all $\mathrm{y} \in E \setminus N$.

For the rest of the proof we fix $y \in E \setminus N$. For $r > 0$ we abbreviate $B(r) := B(\mathrm{y},r)$. Following [48, pp. 190-192] for $j \in \mathbb{N}$ we put

$$W_j := \{\mathrm{x} \in \mathbb{R}^d \setminus N : |D^\beta \mathfrak{u}(\mathrm{x}) - D^\beta \mathfrak{u}(\mathrm{y})| > 1/j\}. \tag{5.6}$$

Thanks to (5.5) for each $j \in \mathbb{N}$ we can choose some $r_j > 0$ such that $|B(r) \cap W_j| < 2^{-j}|B(r)|$ holds for all $r \in]0,r_j]$. Clearly, we can arrange that the sequence $\{r_j\}_j$ is decreasing. Now,

$$W := \bigcup_{j\in\mathbb{N}} \Big\{\big(B(r_j) \setminus B(r_{j+1})\big) \cap W_j\Big\} \tag{5.7}$$

has vanishing Lebesgue density at y, i.e. $r^{-d}|B(r) \cap W|$ vanishes as r tends to 0: Indeed, if $r \in]r_{l+1}, r_l]$, then

$$|B(r) \cap W| \le \Big|\big(B(r) \cap W_l\big) \cup \bigcup_{j \ge l+1} \big(B(r_j) \cap W_j\big)\Big| \le 2^{-l}|B(r)| + \sum_{j\ge l+1} 2^{-j}|B(r_j)| \le 2^{-l+1}|B(r)|.$$

Now, (5.4) allows to conclude

$$\liminf_{r\to 0} \frac{|B(r) \cap \Lambda \cap (\mathbb{R}^d \setminus W))|}{r^d} > 0.$$

Since u is an extension of $\psi \in C_E^\infty(\Lambda)$ and y is an element of E it holds $D^\beta \mathfrak{u} = 0$ a.e. on $B(r) \cap \Lambda$ with respect to the d-dimensional Lebesgue measure if $r > 0$ is small enough. The previous inequality gives $|B(r) \cap \Lambda \cap (\mathbb{R}^d \setminus W))| > 0$ if $r > 0$ is small enough. In particular, there exists a sequence $\{\mathrm{x}_j\}_j$ in $\mathbb{R}^d \setminus W$ approximating y such that $D^\beta \mathfrak{u}(\mathrm{x}_j) = 0$ for all $j \in \mathbb{N}$. Now, the upshot is that the restriction of $D^\beta \mathfrak{u}$ to $\mathbb{R}^d \setminus W$ is continuous at y since if $\mathrm{x} \in \mathbb{R}^d \setminus W$ satisfies $|\mathrm{x} - \mathrm{y}| \leq r_j$ then by construction $|D^\beta \mathfrak{u}(\mathrm{x}) - D^\beta \mathfrak{u}(\mathrm{y})| \leq 1/j$. Hence, $D^\beta \mathfrak{u}(\mathrm{y}) = 0$ and the proof is complete.

If in addition E is a $(d-1)$-set we simply appeal to Theorem 5.10 and the same argument applies. □

Remark 5.12. If Λ is a d-set and E a $(d-1)$-set, then Lemma 5.11 is proved in [21, Sec. VIII.1].

As a corollary we obtain that extension operators for the complete Sobolev spaces $W^{k,p}(\Lambda)$ always preserve the Dirichlet boundary condition on E if the latter is merely closed.

Corollary 5.13. *Let $k \in \mathbb{N}$ and $p \in]1, \infty[$. Let $\Lambda \subset \mathbb{R}^d$ be a domain and let $E \subset \partial\Lambda$ be closed. If there exists a continuous extension operator $\mathfrak{E} : W^{k,p}(\Lambda) \to W^{k,p}(\mathbb{R}^d)$, then $\mathfrak{E}$ automatically maps $W_E^{k,p}(\Lambda)$ into $W_E^{k,p}(\mathbb{R}^d)$.*

Proof. Being a $W^{k,p}$ extension domain, Λ is a d-set [17, Thm. 2] and as such satisfies (5.4) around every $\mathrm{y} \in \partial\Lambda$. Since $C_E^\infty(\Lambda)$ is dense in $W_E^{k,p}(\Lambda)$ the conclusion follows from Lemma 5.11. □

In the case $p > d$ the statement of Lemma 5.11 even holds without any geometric assumptions on Λ around E.

Lemma 5.14. *Let $k \in \mathbb{N}$ and $p \in]d, \infty[$. Let $\Lambda \subset \mathbb{R}^d$ be a domain and $E \subset \partial\Lambda$ be closed. Then given $\psi \in C_E^\infty(\Lambda)$, any Sobolev extension $u \in W^{k,p}(\mathbb{R}^d)$ of ψ satisfies*

$$\lim_{r \to 0} \frac{1}{|B(\mathrm{y}, r)|} \int_{B(\mathrm{y},r)} D^\beta u(\mathrm{x}) \, \mathrm{dx} = 0$$

for every $\mathrm{x} \in E$ and every multiindex β of order $0 \leq |\beta| \leq k-1$. In particular, $u \in W_E^{k,p}(\mathbb{R}^d)$.

Proof. By Sobolev embeddings, for each multiindex β, $|\beta| \leq k-1$, the distributional derivative $D^\beta u$ has a continuous representative $D^\beta \mathfrak{u}$. Since each $\mathrm{x} \in E$ is an accumulation point of $\Lambda \setminus E$ and since $D^\beta u = D^\beta \psi$ holds a.e. on Λ, each function $D^\beta \mathfrak{u}$ vanishes everywhere on E. This proves the claim on the integral averages. In particular $u \in \mathcal{W}_E^{k,p}(\mathbb{R}^d)$, so $u \in W_E^{k,p}(\mathbb{R}^d)$ by Theorem 5.9. □

The main result of this section is the synthesis of the previously discussed lemmas.

Proposition 5.15. *Let $k \in \mathbb{N}$ and $p \in]1, \infty[$. Let $\Lambda \subset \mathbb{R}^d$ be a domain, let $E \subset \partial\Lambda$ be closed and suppose there is a continuous extension operator $\mathfrak{E} : W_E^{k,p}(\Lambda) \to W^{k,p}(\mathbb{R}^d)$. Both of the following conditions guarantee that $\mathfrak{E}$ in fact maps into $W_E^{k,p}(\mathbb{R}^d)$.*

(i) *The asymptotically nonvanishing relative volume condition* (5.4) *is satisfied (k,p)-quasi-everywhere on E. If E is a $(d-1)$-set, then it suffices that* (5.4) *holds $\mathcal{H}_{d-1}$-almost everywhere on E.*

(ii) *There is $q \in]d, \infty[$ such that $\mathfrak{E}$ extends to a continuous operator $W_E^{k,q}(\Lambda) \to W^{k,q}(\mathbb{R}^d)$.*

Proof. Taking into account that $C_E^\infty(\Lambda)$ is dense in $W_E^{k,p}(\Lambda)$ the first part is due to Lemma 5.11. As for the second part, first use Lemma 5.14 and Theorem 5.9 to deduce that $\mathfrak{E}$ maps $C_E^\infty(\mathbb{R}^d)$ into $W_E^{k,p}(\mathbb{R}^d)$ and conclude by density. □

5.4. **Geometric conditions.** In this subsection we finally review common geometric conditions on the boundary part $\overline{\partial\Lambda \setminus E}$ such that the local sets $\Lambda \cap U_\mathrm{x}$ really admit the Sobolev extension property required in Proposition 5.4.

A first condition, completely sufficient for the treatment of most real world problems, is the following Lipschitz condition.

Definition 5.16. A bounded domain $\Lambda \subset \mathbb{R}^d$ is called *bounded Lipschitz domain* if for each $\mathrm{x} \in \partial\Lambda$ there is an open neighborhood U_x of x and a bi-Lipschitz mapping ϕ_x from U_x onto a cube, such that $\phi_\mathrm{x}(\Lambda \cap U_\mathrm{x})$ is the (lower) half cube and $\partial\Lambda \cap U_\mathrm{x}$ is mapped onto the top surface of this half cube.

It can be proved by elementary means that bounded Lipschitz domains are $W^{1,p}$ extension domains for every $p \in [1, \infty[$, cf. e.g. [14] for the case $p = 2$. In fact, already the following (ε, δ)-condition of Jones [20] assures the existence of a universal Sobolev extension operator.

Definition 5.17. Let $\Lambda \subset \mathbb{R}^d$ be a domain and $\varepsilon, \delta > 0$. Assume that any two points $\mathrm{x}, \mathrm{y} \in \Lambda$, with distance not larger than δ, can be connected within Λ by a rectifiable arc γ with length $l(\gamma)$, such that the following two conditions are satisfied for all points z from the curve γ:

$$l(\gamma) \leq \frac{1}{\varepsilon}\|\mathrm{x} - \mathrm{y}\|, \quad \text{and} \quad \frac{\|\mathrm{x} - \mathrm{z}\|\|\mathrm{y} - \mathrm{z}\|}{\|\mathrm{x} - \mathrm{y}\|} \leq \frac{1}{\varepsilon}\mathrm{dist}(\mathrm{z}, \Lambda^c).$$

Then Λ is called (ε, δ)*-domain.*

Theorem 5.18 (Rogers)**.** *Each (ε, δ)-domain is a universal Sobolev extension domain.*

Remark 5.19.
(i) Theorem 5.18 is due to Rogers [42] and generalizes the celebrated result of Jones [20]. *Bounded* (ε, δ)-domains are known to be *uniform domains*, see [45, Ch. 4.2] and also [20, 34, 35, 33] for further information. In particular, every bounded Lipschitz domain is an (ε, δ)-domain [12, Rem. 5.11].
(ii) Although the uniformity property is not necessary for a domain to be a Sobolev extension domain [47] it seems presently to be the broadest class of domains for which this extension property holds – at least if one aims at all $p \in]1, \infty[$. For example Koch's snowflake is an (ε, δ)-domain [20].

Plugging in Rogers extension operator in Proposition 5.4, Proposition 5.15 lets us re-discover [5, Thm. 1.3] in case of bounded domains and p strictly between 1 and ∞. We even obtain a universal extension operator that simultaneously acts on all $W^{k,p}_E$-spaces and at the same time our argument reveals that the preservation of the trace is irrespective of the specific structure of Jones' or Roger's extension operators but just follows from the fact that both operators satisfy (ii) of Proposition 5.15.

We believe that this sheds some more light also on [5, Thm. 1.3] though – of course – our argument cannot disclose the fundamental assertions on the support of the extended functions obtained in [5] by a careful analysis of Jones' extension operator.

Theorem 5.20. *Let Λ be a bounded domain and let E be a closed part of its boundary. Assume that for every $\mathrm{x} \in \overline{\partial\Lambda \setminus E}$ there is an open neighborhood U_x of x such that $\Lambda \cap U_\mathrm{x}$ is a bounded Lipschitz or, more generally, an (ε, δ)-domain for some values $\varepsilon, \delta > 0$. Then there exists a universal operator $\mathfrak{E}$ that restricts to a bounded extension operator $W^{k,p}_E(\Lambda) \to W^{k,p}_E(\mathbb{R}^d)$ for each $k \in \mathbb{N}$ and each $p \in]1, \infty[$.*

6. Poincare's inequality

In this section we will discuss sufficient conditions for Poincaré's inequality, thereby unwinding Assumption (ii) of Theorem 3.1. Our aim is not greatest generality as e.g. in [37] for functions defined on the whole of $\mathbb{R}^d$, but to include the aspect that our functions are only defined on a domain. Secondly, our interest is to give very general, but in some sense *geometric* conditions, which may be checked more or less 'by appearance' – at least for problems arising from applications.

The next proposition gives a condition that assures that a closed subspace of $W^{1,p}$ may be equivalently normed by the L^p-norm of the gradient of the corresponding functions only. We believe that this might also be of independent interest, compare also [48, Ch. 4]. Throughout $\mathbb{1}$ denotes the function that is identically one.

Proposition 6.1. *Let $\Lambda \subset \mathbb{R}^d$ be a bounded domain and suppose $p \in]1,\infty[$. Assume that X is a closed subspace of $W^{1,p}(\Lambda)$ that does not contain $\mathbb{1}$ and for which the restriction of the canonical embedding $W^{1,p}(\Lambda) \hookrightarrow L^p(\Lambda)$ to X is compact. Then X may be equivalently normed by $v \mapsto \left(\int_\Lambda |\nabla v|^p \mathrm{dx}\right)^{1/p}$.*

Proof. First recall that both X and $L^p(\Lambda)$ are reflexive. In order to prove the proposition, assume to the contrary that there exists a sequence $\{v_k\}_k$ from X such that

$$\frac{1}{k}\|v_k\|_{L^p(\Lambda)} \geq \|\nabla v_k\|_{L^p(\Lambda)}.$$

After normalization we may assume $\|v_k\|_{L^p(\Lambda)} = 1$ for every $k \in \mathbb{N}$. Hence, $\{\nabla v_k\}_k$ converges to 0 strongly in $L^p(\Lambda)$. On the other hand, $\{v_k\}_k$ is a bounded sequence in X and hence contains a subsequence $\{v_{k_l}\}_l$ that converges weakly in X to an element $v \in X$. Since the gradient operator $\nabla : X \to L^p(\Lambda)$ is continuous, $\{\nabla v_{k_l}\}_l$ converges to ∇v weakly in $L^p(\Lambda)$. As the same sequence converges to 0 strongly in $L^p(\Lambda)$, the function ∇v must be zero and hence v is constant. But by assumption X does not contain constant functions except for $v = 0$. So, $\{v_{k_l}\}_l$ tends to 0 weakly in X. Owing to the compactness of the embedding $X \hookrightarrow L^p(\Lambda)$, a subsequence of $\{v_{k_l}\}_l$ tends to 0 strongly in $L^p(\Lambda)$. But this contradicts the normalization condition $\|v_{k_l}\|_{L^p(\Lambda)} = 1$. □

Remark 6.2. It is clear that in case $X = W^{1,p}_D(\Omega)$ the embedding $X \hookrightarrow L^p(\Omega)$ is compact, if there exists a continuous extension operator $\mathfrak{E} : X \to W^{1,p}(\mathbb{R}^d)$. Hence, the compactness of this embedding is no additional requirement in view of Theorem 3.1.

In the case that E is a $(d-1)$-set, the following lemma presents two conditions that are particularly easy to check and entail $\mathbb{1} \notin W^{1,p}_E(\Lambda)$. Loosely speaking, some knowledge on the common frontier of E and $\partial\Lambda \setminus E$ is required: Either not every point of E should lie thereon or $\partial\Lambda$ must not be too wild around this frontier.

Lemma 6.3. *Let $p \in]1,\infty[$, let Λ be a bounded domain and let $E \subset \partial\Lambda$ be a $(d-1)$-set. Both of the following conditions assure $\mathbb{1} \notin W^{1,p}_E(\Lambda)$.*

(i) *The set E admits at least one relatively inner point* x. *Here, 'relatively inner' is with respect to $\partial\Lambda$ as ambient topological space.*
(ii) *For every $\mathrm{x} \in \overline{\partial\Lambda \setminus E}$ there is an open neighborhood U_x of* x *such that $\Lambda \cap U_\mathrm{x}$ is a $W^{1,p}$-extension domain, e.g. a Lipschitz- or an (ε,δ)-domain.*

Proof. We treat both cases separately.

(i) Assume the assertion was false and $\mathbb{1} \in W^{1,p}_E(\Lambda)$. Let x be the inner point of E from the hypotheses and let $B := B(\mathrm{x}, r)$ be a ball that does not intersect $\partial\Lambda \setminus E$. Put $\frac{1}{2}B := B(\mathrm{x}, \frac{r}{2})$ and let $\eta \in C^\infty_0(B)$ be such that $\eta \equiv 1$ on $\frac{1}{2}B$. We distinguish whether or not x is an interior point of $\overline{\Lambda}$.

First, assume it is not. For every $\psi \in C^\infty_E(\Lambda)$ the function $\eta\psi$ belongs to $W^{1,p}_0(\Lambda \cap B)$ and as such admits a $W^{1,p}$-extension $\widehat{\eta\psi}$ by zero to the whole of $\mathbb{R}^d$. In particular,

$$\widehat{\eta\psi}(\mathrm{y}) = \begin{cases} \psi(\mathrm{y}), & \text{if } \mathrm{y} \in \frac{1}{2}B \cap \Lambda \\ 0, & \text{if } \mathrm{y} \in \frac{1}{2}B \setminus \Lambda \end{cases}$$

and consequently,

$$\|\nabla\widehat{\eta\psi}\|_{L^p(\frac{1}{2}B)} = \|\nabla\psi\|_{L^p(\frac{1}{2}B\cap\Lambda)}.$$

Since by assumption $\mathbb{1}$ is in the $W^{1,p}(\Lambda)$-closure of $C^\infty_E(\Lambda)$ and since the mappings $W^{1,p}_E(\Lambda) \ni \psi \mapsto \nabla\widehat{\eta\psi} \in L^p(\frac{1}{2}B)$ and $W^{1,p}_E(\Lambda) \ni \psi \mapsto \nabla\psi \in L^p(\Lambda \cap \frac{1}{2}B)$ are continuous, the previous equation extends to $\psi = \mathbb{1}$:

$$\|\nabla\widehat{\eta\mathbb{1}}\|_{L^p(\frac{1}{2}B)} = \|\nabla\mathbb{1}\|_{L^p(\frac{1}{2}B\cap\Lambda)} = 0.$$

On the other hand x is not an inner point of $\overline{\Lambda}$ so that in particular $\frac{1}{2}B\setminus\overline{\Lambda}$ is non-empty. Since this set is open, $|\frac{1}{2}B\setminus\overline{\Lambda}|>0$. Recall that by construction $\widehat{\eta\mathbb{1}}\in W^{1,p}(B)$ vanishes a.e. on $\frac{1}{2}B\setminus\overline{\Lambda}$. Hence, for some $c>0$ the Poincaré inequality

$$\|\widehat{\eta\mathbb{1}}\|_{L^p(\frac{1}{2}B)}\leq c\|\nabla\widehat{\eta\mathbb{1}}\|_{L^p(\frac{1}{2}B)},$$

holds, cf. [48, Thm. 4.4.2]. But we already know that the right hand side is zero, whereas the left hand side equals $|\frac{1}{2}B\cap\Lambda|^{1/p}$, which is nonzero since $\frac{1}{2}B\cap\Lambda$ is nonempty and open – a contradiction.

Now, assume x is contained in the interior of $\overline{\Lambda}$. Upon diminishing B we may assume $B\subset\overline{\Lambda}$. For every $\psi\in C_E^\infty(\mathbb{R}^d)$ we have $\eta\psi\in C_E^\infty(\mathbb{R}^d)$ with an estimate

$$\|\eta\psi\|_{W^{1,p}(\mathbb{R}^d)}\leq c\|\psi\|_{W^{1,p}(B)}=c\Big(\int_B|\psi|^p+|\nabla\psi|^p\,\mathrm{dx}\Big)^{1/p}$$

for some constant $c>0$ depending only on η and p. By our choice of B split

$$B=B\cap\overline{\Lambda}=(B\cap\Lambda)\cup(B\cap\partial\Lambda)=(B\cap\Lambda)\cup(B\cap E).$$

Since E is a $(d-1)$-set, $\mathcal{H}_{d-1}(B\cap E)$ is finite and therefore $|B\cap E|=0$. Consequently,

$$\|\eta\psi\|_{W^{1,p}(\mathbb{R}^d)}\leq c\Big(\int_{B\cap\Lambda}|\psi|^p+|\nabla\psi|^p\,\mathrm{dx}\Big)^{1/p}\leq\|\psi\|_{W^{1,p}(\Lambda)}.$$

By assumption there is a sequence $\{\psi_j\}_j$ tending to $\mathbb{1}$ in the $W^{1,p}(\Lambda)$-topology. Due to the previous estimate and the choice of η, the sequence $\{\eta\psi_j\}_j\subset C_E^\infty(\mathbb{R}^d)$ then tends to some $\chi\in W_E^{1,p}(\mathbb{R}^d)$ satisfying $\chi=1$ a.e. on $\frac{1}{2}B\cap\Lambda$.

On the other hand we obtain $|\frac{1}{2}B|=|\frac{1}{2}B\cap\Lambda|$ from the $(d-1)$-property of E as before. Thus, $\chi=1$ a.e. on $\frac{1}{2}B$ and in particular

$$\lim_{r\to0}\frac{1}{|B(\mathrm{y},r)|}\int_{B(\mathrm{y},r)}\chi\,\mathrm{dx}=1$$

for every $\mathrm{y}\in\frac{1}{3}B\cap E$. But once again by the $(d-1)$-property of E we conclude $\mathcal{H}_{d-1}(\frac{1}{3}B\cap E)>0$. This is a contradiction to Theorem 5.10.

(ii) Again assume the assertion was false. Then by (i) there exists some $\mathrm{x}\in E$ that is not an inner point of E with respect to $\partial\Lambda$. Hence x is an accumulation point of $\partial\Lambda\setminus E$ and by assumption there is a neighborhood $U=U_\mathrm{x}$ of x such that $\Lambda\cap U$ is a $W^{1,p}$ extension domain. Denote the corresponding extension operator by $\mathfrak{E}$. We shall localize the assumption $\mathbb{1}\in W_E^{1,p}(\Lambda)$ within U to arrive at a contradiction.

To this end, let $r_0>0$ be such that $\overline{B(\mathrm{x},r_0)}\subset U$ and let $\eta\in C_0^\infty(U)$ be such that $\eta\equiv1$ on $B(\mathrm{x},r_0)$. Then also $\eta=\eta\mathbb{1}\in W_E^{1,p}(\Lambda)$ and in particular $\eta|_{\Lambda\cap U}$ belongs to $W_F^{1,p}(\Lambda\cap U)$, where $F:=\overline{B(\mathrm{x},r_0/2)}\cap E\subset\partial(\Lambda\cap U)$. Recall from the proof of Corollary 5.13 that $\Lambda\cap U$ satisfies the asymptotically nonvanishing relative volume condition

$$\liminf_{r\to0}\frac{|B(\mathrm{y},r)\cap\Lambda\cap U)|}{r^d}>0.$$

around every $\mathrm{y}\in\partial(\Lambda\cap U)$ and in particular around every $\mathrm{y}\in F$. Thus, Proposition 5.15 yields $u:=\mathfrak{E}(\eta|_{\Lambda\cap U})\in W_F^{1,p}(\mathbb{R}^d)$.

On the other hand, similar to the proof of Lemma 5.11 let $\mathfrak{u}$ be the representative of u that is defined by limits of integral means on the complement of some exceptional set N with $C_{1,p}(N)=0$ and fix $\mathrm{y}\in F\setminus N$. Take W as in (5.6) and (5.7). Repeating the arguments in the proof of Lemma 5.11 reveals that the restriction of $\mathfrak{u}$ to $\mathbb{R}^d\setminus W$ is continuous at y and that $|B(\mathrm{y},r)\cap\Lambda\cap U\cap(\mathbb{R}^d\setminus W)|>0$ if $r>0$ is small enough. By construction $\mathfrak{u}=1$ a.e. on $B(\mathrm{y},r)\cap\Lambda\cap U\cap(\mathbb{R}^d\setminus W)$ if $r<r_0$. Hence, there is

a sequence $\{\mathrm{x}_j\}_j$ approximating y such that $\mathfrak{u}(\mathrm{x}_j) = 1$ for every $j \in \mathbb{N}$. By continuity $\mathfrak{u}(\mathrm{y}) = 1$ follows. This proves that $\mathfrak{u} = 1$ holds $(1,p)$-quasieverywhere on F.

To see that this is in contradiction with Theorem 5.9 it remains to note that F is not a $C_{1,p}$-nullset: Indeed, this follows from Theorem 5.7, since the $(d-1)$-property of E entails $\mathcal{H}_{d-1}(F) > 0$. □

Remark 6.4. (i) In the proof of the first item the $(d-1)$-property of E is only needed if all relatively inner points of E are contained in the interior of $\overline{\Lambda}$.

(ii) Of course the Poincaré inequality holds in the case $E = \partial\Lambda$ irrespective of any geometric considerations as long as Λ is bounded. This can be rediscovered by the results of this section. Indeed, E then only consists of relatively inner points and as $\emptyset \neq \partial\overline{\Lambda} \subset \partial\Lambda = E$ holds, it cannot be contained in the interior of $\overline{\Lambda}$. Hence $\mathbb{1} \notin W_0^{1,p}(\Lambda)$. The compactness of the embedding $W_0^{1,p}(\Lambda) \hookrightarrow L^p(\Lambda)$ is classical and Theorem 6.1 gives the claim.

(iii) If E is an arbitrary closed subset of $\partial\Lambda$, the techniques from the proof of Lemma 6.3 can be used to set up more general conditions that still imply $\mathbb{1} \notin W_E^{1,p}(\Lambda)$. Since for Hardy's inequality we mostly assume the the $(d-1)$-set property of the Dirichlet part anyway, these considerations lie beyond the scope of this article.

Under the second assumption of Lemma 6.3 there exists a linear continuous Sobolev extension operator $\mathfrak{E} : W_E^{1,p}(\Lambda) \to W^{1,p}(\mathbb{R}^d)$, cf. Proposition 5.4. Then the compactness of the embedding $W_E^{1,p}(\Lambda) \hookrightarrow L^p(\Lambda)$ is classical and owing to Theorem 6.1 we can record the following special Poincaré inequality.

Proposition 6.5. *Let $p \in \,]1,\infty[$ and let Λ be a bounded domain. Suppose that $E \subset \partial\Lambda$ is a $(d-1)$-set and that for each $\mathrm{x} \in \overline{\partial\Lambda \setminus E}$ there is an open neighborhood U_x of x such that $\Lambda \cap U_\mathrm{x}$ is a $W^{1,p}$-extension domain, e.g. a Lipschitz- or an (ε,δ)-domain. Then $W_E^{1,p}(\Lambda)$ may equivalently be normed by $v \mapsto \left(\int_\Lambda |\nabla v|^p \mathrm{dx}\right)^{1/p}$.*

Now, also Theorem 3.2 follows. In fact, this result is just the synthesis of Lemma 4.8, Theorem 5.20 and Proposition 6.5 above.

7. The proof of Theorem 3.3

The strategy of proof is to write u as the sum of $v \in W^{1,p}(\Omega)$ with $v/\operatorname{dist}_{\partial\Omega} \in L^p(\Omega)$ and $w \in W^{1,p}$ with support within a neighborhood of $\overline{\partial\Omega \setminus D}$. Then v can be handled by the following classical result.

Proposition 7.1 ([11, Thm. V.3.4]). *Let $\emptyset \subsetneq \Lambda \subsetneq \mathbb{R}^d$ be open and let $p \in \,]1,\infty[$. Then if $u \in W^{1,p}(\Lambda)$ and $u/\operatorname{dist}_{\partial\Lambda} \in L^p(\Lambda)$, it follows $u \in W_0^{1,p}(\Lambda)$.*

For w we can – since local extension operators are available – rely on the techniques developed in Section 5.

To make all this precise, as in the proof of Proposition 5.4 for every $\mathrm{x} \in \overline{\partial\Omega \setminus D}$, let U_x be the open neighborhood of x from the assumption, let $U_{\mathrm{x}_1}, \ldots, U_{\mathrm{x}_n}$ be a finite subcovering of $\overline{\partial\Omega \setminus D}$ and let $\varepsilon > 0$ be such that the sets $U_{\mathrm{x}_1}, \ldots, U_{\mathrm{x}_n}$, together with $U := \{\mathrm{y} \in \Omega : \operatorname{dist}(\mathrm{y}, \overline{\partial\Omega \setminus D}) > \varepsilon\}$, form an open covering of $\overline{\Omega}$. Finally, let $\eta, \eta_1, \ldots, \eta_n$ be a subordinated C^∞-partition of unity. The described splitting is $u = v + w$, where $v := \eta u$ and $w := \sum_{j=1}^n \eta_j u = (1-\eta)u$. The rest of the proof is in five steps.

Step 1: It holds $v \in W_D^{1,p}(\Omega)$. Since

$$\operatorname{dist}_{\partial\Omega}(\mathrm{x}) \geq \min\{\varepsilon, \operatorname{dist}_D(\mathrm{x})\} \geq \min\{\varepsilon \operatorname{diam}(\Omega)^{-1}, 1\} \cdot \operatorname{dist}_D(\mathrm{x})$$

holds for every $\mathrm{x} \in \operatorname{supp}(\eta) \cap \Omega$, the function $v \in W^{1,p}(\Omega)$ satisfies

$$\int_\Omega \left| \frac{v}{\mathrm{dist}_{\partial\Omega}} \right|^p \mathrm{dx} \leq c \int_\Omega \left| \frac{v}{\mathrm{dist}_D} \right|^p \mathrm{dx} \leq c \int_\Omega \left| \frac{u}{\mathrm{dist}_D} \right|^p \mathrm{dx} < \infty$$

by assumption on u. Now, Proposition 7.1 yields the claim $v \in W_0^{1,p}(\Omega) \subset W_D^{1,p}(\Omega)$.

Step 2: Extending w. Recall from Subsection 5.4 that the sets $\Omega \cap U_{\mathrm{x}_j}$, $1 \leq j \leq n$, are universal Sobolev extension domains. Since $w = (1-\eta)u$, where $(1-\eta)$ has compact support in the union of these domains, an extension $\widehat{w} \in W^{1,p}(\mathbb{R}^d)$ of $w \in W^{1,p}(\Omega)$ with compact support within $\bigcup_{j=1}^n U_{\mathrm{x}_j}$ can be constructed just as in the proof of Proposition 5.4.

Now, if we can show $\widehat{w} \in W_D^{1,p}(\mathbb{R}^d)$, then $w \in W_D^{1,p}(\Omega)$ and by Step 1 also $u = v + w$ belongs to this space.

Step 3: Estimating the trace of $\widehat{w}$. To prove $\widehat{w} \in W_D^{1,p}(\mathbb{R}^d)$ we rely once more on the techniques used in the proof of Lemma 5.11. So, let $\widehat{\mathfrak{w}}$ be the representative of $\widehat{w}$ defined on $\mathbb{R}^d \setminus N$ via

$$\widehat{\mathfrak{w}}(\mathrm{y}) := \lim_{r \to 0} \frac{1}{|B(\mathrm{y}, r)|} \int_{B(\mathrm{y},r)} \widehat{w} \, \mathrm{dx},$$

where the exceptional set N is of vanishing $(1,p)$-capacity and hence a $\mathcal{H}_{d-1}$-nullset, cf. Theorem 5.7. Owing to Theorem 5.10 the proof is complete once we have shown that $\widehat{\mathfrak{w}} = 0$ holds $\mathcal{H}_{d-1}$-a.e. on $D \setminus N$.

To this end let $\mathrm{y} \in D \setminus N$. Clearly, we only have to consider the case $\mathrm{y} \in \operatorname{supp}(\widehat{w})$ in which y lies on the boundary of one of the $W^{1,p}$ extension domains $\Omega \cap U_{\mathrm{x}_1}, \ldots, \Omega \cap U_{\mathrm{x}_n}$.

¿From [17, Thm. 2] it follows that these domains are d-sets. Hence, $\Lambda = \Omega$ satisfies the asymptotically nonvanishing relative volume condition (5.4) around y with a lower bound $c > 0$ on the limes inferior that is independent of y and – just as in the proof of Lemma 5.11 – a set $W \subset \mathbb{R}^d$ can be constructed such that the restriction of $\widehat{\mathfrak{w}}$ to $\mathbb{R}^d \setminus W$ is continuous at y and such that $|B(\mathrm{y}, r) \cap \Omega \cap (\mathbb{R}^d \setminus W)| \geq cr^d/2$ if $r > 0$ is small enough. For the sake of completeness let us remark that W is Lebesgue measurable since $\widehat{\mathfrak{w}}$ is a representative of $\widehat{w}$ and since sets of zero $(1,p)$-capacity are of zero $\mathcal{H}_{d-1}$-measure, cf. Theorem 5.7, and hence a fortiori of zero Lebesgue measure on $\mathbb{R}^d$. By these properties of W:

$$\begin{aligned}
|\widehat{\mathfrak{w}}(\mathrm{y})| &= \left| \lim_{r \to 0} \frac{1}{|B(\mathrm{y}, r) \cap \Omega \cap (\mathbb{R}^d \setminus W)|} \int_{B(\mathrm{y},r) \cap \Omega \cap (\mathbb{R}^d \setminus W)} \widehat{\mathfrak{w}} \, \mathrm{dx} \right| \\
&\leq \limsup_{r \to 0} \frac{2}{cr^d} \int_{B(\mathrm{y},r) \cap \Omega} |\widehat{w}| \, \mathrm{dx} \\
&= \limsup_{r \to 0} \frac{2}{cr^d} \int_{B(\mathrm{y},r) \cap \Omega} |w| \, \mathrm{dx}.
\end{aligned}$$

In order to force these mean-value integral to vanish in the limit $r \to 0$, introduce the function $\log(\mathrm{dist}_D)^{-1}$, which is bounded above by $|\log r|^{-1}$ on $B(\mathrm{y}, r)$ if $r < 1$. It follows

$$|\widehat{\mathfrak{w}}(\mathrm{y})| \leq c \limsup_{r \to 0} |\log r|^{-1} \left(\frac{1}{r^d} \int_{B(\mathrm{y},r) \cap \Omega} |w \log(\mathrm{dist}_D)| \, \mathrm{dx} \right). \tag{7.1}$$

So, the upshot of this step is that $\widehat{w} \in W_D^{1,p}(\mathbb{R}^d)$ follows provided the right-hand side of (7.1) is zero for $\mathcal{H}_{d-1}$-almost every $\mathrm{y} \in D$.

Step 4: Intermezzo on $w\log(\operatorname{dist}_D)$. In this step we want to prove the following.

Lemma 7.2. *The function $w\log(\operatorname{dist}_D)$ belongs to $W^{1,r}(\Omega)$ for every $r \in [1,p[$.*

For the proof we need the following notion due to H. Aikawa [3], see [28, 29] for a comparison with other concepts of dimension.

Definition 7.3. Let $F \subset \mathbb{R}^d$ be a closed set with empty interior. The infimum of all $s \in \,]0,d]$ for which there exists a constant $c_s > 0$ such that

$$\int_{B(\mathrm{x},r)} \operatorname{dist}_F(\mathrm{x})^{s-d}\,\mathrm{dx} \leq c_s r^s$$

holds for every $\mathrm{x} \in F$ and all $r > 0$ is called *Aikawa dimension* of F.

Proof of Lemma 7.2. Since by assumption $D \subset \mathbb{R}^d$ is a $(d-1)$-set, it has empty interior and Aikawa dimension not larger than $(d-1)$, cf. [28, Lem. 2.1]. Hence, some negative power of dist_D is locally integrable on $\mathbb{R}^d$ and by subordination of logarithmic growth $\log(\operatorname{dist}_D) \in L^q_{\mathrm{loc}}(\mathbb{R}^d)$ for every $q \in [1,\infty[$. In the following fix $r \in [1,p[$ and let $q \in [1,\infty[$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Now, since Ω is bounded, $w\log(\operatorname{dist}_D) \in L^r(\Omega)$ by Hölder's inequality. As for the derivative, dist_D is Lipschitz continuous on $\mathbb{R}^d$ and $\log(\operatorname{dist}_D)$ is Lipschitz continuous on compact subsets of Ω, so that

$$\nabla(w\log(\operatorname{dist}_D)) = \log(\operatorname{dist}_D)\nabla w + \frac{w}{\operatorname{dist}_D}\nabla \operatorname{dist}_D$$

on Ω in the sense of distributions. Hölder's inequality yields

$$\|\nabla(w\log(\operatorname{dist}_D))\|_r \leq \|\log(\operatorname{dist}_D)\|_q \|\nabla w\|_p + \left\|\frac{w}{\operatorname{dist}_D}\right\|_p |\Omega|^{1/q} \|\nabla \operatorname{dist}_D\|_\infty < \infty,$$

where all norms are on Ω and we have used

$$\int_\Omega \left|\frac{w}{\operatorname{dist}_D}\right|^p \mathrm{dx} \leq \|1-\eta\|_{L^\infty(\Omega)} \int_\Omega \left|\frac{u}{\operatorname{dist}_D}\right|^p \mathrm{dx} < \infty,$$

which holds by assumption on u. Altogether, $w\log(\operatorname{dist}_D) \in W^{1,r}(\Omega)$. ☐

Step 5: The right-hand side of (7.1). To complete the proof of Theorem 3.3 we still have to show that for $\mathcal{H}_{d-1}$-almost every $\mathrm{y} \in D$ the right-hand side of (7.1) is zero.

Fix some $r \in \,]1,p[$. As Lemma 7.2 guarantees $w\log(\operatorname{dist}_D) \in W^{1,r}(\Omega)$, an extension $f \in W^{1,r}(\mathbb{R}^d)$ of $|w\log(\operatorname{dist}_D)|$ can be constructed as before. At this point we use that the domains $\Omega \cap U_{\mathrm{x}_j}$ are not just $W^{1,p}$ extension domains but also $W^{1,r}$ extension domains. Since $W^{1,r}(\mathbb{R}^d)$ is closed under truncation, see e.g. [11, Prop. 2.6], also $|f| \in W^{1,r}(\mathbb{R}^d)$. From Subsection 5.3 it is known that for $(1,r)$-quasievery – and hence for $\mathcal{H}_{d-1}$-almost every $\mathrm{y} \in \mathbb{R}^d$ – it holds

$$\limsup_{r\to 0} \frac{1}{r^d}\int_{B(\mathrm{y},r)\cap\Omega} |w\log(\operatorname{dist}_D)|\,\mathrm{dx} \leq \limsup_{r\to 0} \frac{1}{|B(\mathrm{y},r)|}\int_{B(\mathrm{y},r)} |f|\,\mathrm{dx} < \infty.$$

But $|\log r|^{-1} \to 0$ as $r \to 0$, so this already implies that the right-hand side of (7.1) is zero for all such y.

Remark 7.4. Step 4 and 5 of the proof of Theorem 3.3 reveal that the assumption on $\overline{\partial\Omega \setminus D}$ can be weakened to the following: For every $\mathrm{x} \in \overline{\partial\Omega \setminus D}$ there exists a neighborhood U_x such that $U_\mathrm{x} \cap \Omega$ is a $W^{1,p}$ extension domain and for some $r \in \,]1,p[$ the corresponding extension operator extends to a bounded operator $W^{1,r}(\Omega) \to W^{1,r}(\mathbb{R}^d)$.

8. A Generalization

If one asks: 'What is the most restricting condition in Theorem 3.1?', the answer doubtlessly is the assumption that a *global* extension operator shall exist. Certainly, this excludes all geometries that include cracks not belonging completely to the Dirichlet boundary part as in the subsequent Figure.

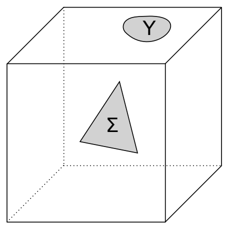

Figure 2. The domain Ω is the cube minus the triangle Σ. The Dirichlet boundary part D consists exactly of the six outer sides of the cube minus the droplet Υ on the cover plate.

Since the distance function dist_D measures only the distance to the Dirichlet boundary part D, points in $\partial\Omega$ that are far from D should not be of great relevance in view of the Hardy inequality (3.1). In the following considerations we intend to make this precise. Let $U, V \subset \mathbb{R}^d$ be two open, bounded sets with the properties

$$D \subset U, \quad \overline{V} \cap D = \emptyset, \quad \overline{\Omega} \subset U \cup V. \tag{8.1}$$

The philosophy behind this is to take U as a small neighborhood of D which – desirably – excludes the 'nasty parts' of $\partial\Omega \setminus D$. More properties of U, V will be specified below. Let $\eta_U \in C_0^\infty(U), \eta_V \in C_0^\infty(V)$ be two functions with $\eta_U + \eta_V = 1$ on $\overline{\Omega}$. Then one can estimate

$$\Big(\int_\Omega |u|^p \,\mathrm{dist}_D^{-p} \,\mathrm{dx}\Big)^{1/p} \le \Big(\int_{U\cap\Omega} |\eta_U u|^p \,\mathrm{dist}_D^{-p} \,\mathrm{dx}\Big)^{1/p} + \Big(\int_{V\cap\Omega} |\eta_V u|^p \,\mathrm{dist}_D^{-p} \,\mathrm{dx}\Big)^{1/p}.$$

Since dist_D is larger than some $\varepsilon > 0$ on $\mathrm{supp}(\eta_V) \subset V$, the second term can be estimated by $\frac{1}{\varepsilon}\big(\int_\Omega |u|^p \,\mathrm{dx}\big)^{1/p}$. If one assumes, as above, Poincaré's inequality, then this term may be estimated as required. In order to provide an adequate estimate also for the first term, we introduce the following assumption.

Assumption 8.1. *The set U from above can be chosen in such a way that $\Lambda := \Omega \cap U$ is again a domain and if one puts $\Gamma := (\partial\Omega \setminus D) \cap U$ and $E := \partial\Lambda \setminus \Gamma$, then there is a linear, continuous extension operator $\mathfrak{F} : W_E^{1,p}(\Lambda) \to W_E^{1,p}(\mathbb{R}^d)$.*

Clearly, this assumption is weaker than Condition (iii) in Theorem 3.1; in other words: Condition (iii) in Theorem 3.1 requires Assumption 8.1 to hold for an open set $U \supset \overline{\Omega}$.

Let us discuss the sense of Assumption 8.1 in extenso. Philosophically spoken, it allows to focus on the extension not of the functions u but the functions $\eta_U u$, which live on a set whose boundary does (possibly) not include the 'nasty parts' of $\partial\Omega \setminus D$ that are an obstruction against a global extension operator. In detail: one first observes that, for $\eta = \eta_U \in C_0^\infty(U)$ and $v \in W_D^{1,p}(\Omega)$, the function $\eta v|_\Lambda$ even belongs to $W_E^{1,p}(\Lambda)$, see [18, Thm. 5.8]. Secondly, we have by the definition of E

$$\partial U \cap \Omega = (\partial U \cap \Omega) \setminus \Gamma \subset \partial\Lambda \setminus \Gamma = E.$$

This shows that the 'new' boundary part $\partial U \cap \Omega$ of Λ belongs to E and is, therefore, uncritical in view of extension. Thirdly, one has $D = D \cap U \subseteq \partial\Omega \cap U \subset \partial\Lambda$, and it is clear that the 'new Dirichlet boundary part' E includes the 'old' one D. Hence, the extension operator $\mathfrak{F}$ may be viewed also as a continuous one between $W^{1,p}_E(\Lambda)$ and $W^{1,p}_D(\mathbb{R}^d)$. Thus, concerning $v := \eta u = \eta_U u$ one is – mutatis mutandis – again in the situation of Theorem 3.1: $\eta u \in W^{1,p}_E(\Lambda) \subset W^{1,p}_D(\Lambda)$ admits an extension $\mathfrak{F}(\eta u) \in W^{1,p}_E(\mathbb{R}^d) \subseteq W^{1,p}_D(\mathbb{R}^d)$, which satisfies the estimate $\|\mathfrak{F}(\eta u)\|_{W^{1,p}_D(\mathbb{R}^d)} \le c\|\eta u\|_{W^{1,p}_D(\Lambda)}$, the constant c being independent from u. This leads, as above, to a corresponding (continuous) extension operator $\mathfrak{F}_\bullet : W^{1,p}_E(\Lambda) \to W^{1,p}_0(\Lambda_\bullet)$. Here, of course, $\Lambda_\bullet$ has again to be defined as the connected component of $B \setminus D$ that contains Λ. Thus one may proceed again as in the proof of Theorem 3.1, and gets, for $u \in W^{1,p}_D(\Omega)$,

$$\begin{aligned}\int_\Omega \Big(\frac{|\eta u|}{\mathrm{dist}_D}\Big)^p \,\mathrm{dx} = \int_\Lambda \Big(\frac{|\eta u|}{\mathrm{dist}_D}\Big)^p \,\mathrm{dx} \le \int_{\Lambda_\bullet} \Big(\frac{|\mathfrak{F}_\bullet(\eta u)|}{\mathrm{dist}_{\partial\Lambda_\bullet}}\Big)^p \,\mathrm{dx} \le c\|\nabla(\mathfrak{F}_\bullet(\eta u))\|^p_{L^p(\Lambda_\bullet)} \\ \le c\|\mathfrak{F}_\bullet(\eta u)\|^p_{W^{1,p}(\Lambda_\bullet)} \le c\|\eta u\|^p_{W^{1,p}(\Lambda)} \le c\big(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\big),\end{aligned}$$

since ηu belongs to $W^{1,p}_E(\Lambda) \subset W^{1,p}_D(\Lambda)$. Exploiting a last time Poincaré's inequality, whose validity will be discussed in a moment, one gets the desired estimate.

When aiming at Poincare's inequality, it seems convenient to follow again the argument in the proof of Proposition 6.1: as pointed out above, the property $\mathbb{1} \notin W^{1,p}_D(\Omega)$ has to do only with the local behavior of Ω around the points of D, cf. Lemma 6.3. Hence, this will not be discussed further here.

Concerning the compactness of the embedding $W^{1,p}_D(\Omega) \hookrightarrow L^p(\Omega)$, one does not need the existence of a *global* extension operator $\mathfrak{E} : W^{1,p}_D(\Omega) \to W^{1,p}(\mathbb{R}^d)$. In fact, writing for every $v \in W^{1,p}_D(\Omega)$ again $v = \eta_U v + \eta_V v$ and supposing Assumption 8.1, one gets the following:

If $\{v_k\}_k$ is a bounded sequence in $W^{1,p}_D(\Omega)$, then the sequence $\{\eta_U v_k|_\Lambda\}_k$ is bounded in $W^{1,p}_E(\Lambda)$. Due to the extendability property, this sequence contains a subsequence $\{\eta_U v_{k_l}|_\Lambda\}_l$ that converges in $L^p(\Lambda)$ to an element v_U. Thus, $\{\eta_U v_{k_l}\}_l$ converges to the function on Ω that equals v_U on Λ and 0 on $\Omega \setminus \Lambda$. The elements $\eta_V v_k$ in fact live on the set $\Pi := \Omega \cap V$ and are zero on $\Omega \setminus V$. In particular they are zero in a neighborhood of D. Moreover, they form a bounded subset of $W^{1,p}(\Pi)$. Therefore it makes sense to require that Π is again a domain, and, secondly that Π meets one of the well-known compactness criteria $W^{1,p}(\Pi) \hookrightarrow L^p(\Pi)$, cf. [37, Ch. 1.4.6]. Keep in mind that such requirements are much weaker than the global $W^{1,p}$-extendability, and in particular include the example in Figure 2, as long as the triangle Σ has a positive distance to the six outer sides of the cube. Resting on these criteria, one obtains again the convergence of a subsequence $\{\eta_V v_{k_l}|_\Pi\}_l$ that converges in $L^p(\Pi)$ towards a function v_V. The sequence $\{\eta_V v_{k_l}\}_l$ then converges in $L^p(\Omega)$ to a function that equals v_V on Π and zero on $\Omega \setminus V$.

Altogether, we have extracted a subsequence of $\{v_k\}_k$ that converges in $L^p(\Omega)$.

Remark 8.2. In fact one does not really need that Π is connected. By similar arguments as above it suffices to demand that it splits up in at most finitely many components $\Pi_1, \dots, \Pi_n$, such that each of these admits the compactness of the embedding $W^{1,p}(\Pi_j) \hookrightarrow L^p(\Pi_j)$.

We summarize these considerations in the following theorem.

Theorem 8.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $D \subset \partial\Omega$ be a closed part of the boundary. Suppose that the following three conditions are satisfied:*

(i) *The set D is $(d-1)$-thick, cf. Theorem 3.1 above.*
(ii) *The space $W^{1,p}_D(\Omega)$ can be equivalently normed by $\|\nabla \cdot\|_{L^p(\Omega)}$.*
(iii) *There are two open sets $U, V \subset \mathbb{R}^d$ that satisfy (8.1) and U satisfies Assumption 8.1.*

Then there is a constant $c > 0$ such that Hardy's inequality

$$\int_\Omega \Big|\frac{u}{\mathrm{dist}_D}\Big|^p \,\mathrm{dx} \le c \int_\Omega |\nabla u|^p \,\mathrm{dx} \tag{8.2}$$

holds for all $u \in W^{1,p}_D(\Omega)$.

Acknowledgment. In 2012, after we asked him a question, V. Maz'ya proposed a proof [38] of Theorem 4.6 that heavily relied on several deep results from his book [37]. Actually, we found, after this, references in the literature [24], [27] with a different approach that apply directly, provided Lemma 4.8 is established. It was again V. Maz'ya who drew our attention to the fact that something like this lemma is needed. We warmly thank him for all that.

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