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Higher L^p regularity for vector fields that satisfy divergence and rotation constraints in dual Sobolev spaces, and application to some low-frequency Maxwell equations

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Abstract

We show that L^p vector fields over a Lipschitz domain are integrable to higher exponents if their generalized divergence and rotation can be identified with bounded linear operators acting on standard Sobolev spaces. A *Div-Curl Lemma*-type argument provides compact embedding results for such vector fields. We investigate the regularity of the solution fields for the low-frequency approximation of the Maxwell equations in time-harmonic regime. We focus on the weak formulation 'in H' of the problem, in a reference geometrical setting allowing for material heterogeneities.

1 Introduction

In a bounded domain $\Omega \subset \mathbb{R}^3$ assume that the electromagnetic properties are determined by the following *low-frequency approximation* ([Bos04], pages 42–46) of Maxwell's equations in *time-harmonic regime*

$$\operatorname{curl} H = J\,, \tag{1}$$

$$\operatorname{div} B = 0\,, \tag{2}$$

$$\imath\,\omega\,B + \operatorname{curl} E = 0\,, \tag{3}$$

where H, J, B, E are complex-valued unknown vector fields called the magnetic field strength, the electrical current density, the magnetic induction and the electric field strength. The constant ω is a characteristic alternating frequency in the assumed time-harmonic oscillation. It is usual to supplement these equations by Ohm's law in the electrical conductors

$$J = \sigma\left(E + v \times B\right) + J_{\mathsf{g}} \text{ in } \Omega_{\mathsf{c}}\,, \tag{4}$$

where the given vector field v represents the velocity of the medium assumed stationary, J_{g} is the given density of a source current, and the proportionality factor σ is called the electrical conductivity. In order to determine the electric field outside of the conductors, the Poisson equation

$$\operatorname{div} D = 0 \text{ in } \Omega_{\mathsf{nc}} \tag{5}$$

is considered. If the involved materials are not ferromagnetic, the constitutive relations between the fields B and H, and between E and D in Ω are linear

$$B = \mu H\,, \qquad D = \varepsilon E\,, \tag{6}$$

with proportionality factors μ, ε called the magnetic permeability and the electrical permittivity of the medium. At interior interfaces in Ω, the fields B, H, E have to satisfy the *natural interface*

conditions: Assuming a partition of $\overline{\Omega} = \bigcup_{i=1}^{m} \overline{\Omega}_i$, where $\Omega_1, \ldots, \Omega_m$ are subdomains that represent heterogeneous materials

$$\left[H \times \nu \right] = 0 \,, \quad \left[B \cdot \nu \right] = 0 \,, \quad \left[E \times \nu \right] = 0 \quad \text{on } \partial \Omega_i \cap \partial \Omega_j, \, i \neq j \tag{7}$$

with $[\cdot]$ denoting as usual the difference between the value of the enclosed quantity from the side of Ω_i to its value from the side of Ω_j (the 'jump' of this quantity), across the surface $\partial \Omega_i \cap \partial \Omega_j$. We denote ν is a unit normal to the corresponding surface. At the outer boundary $\partial \Omega$ the conditions

$$B \cdot \nu = 0 \,, \quad E \times \nu = 0 \quad \text{on } \partial \Omega \tag{8}$$

are considered. These conditions model a magnetic shield, but are also frequently used in practical models in connection with sufficiently large a region Ω to avoid the conditions of vanishing at infinity.

We call (P) the problem of finding fields H, J, B, E, D that satisfy (1), (2), (3), (4) and (5) together with the constitutive relations (6) and the interface and boundary conditions (7) and (8).

The system (1), (2), (3), (4) and (5) of PDEs is a very well known model for electromagnetic processes at low-frequency, for example industrial high-temperatures applications with heating based on the Joule effect (examples in [Bos04, HMRR10, HR11, KPS04, DKS$^+$11]). It is justified to use this simplification of Maxwell's equations under several conditions: The ratio displacement current over ohmic current $|\partial_t D|/|\sigma E| \approx |\epsilon \, \omega|/|\sigma|$ has to be comparatively small throughout all conductors of the system; The hypothesis of charge neutrality $\rho \approx 0$ must be valid; The equation $\mathrm{div}\, D = \rho \approx 0$ must be eliminated in the electrical conductors. There are well-known drawbacks and possible medicines of the model ([Bos04] for an introduction) as well as recent interesting discussions for the range of its validity in the context of complex applications ([DGM12]).

From the point of view of applied analysis, the solvability of the problem (P) has been successfully discussed in the past. In the case of variable, discontinuous and even anisotropic material properties and of the presence of nonsmooth interfaces in the domain, even refined results of classical potential theory cannot be applied. It is necessary to resort to the theory of *generalized electromagnetics* in Hilbert spaces, and the use of decomposition theorems of the space L^2 exposed e. g. in [PM99], that in some cases can even help fixing nonlinear constitutive relations instead of (6) ([Pic84b][1]). Weak approaches of the system (1), (2), (3), (4) and (5) are characterized by the *a priori* choice of a 'main unknown' since it is possible to reduce everything to a system of equations for only one of the fields. Recently the problem 'in E' has raised much interest ([TY12, You12, AvH12, NT13] among others); The current formulation 'in J' remains more or less marginal ([MS96, GK06] among others); the problem 'in H' has a long history (especially in the context of magnetohydrodynamics: [LS60, LS77, DL72, Dru09a, Dru09b, Dru09c] among others) and has the advantage not to involve any kind of degeneracy in the equations; Finally, let us mention numerous models and approaches based on the introduction of vector potentials ([RT92, KPS04, HR11] among others).

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[1]R. Picard wrote 1984: *During the last 25 years research in this field has included more and more the use of Hilbert space settings. The strength of this method consists in the fact that it is capable of dealing with anisotropic, inhomogeneous media and nonsmooth data and boundary in considerable generality and relative ease.*

For a weak solution 'in H' to the problem (P), the generalized theory of electromagnetics gives the following basic informations (see for example [DL76], [PM99] or [Bos04]):

$$H \in \{\psi \in L^2(O; \mathbb{C}^3) \,:\, \operatorname{curl}\psi \in L^2(O; \mathbb{C}^3),\ \operatorname{div}(\mu\psi) = 0,\ \gamma_\nu(\mu\psi) = 0\}, \qquad (9)$$

where the operators curl and div are intended in the generalized sense, and γ_ν is the abstract trace operator. Our interest in the equations of electrotechnics started from a multiphysics application in crystal growth: see [Dru09a]. In this coupled problem involving also the Navier-Stokes equations and the heat equation the use of weak solution fields led to considerable difficulties: The mechanical force influencing the fluid motion (Lorentz force) and the heat source density resulting from the Joule effect are given by the quadratic expressions $J \times B = \operatorname{curl} H \times \mu H$ and $\sigma^{-1} J \cdot \overline{J} = \sigma^{-1} \operatorname{curl} H \cdot \operatorname{curl} \overline{H}$. Therefore, if one's knowledge about the regularity of H is limited to (9), we cannot expect in general more than L^1-terms, yielding a very bad coupling to PDEs for momentum and energy balance. This is to say that the regularity theory in L^p spaces, $p > 2$ for the field H and its rotation is not insignificant for applied analysis in general, and for the analysis of models occurring in industrial applications of basic importance: [Dru09a, DKS$^+$11, DDKS12].

In our previous study [Dru07], the question of the higher integrability of the Lorentz force was asked already. Surveying the recent literature on elliptic problems/interface problems (essentially [Zan00, ERS07, ABDG98, HDKRS08]), we could gather some sufficient conditions for the domain and the (scalar) magnetic permeability μ that yield $|H| \in L^q(\Omega)$ for an exponent $q > 3$. Unfortunately, this result relied on a global $\mathcal{C}^1$ assumption for the interfaces between the different materials in the domain Ω. In the presence of interior polyhedral interfaces and multiple junctions, the optimal exponent of higher integrability for H, and its relationship to the diffusion coefficient and the geometry of the problem, are comparatively very intricated subjects (see the interesting study [NS99]) : concrete answers like in [Mer03, Dau92] seem difficult to provide in this way of investigation for general situations). In this case, due to the existence of so called *singular epxonents*, the higher-integrability for the gradient of solutions to the transmission problem for the operator $-\operatorname{div}(\mu\nabla u)$ – thus also for the field H in the Maxwell equations here considered (see below for details) – is known to turn even arbitrary little: Examples in [Mer03, NS99, ERS07].

Later (essentially in [Dru09a] and in [DKS$^+$11]) we investigated the higher integrability of $J = \operatorname{curl} H$ and of the heat source density. Here also, we used strong regularity assumptions (globally $\mathcal{C}^1$) on the interfaces. Only afterwards we realized that the higher-integrability of $|\operatorname{curl} H|$ is a problem essentially independent on whether $|H|$ itself is higher-integrable, due to the fact that *the field J is localized in the conductors*. For simplicity, we shall restrict our investigation to a model geometrical setting described hereafter. We consider bounded domains $\Omega_i \subset \mathbb{R}^3$, $i = 1, \dots, 4$, that represent disjoint materials with different electromagnetic properties such that $\overline{\Omega} = \bigcup_{i=1}^{4} \overline{\Omega_i}$.
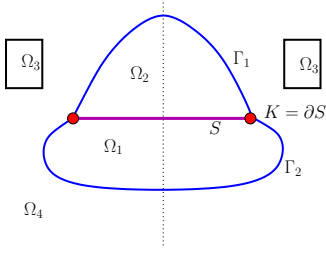
Figure 1: Transversal cut through a model axisymmetric configuration Ω: note the domain G (blue boundary), the surface S separating the conductors (magenta), and the contact line K (red)

We assume that the set Ω is simply connected. We moreover assume that the domains $\Omega_1, \ldots, \Omega_3$ are enclosed by the domain Ω_4, in the sense that the set $\mathbb{R}^3 \setminus \Omega_4$ is disconnected, or equivalently that $\operatorname{dist}(\Omega_i, \partial\Omega) > 0$ for $i = 1, 2, 3$. There is a common interface between the domains Ω_1 and Ω_2, that is, $S := \partial\Omega_1 \cap \partial\Omega_2$ is a nontrivial two-dimensional submanifold. We introduce the domain $G := \Omega_1 \cup S \cup \Omega_2$ as the region that one tries to influence, whereas Ω_3 is the region where the current source is given. There is no common interface between Ω_3 and G.

For the sake of generality, we assume that the coefficients σ, μ and ε take values in the set $\mathbb{C}^{3\times3}_{\mathsf{sym}}$ of symmetric real matrices, and are piecewise uniformly continuous with respect to the partition of Ω. Moreover, there are positive real numbers σ_0, σ_1, μ_0, μ_1 and ε_0, ε_1 such that for all $\eta \in \mathbb{R}^3$ both the real and imaginary part of the coefficients satisfy

$$
\begin{aligned}
\sigma_0 \left|\eta\right|^2 &\le \sigma(x)\eta \cdot \eta \le \sigma_1 \left|\eta\right|^2 && \text{for all } x \in \Omega_{\mathsf{c}} \\
\varepsilon_0 \left|\eta\right|^2 &\le \varepsilon(x)\eta \cdot \eta \le \varepsilon_1 \left|\eta\right|^2 && \text{for all } x \in \Omega_{\mathsf{nc}} \quad\quad (10) \\
\mu_0 \left|\eta\right|^2 &\le \mu(x)\eta \cdot \eta \le \mu_1 \left|\eta\right|^2 && \text{for all } x \in \Omega \,.
\end{aligned}
$$

Theorem 1.1. *Assume that Ω is a simply connected Lipschitz domain, and that Ω_3 is also Lipschitzian. Assume that $S = \partial\Omega_1 \cap \partial\Omega_2$ is a surface of class $\mathcal{C}^1$, and that for $i = 1, 2$, the surface $\Gamma_i := \partial\Omega_i \setminus S$ is of class $\mathcal{C}^1$ as well. Let $\sigma : \Omega_c \to \mathbb{C}^{3\times3}_{sym}$, $\varepsilon : \Omega_{nc} \to \mathbb{C}^{3\times3}_{sym}$ and $\mu : \Omega \to \mathbb{C}^{3\times3}_{sym}$ be piecewise uniformly continuous and satisfy (10). Assume that $v \in W^{1,\infty}_0(\Omega_c; \mathbb{R}^3)$. Assume that $J_g \in L^2(\Omega_{c0}; \mathbb{C}^3)$.*

Then, there are $q > 3$ and $r > 2$ such that if $J_g \in L^q(\Omega_{c0}; \mathbb{C}^3)$, every weak solution to the problem (P) satisfies

$$
\left|J\right|, \left|E\right| \in L^q(\Omega), \quad \left|H\right|, \left|\operatorname{curl} E\right| \in L^r(\Omega) \,.
$$

Moreover, if μ is a sufficiently small perturbation of the identity, the same is valid for a $r > 3$.

The first section in the article is devoted to the proof of a general embedding inequality for vector fields that satisfy a divergence and a rotation constraint. This result might possess interest as an application-independent tool.

In the second section we apply the result to the regularity analysis for the problem (P).

2 Embedding results for vector fields that satisfy a rotation and a divergence constraint

Several embedding results have been stated in the past for vector fields that satisfy a rotation and a divergence constraint, and in general also a constraint on the normal or on the tangential

values taken at the boundary. For a typical example we quote the inequality

$$\|\nabla \psi\|_{[L^2(O)]^9} \le c \left(\|\psi\|_{[L^2(O)]^3} + \| \operatorname{curl} \psi\|_{[L^2(O)]^3} + \| \operatorname{div} \psi\|_{L^2(O)} \right), \tag{11}$$

valid in every domain $O \subset \mathbb{R}^3$ of class $\mathcal{C}^2$ (see [DL76], Ch. 7, Th. 6.1 for a proof) with $c = c(O)$ for all $\psi \in W^{1,2}(O; \mathbb{R}^3)$ such that $\psi \cdot \nu = 0$ on ∂O. The inequality (11) is known in the context of differential geometry as Gaffney's inequality, see [Pic84a]). Inequalities of this type can be generalized in smooth domains to the case $1 < p < +\infty$, as was shown in [vW92], Th. 2.1. For nonsmooth domains, the Gaffney inequality continues to be valid on convex polyhedra (see [GR86] and references), but examples of Lipschitz domains in three space dimensions are known for which (11) fails (singular exponents). One can still hope, though, to prove an embedding result into Sobolev spaces of fractional order: [ABDG98, Cos90], overview in [Mon03]. In this paper we go for an embeddment into higher L^p-spaces allowing curl and div to be abstract (distribution valued) operators.

We first recall basic notions concerning the generalized operators curl and div in Lebesgue spaces and in the dual of a Sobolev space over a *Lipschitz domain*. In the second and third subsection, we then investigate embedding and compact embedding results. We here can restrict to real-valued vector fields: this is completely sufficient for the purpose.

2.1 The generalized operators curl and div

Let $O \subset \mathbb{R}^3$ be a bounded domain, and $1 < p < +\infty$. Throughout the section, we denote $p' := \frac{p}{p-1}$ and the optimal embedding exponent for the space $W^{1,p}(O)$ into a Lebesgue space $L^{p_d^*}(S_d)$, $S_d \subset O$ a d-dimensional Lipschitz submanifold, is given by

$$p_d^* := \begin{cases} \frac{d\,p}{3-p} & \text{for } 1 \le p < 3 \\ 1 \le s < \infty \text{ arbitrary} & \text{for } p = 3 \\ +\infty & \text{for } p > 3 \end{cases}, \quad d = 2, 3. \tag{12}$$

We also consider a mapping $a \in L^\infty(O; \mathbb{R}^{3\times3}_{\text{sym}})$ that satisfies the ellipticity condition

$$a_0 \,|\eta|^2 \le a(x)\eta \cdot \eta \le a_1 \,|\eta|^2 \text{ for all } \eta \in \mathbb{R}^3, \text{ for almost all } x \in O, \tag{13}$$

with two constants $0 < a_0 \le a_1 < +\infty$. We commence recalling the well-known definition and properties of vector fields having a rotation/divergence in L^p-spaces[2].

Definition 2.1. *For a vector field $\psi \in L^1(O; \mathbb{R}^3)$:*

(1) We write $\operatorname{curl} \psi \in L^p(O; \mathbb{R}^3)$ if there exists $\xi \in L^p(O; \mathbb{R}^3)$ such that $\int_O \psi \cdot \operatorname{curl} \Phi = \int_O \xi \cdot \Phi$ for all $\Phi \in C_c^\infty(O; \mathbb{R}^3)$. The uniquely determined vector field ξ is called the generalized rotation of ψ, and we define $\operatorname{curl} \psi := \xi$;

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[2]These spaces are have been used by numerous authors, mostly in the case $p = 2$. It would be of interest to trace the original idea.

(2) We write $\operatorname{div}_a \psi \in L^p(O)$ if there exists a function $\zeta \in L^p(O)$ such that $\int_O a\psi \cdot \nabla \phi = -\int_O \zeta \, \phi$ for all $\phi \in C_c^\infty(O)$. The uniquely determined function ζ is called the generalized a-divergence (the generalized divergence for $a = \operatorname{Id}$) of ψ, and we define $\operatorname{div}_a \psi := \zeta$.

On the basis of Definition 2.1, we then introduce

$$
\begin{aligned}
L^p_{\mathrm{curl}}(O) &:= \left\{ \psi \in L^p(O; \mathbb{R}^3) \ : \ \operatorname{curl} \psi \in L^p(O; \mathbb{R}^3) \right\}, \\
L^p_{\mathrm{div}_a}(O) &:= \left\{ \psi \in L^p(O; \mathbb{R}^3) \ : \ \operatorname{div}_a \psi \in L^p(O) \right\}.
\end{aligned}
\tag{14}
$$

These spaces are Banach spaces with respect to the graph topology. For $p = 2$, they are Hilbert spaces. The notations $H(O; \operatorname{curl}) = L^2_{\mathrm{curl}}(O)$ and $H(O; \operatorname{div}_a) = L^2_{\mathrm{div}_a}(O)$ are also in common use. For vector fields that belong to a space (14), it is possible to define a trace operator on surfaces. Denote ν the outward-pointing unit normal to ∂O. For $\Phi, \psi \in C^\infty(\overline{O}; \mathbb{R}^3)$ the Gauss theorem implies the identity

$$
\int_O \psi \cdot \operatorname{curl} \Phi - \int_O \operatorname{curl} \psi \cdot \Phi = \int_{\partial O} (\nu \times \psi) \cdot \Phi =: \left\langle \gamma_\tau(\psi), \, \Phi \right\rangle.
\tag{15}
$$

Thanks to results for the density of the smooth functions in the spaces (14), it can be shown (see for example [DL76], [PM99] in the case $p = 2$ and Lemma A.5 for the general case) that the operator $\gamma_\tau(\psi)$ extends to a linear bounded operator on the space $L^p_{\mathrm{curl}}(O)$ with values in the dual space $[W^{1/p,p'}(\partial O; \mathbb{R}^3)]^*$ ($\gamma_\tau(\psi)$ extends even over $[L^{p'}_{\mathrm{curl}}(O)]^*$ in a certain sense, but this is more delicate: [Mon03], pages 57–60). Similarly, for $\psi \in C^\infty(\overline{O}; \mathbb{R}^3)$ and $\phi \in C^\infty(\overline{O})$, the Gauss integral theorem implies that

$$
\int_O a\psi \cdot \nabla \phi + \int_O \operatorname{div}(a\psi) \, \phi = \int_{\partial O} \psi \cdot a\nu \, \phi =: \left\langle \gamma_{a\nu}(\psi), \, \phi \right\rangle.
\tag{16}
$$

The operator $\gamma_{a\nu}$ extends to a linear bounded operator on the space $L^p_{\mathrm{div}}(O)$ with values in $[W^{1/p,p'}(\partial O)]^*$. The kernel spaces of these operators are needed. We define

$$
\begin{aligned}
L^p_{\mathrm{div}_a,0}(O) &:= \left\{ \psi \in L^p_{\mathrm{div}_a}(O) \ : \ \operatorname{div}_a \psi = 0 \text{ a. e. in } O \right\}, \\
L^p_{\mathrm{curl},0}(O) &:= \left\{ \psi \in L^p_{\mathrm{curl}}(O) \ : \ \operatorname{curl} \psi = 0 \text{ a. e. in } O \right\}, \\
L^p_{\mathrm{div}_a + \gamma_{a\nu},0}(O) &:= \left\{ \psi \in L^p_{\mathrm{div}_a,0}(O) \ : \ \gamma_{a\nu}(\psi) = 0 \text{ in } [W^{1/p,p'}(\partial O)]^* \right\}, \\
L^p_{\mathrm{curl} + \gamma_\tau,0}(O) &:= \left\{ \psi \in L^p_{\mathrm{curl},0}(O) \ : \ \gamma_\tau(\psi) = 0 \text{ in } [W^{1/p,p'}(\partial O; \mathbb{R}^3)]^* \right\}.
\end{aligned}
\tag{17}
$$

In connection with higher integrability results for solutions to the Maxwell equations, it is now convenient to allow for the rotation or divergence of a vector field being in the dual of a Sobolev space. This idea is already used in the context of the celebrated Div-Curl Lemma (see [GM08] for an overview over recent generalizations of the original ideas by L. Tartar).

Definition 2.2. *Let O be a Lipschitz domain, $X = W^{1,p}(O; \mathbb{R}^3)$ and $Y := W^{1,p}(O)$, $1 \leq p < +\infty$. For a vector field $\psi \in [L^1(O)]^3$*

(1) We write $\operatorname{curl} \psi \in X^*$ if there is $c = c_\psi$ such that $\left| \int_O \psi \cdot \operatorname{curl} \Phi \right| \leq c \, \|\Phi\|_X$ for all $\Phi \in C^\infty(\overline{O}; \mathbb{R}^3)$ such that $\operatorname{curl} \Phi \cdot \nu = 0$ on ∂O.

(2) We write $\operatorname{curl}\psi \in X^*$ *and* $\gamma_\tau(\psi) \in X^*$ *(short:* $\operatorname{curl}+\gamma_\tau \in X^*$*) if there is* $c = c_\psi$ *such that* $\left|\int_O \psi \cdot \operatorname{curl}\Phi\right| \leq c\,\|\Phi\|_X$ *for all* $\Phi \in C^\infty(\overline{O}; \mathbb{R}^3)$*.*

(3) We write $\operatorname{div}_a\psi \in Y^*$ *if there is* $c = c_\psi$ *such that* $|\int_O a\psi \cdot \nabla\phi| \leq c\,\|\phi\|_Y$ *for all* $\phi \in C_c^\infty(O)$*.*

(4) We write $\operatorname{div}_a\psi \in Y^*$ *and* $\gamma_{a\nu}(\psi) \in Y^*$ *(short:* $\operatorname{div}_a+\gamma_{a\nu} \in Y^*$*) if there is* $c = c_\psi$ *such that* $|\int_O a\psi \cdot \nabla\phi| \leq c\,\|\phi\|_Y$ *for all* $\phi \in C^\infty(\overline{O})$*.*

On the basis of Definition 2.2, we introduce

$$
\begin{aligned}
L^p(O \,|\, \operatorname{curl} \in X^*) &:= \{\psi \in L^p(O; \mathbb{R}^3) \,:\, \operatorname{curl}\psi \in X^*\}, \\
L^p(O \,|\, \operatorname{curl}+\gamma_\tau \in X^*) &:= \{\psi \in L^p(O; \mathbb{R}^3) \,:\, (\operatorname{curl}+\gamma_\tau)\psi \in X^*\}, \\
L^p(O \,|\, \operatorname{div}_a \in Y^*) &:= \{\psi \in L^p(O; \mathbb{R}^3) \,:\, \operatorname{div}_a\psi \in Y^*\}, \\
L^p(O \,|\, \operatorname{div}_a+\gamma_{a\nu} \in Y^*) &:= \{\psi \in L^p(O; \mathbb{R}^3) \,:\, (\operatorname{div}_a+\gamma_{a\nu})\psi \in Y^*\}.
\end{aligned}
\tag{18}
$$

These spaces are Banach spaces with respect to the natural graph-topology. The definitions (2), (3) and (4) are natural and straightforward. The choice of the test function Φ in the definition of (1) needs however explanation. Indeed, it seems more natural in analogy to the Definition 2.1 to choose test functions $\Phi \in C_c^\infty(O; \mathbb{R}^3)$ (equivalently: $\Phi \in C^\infty(\overline{O}; \mathbb{R}^3)$ such that $\gamma_\tau(\Phi) = 0$) in order to define the weak curl operator. But in the context of Lipschitz domains, this would provide insufficient information for our purpose: This is mainly related to the following simple statement about the existence of a vector potential and the subsequent Remark 2.4.

Lemma 2.3. *Let* O *be a bounded Lipschitz domain such that* ∂O *is Lipschitz diffeomorphic to the unit sphere, and* $1 < p < +\infty$*. For every* $\psi \in L_{\operatorname{div},0}^p(O)$ *there is a vector potential* $\Phi \in W^{1,p}(O; \mathbb{R}^3)$ *such that* $\operatorname{curl}\Phi = \psi$ *almost everywhere in* O*, and a constant* $c = c(p, O)$ *such that* $\|\Phi\|_{W^{1,p}(O;\mathbb{R}^3)} \leq c\,\|\psi\|_{L^p(O;\mathbb{R}^3)}$*.*

Proof. Let $B_R(x_0)$ be a ball that compactly contains O. Let $\psi \in L_{\operatorname{div},0}^p(O)$. Due to Lemma A.2, we find an extension $\mathscr{E}\psi \in L_{\operatorname{div}+\gamma_\nu,0}^p(B_R)$. The Theorem 3.3 in [Gri90] directly yields the claim. $\square$

Remark 2.4. *If the domain* O *is of class* $\mathcal{C}^1$*, there is instead an 'improved' statement. For* $\psi \in L_{\operatorname{div}+\gamma_\nu,0}^p(O)$*, the vector potential* $\Phi \in W^{1,p}(O; \mathbb{R}^3)$ *can be chosen such that* $\Phi = 0$ *on* ∂O*. This follows from the same argument as Lemma 2.3, since we can apply the Theorem 3.3 of [Gri90] directly in the domain* O*. For a Lipschitz domain we can show that there is a vector potential in* $L_{\operatorname{curl}}^p(O \,|\, \gamma_\tau = 0)$*, but* not *preserve its Sobolev quality. This is the reason that dictates the choice of* Φ *in the Definition 2.2,* (1)*.*

The next Lemma shows that in the Definition 2.2, (1) of the weak curl operator, we can also choose the test vector fields in a Sobolev space, which we shall need later.

Lemma 2.5. *Assume that* $\psi \in L^r(O; \mathbb{R}^3)$ *for a* $1 < r < +\infty$ *and that* $\operatorname{curl}\psi \in [W^{1,p}(O; \mathbb{R}^3)]^*$ *for a* $r' \leq p < +\infty$ *(sense of Definition 2.2). Then, the inequality*

$$
\left|\int_O \psi \cdot \operatorname{curl}\Phi\right| \leq \|\operatorname{curl}\psi\|_{[W^{1,p}(O;\mathbb{R}^3)]^*} \|\operatorname{curl}\Phi\|_{L^p(O;\mathbb{R}^3)},
$$

is valid for all $\Phi \in W^{1,p}(O; \mathbb{R}^3)$ *such that* $\gamma_\nu(\operatorname{curl}\Phi) = 0$*.*

Proof. Assume that $\Phi \in W^{1,p}(O; \mathbb{R}^3)$ satisfies $\gamma_\nu(\operatorname{curl}\Phi) = 0$. Due to the Lemma A.6, there is a sequence $\{u_n\} \subset C_c^\infty(O; \mathbb{R}^3)$, $\operatorname{div} u_n = 0$ such that $u_n \to \operatorname{curl}\Phi$ in the norm of $L_{\mathrm{div}}^p(O)$. Owing to the Helmholtz decomposition, there is an analytic vector potential Φ_n such that $\operatorname{curl}\Phi_n = u_n$ in $\mathbb{R}^3$, and due to the Calderon-Zygmund inequality (see [Gri90], Th. 3.3), also

$$\|\Phi_n\|_{W^{1,p}(O;\mathbb{R}^3)} \le c \, \|u_n\|_{L^p(O;\mathbb{R}^3)} \, .$$

Using the definition of $\operatorname{curl}\psi \in [W^{1,p}(O;\mathbb{R}^3)]^*$, we obtain that

$$\left| \int_O \psi \cdot \operatorname{curl}\Phi_n \right| \le \|\operatorname{curl}\psi\|_{[W^{1,p}(O;\mathbb{R}^3)]^*} \|\Phi_n\|_{W^{1,p}(O;\mathbb{R}^3)}$$

$$\le c \, \|\operatorname{curl}\psi\|_{[W^{1,p}(O;\mathbb{R}^3)]^*} \|u_n\|_{L^p(O;\mathbb{R}^3)}$$

$$\to c \, \|\operatorname{curl}\psi\|_{[W^{1,p}(O;\mathbb{R}^3)]^*} \|\operatorname{curl}\Phi\|_{L^p(O;\mathbb{R}^3)} \, .$$

Extracting a weakly convergent subsequence and using $p \ge r'$ it follows $\int_O \psi \cdot \operatorname{curl}\Phi_n \to \int_O \psi \cdot \operatorname{curl}\Phi$. $\qquad \square$

Finally, we note a localization property of the spaces in (18).

Lemma 2.6. *Let $O \subset \mathbb{R}^3$, and $U \subseteq O$ be Lipschitz domains, and $1 < p < +\infty$. Let $\psi \in L^{q'}(O; \mathbb{R}^3)$, $q = p_3^*$ (cf. (12)) satisfy one of the conditions characterizing the spaces (18): $\operatorname{curl}, \operatorname{curl}+\gamma_\tau \in [W^{1,p}(O, \mathbb{R}^3)]^*$ or $\operatorname{div}_a, \operatorname{div}_a +\gamma_{a\nu} \in [W^{1,p}(O)]^*$. If $\eta \in C_c^\infty(U \cup (\overline{U} \cap \partial O))$, then the vector field $\eta\,\psi$ satisfies the same condition with respect to U.*

Proof. The proof is similar in each of the four cases. Exemplarily, we show that

$$\psi \in L^{q'}(O \mid \operatorname{curl} \in [W^{1,p}(O, \mathbb{R}^3)]^*) \Rightarrow \eta\,\psi \in L^{q'}(U \mid \operatorname{curl} \in [W^{1,p}(U; \mathbb{R}^3)]^*) \, .$$

For $\Phi \in C^\infty(\overline{U}; \mathbb{R}^3)$ such that $\gamma_{\nu_U}(\operatorname{curl}\Phi) = 0$, the trivial extension of the field $\eta\,\Phi$ clearly belongs to $C^\infty(\overline{O}; \mathbb{R}^3)$. Moreover, if $\partial U \cap \partial O$ has positive surface measure, then $\operatorname{curl}\Phi \cdot \nu = 0$ thereon. Thus, due to the choice of η, we obtain that $\gamma_\nu(\operatorname{curl}(\eta\,\Phi)) = 0$ with respect to ∂O. Since it is assumed that $\psi \in L^{q'}(O \mid \operatorname{curl} \in [W^{1,p}(O, \mathbb{R}^3)]^*)$

$$\left| \int_O \psi \cdot \operatorname{curl}(\eta\,\Phi) \right| \le \|\operatorname{curl}\psi\|_{[W^{1,p}(O, \mathbb{R}^3)]^*} \|\eta\,\Phi\|_{W^{1,p}(O, \mathbb{R}^3)}$$

$$\le c_\eta \, \|\operatorname{curl}\psi\|_{[W^{1,p}(O, \mathbb{R}^3)]^*} \|\Phi\|_{W^{1,p}(U, \mathbb{R}^3)} \, .$$

It follows that

$$\left| \int_U \eta\,\psi \cdot \operatorname{curl}\Phi \right| \le \left| \int_O \psi \cdot (\Phi \times \nabla\eta) \right| + c_\eta \, \|\operatorname{curl}\psi\|_{[W^{1,p}(O, \mathbb{R}^3)]^*} \|\Phi\|_{W^{1,p}(U, \mathbb{R}^3)}$$

$$\le c_\eta \, \|\psi\|_{L^{[p_3^*]'}(O;\mathbb{R}^3)} \|\Phi\|_{L^{p_3^*}(U;\mathbb{R}^3)} + c_\eta \, \|\operatorname{curl}\psi\|_{[W^{1,p}(O, \mathbb{R}^3)]^*} \|\Phi\|_{W^{1,p}(U, \mathbb{R}^3)}$$

$$\le c_\eta \, \|\psi\|_{L^{[p_3^*]'}(O \mid \operatorname{curl} \in [W^{1,p}(O, \mathbb{R}^3)]^*)} \|\Phi\|_{W^{1,p}(U, \mathbb{R}^3)} \, .$$

$\qquad \square$

2.2 Embedment into a higher Lebesque space over a Lipschitz domain.

For a symmetric matrix of measurable coefficients a satisfying the ellipticity condition (13), the embedding results that we are going to prove first rely on the following *abstract* assumption:

Assumption 2.7. *There is a number $q_1 > 2$ so that for every $p \in [q_1', q_1]$, every $F \in [W^{1,p'}(O)]^*$ such that $F(1) = 0$ and $G \in [W_0^{1,p'}(O)]^*$ the weak Neumann/Dirichlet problem with homogeneous transmission conditions*

$$\int_O a\nabla u \cdot \nabla \phi = F(\phi), \quad \text{for all } \phi \in W^{1,p'}(O), \tag{19}$$

$$\int_O a\nabla u \cdot \nabla \phi = G(\phi), \quad \text{for all } \phi \in W_0^{1,p'}(O) \tag{20}$$

possesses an up to constants unique solution $u \in W^{1,p}(O)$/a unique solution $u \in W_0^{1,p}(O)$.

Remark 2.8. *If $a = \text{Id}$, the Theorem 1.6 in [Zan00] (Neumann problem) and [JK95] (Dirichlet problem) show that the Assumption 2.7 is satisfied for a $q_1 > 3$ that depends only on the domain O (see [Dau92] for similar results on curvilinear polyhedra). In order to single this case out, we call $q_0 > 3$ the optimal exponent in the case $a = \text{Id}$.*

We begin with elementary decomposition results of Helmoltz type in Lipschitz domains.

Lemma 2.9. *Let $O \subset \mathbb{R}^3$ be a simply connected bounded Lipschitz domain. Let $\psi \in L^p(O; \mathbb{R}^3)$, with $p \in [q_1', q_1]$, where $q_1 > 2$ is the constant of condition 2.7. Then, there are*

(1) A vector field $\Phi \in W^{1,p}(O; \mathbb{R}^3)$ such that $\gamma_\nu(\text{curl}\,\Phi) = 0$, and a function $u \in W^{1,p}(O)$ such that $\psi = \text{curl}\,\Phi + a\nabla u$ almost everywhere in O. Moreover, there is a constant $c = c(p, O, a)$ independent on ψ such that

$$\|\Phi\|_{W^{1,p}(O;\mathbb{R}^3)} + \|u\|_{W^{1,p}(O)} \leq c\,\|\psi\|_{L^p(O;\mathbb{R}^3)}.$$

(2) A vector field $\Phi \in W^{1,p}(O; \mathbb{R}^3)$, and a function $u \in W_0^{1,p}(O)$ such that $\psi = \text{curl}\,\Phi + a\nabla u$ almost everywhere in O, as well as a constant $c = c(p, O, a)$ independent on ψ such that

$$\|\Phi\|_{W^{1,p}(O;\mathbb{R}^3)} + \|u\|_{W^{1,p}(O)} \leq c\,\|\psi\|_{L^p(O;\mathbb{R}^3)}.$$

Proof. Due to the condition 2.7, the weak Neumann problem

$$\int_O (a\nabla u - \psi) \cdot \nabla \phi = 0, \quad \text{for all } \phi \in W^{1,p'}(O),$$

possesses an up to constants unique solution $u \in W^{1,p}(O)$ with continuity estimate. We easily verify that $f \in L_{\text{div}+\gamma_\nu,0}^p(O)$. Applying Lemma 2.3, there is a vector potential $\Phi \in W^{1,p}(O; \mathbb{R}^3)$ such that

$$\|\Phi\|_{W^{1,p}(O;\mathbb{R}^3)} \leq \bar{c}\,\|\psi - a\nabla u\|_{L^p(O;\mathbb{R}^3)} \leq c\,\|\psi\|_{L^p(O;\mathbb{R}^3)}. \tag{21}$$

In addition $\gamma_\nu(\operatorname{curl}\Phi) = \gamma_\nu(\psi - a\nabla u) = 0$. This establishes the validity of the first decomposition.

For the second decomposition, the Assumption 2.7 implies that the weak Dirichlet problem

$$\int_O (a\nabla u - \psi) \cdot \nabla\phi = 0, \quad \text{for all } \phi \in W_0^{1,p'}(O),$$

possesses a unique solution $u \in W_0^{1,p}(O)$ with continuity estimate. The vector field $f := a\nabla u - \psi$ belongs to $L_{\mathrm{div},0}^p(O)$. Thus, there is a vector potential $\Phi \in W^{1,p}(O;\,\mathbb{R}^3)$ such that $\operatorname{curl}\Phi = f$ almost everywhere in O, and such that $\|\Phi\|_{W^{1,p}(O;\mathbb{R}^3)} \leq \bar{c}\,\|f\|_{[L^p(O;\mathbb{R}^3)}$ with a constant $\bar{c}$ that depends only on O and on p. $\qquad\square$

In order to make our main statements independent of the condition of a simply connected domain, note the following remark.

Remark 2.10. *For a bounded Lipschitz domain O, there are $m \in \mathbb{N}$ and a family of simply connected Lipschitz domains $\{U_i\}_{i=1,\ldots,m}$ such that $U_i \subseteq O$ and $O = \bigcup_{i=1}^m U_i$. There are functions $\eta_1, \ldots, \eta_m$ such that $\sum_{i=1}^m \eta_i \equiv 1$ in $\overline{O}$ and $\eta_i \in C_c^\infty(U_i \cup [\partial O \partial U_i])$.*

Proof. The definition of a *Lipschitz boundary* implies that there is a finite covering of ∂O with balls $\{B_{r_i}(x_i)\}_{i=1,\ldots,k}$, where $k \in \mathbb{N}$, and $r_i > 0$ and $x_i \in \partial O$ for $i = 1, \ldots, k$. Since the portion of the boundary $\partial O \cap B_{r_i}(x_i)$ is the graph of a Lipschitz continuous function in some Euclidean coordinates, the set $U_i := B_{r_i}(x_i) \cap O$ is diffeomorphic to a half-ball and therefore simply connected. The set $O_0 := O \setminus \overline{\bigcup_{i=1}^k B_{r_i}(x_i)}$ is compactly included in O, and therefore, there is a finite covering of O_0 with balls $\{B_{r_i}(x_i)\}_{i=k+1,\ldots,k+l}$, where $l \in \mathbb{N}$, and $r_i > 0$ and $x_i \in O$ and $B_{r_i}(x_i) \subset O$ compactly for $i = k+1, \ldots, k+l$. We set $m = k+l$. We choose a smooth partition of unity $\eta_1, \ldots, \eta_m$ subordinated to the covering $O \subset \bigcup_{i=1}^m B_{r_i}(x_i)$. $\qquad\square$

Our embedding result for vector fields with rotation *and* a divergence constraint is next stated.

Proposition 2.11. *Let $O \subset \mathbb{R}^3$ be a bounded Lipschitz domain, and $q_1 > 2$ the same constant as in Lemma 2.9. Then, for all $p \in \,]2,\ q_1]$*

(1) The space $L^{q_1'}(O \,|\, \operatorname{curl} \in [W^{1,p'}(O;\mathbb{R}^3)]^*) \cap L^{q_1'}(O \,|\, \operatorname{div}_a + \gamma_{a\nu} \in [W^{1,p'}(O)]^*)$ embeds continuously in $[L^p(O)]^3$;

(2) The space $L^{q_1'}(O \,|\, \operatorname{curl} + \gamma_\tau \in [W^{1,p'}(O;\mathbb{R}^3)]^*) \cap L^{q_1'}(O \,|\, \operatorname{div}_a \in [W^{1,p'}(O)]^*)$ embeds continuously in $[L^p(O)]^3$;

Proof. We first assume that O is simply connected.

(1): Let $\psi \in L^{q_1'}(O \,|\, \operatorname{curl} \in [W^{1,p'}(O;\mathbb{R}^3)]^*) \cap L^{q_1'}(O \,|\, \operatorname{div}_a + \gamma_{a\nu} \in [W^{1,p'}(O)]^*)$. For $V \in L^{q_1}(O;\,\mathbb{R}^3)$, we find according to Lemma 2.9 a Helmoltz decomposition $V = \operatorname{curl}\Phi + a\nabla u$ with $\Phi \in W^{1,q_1}(O;\,\mathbb{R}^3)$ such that $\gamma_\nu(\operatorname{curl}\Phi) = 0$, and a function $u \in W^{1,q_1}(O)$.

Therefore, using only the definition of the generalized operators curl (see also Lemma 2.5) and div_a and the continuity of the decomposition of V for all $p \in [q_1', q_1]$ it follows that

$$\left| \int_O \psi \cdot V \right| \leq \left| \int_O \psi \cdot \operatorname{curl} \Phi \right| + \left| \int_O a^T \psi \cdot \nabla u \right|$$

$$\leq \| \operatorname{curl} \psi \|_{[W^{1,p'}(O;\mathbb{R}^3)]^*} \| \Phi \|_{W^{1,p'}(O;\mathbb{R}^3)} + \| (\operatorname{div}_a + \gamma_{a\nu}) \psi \|_{[W^{1,p'}(O)]^*} \| u \|_{W^{1,p'}(O)}$$

$$\leq c \left(\| \operatorname{curl} \psi \|_{[W^{1,p'}(O;\mathbb{R}^3)]^*} + \| (\operatorname{div}_a + \gamma_{a\nu}) \psi \|_{[W^{1,p'}(O)]^*} \right) \| V \|_{L^{p'}(O;\mathbb{R}^3)} \, .$$

Elementary arguments show that $\psi \in L^p(O; \mathbb{R}^3)$ satisfies the embedding inequality.

(2): Let $\psi \in L^{q_1'}(O \mid \operatorname{curl} + \gamma_\tau \in [W^{1,p'}(O;\mathbb{R}^3)]^*) \cap L^{q_1'}(O \mid \operatorname{div}_a \in [W^{1,p'}(O)]^*)$. For $V \in L^{q_1}(O; \mathbb{R}^3)$, we find according to Lemma 2.9 a Helmoltz decomposition $V = \operatorname{curl} \Phi + a \nabla u$ with $\Phi \in W^{1,q_1}(O; \mathbb{R}^3)$, and a function $u \in W_0^{1,q_1}(O)$. Therefore

$$\left| \int_O \psi \cdot V \right| \leq \left| \int_O \psi \cdot \operatorname{curl} \Phi \right| + \left| \int_O a^T \psi \cdot \nabla u \right|$$

$$\leq \| \operatorname{curl} \psi + \gamma_\tau \|_{[W^{1,p'}(O;\mathbb{R}^3)]^*} \| \Phi \|_{W^{1,p'}(O;\mathbb{R}^3)} + \| \operatorname{div} a\psi \|_{[W_0^{1,p'}(O)]^*} \| u \|_{W_0^{1,p'}(O)}$$

$$\leq c \left(\| \operatorname{curl} \psi + \gamma_\tau \|_{[W^{1,p'}(O;\mathbb{R}^3)]^*} + \| \operatorname{div} a\psi \|_{[W_0^{1,p'}(O)]^*} \right) \| V \|_{L^{p'}(O;\mathbb{R}^3)} \, .$$

Assume now that the domain O is not simply connected. Then, recalling the Remark 2.10, we denote $\psi^i := \eta_i \psi$ for $i = 1, \ldots, m$, and the Lemma 2.6 in the respective case (1), (2) implies that

(1) $\psi^i \in L^{q_1'}(U_i \mid \operatorname{curl} \in [W^{1,p'}(U_i;\mathbb{R}^3)]^*) \cap L^{q_1'}(U_i \mid \operatorname{div}_a + \gamma_{a\nu} \in [W^{1,p'}(U_i)]^*)$;

(2) $\psi^i \in L^{q_1'}(U_i \mid \operatorname{curl} + \gamma_\tau \in [W^{1,p'}(U_i;\mathbb{R}^3)]^*) \cap L^{q_1'}(U_i \mid \operatorname{div}_a \in [W^{1,p'}(U_i)]^*)$;

As U_i is a simply connected domain for all $i = 1, \ldots, m$, the first step of the proof implies the embedding for U_i, and since $\psi = \sum_{i=1}^m \psi^i$, the claim follows. $\qquad \square$

We now note a result concerning the relationship between the generalized operators in Definition 2.1 and the abstract operators in Definition 2.2.

Lemma 2.12. *Let O be a bounded, simply connected Lipschitz domain, $q_0' \leq p \leq q_0$ for the $q_0 > 3$ of Remark 2.8, and $q := \min\{\frac{3p}{4p-3}, 1\}$. Then, every $\psi \in L_{\operatorname{curl}}^p(O)$ belongs to $L^p(O \mid \operatorname{curl} \in [W^{1,q}(O; \mathbb{R}^3)]^*)$. There is $\widetilde{\psi} \in W^{1/p',p}(\partial O; \mathbb{R}^3)$ such that $\| \widetilde{\psi} \|_{W^{1/p',p}(\partial O;\mathbb{R}^3)} \leq c \, \| \psi \|_{L_{\operatorname{curl}}^p(O)}$, and such that the functional $\operatorname{curl} \psi$ possesses the representation*

$$(\operatorname{curl} \psi)(\Phi) := \int_O \operatorname{curl} \psi \cdot \Phi + \int_{\partial O} (\nu \times \tilde{\psi}) \cdot \Phi, \quad \Phi \in W^{1,q}(O; \mathbb{R}^3), \, \gamma_\nu(\operatorname{curl} \Phi) = 0 \, .$$

Proof. Let $\psi \in L_{\operatorname{curl}}^p(O)$. The vector field $f := \operatorname{curl} \psi$ belongs by assumption to $L_{\operatorname{div},0}^p(O)$. Thus, there is a vector potential $\widetilde{\psi} \in W^{1,p}(O; \mathbb{R}^3)$ such that $\operatorname{curl} \widetilde{\psi} = f = \operatorname{curl} \psi$ almost everywhere in O, and such that $\| \widetilde{\psi} \|_{W^{1,p}(O;\mathbb{R}^3)} \leq \bar{c} \, \| f \|_{L^p(O;\mathbb{R}^3)}$ (Lemma 2.3). Since we assume that O is simply connected, we find a function $u \in W^{1,p}(O)$ such that $\psi - \widetilde{\psi} = \nabla u$.

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For arbitrary $\Phi \in W^{1,q}(O; \mathbb{R}^3)$ such that $\gamma_\nu(\operatorname{curl}\Phi) = 0$, the definition of γ_ν yields $\int_O \nabla u \cdot \operatorname{curl}\Phi = 0$. Therefore

$$\int_O \psi \cdot \operatorname{curl}\Phi = \int_O \widetilde{\psi} \cdot \operatorname{curl}\Phi = \int_O \operatorname{curl}\widetilde{\psi} \cdot \Phi + \int_{\partial O} (\nu \times \widetilde{\psi}) \cdot \Phi$$
$$= \int_O \operatorname{curl}\psi \cdot \Phi + \int_{\partial O} (\nu \times \widetilde{\psi}) \cdot \Phi \,.$$

Using the Sobolev embedding Theorems, it follows that

$$\left| \int_O \psi \cdot \operatorname{curl}\Phi \right| \leq \|\operatorname{curl}\psi\|_{L^p(O;\mathbb{R}^3)} \|\Phi\|_{L^{p'}(O;\mathbb{R}^3)} + \|\nu \times \widetilde{\psi}\|_{L^{p_2^*}(\partial O;\mathbb{R}^3)} \|\Phi\|_{L^{[p_2^*]'}(\partial O;\mathbb{R}^3)}$$
$$\leq c \|\psi\|_{L^p_{\operatorname{curl}}(O)} \|\Phi\|_{W^{1,q}(O;\mathbb{R}^3)} \,.$$

which proves the claim. $\qquad\square$

Remark 2.13. *It is only a matter of definition to show that a vector field $\psi \in L^p_{\operatorname{div}_a}(O)$ belongs to $L^p(O \mid \operatorname{div}_a \in [W^{1,q}(O)]^*)$ for $q := \min\{1, \frac{3p}{4p-3}\}$.*

In the paper [Dru07], we considered for $1 < p, r < \infty$ and $a = \operatorname{Id}$ the spaces

$$\mathcal{W}^{p,r}_{a,\nu}(O) := \left\{ \psi \in L^p_{\operatorname{curl}}(O) \cap L^p_{\operatorname{div}_a}(O) : \gamma_\nu(a\psi) \in L^r(\partial O) \right\}. \tag{22}$$

For the sake of generality, we also introduce

$$\mathcal{W}^{p,r}_{a,\tau}(O) := \left\{ \psi \in L^p_{\operatorname{curl}}(O) \cap L^p_{\operatorname{div}_a}(O) : \gamma_\tau(\psi) \in L^r(\partial O; \mathbb{R}^3) \right\}. \tag{23}$$

For these spaces, we now give a more general proof of the embedding result than in [Dru07].

Corollary 2.14. *Let $O \subset \mathbb{R}^3$ be a bounded Lipschitz domain such that the Assumption 2.7 is valid for a $q_1 \leq q_0$ (with q_0 sastisfying Remark 2.8). For all $p \in [q_1', q_1]$, the spaces $\mathcal{W}^{p,r}_{a,\nu}(O)$, $\mathcal{W}^{p,r}_{a,\tau}(O)$ embed continuously in $[L^s(O)]^3$ for $s := \min\left\{ \frac{3r}{2}, p_3^*, q_1 \right\}$.*

Proof. Without loss of generality, we can assume that the domain O itself is simply connected. Otherwise we use the argument of Proposition 2.11, covering O with simply connected Lipschitz domains $U_1, \ldots, U_m$, and localizing $\psi = \sum_{i=1}^m \eta_i \psi$, with $\eta_i \psi \in \mathcal{W}^{p,r}_{a,\nu}(U_i)$ or in $\mathcal{W}^{p,r}_{a,\tau}(U_i)$.

For ψ in $\mathcal{W}^{p,r}_{a,\nu}(O)$, Lemma 2.12 yields $\psi \in L^p(O \mid \operatorname{curl} \in [W^{1,q}(O)]^*)$, $q = \min\{3p/(4p - 3), 1\}$. Owing to the estimate

$$|\langle \gamma_\nu(a\psi), \phi \rangle| \leq \|\gamma_\nu(a\psi)\|_{L^r(\partial O)} \|\phi\|_{L^{r'}(\partial O)} \leq c \|\gamma_\nu(a\psi)\|_{L^r(\partial O)} \|\phi\|_{W^{1,3r/(3r-2)}(O)} \,,$$

Lemma 2.12 also yields $\psi \in L^p(O \mid \operatorname{div}_a + \gamma_{a\nu} \in [W^{1,\max\{q, 3r/(3r-2)\}}(O)]^*)$. The Proposition 2.11 yields the claim. $\qquad\square$

2.3 Compact embedding

In view of the embedding inequalities of the previous section, it is also possible to throw a bridge to the DIV-CURL Lemma: [RRT87, GM08]. We assume that O is a Lipschitz domain, and define $q_1 > 2$ the exponent of Assumption 2.7. Then, the following basic statement is valid:

Lemma 2.15. *Let $p \in [q_1', \, q_1]$. Consider sequences $\{w_n\}_{n \in \mathbb{N}} \subset L^p(O; \mathbb{R}^3)$ and $\{v_n\}_{n \in \mathbb{N}} \subset L^{p'}(O; \mathbb{R}^3)$ such that $w_n \to w$ in $L^p(O; \mathbb{R}^3)$ and $v_n \to v$ in $L^{p'}(O; \mathbb{R}^3)$ weakly. Assume that one of the following is valid:*

(1) The sequence $\{\operatorname{curl} w_n\}$ is compact in $[W^{1,p'}(O; \, \mathbb{R}^3)]^*$ and the sequence $\{(\operatorname{div}_a + \gamma_{a\nu})(v_n)\}$ is compact in $[W^{1,p}(O; \, \mathbb{R}^3)]^*$;

(2) The sequence $\{(\operatorname{curl} + \gamma_\tau)(w_n)\}$ is compact in $[W^{1,p'}(O; \, \mathbb{R}^3)]^*$ and the sequence $\{\operatorname{div}_a v_n\}$ is compact in $[W^{1,p}(O; \, \mathbb{R}^3)]^*$;

Then $w_n \cdot v_n \to w \cdot v$ weakly as measures in $\overline{O}$.

Proof. We prove only (1) since the proof of the other statement is completely similar. Owing to the covering/localization argument of the proof of Proposition 2.11, it is always possible to cover O with simply connected Lipschitz domains $U_1, \ldots, U_m$, such that the sequences $w_n^i := \eta_i \, w_n$ and $v_n^i = \eta_i \, v_n$ possess the same properties (1) or (2) with respect ot U_i (Lemma 2.6). Thus, it is no loss of generality to assume O simply connected.

Then, due to Lemma 2.9, there are decompositions $v_n = \operatorname{curl} \Phi_n + \nabla u_n$ and $w_n = \operatorname{curl} \Psi_n + \nabla h_n$ such that $\gamma_\nu(\operatorname{curl} \Phi_n) = 0 = \gamma_\nu(\operatorname{curl} \Psi_n)$ and moreover

$$\|\Phi_n\|_{W^{1,p'}(O; \mathbb{R}^3)} + \|\nabla u_n\|_{W^{1,p'}(O)} \leq c, \quad \|\Psi_n\|_{W^{1,p}(O; \mathbb{R}^3)} + \|\nabla h_n\|_{W^{1,p}(O)} \leq c \,.$$

Owing to standard theorems of functional analysis, we can extract subsequences and find $\Phi \in W^{1,p'}(O; \mathbb{R}^3)$, $\Psi \in W^{1,p}(O; \mathbb{R}^3)$, $u \in W^{1,p'}(O)$ and $h \in W^{1,p}(O)$ such that

$$
\begin{aligned}
&\nabla \Phi_n \to \nabla \Phi, \ \nabla u_n \to \nabla u \text{ weakly in } L^{p'}(O) \\
&\nabla \Psi_n \to \nabla \Psi, \ \nabla h_n \to \nabla h \text{ weakly in } L^{p}(O) \\
&\Phi_n \to \Phi, \ u_n \to u \text{ in } L^{p'}(O) \\
&\Psi_n \to \Psi, \ h_n \to h \text{ in } L^{p}(O) \,.
\end{aligned}
\tag{24}
$$

For $\zeta \in C^\infty(\overline{O})$ arbitrary

$$\int_O \zeta \, w_n \cdot v_n = \int_O \zeta \, w_n \cdot \operatorname{curl} \Phi_n + \int_O \zeta \, w_n \cdot \nabla u_n \tag{25}$$

$$= \int_O w_n \cdot \operatorname{curl}(\zeta \, \Phi_n) - \int_O w_n \cdot (\Phi_n \times \nabla \zeta) + \int_O \zeta \, (\operatorname{curl} \Psi_n + \nabla h_n) \cdot \nabla u_n \,.$$

Using integration by parts,

$$\int_O \zeta \, \operatorname{curl} \Psi_n \cdot \nabla u_n = -\int_O u_n \, \operatorname{curl} \Psi_n \cdot \nabla \zeta \,. \tag{26}$$

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Moreover

$$\int_O \zeta\, \nabla h_n \cdot \nabla u_n = \int_O \zeta\, v_n \cdot \nabla h_n - \int_O \zeta\, \operatorname{curl}\Phi_n \cdot \nabla h_n$$

$$= \int_O v_n \cdot \nabla(\zeta\, h_n) - \int_O v_n \cdot \nabla\zeta\, h_n + \int_O \operatorname{curl}\Phi_n \cdot \nabla\zeta\, h_n\,. \qquad (27)$$

Thus

$$\int_O \zeta\, w_n \cdot v_n = \int_O w_n \cdot \operatorname{curl}(\zeta\, \Phi_n) + \int_O v_n \cdot \nabla(\zeta\, h_n)$$

$$+ \int_O \nabla\zeta \cdot \{\Phi_n \times w_n - u_n\, \operatorname{curl}\Psi_n - h_n\, \nabla u_n\}\,.$$

Owing to (24), the products $w_n \times \Phi_n$, $u_n\, \operatorname{curl}\Psi_n$ and $h_n\, \nabla u_n$ all weakly convege in $L^1(O)$ to the natural limit, that is

$$\int_O \nabla\zeta \cdot \{\Phi_n \times w_n - u_n\, \operatorname{curl}\Psi_n - h_n\, \nabla u_n\} \to \int_O \nabla\zeta \cdot \{\Phi \times w - u\, \operatorname{curl}\Psi - h\, \nabla u\}\,.$$

By assumption, there are a functional $W \in [W^{1,p'}(O; \mathbb{R}^3)]^*$ and a subsequence such that $\operatorname{curl} w_n \to W$ strongly in $[W^{1,p'}(O; \mathbb{R}^3)]^*$. As $w_n \to w$ weakly in L^p, we easily obtain the representation $W(\Phi) = \int_O w \cdot \operatorname{curl}\Phi$ for all $\Phi \in W^{1,p'}(O; \mathbb{R}^3)$ such that $\gamma_\nu(\operatorname{curl}\Phi) = 0$. Similarily, there is $V \in [W^{1,p}(O; \mathbb{R}^3)]^*$ and a subsequence such that $\operatorname{div} v_n + \gamma_\nu v_n \to V$ strongly in $[W^{1,p'}(O)]^*$, and again, we easily obtain the representation $V(h) = \int_O v \cdot \nabla h$. It follows that

$$\int_O w_n \cdot \operatorname{curl}(\zeta\, \Phi_n) \to \int_O w \cdot \operatorname{curl}(\Phi\, \zeta), \quad \int_O v_n \cdot \nabla(\zeta\, h_n) \to \int_O v \cdot \nabla(\zeta\, h)\,.$$

Thus, performing backward the manipulations (25), (26), (27), we see that $\int_O \zeta\, w_n \cdot v_n \to \int_O \zeta\, w \cdot v$ which is the claim. $\qquad \square$

The compact embedding result for the usual spaces $L^p_{\operatorname{curl}}(O) \cap L^p_{\operatorname{div}_a}(O)$ is next given, and generalizes well known properties ([Pic84a] or [Mon03], Th. 4.7).

Corollary 2.16. *Let $O \subset \mathbb{R}^3$ be a bounded Lipschitz domain such that the Assumption 2.7 is valid for a $q_1 \leq q_0$ ($q_0 =$ the exponent of Remark 2.8). Then, for all $p \in [q_1', q_1]$, such that $p > 6/5$, and all $4/3 < r < +\infty$, the spaces $\mathcal{W}^{p,r}_{a,\nu}(O)$, $\mathcal{W}^{p,r}_{a,\tau}(O)$ embed compactly into $L^2(O; \mathbb{R}^3)$.*

Proof. For $p \geq 6/5$ and $r \geq 4/3$, the spaces $\mathcal{W}^{p,r}_{a,\nu}(O)$, $\mathcal{W}^{p,r}_{a,\tau}(O)$ are embedded in $L^2(O; \mathbb{R}^3)$ (Lemma 2.14).

We at first assume that O is simply connected. Let $\{\psi_n\}$ be a uniformly bounded sequence in $\mathcal{W}^{p,r}_{a,\nu}(O)$. For a subsequence, $\psi_n \to \psi$ weakly in $L^p_{\operatorname{curl}}(O)$. Define $A_n := \psi_n - \psi$. Due to Lemma 2.12 (see also the proof of Lemma 2.14), it follows that A_n belongs to $L^p(O \mid \operatorname{curl} \in [W^{1,q}(O; \mathbb{R}^3)]^*)$, $q = \min\{1, 3p/(4p-3)\}$, and the sequence $\{A_n\}$ is uniformly bounded

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in this space. Let $\delta > 0$ be an arbitrarily 'small' number. Owing to the Hahn-Banach theorem, there is for each $n \in \mathbb{N}$ a $\Phi_n \in W^{1,q+\delta}(O; \mathbb{R}^3)$ such that

$$\| \operatorname{curl} A_n \|_{[W^{1,q+\delta}(O;\mathbb{R}^3)]^*} = (\operatorname{curl} A_n)(\Phi_n), \quad \|\Phi_n\|_{W^{1,q+\delta}(O;\mathbb{R}^3)} = 1 \,. \tag{28}$$

Owing to well-known properties of Sobolev spaces, we can extract a subsequence such that $\Phi_n \to \Phi$ in $L^s(O; \mathbb{R}^3)$ for all $s \le q_3^*$ and in $L^t(\partial O; \mathbb{R}^3)$ for all $t \le q_2^*$. Using the representation statement in Lemma 2.12, there is moreover a sequence $\{\widetilde{A}_n\}$ uniformly bounded in $W^{1,p}(O; \mathbb{R}^3)$, and a sequence $\{\nabla u_n\}$ bounded in $L^p(O; \mathbb{R}^3)$ such that $A_n = \widetilde{A}_n + \nabla u_n$, and such that

$$(\operatorname{curl} A_n)(\Phi_n) = \int_O \operatorname{curl} A_n \cdot \Phi_n + \int_{\partial O} (\nu \times \widetilde{A}_n) \cdot \Phi_n \to \int_{\partial O} (\nu \times \widetilde{A}) \cdot \Phi \,.$$

Now, since $A_n \to 0$ weakly in L^p, we realize that $\widetilde{A} = \nabla u$, and therefore, the condition $\gamma_\nu(\operatorname{curl} \Phi) = 0$ garanties that

$$\int_{\partial O} (\nu \times \widetilde{A}) \cdot \Phi = \int_O \nabla u \cdot \operatorname{curl} \Phi = 0 \,.$$

Thus, the identity (28) and the weak convergence of ψ_n in L^p yield

$$\limsup_{n \to \infty} \| \operatorname{curl}(\psi_n - \psi)) \|_{[W^{1,q+\delta}(O;\mathbb{R}^3)]^*} = 0 \,.$$

We thus see that $\{\operatorname{curl} \psi_n\}$ is compact in $[W^{1,q+\delta}(O; \mathbb{R}^3)]^*$.

On the other hand, $\psi_n \in \mathcal{W}_{a,\nu}^{p,r}(O)$ also implies as in the Lemma 2.14 that the sequence $\{\psi_n\}$ is uniformly bounded in $L^p(O \,|\, \operatorname{div}_a + \gamma_{a\nu} \in [W^{1,s}(O)]^*)$, $s = \max\{q, 3r/(3r-2)\}$. Using the representation

$$(\operatorname{div}_a \psi + \gamma_{a\nu}\psi)(\phi) = -\int_O \operatorname{div} a\psi \, \phi + \int_O a\psi \cdot \nu \, \phi, \quad \phi \in W^{1,s}(O) \,,$$

We can show that $\{(\operatorname{div}_a + \gamma_{a\nu})(\psi_n)\}$ is compact in $[W^{1,s+\delta}(O)]^*$.

We now want to apply the Lemma 2.15 with $w_n := \psi_n$, $v_n := a\psi_n$ and $p = 2$. In order to obtain that $\{\operatorname{curl} \psi_n\}$ is compact in $[W^{1,2}(O; \mathbb{R}^3)]^*$ and $\{(\operatorname{div}_a + \gamma_{a\nu})(\psi_n)\}$ is compact in $[W^{1,2}(O)]^*$, it is sufficient to verify that

$$q < 2, \, s < 2 \,,$$

which exactly corresponds to the condition $p > 6/5$, $r > 4/3$. Owing to the Lemma 2.15, we now obtain for $\zeta \equiv 1$ that $\int_O \psi_n \cdot a\psi_n \to \int_O \psi \cdot a\psi$, and the strong convergence in L^2 easily follows.

If the domain O is not simply connected, we use a localization strategy as in the proof of Proposition 2.11 or Lemma 2.15. For each domain of the covering $U_1, \ldots, U_m$, the compact embedding of $\mathcal{W}_{a,\nu}^{p,r}(U_i)$, $\mathcal{W}_{a,\tau}^{p,r}(U_i)$ into $L^2(U_i; \mathbb{R}^3)$ is valid, and the claim follows. $\qquad \square$

2.4 Verification of the abstract condition

In order to obtain the embedding result 2.11, 2.14, and the compactness result 2.16 it remains to verify the abstract condition 2.7. Its validity and the size of the optimal exponent q_1 depend on the matrix a and on the regularity of the domain. In the case of a general L^∞ matrix, it is possible to estimate q_1 from below in function of the two numbers a_0 and a_1 of the condition (13) only.

Proposition 2.17. *Let O be a simply connected Lipschitz domain, and assume that $a \in L^\infty(O; \mathbb{R}^{3\times3}_{sym})$ satisfies (13). Then the following is valid:*

(1) The Assumption 2.7 is always valid for a $q_1 = q_1(O, a_0/a_1) > 2$;

(2) For all $q \in [q'_0, q_0]$, there is a constant $c = c(O, q)$ such that under the restriction $c(1 - a_0/a_1) < 1$, the Assumption 2.7 is valid with $q_1 = q$.

Proof. These are well-known properties. The first one relies on a perturbation argument originally exposed in [Mey63]. The second one is a simple application of the Banach perturbation argument. See [Dru07] for details. $\square$

If the matrix a is *piecewise uniformly continuous*, it is often possible to refine the estimate in function of the structure of the surface where a is discontinuous. This is a topic of general interest, but in our context we shall restrict to a few geometrical structures of relevance for the model setting described in the Introduction. The following statement is proved in [ERS07].

Proposition 2.18. *Let O be a bounded Lipschitz domain. Assume that there is a domain $O_1 \subset O$ compactly, O_1 of class $\mathcal{C}^1$, and set $O_2 := O \setminus \overline{O_1}$. Assume that the matrix a is symmetric, satisfies (13), and belongs to $C(\overline{O_i}; \mathbb{R}^{3\times3}_{sym})$ for $i = 1, 2$. Then, there is $q_1 > 3$ such that the Assumption 2.7 is valid. If O itself is of class $\mathcal{C}^1$, then every $q_1 < +\infty$ satisfies the Assumption 2.7.*

Proposition 2.19. *Let O be a bounded Lipschitz domain of the following structure:*

(1) The boundary of O consists of two open surfaces Γ_1, Γ_2 of class $\mathcal{C}^1$ meeting at a closed line K;

(2) The domain O consists of two subdomains O_1, O_2 separated by an open surface S of class $\mathcal{C}^1$ with K as its boundary.

Assume that the matrix a satisfies (13), and belongs to $C(\overline{O_i}; \mathbb{R}^{3\times3}_{sym})$ for $i = 1, 2$. Then, there is $q_1 > 3$ such that the Assumption 2.7 is valid.

Proof. We observe that there is a diffeomorphism of class $C^{0,1}(O)$ as well as $C^1(\overline{O_i})$, $i = 1, 2$ that maps the domain O onto the reference configuration of the paper [HDKRS08]. We apply the results of this paper. $\square$

Remark 2.20. *These statements provide sufficient conditions for the validity of the fundamental assumption. In particular the Proposition 2.19 allows for a geometrical setting of a certain complexity. However for the setting of the introduction, characterized by the presence of an interior triple line of contact, and assume that the matrix a belongs to $C(\overline{\Omega_i}; \mathbb{R}^3 \times \mathbb{R}^3)$ for $i = 1, \ldots, 4$. Then, only the statements of Proposition 2.17 are in general available, and there is a counterexample in the paper [ERS07] that shows that q_1 in general does not exceed 2 of more than an aritrarily little quantity. Note that the latest mentioned and other similar counterexamples (see [Mer03], [Mey63] and references) are all based on letting the ratio a_0/a_1 tend to zero. Therefore, it is reasonable though not justified from the purely scientific point of view to expect that many a particular situation in which the ratio is moderate or the geometry favourable will allow for a significant higher integrability of the fields. This will require further investigation.*

3 Application to the Maxwell equations

Throughout the section, we consider the model geometrical setting of the Introduction. Since we want to allow for the motion of some of the conductors, we assume that a velocity vector $v : \Omega_{\mathrm{c}} \longrightarrow \mathbb{R}^3$ is given. We denote by $\bar{v}$ the extension by zero of v to Ω_{nc}. We derive a variational formulation of (P) in the fashion of [LS60] starting from the equation (1) and (4): They yield

$$\operatorname{curl} H = \sigma \left(E + v \times B \right) + J_g \quad \text{in } \Omega_{\mathrm{c}} \,. \tag{29}$$

We introduce the space of real-valued vector fields

$$\mathcal{H}(\Omega) := \{ \psi \in L^2_{\mathrm{curl}}(\Omega) \: : \: \operatorname{curl} H = 0 \text{ in } \Omega_{\mathrm{nc}} \} \,.$$

as the standard test space for the problem. The relation (29) yields for $\psi \in \mathcal{H}(\Omega)$ arbitrary

$$\int_{\Omega_{\mathrm{c}}} \sigma^{-1} \operatorname{curl} H \cdot \operatorname{curl} \psi = \int_{\Omega_{\mathrm{c}}} E \cdot \operatorname{curl} \psi + \int_{\Omega_{\mathrm{c}}} \{ v \times \mu H + \sigma^{-1} J_g \} \cdot \operatorname{curl} \psi \,.$$

Since $\psi \in \mathcal{H}(\Omega)$ has a vanishing rotation in the non-conductors, the conditions (3) and (7), (8) yield for ψ smooth enough

$$\int_{\Omega_{\mathrm{c}}} E \cdot \operatorname{curl} \psi = \int_{\Omega} E \cdot \operatorname{curl} \psi = \int_{\Omega} \operatorname{curl} E \cdot \psi \,.$$

Therefore, all $\psi \in \mathcal{H}(\Omega)$ satisfy the integral identity

$$\imath \omega \int_{\Omega} \mu H \cdot \psi + \int_{\Omega} \sigma^{-1} \operatorname{curl} H \cdot \operatorname{curl} \psi = \int_{\Omega} (\bar{v} \times \mu H) \cdot \operatorname{curl} \psi + F_g(\psi)$$

$$F_g(\psi) := \int_{\Omega_{\mathrm{c}}} \sigma^{-1} J_g \cdot \operatorname{curl} \psi \,. \tag{30}$$

Remark 3.1. *For the sake of generality, we can consider also in* (30) *arbitrary an abstract element $F_g \in \left[\mathcal{H}(O) \right]^*$ that satisfies the condition*

$$F_g(\nabla \phi) = 0 \quad \text{for all } \phi \in C_c^\infty(\Omega) \,. \tag{31}$$

Definition 3.2. *We call a vector field $H \in \mathcal{H}(\Omega)$ a weak solution to the problem (P) if the relation* (30) *is satisfied for all $\psi \in \mathcal{H}(\Omega)$.*

The existence of weak solutions is a well-known consequence of the Lax-Milgram Lemma and its generalizations (see [DKS$^+$11]). Uniqueness is valid if $\|v\|_{L^\infty(\Omega_c; \mathbb{R}^3)}$ is comparatively not too large: [Dru09a]. For the existence proof, it is important to ensure that the conditions $\operatorname{div} \mu H = 0$ and $\gamma_\nu(\mu H) = 0$ are implicitly satisfied in the weak sense. Under the assumption (31) for F_g, this is always the case. Since the test space $\mathcal{H}(\Omega)$ contains the L^2-gradient fields, we can insert into (30) the field $\psi = \nabla\phi$, $\phi \in C^1(\overline{\Omega}; \mathbb{R}^3)$ arbitrary, and obtain with the aid of Lemma 3.1 that $\imath\omega \int_\Omega \mu H \cdot \nabla\phi = F_g(\nabla\phi) = 0$.

We now turn to the regularity topic and the proof of Theorem 1.1. The Remark 2.20 shows that even in the case of a scalar coefficient function μ, a 'large' higher-integrability of the magnetic field H is not to expect from the sole informations $\operatorname{div}(\mu H) \in L^2(\Omega)$, $|\operatorname{curl} H| \in L^2(\Omega)$ and $\gamma_\nu(\mu H) = 0$.

Lemma 3.3. *Let $H \in \mathcal{H}(\Omega)$ be a weak solution to (P) in the sense of Definition 3.2 with $J_g \in L^2(\Omega_c; \mathbb{C}^3)$. Then, there is $r > 2$ such that $H \in L^r(\Omega; \mathbb{R}^3)$. If μ is a sufficiently small perturbation of the identity, the same is valid for a $r > 3$.*

Proof. Obviously, $H \in \mathcal{W}_{\mu,\nu}^{2;\infty}(\Omega)$ (cf. (22)). In view of Proposition 2.17, the Assumption 2.7 for $a = \mu$ is valid in general for a $q_1 > 2$, and for a $q_1 > 3$ if μ is a small perturbation of the identity. Thus, Corollary 2.14 yields the statement. $\qquad\square$

It turns out that the problem of the higher integrability of the current $J = \operatorname{curl} H$ has more often a positive answer, due to the following Remark.

Remark 3.4. *For the setting of the introduction, there is always a $q_1 > 3$ such that the Assumption 2.7 is valid for $O = \Omega_c$ and every piecewise continuous a satisfying* (13). *Indeed, the set Ω_c has the structure $\Omega_c = G \cup \Omega_3$ ($G = \Omega_1 \cup \Omega_2 \cup S$). The set Ω_3 is a domain with Lipschitz boundary. Thus, owing to Proposition 2.18, there is $q_1 > 3$ such that the Assumption 2.7 is valid for $O = \Omega_3$. The set G has precisely the structure described in Proposition 2.19. Again, there is $q_1 > 3$ such that the Assumption 2.7 is valid for $O = G$.*

Lemma 3.5. *Assume that $H \in \mathcal{H}(\Omega)$ is a weak solution to (P) in the sense of Definition 3.2 under the assumptions of Theorem 1.1. Assume that $J_g \in L^q(\Omega_{c0}; \mathbb{C}^3)$ for a $2 < q \leq \min\{6, q_1\}$ with $q_1 > 3$ according to Remark 3.4. Assume moreover that that $v \in W_0^{1,\infty}(\Omega_c; \mathbb{R}^3)$. Then the auxiliary vector field $w := \sigma^{-1} \operatorname{curl} H \in L^2(\Omega; \mathbb{C}^3)$ belongs to $L^2(\Omega; \mathbb{C}^3 \mid \operatorname{curl} \in [W^{1,q'}(\Omega_c; \mathbb{C}^3)]^*$ and to $L_{\operatorname{div}_\sigma + \gamma_{\sigma\nu}, 0}^2(\Omega; \mathbb{C}^3)$. Moreover*

$$\|\operatorname{curl} w\|_{[W^{1,q'}(\Omega_c; \mathbb{C}^3)]^*} \leq c\left((1 + \|v\|_{W_0^{1,\infty}(\Omega_c; \mathbb{R}^3)})\|F_g\|_{[\mathcal{H}(\Omega)]^*} + \|F_g\|_{[W^{1,q'}(\Omega; \mathbb{R}^3)]^*}\right).$$

Proof. For all $\psi \in \mathcal{H}(\Omega)$, we rewrite the relation (30) in the form

$$\int_\Omega w \cdot \operatorname{curl} \psi = -\int_\Omega \imath\omega\,\mu H \cdot \psi + \int_\Omega \bar{v} \times \mu H \cdot \operatorname{curl}\psi + F_g(\psi)\,.$$

Consider $\Phi \in W^{1,2}(\Omega_{\mathrm{c}};\, \mathbb{R}^3)$ such that $\gamma_{\nu_{\mathrm{c}}}(\operatorname{curl}\Phi) = 0$. We extend $\operatorname{curl}\Phi$ by zero outside of Ω_{c} and we find a potential $\tilde{\Phi} \in W^{1,2}(\mathbb{R}^3;\, \mathbb{R}^3)$ such that $\operatorname{curl}\tilde{\Phi} = \operatorname{curl}\Phi$ in Ω_{c} and $\operatorname{curl}\tilde{\Phi} = 0$ in $\mathbb{R}^3 \setminus \Omega_{\mathrm{c}}$. For all $1 < p \leq 2$, the Lemma 2.3 moreover implies that

$$\|\tilde{\Phi}\|_{W^{1,p}(\mathbb{R}^3;\, \mathbb{R}^3)} \leq c \, \|\operatorname{curl}\Phi\|_{L^p(\Omega_{\mathrm{c}};\, \mathbb{R}^3)} \,.$$

Observe in particular that $\tilde{\Phi} \in \mathcal{H}(\Omega)$, and follows that

$$\left| \int_{\Omega_{\mathrm{c}}} w \cdot \operatorname{curl}\Phi \right| = \left| \int_{\Omega_{\mathrm{c}}} w \cdot \operatorname{curl}\tilde{\Phi} \right| \leq \omega \, \|\mu H\|_{L^2(\Omega)} \, \|\tilde{\Phi}\|_{L^2(\Omega)}$$

$$+ \|v \times \mu H\|_{L^q(\Omega_{\mathrm{c}})} \, \|\operatorname{curl}\tilde{\Phi}\|_{L^{q'}(\Omega_{\mathrm{c}})} + \|F_g\|_{[W^{1,q'}(\Omega;\, \mathbb{R}^3)]^*} \, \|\tilde{\Phi}\|_{W^{1,q'}(\Omega;\, \mathbb{R}^3)}$$

$$\leq c \, (\|\mu H\|_{L^2(\Omega)} + \|v \times \mu H\|_{L^q(\Omega_{\mathrm{c}})} + \|F_g\|_{[W^{1,q'}(\Omega;\, \mathbb{R}^3)]^*}) \, \|\Phi\|_{W^{1,q'}(\Omega_{\mathrm{c}})} \,. \qquad (32)$$

Consider for aribtary $i \in \{1, 2, 3\}$ the auxiliary vector field $b := v_i \, H$.

$$\|\operatorname{curl} b\|_{L^2(\Omega_{\mathrm{c}})} \leq \|v\|_{L^\infty} \, \|\operatorname{curl} H\|_{L^2} + \|\nabla v_i \times H\| \leq \|v_i\|_{W_0^{1,\infty}(\Omega_{\mathrm{c}})} \, \|H\|_{L^2_{\operatorname{curl}}(\Omega)}$$

$$\|\operatorname{div} \mu b\|_{L^2(\Omega_{\mathrm{c}})} = \|\mu H \cdot \nabla v_i\|_{L^2(\Omega_{\mathrm{c}})} \leq \|v_i\|_{W_0^{1,\infty}(\Omega_{\mathrm{c}})} \, \|H\|_{L^2(\Omega_{\mathrm{c}})}$$

$$\gamma_{\nu_c}(b) = v_i \, \gamma_{\nu_c}(\mu H) = 0 \,.$$

Thus, $b \in \mathcal{W}^{2,\infty}_{\mu,\nu}(\Omega_{\mathrm{c}})$. Due to Remark 3.4, the Lemma 2.14 yields

$$\|b\|_{L^q(\Omega_{\mathrm{c}})} \leq c \, \|v\|_{W_0^{1,\infty}(\Omega_{\mathrm{c}};\, \mathbb{R}^3)} \, \|H\|_{L^2_{\operatorname{curl}}(\Omega_{\mathrm{c}})} \,.$$

Thus vH belongs to $L^q(\Omega_{\mathrm{c}};\, \mathbb{C}^3)$ with corresponding continuity estimate, and (32) implies that

$$\left| \int_O w \cdot \operatorname{curl}\Phi \right|$$

$$\leq c \, ((1 + \|v\|_{W_0^{1,\infty}(\Omega_{\mathrm{c}};\, \mathbb{R}^3)}) \, \|H\|_{L^2_{\operatorname{curl}}(\Omega)} + \|F_g\|_{[W^{1,q'}(\Omega;\, \mathbb{R}^3)]^*}) \, \|\Phi\|_{W^{1,q'}(\Omega_{\mathrm{c}};\, \mathbb{R}^3)}$$

$$\leq c \, ((1 + \|v\|_{W_0^{1,\infty}(\Omega_{\mathrm{c}};\, \mathbb{R}^3)}) \, \|F_g\|_{[\mathcal{H}(\Omega)]^*} + \|F_g\|_{[W^{1,q'}(\Omega;\, \mathbb{R}^3)]^*}) \, \|\Phi\|_{W^{1,q'}(\Omega_{\mathrm{c}};\, \mathbb{R}^3)}$$

According to the Definition 2.2 of the weak rotation operator, we obtain that $w \in L^2(\Omega_{\mathrm{c}};\, \mathbb{C}^3 \mid \operatorname{curl} \in [W^{1,q'}(\Omega_{\mathrm{c}};\, \mathbb{R}^3)]^*)$ with inequality. On the other hand, $\operatorname{div}(\sigma\, w) = \operatorname{div} \operatorname{curl} H = 0$ in the weak sense, and since $\operatorname{curl} H = 0$ in $\Omega \setminus \Omega_{\mathrm{c}}$, $\gamma_\nu(\operatorname{curl} H) = 0$ and we obtain that $(\operatorname{div}_\sigma + \gamma_{\sigma\nu_c})w = 0$. $\qquad \square$

Thus, we obtain the following result directly from Lemma 2.11.

Corollary 3.6. *Assumptions of Lemma 3.5. Then, there is $q > 3$ such that for all $p \in [2, q]$, every weak solution to (P) with $J_g \in L^p(\Omega_{c0};\, \mathbb{C}^3)$ satisfies $\operatorname{curl} H \in L^p(\Omega;\, \mathbb{C}^3)$ with estimate*

$$\|\operatorname{curl} H\|_{L^p(\Omega;\, \mathbb{C}^3)} \leq c \, \|J_g\|_{L^p(\Omega_{\mathrm{c}};\, \mathbb{C}^3)} \,.$$

We now investigate the regularity of the electric field E. The recovering method for the field E from the weak problem 'in H' was exposed already in [LS60].

We first find a vector potential $A \in W^{1,2}(\mathbb{R}^3; \mathbb{C}^3)$ such that $\operatorname{curl} A = \mu H$ in Ω and $\operatorname{curl} A = 0$ in $\mathbb{R}^3 \setminus \Omega$. Since $\mathbb{R}^3 \setminus \Omega$ is simply connected, the latest implies that $A = \nabla p$ in $\mathbb{R}^3 \setminus \Omega$, with a function $p \in W^{2,2}(\mathbb{R}^3 \setminus \Omega)$. Thus, for $\psi \in \mathcal{H}(\Omega)$ arbitrary,

$$
\int_\Omega \mu H \cdot \psi = \int_\Omega \operatorname{curl} A \cdot \psi = \int_\Omega A \cdot \operatorname{curl} \psi + \langle \gamma_\tau(\psi), A \rangle
$$
$$
= \int_\Omega A \cdot \operatorname{curl} \psi + \langle \gamma_\tau(\psi), \nabla p \rangle .
$$

If we now choose a smooth function ζ which on $\partial\Omega$ is equal to one, and vanishes uniformly outside of a neighbourhood that does not intersect $\Omega_{\mathrm c}$, and an extension $\tilde{p} \in W^{2,2}(\mathbb{R}^3)$ such that $\tilde{p} = p$ in $\mathbb{R}^3 \setminus \Omega$ (Extension theorem: [KJF77], 6.5.1), we can use the representation (15) for the operator γ_τ yields

$$
\langle \gamma_\tau(\psi), \nabla p \rangle = \langle \gamma_\tau(\psi), \nabla(\zeta\,\tilde{p}) = \rangle = \int_O \underbrace{\operatorname{curl}\psi}_{=0 \text{ on } \operatorname{supp}(\zeta)} \cdot \nabla(\tilde{p}\,\zeta) = 0 .
$$

Thus $\int_\Omega \mu H \cdot \psi = \int_\Omega A \cdot \operatorname{curl} \psi$. Owing to the weak formulation (30), we easily show using Lemma 2.3 that

$$
\int_{\Omega_c} \{ \imath\,\omega\, A + \sigma^{-1} \operatorname{curl} H - \bar{v} \times \mu H - \sigma^{-1} J_g \} \cdot \jmath = 0
$$
for all $\jmath \in L^2(\Omega_c; \mathbb{R}^3) \,:\, \operatorname{div} \jmath = 0, \; \gamma_\nu(\jmath) = 0$ weakly with respect to Ω_c .

The classical Helmoltz decomposition of $L^2(\Omega_c; \mathbb{R}^3)$ implies that there is $\chi^c \in W^{1,2}(\Omega_c)$ such that

$$
\imath\,\omega\, A + \sigma^{-1} \operatorname{curl} H - \bar{v} \times \mu H - \sigma^{-1} J_g = \nabla \chi^c \text{ a. e. in } \Omega_{\mathrm c} . \tag{33}
$$

To obtain the electric field outside of the conductors, we note that $\gamma_\nu(\operatorname{curl} A) = \gamma_\nu(\mu H) = 0$ with respect to $\partial\Omega$. Thus $A = \nabla\chi_0$ on $\partial\Omega$ with $\chi_0 \in W^{3/2,2}(\partial\Omega; \mathbb{C})$. We introduce the weak solution $\chi^{\mathrm{nc}} \in W^{1,2}(\Omega \setminus \Omega_{\mathrm c}; \mathbb{C})$ to the Dirichlet problem $\chi^{nc} = \chi^c$ on $\partial\Omega_{\mathrm c}$, $\chi^{nc} = \chi_0$ on $\partial\Omega$ and

$$
\int_{\Omega_{\mathrm{nc}}} \varepsilon \nabla \chi^{\mathrm{nc}} \cdot \nabla\phi = \int_{\Omega \mathrm{nc}} \imath\,\omega\,\varepsilon A \cdot \nabla\phi, \quad \text{for all } \phi \in W_0^{1,2}(\Omega\mathrm{nc}) . \tag{34}
$$

We then define $E := -\imath\,\omega\, A + \nabla\chi_{\mathrm{nc}}$ in the nonconductors.

Corollary 3.7. *Assumptions of Theorem 1.1. Define $r > 2$ as in Lemma 3.3 and $q > 3$ as in Lemma 3.5. The electric field belongs to $L^s(\Omega; \mathbb{C}^3)$, $s = \min\{q, p, r_3^*\}$ whenever* $\operatorname{curl} H, J_g \in L^p(O; \mathbb{C}^3)$. *Moreover $\operatorname{curl} E \in L^r(\Omega; \mathbb{C}^3)$.*

Proof. Owing to Lemma 3.3, the field μH belongs to $L^r(\Omega; \mathbb{C}^3)$. Therefore, the vector potential A can be chosen in $W^{1,r}(\mathbb{R}^3; \mathbb{C}^3)$. Since the identity (4), (33) is valid in the conductors, we directly obtain that $E \in L^p(\Omega_{\mathrm c}; \mathbb{C}^3)$ whenever $\operatorname{curl} H, J_g \in L^p(O; \mathbb{C}^3)$. On the other hand, we also obtain from (33) for the scalar potential that $\nabla\chi^c \in L^s(\Omega_{\mathrm c}; \mathbb{C}^3)$ with $s = \min\{p, r_3^*\}$.

The trace of χ^c on the boundary of the domain G remains of class $W^{1/s',s}(\partial G; \mathbb{C})$ for $s = \min\{p, r_3^*\}$.

Since the set Ω_{nc} is Lipschitz and $\varepsilon \in C(\overline{\Omega_{\mathsf{nc}}}; \mathbb{C}^3 \times \mathbb{C}^3)$, the Proposition 2.18 implies that the Asssumption 2.7 is valid for $O = \Omega_{\mathsf{nc}}$ and $a = \varepsilon$ with a $q_1 > 3$. Thus, the gradient of the solution χ_{nc} to the problem (34) belongs to $L^{\bar{s}}(\Omega_{\mathsf{nc}}; \mathbb{C}^3)$ with $\bar{s} = \min\{s, q_1\}$, and we obtain that $E \in L^{\bar{s}}(\Omega^{\mathsf{nc}}; \mathbb{C}^3)$. Since $\operatorname{curl} E = -\imath \omega \mu H$, the last claim is obvious. $\qquad \square$

A   Extension and approximation properties

Let $O \subset \mathbb{R}^3$ be a bounded Lipschitz domain, and $1 < p < +\infty$. In this section we prove a few elementary properties extension and density properties for the spaces $L_{\mathrm{div}}^p(O)$ and $L_{\mathrm{curl}}^p(O)$. The most of it is of course well known from all insiders: see [Mon03], Chapter 3 for an impressive collection of theorems about the Hilbert space case. Nevertheless it is often difficult to quote books our papers that exactly contain the statements needed, and moreover often interesting to provide alternative proofs.

We begin with the space $L_{\mathrm{div}}^p(O)$. Due to the close relation of the divergence operator to the gradient operator, it turns out that everything follows from the basic properties of Sobolev spaces.

Lemma A.1. *Let $U \subseteq \mathbb{R}^3$ be a Lipschitz domain.*

(1) For all $f \in [W^{1,p}(U)]^*$, there are $g \in L^{p'}(U)$ and $h \in L^{p'}(U; \mathbb{R}^3)$ such that $\|g\|_{L^{p'}(U)}^{p'} + \|h\|_{L^{p'}(U; \mathbb{R}^3)}^{p'} = \|f\|_{[W^{1,p}(U)]^*}^{p'}$ and such that

$$f(\phi) = \int_U \{g\, \phi + h \cdot \nabla \phi\}, \quad \text{for all } \phi \in W^{1,p}(U).\tag{35}$$

(2) For all $f \in [W^{1,p}(U)]^*$, such that $f(1) = 0$ there is $h \in L^{p'}(U; \mathbb{R}^3)$ and a constant $c = c(U, p)$ such that $\|h\|_{L^{p'}(U; \mathbb{R}^3)} \leq c \, \|f\|_{[W^{1,p}(U)]^*}$ and such that

$$f(\phi) = \int_U h \cdot \nabla \phi, \quad \text{for all } \phi \in W^{1,p}(U).\tag{36}$$

Proof. This is a standard exercise of functionalanalysis. We prove only the second statement. We denote $W_M^{1,p}(U) := \{\phi \in W^{1,p}(U) : \int_U \phi = 0\}$. Due to the Poincaré inequality, the $W^{1,p}$-gradient half-norm $\|\phi\|_{W_M^{1,p}(U)} := \|\nabla \phi\|_{L^p(U; \mathbb{R}^3)}$ is a norm on $W_M^{1,p}(U)$. Consider a mapping $T \in \mathscr{L}(W_M^{1,p}(U), L^p(U; \mathbb{R}^3))$ defined via $T\phi := \nabla \phi$. The range of T is a closed subspace of $L^p(O; \mathbb{R}^3)$. For $y \in \operatorname{Range}(T)$, there is a unique $\phi \in W_M^{1,p}(U)$ such that $y = T\phi$, and moreover $\|y\|_{L^p(U; \mathbb{R}^3)} = \|\phi\|_{W_M^{1,p}(U)}$. Thus, calling c_0 the constant of the Poincaré inequality the functional $G(y) := f(\phi)$ satisfies

$$\begin{aligned}
|G(y)| = |f(\phi)| &\leq \|f\|_{[W^{1,p}(U)]^*} \|\phi\|_{W^{1,p}(U)} \\
&\leq c_0 \|f\|_{[W^{1,p}(U)]^*} \|\nabla \phi\|_{L^p(U; \mathbb{R}^3)} = c \|f\|_{[W^{1,p}(U)]^*} \|y\|_{L^p(U; \mathbb{R}^3)}.
\end{aligned}$$

It follows that $\|G\|_{[\mathrm{Range}(T)]^*} \leq c \|f\|_{[W^{1,p}(U)]^*}$. Owing to the Hahn-Banach Theorem, there is a norm preserving extendion for the functional G to the entire $L^p(U; \mathbb{R}^3)$. Thus, applying the reprentation theorem for $[L^p(U; \mathbb{R}^3)]^*$, we find $h \in L^{p'}(U; \mathbb{R}^3)$ such that $\|h\|_{L^{p'}} = \|G\|_{[L^p]^*} = \|G\|_{[\mathrm{Range}(T)]^*}$ and such that $G(y) = \int_O h \cdot y$ for all $y \in L^p(O; \mathbb{R}^3)$. Thus, for $y = T\phi$, $\phi \in W^{1,p}(U)$, we obtain the representation (36). $\square$

Corollary A.2. *Let O be a Lipschitz domain, $1 < p < \infty$, and assume that $\tilde{O}$ is a bounded domain such that $O \subset \tilde{O}$ compactly. There are*

(1) A linear bounded extension operator $\mathscr{E}_p \in \mathscr{L}(L^p_{\mathrm{div}}(O), L^p_{\mathrm{div}}(\tilde{O}))$, such that $\mathscr{E}_p\psi = \psi$ in O and $\gamma_{\tilde{\nu}}(\mathscr{E}_p\psi) = 0$.

(2) A linear bounded extension operator $\mathscr{E}_p \in \mathscr{L}(L^p_{\mathrm{div},0}(O), L^p_{\mathrm{div}+\gamma_\nu,0}(\tilde{O}))$, such that $\mathscr{E}_p\psi = \psi$ in O.

Proof. (1): Let $\psi \in L^p_{\mathrm{div}}(O)$. Owing to the definition of the operator γ_ν, the operator $\gamma_\nu(\psi)$ belongs to $[W^{1/p,p'}(\partial O)]^*$. Due to the trace theorem, the operator $\gamma_\nu(\psi) \circ \mathrm{trace}$ belongs to $W^{1,p'}(\tilde{O} \setminus O)$. Applying the Lemma A.1, we find $g \in L^p(\tilde{O} \setminus O)$ and $h \in L^p(\tilde{O} \setminus O; \mathbb{R}^3)$ such that

$$\int_{\tilde{O}\setminus O} \{g\,\phi + h \cdot \nabla\phi\} = -\langle\gamma_\nu(\psi),\,\phi\rangle, \quad \text{for all } \phi \in W^{1,p'}(\tilde{O} \setminus O)\,.$$

In particular for $\phi \in W^{1,p'}(\tilde{O})$, the representation formula (16) implies that

$$\int_{\tilde{O}\setminus O} \{g\,\phi + h \cdot \nabla\phi\} = -\langle\gamma_\nu(\psi),\,\phi\rangle = -\int_O \{\mathrm{div}\,\psi\,\phi + \psi \cdot \nabla\phi\}\,.$$

By definition, the vector field $\mathscr{E}_p\psi := \psi$ in O, $\mathscr{E}_p\psi := h$ in $\tilde{O}\setminus O$ possesses the weak divergence $\mathrm{div}\,\mathscr{E}_p\psi = \chi_O\,\mathrm{div}\,\psi + \chi_{\tilde{O}\setminus O}\,g$. Thus, $\mathscr{E}_p\psi \in L^p_{\mathrm{div}}(\tilde{O} \setminus O)$. Choosing functions with compact support in $\overline{\tilde{O}} \setminus \overline{O}$, we easily show that $\gamma_{\tilde{\nu}}(\mathscr{E}_p\psi) = 0$. It follows from the estimate in Lemma A.1 that

$$\|g\|^p_{L^p(\tilde{O}\setminus O)} + \|h\|^p_{L^p(\tilde{O}\setminus O;\mathbb{R}^3)} = \|\gamma_\nu(\psi)\|^p_{[W^{1,p'}(\tilde{O}\setminus O)]^*} \leq c\,\|\psi\|^p_{L^p_{\mathrm{div}}(O)}\,,$$

which prove that $E_p \in \mathscr{L}(L^p_{\mathrm{div}}(O), L^p_{\mathrm{div}}(\tilde{O}))$.

(2): Let $\psi \in L^p_{\mathrm{div},0}(O)$. Then, the operator $\gamma_\nu(\psi)$ satisfies $-\langle\gamma_\nu(\psi),\,\mathbb{1}\rangle = 0$. Applying the Lemma A.1 we find $h \in L^p(\tilde{O} \setminus O; \mathbb{R}^3)$ such that

$$\int_{\tilde{O}\setminus O} h \cdot \nabla\phi = -\langle\gamma_\nu(\psi),\,\phi\rangle = -\int_O \psi \cdot \nabla\phi \quad \phi \in W^{1,p'}(\tilde{O})\,.$$

The claim follows. $\square$

Corollary A.3. *The set $C^\infty(\overline{O}; \mathbb{R}^3)$ is dense in $L^p_{\mathrm{div}}(O)$.*

Proof. For $\epsilon > 0$, denote ϕ_ϵ the kernel of the standard Dirac sequence. For $\psi \in L^p_{\mathrm{div}}(O)$, we define $\psi_\epsilon := \phi_\epsilon \star \mathscr{E}_p\psi \in C^\infty_c(\mathbb{R}^3)$. Then $\psi_\epsilon \to \psi$ in $L^p(O; \mathbb{R}^3)$. Since $\mathrm{div}\, \mathscr{E}_p\psi \in L^p(\mathbb{R}^3)$ has also compact support, we easily verify that

$$\mathrm{div}\, \psi_\epsilon = \phi_\epsilon \star (\mathrm{div}\, \mathscr{E}_p\psi) \to \mathrm{div}\, \mathscr{E}_p\psi \text{ in } L^p(\mathbb{R}^3; \mathbb{R}^3) \,.$$

$\square$

For the spaces $L^p_{\mathrm{curl}}(O)$, we do not see the means of reducing the elementary extension and approximation properties to the ones of Sobolev functions.

Proposition A.4. *Assume that $\tilde{O}$ is a bounded domain such that $O \subset \tilde{O}$ compactly. Then there is a linear bounded extension operator $\mathscr{E}_p \in \mathscr{L}(L^p_{\mathrm{curl}}(O), L^p_{\mathrm{curl}}(\tilde{O}))$, such that $\mathscr{E}_p\psi = \psi$ in O and $\gamma_\tau(\mathscr{E}_p\psi) = 0$ with respect to $\partial\tilde{O}$.*

Proof. Since O is a Lipschitz domain, we can find a finite covering $U_1, \ldots, U_m$ of a neighbourhood of $B_\rho(\partial O)$ with open smooth sets $\{U_i\}$, and a family of bi-Lipschitz transformations $\{F^i\}_{i=1,\ldots,m}$, $F^i \in C^{0,1}(\overline{U_i}, \mathbb{R}^3)$ such that the portion $\Gamma_i := \partial O \cap U_i$ of the boundary is mapped onto the two-dimensional unit ball $B_1(0; \mathbb{R}^2)$, such that the domain $O_i^- := O \cap U_i$ is mapped onto the cylinder $Z^- := B_1(0; \mathbb{R}^2) \times]-1, 0[$ and the domain $O_i^+ := U_i \setminus O$ is mapped onto $Z^+ := B_1(0; \mathbb{R}^2) \times]0, 1[$. Define $U_0 := O \setminus \overline{\bigcup_{i=1}^m U_i}$, and $\eta_0, \ldots, \eta_m$ be a partition of unity for $U_0, \ldots, U_m$.

For $\psi \in L^p_{\mathrm{curl}}(O)$, we denote $\psi^i := \eta_i\,\psi \in L^p_{\mathrm{curl}}(O)$. Using for $i \geq 1$ the mapping F^i and the formula (3.77) in [Mon03], we can introduce a vector field $\hat{\psi}_i \in L^p(Z^-; \mathbb{R}^3)$ via

$$\hat{\psi}^i(\hat{x}) := (dF^i(\hat{x}))^T \psi^i(F^i(\hat{x})) \,, \tag{37}$$

where $\hat{x}$ denote the reference coordinates in the zylinder $Z := B_1(0; \mathbb{R}^2) \times]-1, 1[$. The transformation (37) is known to be curl conforming. Indeed, [Mon03], Corollary 3.58 shows that

$$\widehat{\mathrm{curl}}\hat{\psi}^i = \det(dF^i)\,(dF^i)^{-1}\psi^i \circ F^i \in L^p(Z^-; \mathbb{R}^3) \,.$$

Thus, $\hat{\psi}_i \in L^p_{\widehat{\mathrm{curl}}}(Z^-)$. In order to extend the vector field, we define a reflexion $R(\hat{x}) : Z^- \to Z^+$ via $R_i(\hat{x}) := \hat{x}_i$ for $i = 1, 2$ and $R_3(\hat{x}) := -\hat{x}_3$. We define an extended vector $\hat{\psi}^i \in L^p(Z; \mathbb{R}^3)$ setting

$$\hat{\psi}^i(R\hat{x}) := (dR(\hat{x}))^{-T}\hat{\psi}^i(\hat{x}) = \begin{cases} \hat{\psi}^i_j(\hat{x}) & \text{for } j = 1, 2 \\ -\hat{\psi}^i_{\,j}(\hat{x}) & \text{for } j = 3\,. \end{cases} \tag{38}$$

This is again a curl conforming transformation, and we verify that

$$\int_Z \hat{\psi}^i \cdot \widehat{\mathrm{curl}}\hat{\Phi} = \int_Z \xi \cdot \hat{\Phi} \text{ with } \xi := \begin{cases} \widehat{\mathrm{curl}}\hat{\psi}^i & \text{in } Z^- \\ \det(dR)^{-1}\,(dR)\widehat{\mathrm{curl}}\hat{\psi}^i \circ R & \text{in } Z^+ \,. \end{cases}$$

Thus, $\hat{\psi}^i \in L^p_{\widehat{\mathrm{curl}}}(Z)$. We transfom back according to (37), and obtain an extension vector $\psi^i \in L^p_{\mathrm{curl}}(U_i)$. In order to obtain $\mathscr{E}_p \in \mathscr{L}(L^p_{\mathrm{curl}}(O), L^p_{\mathrm{curl}}(\tilde{O}))$, we choose a smooth function ζ_ρ that vanishes outside $B_\rho(O) \cap \tilde{O}$ and is equal to one on O. We define $\mathscr{E}_p\psi := \zeta_\rho \sum_{i=0}^m \psi^i \in L^p_{\mathrm{curl}}(\mathbb{R}^3)$. $\square$

Exactly in the manner of Corollary A.3, we can now prove:

Corollary A.5. *The set $C^\infty(\overline{O};\,\mathbb{R}^3)$ is dense in $L^p_{\mathrm{curl}}(O)$.*

Of importance are also approximation results for functions with compact support. We prove only the one statement which we use in the paper.

Lemma A.6. *Assume that there is a Lipschitz continuous diffeomorphism $F \in C^{0,1}(\mathbb{R}^3;\,\mathbb{R}^3)$ such that $\partial O = F(\partial B_1(0))$. The set $C^\infty_c(O;\,\mathbb{R}^3) \cap L^p_{\mathrm{curl},0}(O)$ is dense in $L^p_{\mathrm{curl}+\gamma_\tau,0}(O)$, and $C^\infty_c(O;\,\mathbb{R}^3) \cap L^p_{\mathrm{div},0}(O)$ is dense in $L^p_{\mathrm{div}+\gamma_\nu,0}(O)$.*

Proof. We prove for $1 < p < +\infty$ that every vector field $\psi \in L^p_{\mathrm{div}}(O)$ such that $\mathrm{div}\,\psi = 0$ and $\gamma_\nu(u) = 0$ can be approximated by a sequence $\{u_n\} \subset C^\infty_c(O;\,\mathbb{R}^3)$ in the norm of $L^p_{\mathrm{div}}(O)$. The proof is again based on divergence/rotation preserving coordinate-transformations (see [Mon03], pages 77–80). By assumption, there is a Lipschitz continuous diffeomorphism F mapping the unit ball onto O. For $\psi \in L^p_{\mathrm{div}}(O)$, the vector field $\hat{\psi} := \det(dF)\,dF^{-1}\psi \circ F$ belongs to $L^p_{\mathrm{div}+\gamma_{\hat{\nu}},0}(B_1(0))$. We extend $\hat{\psi}$ by zero to $\mathbb{R}^3 \setminus B_1(0)$, and for $\hat{x} \in \mathbb{R}^3$ and $n \in \mathbb{N}$, we define $\hat{\psi}_n(\hat{x}) := \hat{\psi}((1 - 1/n)\,\hat{x})$, which is a divergence-free vector field in the reference coordinates, whose support is contained in $B_{1-1/n}(0)$. We obtain a divergence free L^p vector field with compact support in O via $w_n \circ F := \frac{1}{\det(dF)}\,dF\hat{\psi}_n$, and a smooth solenoidal vector field in $\mathbb{R}^3$ via

$$\psi_n(x) := (\phi_{1/4n} \star w_n)(x)\,,$$

where ϕ is the standard convolution kernel. Using standard properties of the Dirac-convolution sequence, it is readily verified that $\psi_n \to \psi$ in $L^p(O;\,\mathbb{R}^3)$ and $L^p_{\mathrm{div}}(O)$. Similar arguments are valid for the density in $L^p_{\mathrm{curl}+\gamma_\tau,0}(O)$. $\qquad\square$

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