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Solitons on a background, rogue waves, and classical soliton solutions of Sasa-Satsuma equation

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Abstract

We present the most general multi-parameter family of a soliton on a background solutions to the Sasa-Satsuma equation. The solution contains a set of several free parameters that control the background amplitude as well as the soliton itself. This family of solutions admits nontrivial limiting cases, such as rogue waves and classical solitons, that are considered in detail.

1 Introduction

Sasa-Satsuma equation [16] (SSE) is one of the integrable extensions of the nonlinear Schrödinger equation (NLSE) which has a wider applicability than the NLSE itself. It includes higher order terms, that contain contributions important in describing pulse propagation in optical fibres [15, 5]. In particular, it contains the term with third order dispersion, the term with self-frequency shift and the term describing self steepening [17]. These are the most general terms that have to be taken into account when extending the applicability of the NLSE. According to the original work of Sasa and Satsuma [16] the equation can be written as:

$$i\psi_\tau + \frac{\psi_{xx}}{2} \quad + \quad |\psi|^2\psi + i\epsilon\left[\psi_{xxx} + 3(|\psi|^2)_x\psi + 6|\psi|^2\psi_x\right] = 0. \tag{1}$$

Here, an arbitrary real parameter ϵ scales the integrable perturbations of the NLSE. When $\epsilon = 0$, Eq. (1) reduces to the standard NLSE which has only the terms describing lowest order dispersion and self-phase modulation. This form of equation has been used in the series of works by Mihalache et al [13, 11, 12]. There is a number of publications dealing with the solutions of SSE [7, 8, 9]. Solutions with nonzero boundary conditions have been presented by Wright III [14] although the form of the SSE in his work and the technique used are different from the original version [16] and from the technique presented in [13, 11, 12].

Here, we are interested in solutions on a background. Thus, we start with the plane-wave solutions of the Sasa-Satsuma equation in the form

$$\psi_0(x,\tau) \quad = \quad \frac{c}{2\epsilon}\exp\left[-\frac{i}{2\epsilon}\left(kx - \frac{\omega}{4\epsilon}\tau\right)\right] \tag{2}$$

where the amplitude c, the wavenumber k and the frequency ω are related through

$$\omega = 6c^2k + 2c^2 - k^3 - k^2. \tag{3}$$

This solution looks singular at the limit $\epsilon \longrightarrow 0$. However, the correct choice of parameters $c \sim \epsilon$, $\omega \sim \epsilon^2$ and $k \sim \epsilon$ eliminates this singularity and in the $\epsilon \longrightarrow 0$ limit, we obtain the plane wave of the NLSE with finite amplitude.

The stability issues of these solutions have been discussed in [14], [3], showing, that the plane-waves become unstable under the condition

$$4(1+3k)^2 > 9c^2. \tag{4}$$

In such cases the plane waves evolve into modulated patterns, which we call *solitons on a background*. Mathematically, these patterns represent heteroclinic connections between plane waves in the initial and the final state, firstly described in [14] for the SSE.

Below, we present such solutions of the SSE in the form of a soliton on a background. This is a multi-parameter solution with variable background, arbitrary velocity for arbitrary real ϵ. As a result, there are several limiting cases that can be calculated using the general expression. Usually, solitons on a background are pulsating formations for the NLSE. This happens due to the nonlinear interference between the soliton and the background that have different propagation constants [2]. The same can be said about the solitons on a background for SSE. They are oscillating along the direction of propagation. This particular oscillation usually disappears in the limit of zero background. Such special limit contains soliton solutions that have been obtained earlier [16]. We provided this correspondence explicitly in [4]. However, for the case of SSE this is not the only possibility. It has been shown in [11] that there is a class of oscillating solitons in the case of zero background, which contain the former ones [16] as a special case. In this paper, we provide the correspondence of these oscillating solitons to a limit case of soliton solutions of the SSE on a background.

2 Soliton on a background

The technique we use is similar to the one employed in [14]. We omit this cumbersome part and just present the most general solution to the SSE (1) of the class discussed above. We are more concerned about the physical applications and illustrations of these solutions. The solutions themselves can be checked using any modern software with symbolic computation facilities. Thus, we start with the solution in general form:

$$\psi(x,\tau) = \psi_0(x,\tau)\left[1 + \frac{i(\zeta-\bar{\zeta})G(x,\tau)}{c|\zeta|^2 f_3(x,\tau)}\right] \tag{5}$$

with

$$\begin{aligned} G(x,\tau) &= (\bar{\zeta} f_1(x,\tau) + \zeta \bar{f}_2(x,\tau))(\zeta|f_1(x,\tau)|^2 + \bar{\zeta}|f_2(x,\tau)|^2) \\ &\quad + \frac{1}{2}(\zeta+\bar{\zeta})(\zeta f_1(x,\tau) + \bar{\zeta}\bar{f}_2(x,\tau)) \end{aligned}$$

where

$$\begin{aligned} f_1(x,\tau) &= \frac{r_{11} + r_{12}\Gamma_1 e^{iM_1x+iN_1\tau} + r_{13}\Gamma_2 e^{iM_2x+iN_2\tau}}{1 + \Gamma_1 e^{iM_1x+iN_1\tau} + \Gamma_2 e^{iM_2x+iN_2\tau}} \\ f_2(x,\tau) &= \frac{r_{21} + r_{22}\Gamma_1 e^{iM_1x+iN_1\tau} + r_{23}\Gamma_2 e^{iM_2x+iN_2\tau}}{1 + \Gamma_1 e^{iM_1x+iN_1\tau} + \Gamma_2 e^{iM_2x+iN_2\tau}} \end{aligned} \tag{6}$$

$$M_1 = (m_2 - m_1)/2\epsilon \tag{7}$$

$$M_2 = (m_3 - m_1))/2\epsilon \tag{8}$$

and

$$r_{nj} \;=\; \frac{-i3c}{3m_j + (-1)^n K - \zeta},$$

where indices $n = 1, 2$, $j = 1, 2, 3$, and $K = 1 + 3k$. Complex conjugation is denoted by "$\bar{z}$" throughout this paper. The parameter $\zeta \in \mathbb{C}$ is the complex eigenvalue of the spectral problem, while Γ_1, Γ_2 are complex numbers which are related to translations x_i and τ_i of the solution along the x and τ -axes. This can be seen if we write Γ_i in the form $\Gamma_i = \exp\left(-iMx_i - iN\tau_i\right)$. Each Γ_i can also be equal to zero. Then the solution takes a simpler form.

The spatial eigenvalues m_j in (6) and (7) are the solutions of the 3-rd order polynomial

$$m^3 - m\left[2c^2 + \frac{\zeta^2}{3} + \left(\frac{K}{3}\right)^2\right] + \frac{2}{3}\zeta\left(c^2 - \left(\frac{K}{3}\right)^2\right) + \frac{2\zeta^3}{27} = 0, \tag{9}$$

such that each m_j depends on the free parameters i.e. $m_j = m_j(\zeta, c, k)$. We sort them in the following way:

$$m_1 \;=\; \frac{i\left[\sqrt[3]{3}\left(-\sqrt{3}+i\right)u_1 + \left(\sqrt{3}+i\right)u_9^2\right]}{6\,3^{2/3}u_9} \tag{10}$$

$$m_2 \;=\; \frac{i\left[\sqrt[3]{3}\left(\sqrt{3}+i\right)u_1 + \left(-\sqrt{3}+i\right)u_9^2\right]}{6\,3^{2/3}u_9} \tag{11}$$

$$m_3 \;=\; \frac{u_9^2 + \sqrt[3]{3}u_1}{3\,3^{2/3}u_9} \tag{12}$$

where

$$\begin{aligned}
u_1 &= 18c^2 + 3\zeta^2 + K^2 \\
u_2 &= 9c^2 + \zeta^2 - K^2 \\
u_5 &= 18c^2 + \frac{1}{3}\left(4 - K^2\right) \\
u_6 &= 36c^2 - K^2 + 3 \\
u_7 &= 18\left(1 - 9c^2\right)\zeta^3 + 2\zeta u_6\left(18c^2 + K^2\right) - 54\zeta^5 \\
u_9 &= \sqrt[3]{\sqrt{3}\sqrt{27\zeta^2u_2^2 - u_1^3} - 9\zeta u_2} \\
u_4 &= 27\zeta^4 - 27\zeta^2u_5 + u_6^2 \\
u_0 &= u_6 - 6\zeta^2.
\end{aligned}$$

Using these expressions, we can specify the two spatial frequencies $M_i = (m_{i+1} - m_1)/2\epsilon$:

$$M_1 \;=\; -\frac{i\left(u_9^2 - \sqrt[3]{3}u_1\right)}{6\sqrt[6]{3}u_9\epsilon} \tag{13}$$

$$M_2 \;=\; \frac{3\sqrt[6]{-3}u_1 - (-3)^{5/6}u_9^2}{18u_9\epsilon}, \tag{14}$$

and the two temporal frequencies N_i:

$$N_1 = \frac{(3\zeta u_7 - u_0 u_4)\, M_1}{72\epsilon^2\,(u_0 - 9\zeta m_2)\,(u_0 - 9\zeta m_1)} \tag{15}$$

$$N_2 = \frac{(3\zeta u_7 - u_0 u_4)\, M_2}{72\epsilon^2\,(u_0 - 9\zeta m_3)\,(u_0 - 9\zeta m_1)}. \tag{16}$$

The solution $\psi(x, \tau; c, k, \zeta; \Gamma_1, \Gamma_2, \epsilon)$ (5) is thereby completely determined by the parameters of the plane wave ψ_0 (2) with c, ω and k, restricted by (3), as well as $\zeta \in \mathbb{C}$, and $\Gamma_1, \Gamma_2 \in \mathbb{C}$.

A few examples of the solution are shown in Fig. 1. In each case, the background plane wave is controlled by the amplitude $c/(2\epsilon)$ which is equal to 1 in these examples. The complexity of the solitonic part of the solution is defined by other parameters. We can choose one of the parameters Γ_i to be zero. In this case, we obtain a soliton with periodicity along its propagation direction that can be clearly seen in Figs.1a and 1b. From the form of exponential functions in (6) and (7) it follows, that the solution (5) is periodic along each of the parallel set of lines

$$\Im[M_1 x + N_1 \tau + const] = 0 \tag{17}$$

for $\Gamma_2 = 0$, and periodic along each of the parallel set of lines

$$\Im[M_2 x + N_2 \tau + const] = 0 \tag{18}$$

for $\Gamma_1 = 0$, i.e. along the direction of propagation, where the imaginary part, denoted by "$\Im[z]$", of the above expressions vanishes. The phase and the location of the periodic function is given by Γ_1 and Γ_2 in either case. The shape of the maxima in each period also varies. Each period may have a double peak structure as in Fig.1a. The value of the period as well as the direction of propagation can also be controlled by the parameters of the solution. The case when both parameters Γ_i are nonzero is shown in Fig.1c. This case can be considered as a nonlinear superposition of the two previous solutions. The background stays the same as before, i.e. 1.

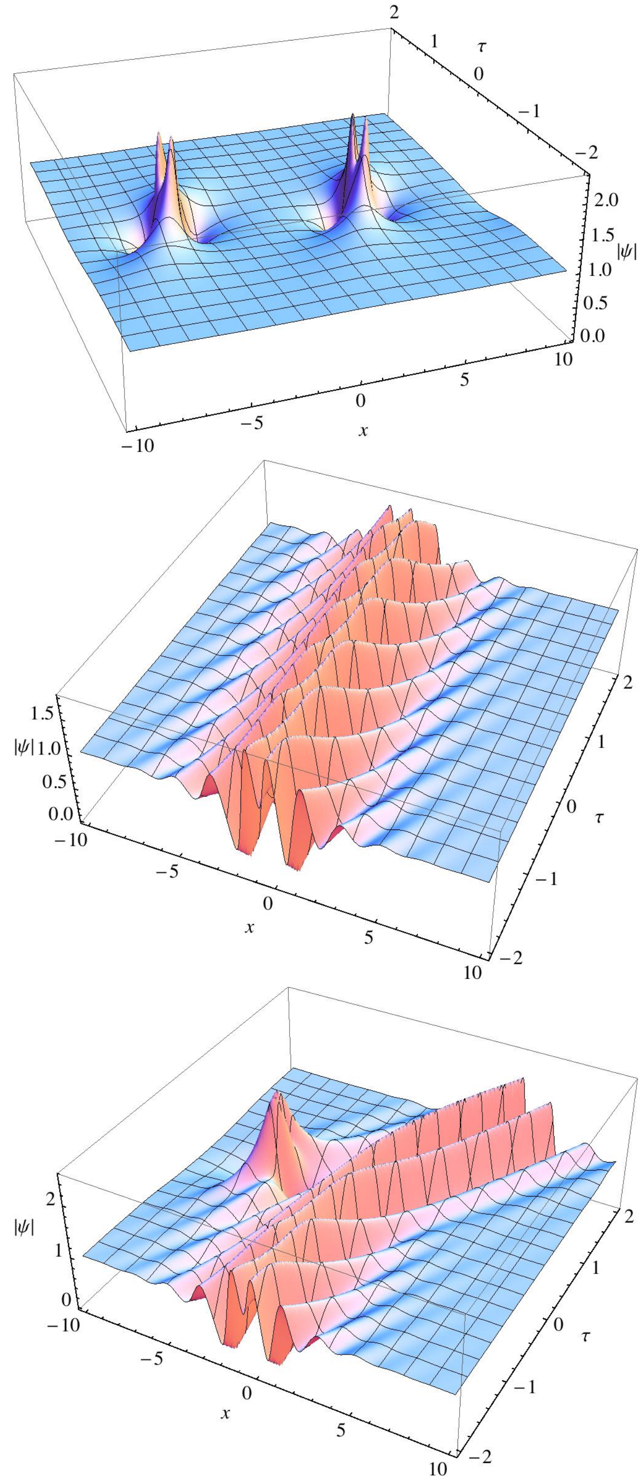

Figure 1: Soliton solutions on a background defined by Eq.(5). Parameters are: $\epsilon = 1/2$, $\zeta = 1. - 2i$, $k = 1$ $(K = 4)$, $c = 1$, resulting in $M_1 = 0.6483 - 0.04i$, $2N_1 = -2.56767 - 4.39656i$, $M_2 = 3.2365 - 0.7i$, $2N_2 = -16.32 - 0.4232i$. Top: $\Gamma_1 = 1, \Gamma_2 = 0$. Middle: $\Gamma_1 = 0, \Gamma_2 = 1$. Bottom: a composite solution when $\Gamma_1 = 1, \Gamma_2 = 1$.

3 Temporal periodic breather on a background

The soliton solution with $\Gamma_2 = 0$ has been investigated in detail in [4]. Analogous considerations can be given for the case $\Gamma_1 = 0$ and $\Gamma_2 \neq 0$. For solutions of this branch which are periodic along the τ-axis, the period is defined by $2\pi/N_2$, where N_2 has to be real. We give an example in Fig. 2, with the same parameters as chosen in Fig. 1, but where ζ has been adjusted numerically to fulfill the condition of periodicity.

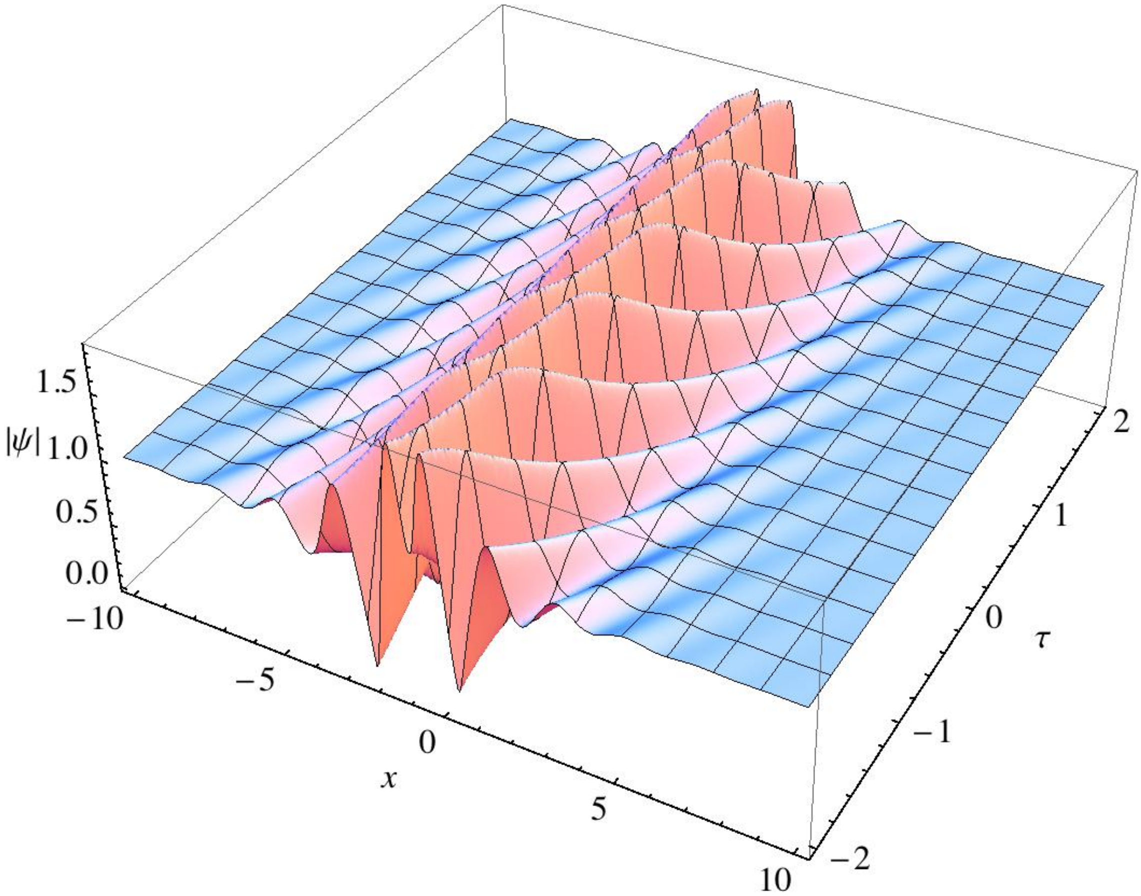

Figure 2: Soliton solution on a background of SSE (1) periodic along the τ-axis when $\Gamma_1 = 0$ and $\Gamma_2 = 1$. The other parameters are: $c = 1$, $\epsilon = 1/2$, $\zeta = 1. - 2.0326i$, $k = 1$ $(K = 4)$. This choice produces: $M_1 = 0.55826 - 0.0346i$, $2N_1 = -2.28153 - 3.8442i$, $M_2 = 3.1862 - 0.7115i$ and $2N_2 = -16.3839$.

4 Spatially periodic breather on a background.

For solutions of the branch $\Gamma_1 = 0$ and $\Gamma_2 \neq 0$ which are periodic along the x-axis the period is defined by $2\pi/M_2$, where M_2 has to be real. The condition of zero imaginary part of M_2 leads to the eigenvalues of inverse scattering ζ being given by [4]:

$$\zeta = \pm\sqrt{\alpha \pm \beta} \tag{19}$$

with $M = M_2$

$$\begin{aligned}
\alpha &= \frac{c^4 + 10c^2\left[\left(\frac{K}{3}\right)^2 - \frac{3M^2}{10}\right) + 2Q\left(M^2 + \left(\frac{K}{3}\right)^2\right]}{2Q} \\
\beta &= \frac{2w}{3Q}\left[\frac{1}{4}\left(c^2 + M^2\right) - \left(\frac{K}{3}\right)^2\right],
\end{aligned}$$

where $w = \sqrt{9c^4 + K^2\left(M^2 - 4c^2\right)}$ and $Q = \frac{M^2}{4} - \left(\frac{K}{3}\right)^2$.

Any choice of signs in (19) provides us with a valid complex eigenvalue. In all four cases ζ is a function of c, k and M. In [4] we have choosen both signs in (19) to be positive, to obtain, explicitly:

$$\zeta_1 = \sqrt{\frac{-162c^4 - 18c^2\left(3\left(w - 9M^2\right) + 10K^2\right) + \left(4K^2 - 9M^2\right)\left(6w + 9M^2 + K^2\right)}{9(4K^2 - 9M^2)}}. \tag{20}$$

The breather solution of branch $\Gamma_2 = 0, \Gamma_1 \neq 0$ is periodic in x in the case $\zeta = \zeta_1$ and has a single growth-decay cycle along the τ-axis. It starts with modulation instability, grows to maximum amplitude and decays the same way as it grew. This solution is similar to the Akhmediev breather solution of the NLSE [6, 10, 1]. However, there are more parameters involved in the SSE case.

For the branch $\Gamma_2 \neq 0, \Gamma_1 = 0$ considered here the breather solution is periodic in x and has a single growth-decay cycle along the τ-axis if we choose $\zeta = -\zeta_1$. We give two examples in Fig. 3. As can be seen from the figure, a double peak structure appears, which is characteristic for many SSE solutions. Typically, the double peak structure disappears for larger values of k.

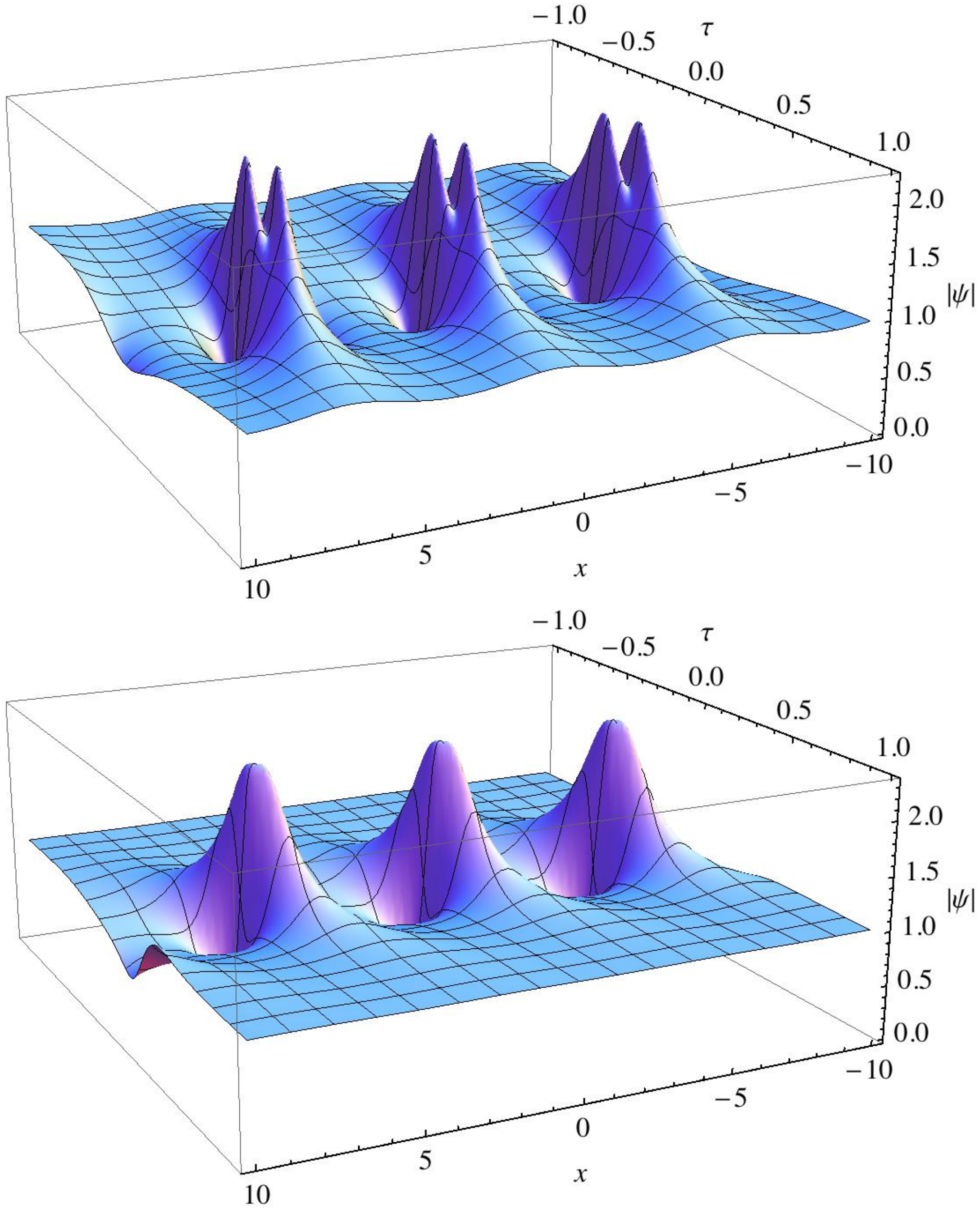

Figure 3: Spatially periodic solutions on a background of SSE (1) for $\Gamma_1 = 0$, $\Gamma_2 = -1$ ($c = 1$, $\epsilon = 1/2$). Top: $\zeta = -0.957906 - i1.81754$, $k = 1$ ($K = 4$), resulting in $M_2 = 1$, $2N_2 = -3. + i6.245$. Bottom: $\zeta = -2.11903 - i1.75897$, $k = 2.$ ($K = 7$), resulting in $M_2 = 1$ and $2N_2 = 8. + i11.7473$.

5 Rogue wave solution

We have demonstrated the existence of rogue wave solutions to the SSE for the branch $\Gamma_1 \neq 0$, $\Gamma_2 = 0$ in [4]. In order to do this, we took the long period limit of the spatially periodic breather solution and used the specific value of $\Gamma_1 = -1$ required to keep the central maximum close to the origin, such that we have been left with just one infinite period. This way, we obtained the solution describing the rogue wave of the SSE (1). Furthermore, in [3], we have demonstrated a proper NLSE limit $\epsilon \to 0$ for these rogue waves which is the Peregrine soliton.

Here, we found similar solutions for the branch $\Gamma_1 = 0, \Gamma_2 \neq 0$. To see this we put $\Gamma_1 = 0$ and $\Gamma_2 = -1$, $\zeta = -\zeta_1$, and take the limit $M_2 \to 0$ for given c and ϵ. This choice gives a different family of rogue wave solutions, compared to those in [3]. The analytic form for this rogue wave is given by a rational function containing 4th-order polynomials in x and τ:

$$\psi(x,\tau) = \frac{1}{2\epsilon}\left[1 + 12i\Im[\zeta]\mu_1^2\nu_2^2\frac{p_{\mathrm{mod}}(x,\tau)}{q_{\mathrm{mod}}(x,\tau)}\right] \tag{21}$$

with

$$\begin{aligned} p_{\mathrm{mod}}(x,\tau) &= \zeta\xi\chi^2\mathcal{M}_1^2\left|V_2\right|^2\left(\zeta\mu_2^2VW_1 + \nu_1^2V_2Wz\right) + \mu_2^2VW_1z + \zeta\nu_1^2V_2W \\ &\quad + \xi\chi^2z\mathcal{M}_2^2\left|V_1\right|^2\left(\zeta\mu_2^2VW_1 + \nu_1^2V_2Wz\right), \end{aligned}$$

and

$$\begin{aligned} q_{\mathrm{mod}}(x,\tau) &= 2M^2\xi\Big(2\chi^2\left(-2\eta^2\Re\left(\mu_1^2\mu_2^2V^2\bar{\Omega}\right) + \mathcal{M}_1^2\left|V\zeta V_2\right|^2 + \mathcal{M}_2^2\left|V\zeta V_1\right|^2\right) \\ &\quad + |V|^2\left(\xi^2\chi^4\left|z\left|V_2\right|^2\mathcal{M}_1^2 + \zeta\left|V_1\right|^2\mathcal{M}_2^2\right|^2 + 1\right)\Big), \end{aligned}$$

where

$$\xi = \Re(\zeta),\ \ \eta = \Im(\zeta),\ \ \chi = \frac{3c}{M\xi\left|V\right|},\ \ \Omega = V_1V_2,$$

$$M = \mathcal{M}_1\mathcal{M}_2, \quad \mathcal{M}_1 = |K-\mu|^2, \quad \mathcal{M}_2 = |K+\mu|^2$$

$$V_1 = V\mu_1 - 24i\epsilon^2, \quad W_1 = \bar{V}_1,$$

$$V_2 = V\mu_2 + 24i\epsilon^2, \quad W_2 = \bar{V}_2,$$

$$V = v\tau + 4x\epsilon, \qquad w = \bar{v}$$

$$\mu = 3m_1 - \zeta, \qquad \nu = \bar{\nu}$$

$$\mu_1 = K - \mu; \qquad \nu_1 = \bar{\mu_1}$$

$$\mu_2 = K + \mu; \qquad \nu_2 = \bar{\mu_2}$$

$$v = \frac{3\zeta u_7 - u_0u_4}{9\left(u_0 - 9m_1\zeta\right)^2}$$

and

$$\zeta = -\frac{i\sqrt{81c^4 + 90c^2K^2 - 2K^4 + 3c\left(9c^2 - 4K^2\right)^{3/2}}}{3\sqrt{2}K}.$$

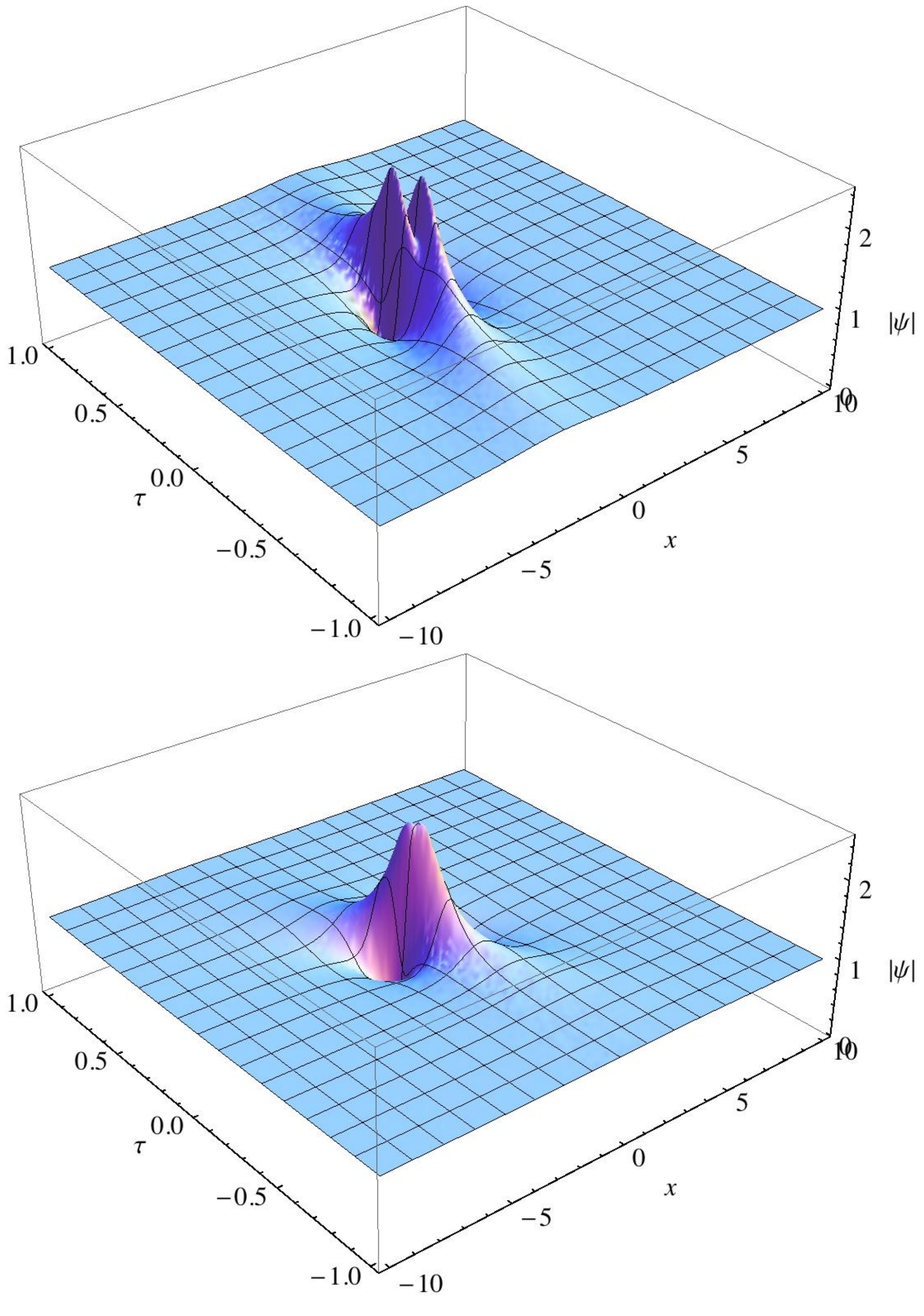

Figure 4: Rogue wave solutions of SSE (1) for the branch $\Gamma_1 = 0$, $\Gamma_2 = -1$. Parameters are as in Fig. 3: $c = 1$, $\epsilon = 1/2$. ζ has been chosen for each k such that $M_2 = 0$, resulting also in $N_2 = 0$. Top: $k = 1$ $(K = 4)$, $\zeta = -1.00088 - i2.12256$. Bottom: $k = 2$, $(K = 7)$, $\zeta = -4.84676 - i0.897292$.

These rogue waves are similar to those found in [3], except for a different localization in the x, τ-plane. Two examples are shown in Fig. 4. Just like for soliton solutions in Fig. 3, the characteristic double peak structure disappears for larger values of k.

We will show now, that for these new rogue waves proper NLSE-limits $\epsilon \to 0$ do exist, if they are taken carefully. For this we scale $k \to 2\epsilon k$ and choose $c = 2\epsilon$. In the limit $\epsilon \to 0$ the solution (21) becomes

$$\psi(x,\tau) = \exp\left(-\frac{1}{2} i \left(\left(k^2-2\right)\tau + 2kx\right)\right) \times \tag{22}$$

$$\left(1 - \frac{4\left(\left(k^2+1\right)\tau^2 + \tau(k(2x-1)-i) + (x-1)x\right)}{2\left(k^2+1\right)\tau^2 + 2k\tau(2x-1) + 2x^2 - 2x + 1}\right)$$

which is a one-parameter family of rational solutions to the NLSE. The degree of the involved polynomials has reduced from four to two, compared to (21). The parameter k tilts and shifts the solution along the x, τ-plane. In particular, by choosing $k = 0$ we obtain

$$\psi(x,\tau) = e^{i\tau}\left(1 - \frac{4\left(\tau^2 - i\tau + (x-1)x\right)}{2\tau^2 + 2x^2 - 2x + 1}\right). \tag{23}$$

A shift of this solution along x by $1/2$, $x \to x + 1/2$, finally provides the Peregrine solution to the NLSE

$$\psi(x,\tau) = e^{i\tau}\left(1 - \frac{4 + 8i\tau}{4\tau^2 + 4x^2 + 1}\right) \tag{24}$$

as a correct limit of the family (21). When $\Gamma_1 \neq 0$ and $\Gamma_2 \neq 0$, it is difficult to find rogue wave solutions, because M_1 and M_2 do not vanish simultaneously. This case needs further studies.

6 Zero background limit.

The background of the solution (5) $c/(2\epsilon)$ is controlled by the parameter c. The limit of zero background is obtained when $c \to 0$. Despite being seemingly simple, this limit is far from being trivial. The difficulty is in finding the limits for the m_j-values that enter the expressions for r_{nj} coefficients. These can be calculated using a series expansion of m_j at small c:

$$r_{nj} = \frac{-ic}{m_j^{(0)} + (-1)^n \frac{K}{3} - \frac{\zeta}{3} + m_j^{(2)} c^2} \tag{25}$$

where $m_j^{(0)} = \lim_{c \to 0} m_j$ and $m_j^{(2)} = \left.\frac{\partial^2 m_j}{\partial c^2}\right|_{c=0}$.

In the limit $c \to 0$, the m_j coefficients become:

$$m_1^{(0)} = -\frac{\left(\sqrt{3}+i\right)R^+ + (\sqrt{3}-i)(\sigma)^{2/3}\sigma_2^2 R}{6\sqrt{3}\sqrt[3]{\sigma}\sigma_2}$$

$$m_2^{(0)} = i\,\frac{(R^+ - (\sigma)^{2/3}\sigma_2^2 R)}{3\sqrt{3}\sqrt[3]{\sigma}\sigma_2}$$

$$m_3^{(0)} = \frac{R^+ + \sigma^{2/3}\sigma_2^2 R}{3\sqrt{3}\sqrt[3]{i\sigma}\sigma_2}$$

where

$$R = \left(K - i\sqrt{3}\zeta\sigma\right), \;\; R^{+} = \left(K + i\sqrt{3}\zeta\sigma\right)$$

$$\sigma = \text{sgn}\left[\Im\left(\zeta^2 K\right)\right] \tag{26}$$

$$\sigma_2 = \begin{cases} e^{\frac{2i\pi}{3}} & \Leftrightarrow \quad -\frac{\pi}{3} > \text{Arg}(w_{3r}) > -\pi \\ e^{-\frac{2i\pi}{3}} & \Leftrightarrow \quad \pi > \text{Arg}(w_{3r}) > \frac{\pi}{3} \\ 1 & \text{else.} \end{cases} \tag{27}$$

$$w_{3r} = \sqrt[6]{3}\sqrt[3]{i\sigma}R. \tag{28}$$

It follows from the expressions above that various branches should be considered. Each of them provides a different limit for the r_{nj}-coefficients. Here, we restrict ourselves in one of these branches given by $\sigma = +1$ and $\sigma_2 = +1$. This requires the following two conditions:

1. $\text{sgn}\left[\Im\left(\zeta^2 K\right)\right] = 1$
2. $-\pi/3 < \arg\left(\sqrt[6]{-3}K - (-3)^{2/3}\zeta\right) < \pi/3.$

In this case, the 2-nd order coefficients $m_1^{(2)}$, $m_2^{(2)}$ in the expression (25) are

$$m_1^{(2)} = -\frac{3}{K - 3\zeta}, \tag{29}$$

$$m_2^{(2)} = \frac{18\zeta}{K^2 - 9\zeta^2}, \tag{30}$$

$$m_3^{(2)} = \frac{3}{K + 3\zeta}. \tag{31}$$

The spatial frequencies $M_i^{(0)} = m_{i+1}^{(0)} - m_1^{(0)}$ then become

$$M_1^{(0)} = K/3 - \zeta, \quad M_2^{(0)} = 2K/3. \tag{32}$$

The corresponding temporal frequencies $N_i^{(0)} = N_{i+1}^{(0)} - N_1^{(0)}$ are

$$N_1^{(0)} = \frac{K}{6}\left(\frac{K^2}{9} - \frac{1}{3}\right) - \frac{\zeta}{2}\left(\zeta^2 - \frac{1}{3}\right) \tag{33}$$

$$N_2^{(0)} = \frac{1}{27}K\left(K^2 - 3\right). \tag{34}$$

For brevity, we denote the exponents appearing in (5) as A and B:

$$A = i\left(M_1 x + N_1 \tau\right) = \frac{i}{2\epsilon}\left(M_1^{(0)}x + \frac{N_1^{(0)}}{\epsilon}\tau\right) \tag{35}$$

$$B = i\left(M_2 x + N_2 \tau\right) = \frac{i}{2\epsilon}\left(M_2^{(0)}x + \frac{N_2^{(0)}}{\epsilon}\tau\right). \tag{36}$$

Along the branch considered here these functions take the form

$$
\begin{aligned}
A &= \frac{x}{2\epsilon}\left[\eta + i\left(\frac{K}{3}-\xi\right)\right] - \frac{\tau}{8\epsilon^2}\left[\eta\left(\eta^2-3\xi^2+\frac{1}{3}\right)\right. \\
&\left. -i\xi\left(3\eta^2-\xi^2+\frac{1}{3}\right) - i\frac{K}{3}\left(\frac{K^2}{9}-\frac{1}{3}\right)\right], \\
B &= \frac{iK\left((K^2-3)\,t+36x\epsilon\right)}{108\epsilon^2},
\end{aligned}
$$

with the real and the imaginary parts of A given explicitly by setting $\zeta = \xi + i\eta$. The leading contributions at small c are:

$$
r_{11} = \frac{3ic}{2K}, \quad r_{12} = \frac{3ic}{3\zeta+K}, \quad r_{13} = -\frac{i(3\zeta+K)}{3c}, \tag{37}
$$

$$
r_{21} = \frac{i(K-3\zeta)}{3c}, \quad r_{22} = \frac{3ic}{3\zeta-K}, \quad r_{23} = -\frac{3ic}{2K}. \tag{38}
$$

Motivated by these expressions we introduce the following notations

$$
Y = (3\zeta - K)/3, \quad W = (3\zeta + K)/3, \quad S = WY,
$$

which will be used in the formulae below.

Within the approximations considered above, the form of the solution can be written explicitly in terms of c:

$$
\psi(x,\tau) = \frac{1}{2\epsilon}\exp\left[-\frac{i}{2\epsilon}\left(kx-\frac{\omega}{4\epsilon}\tau\right)\right]\left[c+\frac{i(\zeta-\bar{\zeta})}{|\zeta|^2}\frac{G(x,\tau)}{f_3(x,\tau)}\right] \tag{39}
$$

where in the lowest order of c:

$$
\frac{G(x,\tau)}{f_3(x,\tau)} = -2icKY\bar{W}|\zeta|^2\frac{G_0}{f_{30}+f_{32}c^2} \tag{40}
$$

with the coefficients G_0, f_{30}, f_{32} given in the appendix. This solution is shown for $\Gamma_1 = 1,\ \Gamma_2 = 0$ and $\Gamma_1 = 1, \Gamma_2 = 1$, in Figs. 5(a) and 5(b) respectively. When decreasing c, the shape and the height of the soliton doesn't change. However, it moves in the (x,τ)-plane, as shown in Fig. 6.

This can be understood if we replace c by $c = e^{-\rho}$ and notice, that the variable ρ adds up to the real part of the exponent A, causing the shift in the (x,τ)-plane.

In the next order of c we include the term $\sim G_2c^2$ in the nominator and the term $\sim f_{34}c^4$ in the denominator:

$$
\frac{G(x,\tau)}{f_3(x,\tau)} = -2ic|\zeta|^2K\bar{W}Y\frac{G_0+G_2c^2}{f_{30}+f_{32}c^2+f_{34}c^4}
$$

with coefficients G_2, f_{34} also given in the appendix. The solution then becomes:

$$
\psi(x,\tau) = \frac{1}{2\epsilon}\exp\left[-i\left(\frac{k}{2\epsilon}x-\frac{\omega\tau}{8\epsilon^2}\tau\right)\right]\left[c-\frac{2ic|\zeta|^2K\bar{W}Y\,(G_0+c^2G_2)}{f_{30}+f_{32}c^2+f_{34}c^4}\right]. \tag{41}
$$

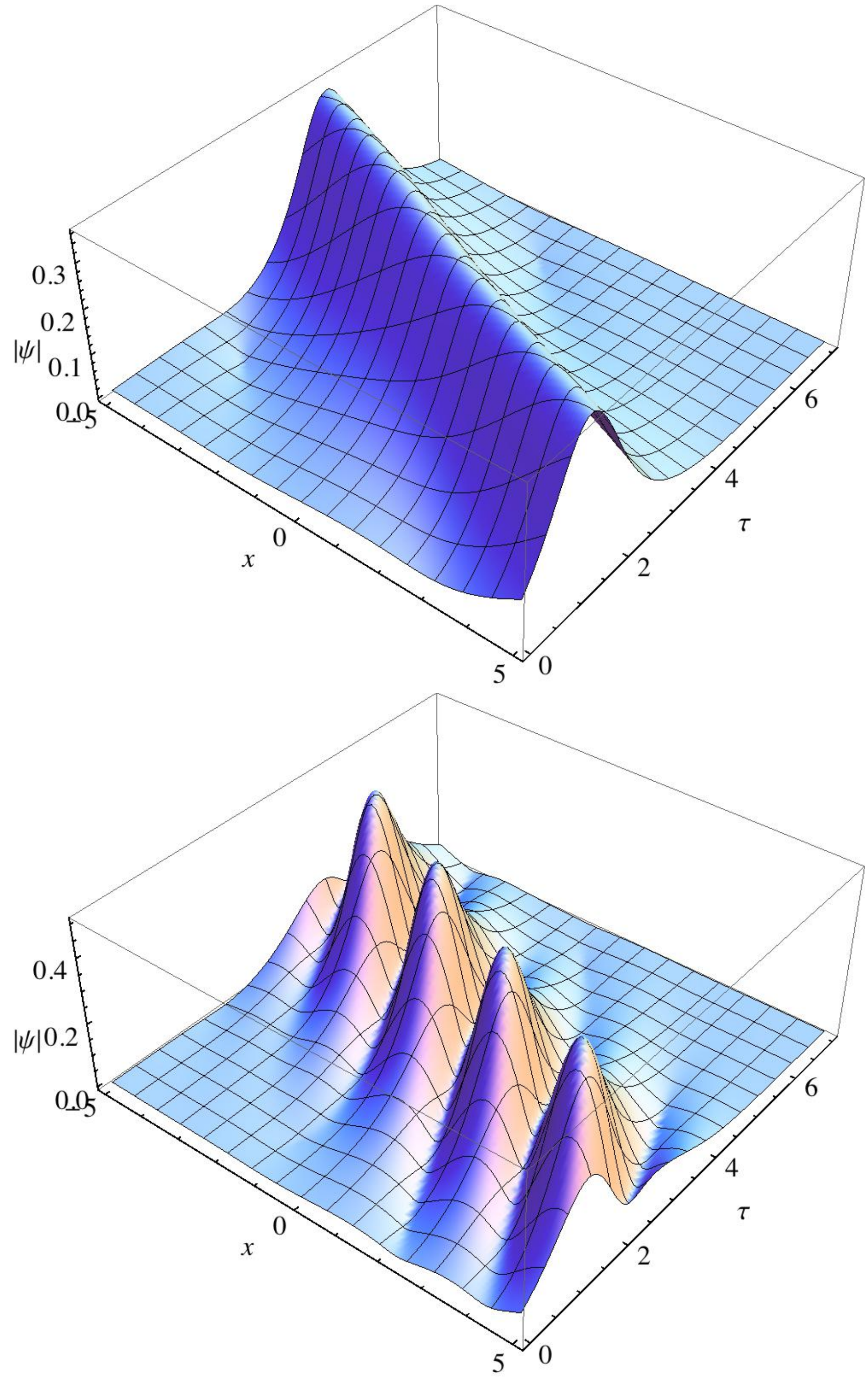

Figure 5: Soliton solutions on a small background with $c = 10^{-2}$, defined by Eq.(39). Parameters are: $\epsilon = 1/2$, $\zeta = 1.5 + i0.5$, $k = 0.1$ $(K = 1.3)$, resulting in $M_1 = -1.06687 - i0.499915$, $2N_1 = -1.81338 - i3.08331$, $M_2 = 0.866638 + i0.0000234777$, $2N_2 = -0.126675 + i5.377 \cdot 10^{-6}$. Fundmental Soliton for (a) $\Gamma_1 = 1$, $\Gamma_2 = 0$. (b) $\Gamma_1 = 1$, $\Gamma_2 = 1$.

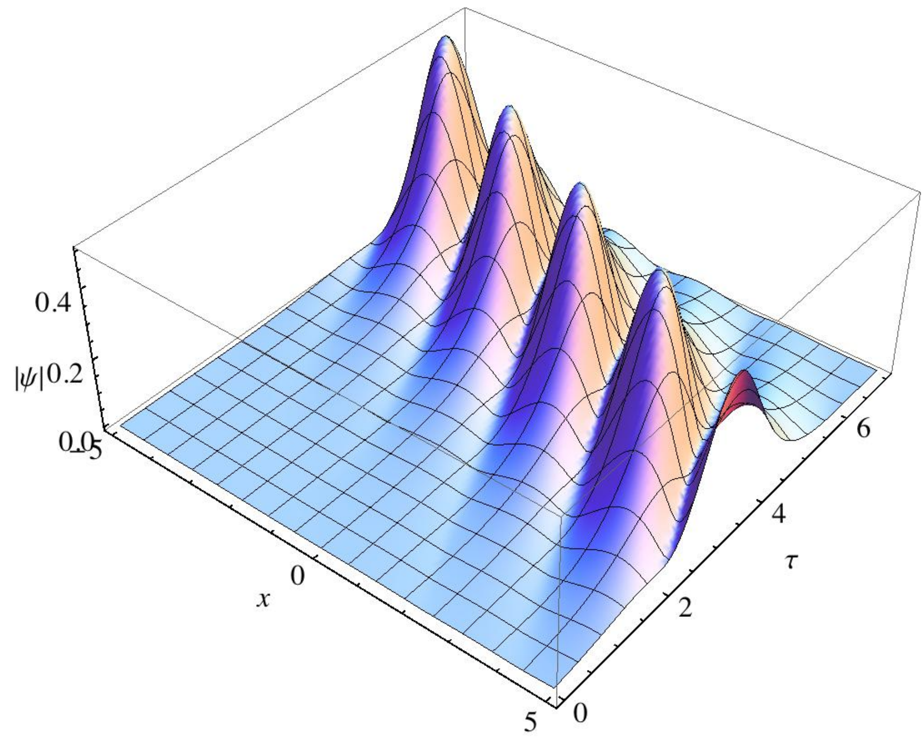

Figure 6: Soliton solution defined by Eq.(39) on a smaller background $c = 10^{-3}$ than in Fig.5(b). The parameters are the same as in Fig. 5, and $\Gamma_1 = 1, \Gamma_2 = 1$.

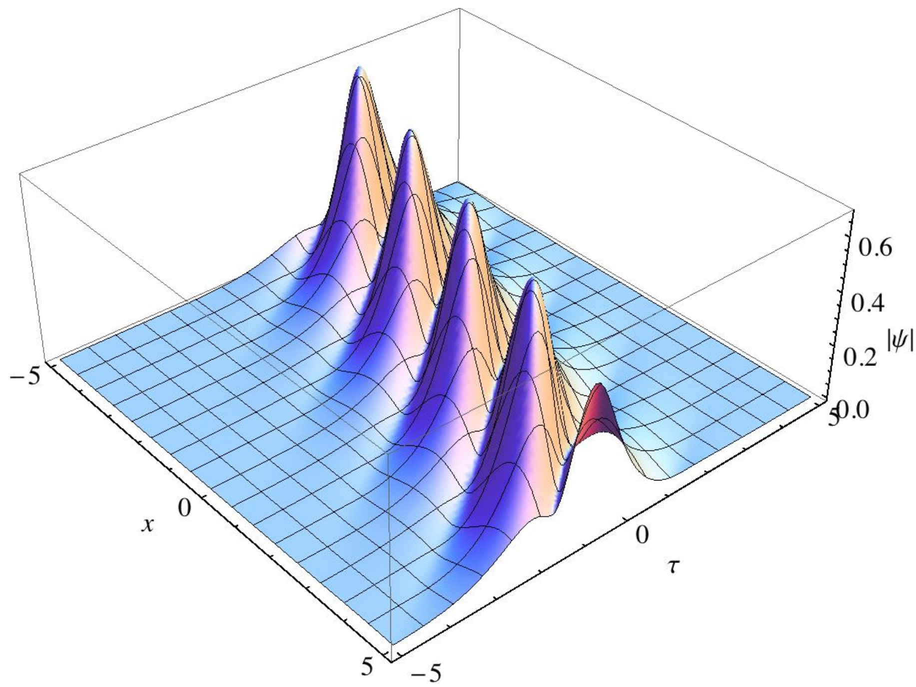

Figure 7: Exact soliton solution (42) of the SSE (1) on zero background $c = 0$. The parameters are the same as in Fig. 5 and $\Gamma_1 = 1, \Gamma_2 = 1$.

To obtain the zero background limit analytically we set $c = e^{-\rho}$ and shift accordingly the real part of the exponent A by ρ. This arrangement moves the solution back to the origin, thus removing the dependence of A on ρ. Taking the limit $\rho \to \infty$ in (41) we obtain:

$$\psi(x,\tau) = -\frac{i}{\epsilon}\Im(\zeta)e^{\left(\frac{i(1-K)\left(\left(K^2+K-2\right)t+36x\epsilon\right)}{216\epsilon^2}\right)}\frac{P}{Q} \tag{42}$$

with

$$\begin{aligned} P &= \Gamma_2 W\bar{\Gamma}_1 e^{\bar{A}+B}\left(\bar{\zeta}^2\left|Y\right|^2+\zeta e^{2\Re(A)}\Re(\zeta)\left|\Gamma_1\right|^2+\left|\Gamma_2\zeta W\right|^2\right)- \\ & \quad e^{A}\Gamma\bar{Y}\left(\bar{\zeta}e^{2\Re(A)}\Re(\zeta)\left|\Gamma_1\right|^2+\zeta^2\left|\Gamma_2 W\right|^2+\left|\zeta Y\right|^2\right) \end{aligned}$$

and

$$\begin{aligned} Q &= 4\Im(\zeta)^2\Re\left(\Gamma_2 S\bar{\Gamma}_1^2 e^{2\bar{A}+B}\right)+\left|\bar{\zeta}\left|\Gamma_2 W\right|^2+\zeta\left|Y\right|^2\right|^2+ \\ & \quad e^{4\Re(A)}\Re(\zeta)^2\left|\Gamma_1\right|^4+2e^{2\Re(A)}\left|\Gamma_1\zeta\right|^2\left(\left|\Gamma_2 W\right|^2+\left|Y\right|^2\right). \end{aligned}$$

We end up with the same solution (42) if we repeat the limiting process $c \to 0$ including all higher order terms G_4c^4, G_6c^6, $f_{36}c^6$, and $f_{38}c^8 \neq 0$ in the expressions above. Solution (42) is therefore the exact zero background ($c = 0$) limit of the solution (5) along the branch $\sigma = 1$, $\sigma_2 = 1$. Thus, it is a particular exact solution of the SSE (1). It is illustrated in Fig. 7 for the same set of parameters as in Fig. 6.

7 The limit of Mihalache soliton solution.

If we replace the real and imaginary parts of the eigenvalue ζ using the following transformations

$$\eta = 2\epsilon\eta_{\mathrm{M}} \quad \text{and} \quad \xi = 2\epsilon\xi_{\mathrm{M}} \tag{43}$$

and choose in particular

$$\begin{aligned} K &= 0, \quad (\text{implies } B = 0) \\ A &= -A_0 + iB_0 \\ \bar{A} &= -A_0 - iB_0 \\ \Gamma_1 &= -2i\epsilon(\xi_{\mathrm{M}}+i\eta_{\mathrm{M}})/\bar{a} \\ \Gamma_2 &= \bar{b}/\bar{a} \end{aligned}$$

we obtain the following exact solution to the SSE:

$$\begin{aligned} \psi(x,\tau) &= 2\eta_{\mathrm{M}} e^{A_0-\frac{i(t-18x\epsilon)}{108\epsilon^2}}\frac{1}{\tilde{\triangle}}\times \\ & \left[\bar{b}e^{2A_0-iB_0}\left(\left(\eta_{\mathrm{M}}^2+\xi_{\mathrm{M}}^2\right)|b|^2-(\eta_{\mathrm{M}}+i\xi_{\mathrm{M}})^2|a|^2\right)+\bar{b}e^{-iB_0}\xi_{\mathrm{M}}(\xi_{\mathrm{M}}+i\eta_{\mathrm{M}})\right. \\ & \left.+ae^{iB_0}\xi_{\mathrm{M}}(\xi_{\mathrm{M}}-i\eta_{\mathrm{M}})+ae^{2A_0+iB_0}\left(\left(\eta_{\mathrm{M}}^2+\xi_{\mathrm{M}}^2\right)|a|^2+(\xi_{\mathrm{M}}+i\eta_{\mathrm{M}})^2|b|^2\right)\right] \end{aligned} \tag{44}$$

with

$$\tilde{\triangle} = -2\eta_{\rm M}^2 e^{2A_0}\left(abe^{2iB_0} + \bar{a}\bar{b}e^{-2iB_0}\right) + 2e^{2A_0}\left(\eta_{\rm M}^2 + \xi_{\rm M}^2\right)\left(|a|^2 + |b|^2\right)$$
$$+e^{4A_0}\left(\eta_{\rm M}^2\left(|a|^2 - |b|^2\right)^2 + \xi_{\rm M}^2\left(|a|^2 + |b|^2\right)^2\right) + \xi_{\rm M}^2.$$

This solution still depends on arbitrary parameters $\xi_{\rm M}, \eta_{\rm M} \in \mathbb{R}$, and $a, b \in \mathbb{C}$. By introducing the coordinates

$$T = \tau \tag{45}$$
$$X = x - \frac{\tau}{12\epsilon} \tag{46}$$

and defining

$$\begin{aligned}
u(X,T) &= \frac{2i}{\Delta}\left(\frac{i\Omega_0}{\eta_{\rm M}}\left(ae^{iB_0} + b^* e^{-iB_0}\right) - \frac{a\Omega}{2\lambda_0^*} + \frac{b^*\Omega^*}{2\lambda_0}\right)\\
\Omega_0 &= e^{A_0}(|a|^2 + |b|^2) + e^{-A_0}\\
\Omega &= 2\bar{a}\bar{b}e^{A_0 - iB_0} + e^{-A_0 + iB_0}\\
A_0 &= \eta_{\rm M}\left(X - T\epsilon\left(\eta_{\rm M}^2 - 3\xi_{\rm M}^2\right)\right)\\
B_0 &= \xi_{\rm M}\left(T\epsilon\left(\xi_{\rm M}^2 - 3\eta_{\rm M}^2\right) + X\right)\\
\lambda_0 &= \frac{1}{2}(-\xi_{\rm M} + i\eta_{\rm M})\\
\Delta &= \frac{|\Omega|^2}{4|\lambda_0|^2} + \frac{\Omega_0^2}{(\lambda_0^* - \lambda_0)^2}
\end{aligned}$$

we arrive at the solution obtained by Mihalache et al. [13, 11, 12]. This solution has been presented in [11] in the form:

$$\psi(x,\tau) = u(X,T)\exp\left(\frac{i}{6\epsilon}\left(x - \frac{\tau}{18\epsilon}\right)\right). \tag{47}$$

It appears to be a special case of the solution (42). Due to presence of Γ_1 and Γ_2 it has an oscillating structure, even with zero background. We give two examples in Fig. 8 and in Fig. 9.

Mihalache et al. have obtained their solutions using the Riemann problem method [11]. It is different from the usual inverse scattering theory, used in [16]. The solution (47) is still more general than the solution of Sasa and Satsuma [16]. In particular, solution (47) contains the latter as the special case when $b = 0$. This is clearly the branch $\Gamma_2 = 0$ in our case. The connection between the zero background limit along the branch $\Gamma_2 = 0$ and the solution of Sasa and Satsuma [16] has been proven in [4].

For convenience we give this solution in explicit form:

$$\psi(x,\tau) = 2\eta_{\rm S}e^{A_0 + iB_0}\exp\left[\frac{-i}{6\epsilon}\left(x - \frac{\tau}{18\epsilon}\right)\right]\frac{a\left(e^{2A_0}|a|^2\left(\eta_{\rm S}^2 + \xi_{\rm S}^2\right) + \xi_{\rm S}(\xi_{\rm S} - i\eta_{\rm S})\right)}{e^{2A_0}|a|^2\left(\eta_{\rm S}^2 + \xi_{\rm S}^2\right)\left(e^{2A_0}|a|^2\eta_{\rm S}^2 + 2\right) + \xi_{\rm S}^2}. \tag{48}$$

We can also rewrite the latter expression in the original form as given in [16]:

$$\psi(x,\tau) = \eta_{\rm S}e^{iB}\frac{2\cosh A + (c-1)e^{-A}}{\cosh(2A - \log|c|) + |c|} \tag{49}$$

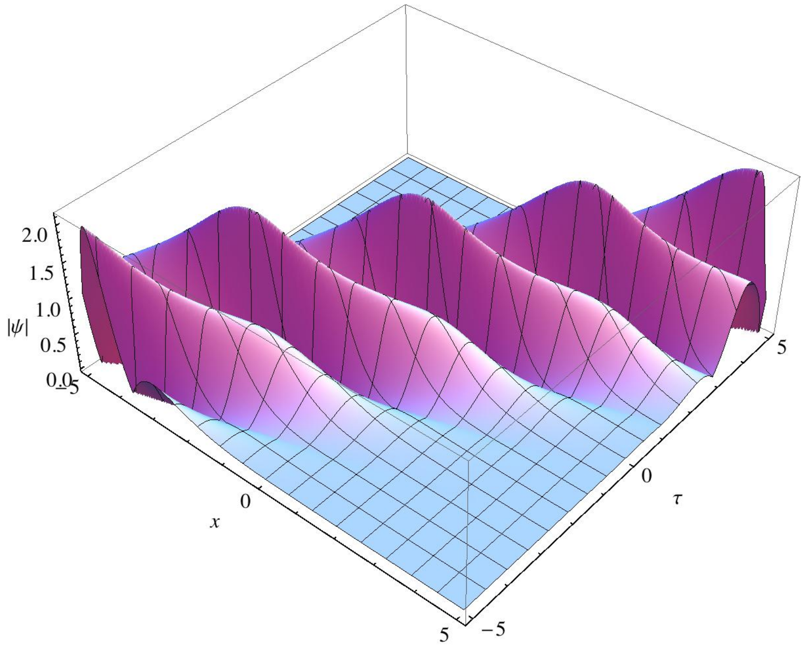

Figure 8: Solution (47) for $a = 1$, $\epsilon = 1/2$, $\xi_{\rm M} = 1/2$, $\eta_{\rm M} = 1.5$, $a = b$.

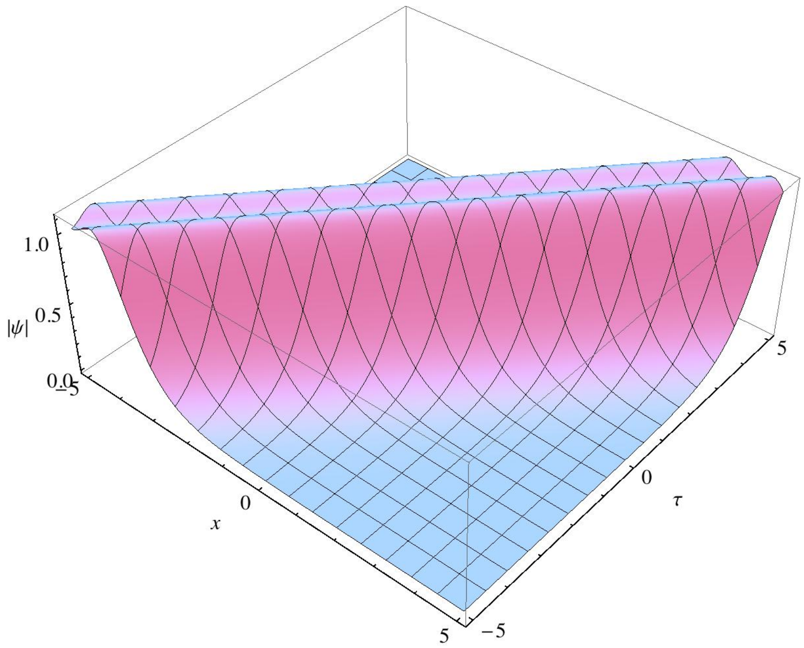

Figure 9: Solution (47) for $a = 1$, $\epsilon = 1/2$, $\xi_{\rm M} = 1/2$, $\eta_{\rm M} = 1.5$. This is a special limit (49) of the solution (47) when $b = 0$.

with

$$A = \eta_{\rm S}\left(x - x_0 - \left(\xi_{\rm S} + \epsilon\left(\eta_{\rm S}^2 - 3\xi_{\rm S}^2\right)\right)\tau\right)$$
$$B = \xi_{\rm S}\left(x - x_1 + \left(\frac{\eta_{\rm S}^2 - \xi_{\rm S}^2}{2\xi_{\rm S}} + \epsilon\left(\xi_{\rm S}^2 - 3\eta_{\rm S}^2\right)\right)\tau\right)$$

$$c = 1 - \frac{i\eta_{\rm S}}{\xi_{\rm S} - \frac{1}{6\epsilon}}; \quad |c| = \sqrt{1 + \frac{36\eta_{\rm S}^2\epsilon^2}{(1 - 6\xi_{\rm S}\epsilon)^2}}.$$

One characteristic example of solution (49) is shown in Fig. 9. This solution is more complicated than a sech-shaped solution. Namely, it exhibits a double peak structure [4]. The peak separation and the velocity can be adjusted via the free parameters $\xi_{\rm S}$ and $\eta_{\rm S}$. For $\epsilon \to 0$ the solution (49) has the NLSE limit which is the fundamental soliton of the NLSE, and, in addition, a singular limit when $|c| \to \infty$, see [4].

As a final remark we note, that Mihalache et al. [11] gave a breather-type solution for the case $a = b$:

$$\psi(x,\tau) = \frac{2\eta_{\rm M}}{D} e^{A_0 + iB_0} \exp\left(\frac{-i}{6\epsilon}\left(x - \frac{\tau}{18\epsilon}\right)\right) \times \tag{50}$$
$$\Big(a|a\zeta|^2 - \bar{a}^3\eta_{\rm M}(\eta_{\rm M} - i\xi_{\rm M}) + a\xi_{\rm M}(\xi_{\rm M} - i\eta_{\rm M})e^{-2A_0}$$
$$+ e^{-2iB_0}\bar{a}\xi_{\rm M}\left(2|a|(\xi_{\rm M} - i\eta_{\rm M}) + (\xi_{\rm M} + i\eta_{\rm M})e^{-2A_0}\right)\Big)$$
$$D = -4e^{2A_0}\eta_{\rm M}^2\left(\bar{a}^4 - |a|^4|\zeta|^2\right) + 4\left(|a|^2|\zeta|^2 - \bar{a}^2\eta_{\rm M}^2\cos(2B_0)\right) + \xi_{\rm M}^2 e^{-2A_0}$$

which has indeed an oscillating behavior. In fact, such oscillations naturally occur if $a \neq 0$ and $b \neq 0$, i.e. $\Gamma_1 \neq 0$ and $\Gamma_2 \neq 0$ in our setting.

8 Conclusions

In this paper, we considered the most general case of a soliton on a background solutions to the Sasa-Satsuma equation. The solution has rich structure and admits several limiting cases that are important for applications. Among them, rogue waves and classical solitons. The zero background limit is also found. It contains, as particular cases, previously known soliton solutions.

A Expressions for Coefficients

In this Appendix, we provide the coefficients used in Eq.(39):

$$G_0 = -8e^{2B}K^3W\bar{Y}|S|^2\left(\bar{\zeta}|Y|^2 + \zeta\,|\Gamma_2|^2|W|^2\right) \times$$
$$\left\{\Gamma_2 W\bar{\zeta}\left(\bar{\Gamma}_1 e^{\bar{A}+B} + \bar{\Gamma}_2 + e^B\right) - \zeta\bar{Y}\left(e^A\Gamma_1 + e^B\Gamma_2 + 1\right)\right\}$$
$$f_{30} = 16e^{2B}K^4\left||S|^2\left(\bar{\zeta}|Y|^2 + \zeta\,|\Gamma_2|^2|W|^2\right)\right|^2$$

$f_{32} =$

$$
\begin{aligned}
&-8e^{B}K^{3}|S|^{2}\times\\
&\Big\{2\Gamma_1 K\bar{Y}e^{A+B}|W|^2\left(2\zeta^3\bar{S}\left|\Gamma_2\right|^2+2|\zeta|^2\left(\bar{W}\left|\Gamma_2\right|^2\left(\bar{\zeta}\bar{Y}-2Y\Re(\zeta)\right)-2i|Y|^2\Im(\zeta)\right)\right)+\\
&2KY\bar{\Gamma}_1e^{\bar{A}+B}|W|^2\left(2W\left|\Gamma_2\right|^2\left(|\zeta|^2\left(\zeta Y-2\bar{Y}\Re(\zeta)\right)+Y\bar{\zeta}^3\right)+4i\Im(\zeta)|\zeta Y|^2\right)+\\
&4\Gamma_2 S\bar{\Gamma}_1e^{\bar{A}+2B}|Y|^2\left(S\bar{\zeta}^3+|\zeta|^2\left(2i|W|^2\Im(\zeta)\left|\Gamma_2\right|^2-2Y\bar{W}\Re(\zeta)+\zeta S\right)\right)-\\
&4\Gamma_2 KS\bar{\Gamma}_1^2\Im(\zeta)^2e^{2\left(\bar{A}+B\right)}|S|^2-\\
&4Ke^{\bar{A}+A+B}\left|\Gamma_1\zeta S\right|^2\Big(\left|\Gamma_2\right|^2\left|W\right|^2+|Y|^2\Big)+\\
&2e^{A}\Gamma_1K\bar{\Gamma}_2\bar{S}Y\left(2\zeta^3\bar{S}+2|\zeta|^2\left(\bar{Y}\left(\bar{\zeta}\bar{W}-2W\Re(\zeta)\right)-2i|W|^2\Im(\zeta)\left|\Gamma_2\right|^2\right)\right)-\\
&4e^{2A}\Gamma_1^2\bar{\Gamma}_2K\bar{S}\Im(\zeta)^2|S|^2+\\
&e^{2B}\Gamma_2|S|^2 6\Big(\bar{Y}\Big(\Re[\zeta^2]\Big(S-|W\Gamma_2|^2\Big)-W|\zeta Y|^2\Big)+W|\Gamma_2\zeta W|^2\Big)-\\
&e^{2B}\Gamma_2|S|^2 4K\Big(\Im[\zeta]^2\left(\bar{S}\left|\Gamma_2\right|^2+S\right)+|\zeta|^2\Big(|W\Gamma_2|^2+|Y|^2\Big)\Big)-\\
&4e^{B}K|S|^2\Big(\left|\Gamma_2\right|^2\left(4\Im[\zeta]^2\Re[S]+\left(|W|^2+|Y|^2\right)|\zeta|^2\right)+|W|^2\left|\Gamma_2\right|^4|\zeta|^2+|Y\zeta|^2\Big)-\\
&\bar{\Gamma}_2|S|^2\big(\left|\Gamma_2\right|^2\left(\bar{W}\left(6S\Re[\zeta^2]+|\zeta|^2\left(4S-6|W|^2\right)\right)+4KS\Im(\zeta)^2\right)\\
&-\left(\bar{S}\left(6Y\Re[\zeta^2]-4K\Im(\zeta)^2\right)-4K|Y\zeta|^2\right)\big)+\bar{\Gamma}_2|S|^2 6Y|Y\zeta|^2\Big\}
\end{aligned}
$$

where "$\Re[z]$" means the real part of z.

The following set of coefficients is used in Eq.(41):

$G_2 =$

$$
\begin{aligned}
&4e^{B}K^2W\bar{Y}\times\Bigg(3e^{A+2B}W\left|\zeta\right|^2\bar{Y}\left(\bar{\zeta}^2-\zeta^2\right)\Gamma_1\Gamma_2\left|W\right|^2+2e^{2A+B+\bar{A}}K\left|S\right|^2\bar{Y}\bar{\zeta}\Re(\zeta)\Gamma_1^2\bar{\Gamma}_1\\
&-e^{B}\left|S\right|^2\left|\Gamma_2\right|^2\left(\frac{K^2}{3}\left(\zeta^2+\left|\zeta\right|^2\left|\Gamma_2\right|^2+\bar{\zeta}^2\right)+16i\Im(\zeta)\Re(\zeta)^2K-3\left|\zeta\right|^2\Big(2\left|\zeta\right|^2+\zeta^2\left|\Gamma_2\right|^2\Big)\right)\\
&+2e^{2B}\left|S\right|^2\left(3\left(\bar{Y}-3\zeta\right)\left(\zeta\bar{Y}-W\left|\Gamma_2\right|^2\bar{\zeta}\right)+K\left(\left(\left(\left|\Gamma_2\right|^2+2\right)\bar{Y}\bar{\zeta}-W\zeta\left(2\left|\Gamma_2\right|^2+1\right)\right)\right)\Re(\zeta)\Gamma_2\right.\\
&+e^{B+\bar{A}}\Big(\left|Y\right|^2\left(K^2-4\zeta K+3\bar{\zeta}^2\right)\left|\zeta\right|^2+S\left|\Gamma_2\right|^2\bar{\zeta}\Big(-4K\zeta^2+3\left|\zeta\right|^2\bar{W}+K\bar{\zeta}\left(\bar{\zeta}-K\right)\Big)\Big)\bar{\Gamma}_1\left|W\right|^2\\
&-2e^{A+2\left(B+\bar{A}\right)}KW\zeta\left|S\right|^2\left|\Gamma_1\right|^2\Re(\zeta)\Gamma_2\bar{\Gamma}_1+e^{B}\left|S\right|^2 3\bar{W}\bar{Y}\left|\zeta\right|^2\\
&+e^{B}KW\Big(2ie^{2A}\Im(\zeta)\bar{Y}^2\bar{\zeta}\Gamma_1^2+e^{B+2\bar{A}}Y\left(2Y\bar{\zeta}^2-2\zeta\bar{Y}\Re(\zeta)\right)\Gamma_2\bar{\Gamma}_1^2\Big)\left|W\right|^2\\
&+6\left|S\right|^2\left(W^2\bar{\zeta}\left|\Gamma_2\right|^2+\zeta\bar{Y}^2\right)\Re(\zeta)\bar{\Gamma}_2-2e^{A+B}K\bar{W}\bar{Y}Y\zeta^2\left|\Gamma_2\right|^2\left(W^2+\bar{S}\right)\Gamma_1\\
&+2e^{A+B+\bar{A}}K\left|\Gamma_1\right|^2\Big(\left|S\right|^4\left(\bar{Y}-\left(W-2\bar{W}\right)\left|\Gamma_2\right|^2-2Y\right)\left|\zeta\right|^2+2|\zeta Y|^2\bar{S}\bar{\zeta}-2S\zeta^2\left|W\Gamma_2\right|^2\bar{\zeta}\Big)\\
&+2e^{A+B}K\bar{W}\bar{Y}\left|\zeta\right|^2\Big(S\left(3\bar{\zeta}-\zeta\right)\left|\Gamma_2\right|^2+2i\Im[Y]W\bar{Y}+2\left(W\left|\Gamma_2\right|^2+Y\right)\bar{Y}\bar{\zeta}\Big)\Gamma_1\\
&+2e^{A+2B+\bar{A}}KW\left|\Gamma_1\right|^2\bar{Y}\Big(Y\left|\zeta\right|^2\bar{W}\left(\bar{Y}-W\right)-2i\Im\left(S\zeta^2 2\bar{\zeta}\right)\Big)\Gamma_2\\
&+3e^{3B+\bar{A}}W\left|Y\right|^2\left|\zeta\right|^2\bar{Y}\left(\zeta^2-\bar{\zeta}^2\right)\Gamma_2^2\bar{\Gamma}_1+2e^{2B+\bar{A}}KS\bar{W}\left(\bar{Y}^2+S\right)\bar{\zeta}^2\Gamma_2\bar{\Gamma}_1\\
&+2e^{2B+\bar{A}}KS\Big(\left|\zeta\right|^2\bar{Y}\Big(W2i\Im[W]\left|\Gamma_2\right|^2-\bar{W}\left(3\zeta-\bar{\zeta}\right)-2W\bar{\zeta}\Big(\bar{W}\left|\Gamma_2\right|^2+\bar{Y}\Big)\zeta^2\Big)\Big)\Gamma_2\bar{\Gamma}_1\\
&+e^{A}\left|Y\right|^2\bar{W}\Big(W\left|\zeta\right|^2\left(3S+4K\bar{W}\right)\left|\Gamma_2\right|^2+\zeta\bar{Y}\left(3\bar{Y}\zeta^2-K\left(K+\bar{\zeta}\right)\zeta+4K\bar{\zeta}^2\right)\Big)\Gamma_1\bar{\Gamma}_2\\
&+K\left|Y\right|^2\bar{Y}\Big(e^{2A}\bar{W}\left(2W\bar{\zeta}\Re(\zeta)-2\zeta^2\bar{W}\right)\bar{\Gamma}_2\Gamma_1^2+2ie^{3B+2\bar{A}}W^2\zeta\Im(\zeta)\Gamma_2^2\bar{\Gamma}_1^2\Big)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
f_{34} = &\\
& 4e^{2B}K^2 \times \Big(4e^{2(A+\bar{A})}K^2 |S|^4 |\Gamma_1|^4 \Re(\zeta)^2 - 32K^2 |S|^2 |\Gamma_1|^2 \Re \left(e^{A+2\bar{A}} Y \zeta \Im(\zeta)^2 \bar{W}^2 \bar{\Gamma}_1 \right) \\
& + |S|^4 \left(9 (\zeta^2 + \bar{\zeta}^2) \left| \bar{Y} - W |\Gamma_2|^2 \right|^2 + 36 \left(|W|^2 + |Y|^2 \right) |\zeta|^2 |\Gamma_2|^2 \right) \\
& +4K^2 |S|^4 \left(|\Gamma_2|^4 + 4 |\Gamma_2|^2 + 1 \right) \Re(\zeta)^2 \\
& +48 |S|^4 \Im \left(e^B K \Im(\zeta) \left(\zeta |\Gamma_2|^2 + \bar{\zeta} \right) \Re(\zeta) \Gamma_2 \right) \\
& +4K |S|^4 \left(3\Re[Y] \left(1 - |\Gamma_2|^4 \right) \left(\Re[\zeta^2] - |\zeta|^2 \right) - 4K |\zeta|^2 |\Gamma_2|^2 \right) \\
& +2K \Re \left(e^{2A} \bar{Y}^2 \left(K \left(4 \left(\bar{Y}^2 - 2 |Y|^2 \right) |\zeta|^2 + 4Y^2 \Re(\zeta)^2 \right) - 12Y |Y|^2 \Im(\zeta)^2 \right) \Gamma_1^2 \right) |W|^4 \\
& +48 e^{A+\bar{A}} K |S|^2 |\zeta|^2 |\Gamma_1|^2 \Im(\zeta) \Im \left(e^B W^2 \bar{Y}^2 \Gamma_2 \right) \\
& +16 e^{A+\bar{A}} K^2 |S|^2 |\Gamma_1|^2 \Im (\zeta^2) \Im \left(e^{B+\bar{A}} W \zeta \bar{Y}^2 \Gamma_2 \bar{\Gamma}_1 \right) \\
& +8 e^{A+\bar{A}} K^2 |S|^2 |\zeta|^2 |\Gamma_1|^2 \Big(\Big(-2\Re[Y^2] + 3 |Y|^2 \left(|\Gamma_2|^2 + 1 \right) + |\Gamma_2|^2 \left(2 (\zeta^2 + \bar{\zeta}^2) - Y\bar{W} \right) \Big) |W|^2 \\
& \quad - |Y|^2 \left(|\Gamma_2|^2 \bar{W}^2 + Y\bar{W} + W\bar{Y} - 2 (\zeta^2 + \bar{\zeta}^2) \right) - 2W^2 |\Gamma_2|^2 \bar{Y} \Re(\zeta) \Big) \\
& +4K |S|^2 \Re \Big(e^{\bar{A}} S \big((\zeta^2 + \bar{\zeta}^2) \Big((6\bar{\zeta} - 3W |\Gamma_2|^2) |Y|^2 + 2K \left(2\bar{W} |\Gamma_2|^2 + \bar{Y} \right) \bar{\zeta} \Big) \\
& \quad -2 |\zeta|^2 \Big(3\bar{W} |Y|^2 + K (\bar{Y}^2 - \bar{W}\bar{Y} + 2Y\bar{W}) + |\Gamma_2|^2 \left(4KW\bar{\zeta} + 3\bar{W}\bar{Y} (\bar{\zeta} - 3\zeta) \right) \Big) \big) \bar{\Gamma}_1 \Big) \\
& -16K^2 |Y|^2 \Re \left(e^{B+2\bar{A}} S \Im(\zeta)^2 \left(-\zeta^2 + 4 |\zeta|^2 + \bar{\zeta}^2 + 3S \right) \Gamma_2 \bar{\Gamma}_1^2 \right) |W|^2 \\
& +8K |S|^2 \Re \Big(e^{A-B} \bar{W}\bar{Y} \big(3 \left(Y |\zeta \Gamma_2|^2 + (\bar{Y} - 2\zeta |\Gamma_2|^2) \Re[\zeta^2] \right) |W|^2 \\
& \qquad - |\zeta|^2 \left(2K \left(Y\bar{W} |\Gamma_2|^2 + 2\Re[Y\bar{W}] \right) + 3S (3\bar{\zeta} - \zeta) \right) \\
& \qquad +4KY \left(W\Re(\zeta)^2 - Y\Im(\zeta)^2 \right) + 2KW |\Gamma_2|^2 \left(Y\Re(\zeta)^2 - W\Im(\zeta)^2 \right) \big) \Gamma_1 \bar{\Gamma}_2 \Big) \\
& +2K |Y|^4 \Re \left\{ e^{2A-2B} \bar{W}^2 \left(12W |W|^2 \Im(\zeta)^2 + K \left(4 \left(\bar{W}^2 - 2 |W|^2 \right) |\zeta|^2 + 4W^2 \Re(\zeta)^2 \right) \right) \Gamma_1^2 \bar{\Gamma}_2^2 \right\} \Big)
\end{aligned}
$$

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