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Domain expression of the shape derivative and application to electrical impedance tomography

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Abstract

The well-known structure theorem of Hadamard-Zolésio states that the derivative of a shape functional is a distribution on the boundary of the domain depending only on the normal perturbations of a smooth enough boundary. However a volume representation (distributed shape derivative) is more general than the boundary form and allows to work with shapes having a lower regularity. It is customary in the shape optimization literature to assume regularity of the domains and use the boundary expression of the shape derivative for numerical algorithm. In this paper we describe the numerous advantages of the distributed shape derivative in terms of generality, easiness of computation and numerical implementation. We give several examples of numerical applications such as the inverse conductivity problem and the level set method.

Introduction

In his research on elastic plates [12] in 1907, Hadamard showed how to obtain the derivative of a shape functional $J(\Omega)$ by considering normal perturbations of the boundary $\partial\Omega$ of a smooth set Ω. This fundamental result of shape optimization was made rigorous later by Zolesio [8] in the so-called "structure theorem". When $J(\Omega)$ and the domain are smooth enough, one may also write the shape derivative as an integral over $\partial\Omega$, which is the canonical form in the shape optimization literature.

However, when Ω is less regular, the shape derivative can often be written as a domain integral even when the boundary expression is not available. The *domain expression* or *distributed shape derivative* has been generally ignored in the shape optimization literature for several reasons: firstly the boundary representation provides a straightforward way of determining an explicit descent direction since it depends linearly on the boundary perturbation θ and not on its gradient, secondly this descent direction only needs to be defined on the boundary. When considering the domain expression, these two advantages disappear as the shape derivative is defined on Ω and depends on the gradient of θ, so that a partial differential equation needs to be solved to obtain a descent direction θ on Ω.

It seems that these drawbacks would definitely rule out the distributed shape derivative, however they turn out to be less dramatic than expected in many situations and the domain formulation has other less foreseeable advantages over the boundary representation. In this paper we advocate for the use of the distributed shape derivative and discuss the advantages of this formulation.

The boundary representation has the following drawbacks. First of all if the data is not smooth enough the integral representation does not exist so that the more general domain representation is the only rigorous alternative. Even when the boundary representation exists and has the form $\int_{\partial\Omega} g\,\theta\cdot n$, it is usually not legitimate to choose $\theta\cdot n = -g$ on $\partial\Omega$ for a descent direction if g is not smooth enough, for instance if $g\in L^1(\partial\Omega)$. Therefore, a smoother θ must be chosen, which requires to solve a partial differential equation on the boundary $\partial\Omega$. When taking $\theta\cdot n = -g$ is legitimate, it might still not be desirable as this may yield a θ with low regularity, in which case one needs to regularize θ on the boundary as well. Therefore the first advantage of the boundary representation disappears.

The second advantage of the boundary representation is that the perturbation field only needs to be defined on the boundary instead of on the whole domain, reducing the cost of the computation. Actually, the distributed shape derivative also has its support on the boundary, and may be computed in a small neighbourhood of the boundary so that the additional cost is minimal. In addition, in most shape optimization applications, g is the restriction from a function defined in a neighborhood of the boundary and not a quantity depending only on the boundary such as the curvature. Therefore from a practical point of view, g must be evaluated in a neighbourhood of $\partial\Omega$ anyway. Also, in many numerical applications, θ must be extended to a neighborhood of Γ or even to the entire domain Ω. This is the case for level set methods for instance, where the level set function must be updated on Ω, or when one wishes to update the mesh along with the domain update, to avoid re-meshing the new domain. The distributed shape derivative then directly gives an extension of θ well-suited to the optimization problem.

Recent results have shown that the distributed shape derivative is also more accurate than the boundary representation from a numerical point of view; see [16] for a comparison. Indeed functions such as gradients of the state and adjoint state appearing in the distributed shape derivative only need to be defined at grid points and not on the interface. Therefore one avoids interpolation of these irregular terms. This is particularly useful for transmission problems where the boundary representation requires to compute the jump of a function over the interface, a delicate and error-prone operation from the numerical point of view.

Two other novel aspects of this paper are the introduction of a new Lagrangian method to compute the shape derivative of a cost function depending on a transmission problem in an efficient way, i.e. bypassing the computation of the shape derivative of the state equation, and the extension of the level set method to the case of the distributed shape derivative. We apply the Lagrangian approach to the problem of electrical impedance tomography in Section 4. Combining these techniques, we obtain a straightforward and general way of solving the shape optimization problem, from the computation of the shape derivative to the numerical implementation.

In Section 1 we recall the concept of shape derivative and the structure theorem. In Section 2 a Lagrangian method for transmission problems is described. In Section 3 we discuss gradient methods based on the distributed shape derivative and give several examples. In Section 4 we apply the results of Section 2 to the inverse problem of electrical impedance tomography. In Section 5 we extend the level set method to the case of the distributed shape derivative and finally in Section 6 we show numerical results for various problems including the problem of electrical impedance tomography.

1 Differentiation of shape functions

Let $\mathcal{P}(D)$ be the set of all subsets of $D \subset \mathbf{R}^d$, where the so-called "universe" $D \subset \mathbf{R}^d$ is assumed to be open. Henceforth we will work with the space of all $k-$times differentiable functions with compact support in D, i.e.

$$\mathrm{Lip}(D, \mathbf{R}^d) := \{f \in C^{0,1}(D, \mathbf{R}^d)| \quad \overline{\{x \in \mathbf{R}^d : \ f \neq 0\}} \subset D\}, \tag{1.1}$$

$$\mathcal{D}^k(D, \mathbf{R}^d) := \{f \in C^k(D, \mathbf{R}^d)| \quad \overline{\{x \in \mathbf{R}^d : \ f \neq 0\}} \subset D\}. \tag{1.2}$$

We have $\mathcal{D}^k(D, \mathbf{R}^d) \subset \mathrm{Lip}(D, \mathbf{R}^d)$ for $k \geq 1$. When $k = \infty$ one uses the notation $\mathcal{D}(D, \mathbf{R}^d) := \mathcal{D}^\infty(D, \mathbf{R}^d)$. Consider a vector field $\theta \in \mathrm{Lip}(D, \mathbf{R}^d)$ and the associated flow $\Phi_t(\theta) : \overline{D} \to D$,

$t \in [0, \tau]$ defined for each $x_0 \in \overline{D}$ as $\Phi_t(\theta)(x_0) := x(t)$, where $x : [0, \tau] \to \mathbf{R}$ solves

$$\begin{aligned} \dot{x}(t) &= \theta(x(t)) \quad \text{for } t \in (0, \tau), \\ x(0) &= x_0. \end{aligned} \tag{1.3}$$

We will sometimes use the simpler notation $\Phi_t = \Phi_t(\theta)$ when no confusion is possible. When $\theta \in \text{Lip}(D, \mathbf{R}^d)$ we have by Nagumo's theorem [21] that for fixed $t \in [0, \tau]$ the flow Φ_t is a homeomorphism on $\overline{D}$ and maps boundary onto boundary and interior onto interior. Further, we consider the family

$$\Omega_t := \Phi_t(\theta)(\Omega) \tag{1.4}$$

of perturbed domains. Let $\mathfrak{P}$ be a subset of $\mathcal{P}(D)$.

Definition 1.1 *Let $J : \mathfrak{P} \to \mathbf{R}$ be a shape functional and Θ be a topological vector subspace of $\mathrm{Lip}(D, \mathbf{R}^d)$. The Eulerian semi-derivative of J, when it exists, is defined by*

$$dJ(\Omega)[\theta] \stackrel{\text{def}}{=} \lim_{t \searrow 0} \left(J(\Omega_t) - J(\Omega) \right) / t \quad \text{exists for all } \theta \in \Theta. \tag{1.5}$$

(i) J is said to be shape differentiable at Ω in Θ' if it has a Eulerian semiderivative at Ω for all $\theta \in \Theta$ and the mapping

$$\begin{aligned} G : \Theta &\to \mathbf{R} \\ \theta &\mapsto dJ(\Omega; \theta) \end{aligned}$$

is linear and continuous, in which case $G(\theta)$ is called the shape derivative at Ω.

(ii) The smallest integer $k \geq 0$ for which G is continuous with respect to the $\mathcal{D}^k(D, \mathbf{R}^d)$-topology is called the order of G.

We have the following fundamental result of shape optimization which gives information about the structure of the shape derivative:

Theorem 1.2 (structure theorem) *Assume $\Gamma := \partial\Omega$ is compact and J is shape differentiable. Denote the shape derivative by*

$$G : \mathcal{D}(D, \mathbf{R}^d) \to \mathbf{R}, \quad G(\theta) := dJ(\Omega)[\theta]. \tag{1.6}$$

(i) Assume that G is of order $k \geq 0$, then G belongs to the Hilbert space $H^{-s}(D, \mathbf{R}^d)$ for some $s \geq 0$ depending on k. Moreover, by the representation theorem of Riesz there exists an element $g \in H^s(\mathbf{R}^d)$ such that

$$G(\theta) = \langle g, \theta \rangle_{H^s(\mathbf{R}^d)}. \tag{1.7}$$

(ii) If G is of order $k \geq 0$ and Γ of class C^{k+1} then there exists a distribution $g \in C^k(\Gamma)'$ such that

$$dJ(\Omega)[\theta] = < g, \gamma(\theta) \cdot n >_{C^k(\Gamma)', C^k(\Gamma)}, \tag{1.8}$$

where $\gamma : \mathcal{D}^k(D, \mathbf{R}^d) \to C^k(\Gamma \cap D, \mathbf{R}^d)$ is the trace operator.

(iii) Moreover, if $g \in L_1(\Gamma)$ then

$$dJ(\Omega)[\theta] = \int_\Gamma g\, \theta \cdot n\, ds. \tag{1.9}$$

Proof: See [8, pp. 480-481]. □

Remark 1.3 *The assumption "Γ must be at least of class C^1" of Theorem 1.2 (ii) may be lowered. It is sufficient for Ω to be of finite perimeter [18, p. 3 Theorem 1.3], in which case the shape derivative depends on $(\theta \cdot \nu)|_{\partial^*\Omega}$, where ν is the generalized normal and $\partial^*\Omega$ denotes the reduced boundary of Ω [31, p.233, Definition 5.5.1].*

In this paper we are interested in numerical methods for shape optimization problems of the type

$$\min_{\Omega \in \mathfrak{P}} J(\Omega), \tag{1.10}$$

where $\mathfrak{P} \subset \mathcal{P}(D)$ is the admissible set. Assume that the shape function $J : \mathfrak{P} \to \mathbf{R}$ is shape differentiable at $\Omega \subset D \subset \mathbf{R}^d$ i.e.

$$\frac{dJ(\Omega_t)}{dt}|_{t=0} = dJ(\Omega)[\theta] \text{ exists for all } \theta \in \mathrm{Lip}(D, \mathbf{R}^d) \tag{1.11}$$

and $G(\theta) = dJ(\Omega)[\theta]$ is of order $k \geq 0$.

Definition 1.4 (descent direction) *The vector field $\theta \in \Theta$ is called a descent direction for J at Ω if there exists a $\bar{t}$ such that*

$$J(\Omega_t) < J(\Omega) \text{ for all } t \in (0, \bar{t}].$$

If J is shape differentiable at Ω in Θ', then θ is a descent direction if

$$dJ(\Omega)[\theta] < 0. \tag{1.12}$$

Descent directions are used in iterative methods for finding approximate (possibly local) minimizers of $J(\Omega)$. Typically, at a given starting point Ω, one determines a descent direction θ and proceeds along this direction as long as the cost functional J reduces sufficiently using a step size strategy.

2 Shape derivatives via Lagrangian method

The Lagrangian method in shape optimization allows to compute the shape derivative of functions depending on the solution of partial differential equations without the need to compute the material derivative of the partial differential equations; see [8] for a description of the method in the linear case. With this approach the computation of the domain representation of the shape derivative is fast and the retrieval of the boundary form is also convenient. We extend here a result from [29], which allows to use the Lagrangian method without any saddle point assumptions unlike in [8].

Let E_1, E_2 and F_1, F_2 be linear spaces and introduce the function

$$\begin{aligned} G : [0,\tau] \times E_1 \times E_2 \times F_1 \times F_2 \to \mathbf{R}, \\ (t, \varphi_1, \varphi_2, \psi_1, \psi_2) \mapsto G(t, \varphi_1, \varphi_2, \psi_1, \psi_2). \end{aligned} \tag{2.1}$$

We make the following assumption for G.

Assumption 2.1 *For every* $(t, u_1, u_2, p_1, p_2) \in [0, \tau] \times E_1 \times E_2 \times F_1 \times F_2$ *the mappings*

$$\psi_2 \mapsto G(t, u_1, u_2, p_1, \psi_2), \qquad \psi_1 \mapsto G(t, u_1, u_2, \psi_1, p_2)$$

are affine-linear.

For $p = (p_1, p_2) \in F_1 \times F_2$ consider the problem: Find $(u_1, u_2) \in E_1 \times E_2$:

$$\begin{aligned} \partial_{\psi_1} G(t, u_1, u_2, p_1, p_2)(\hat{\psi}_1) = 0 \qquad \forall \hat{\psi}_1 \in F_1, \\ \partial_{\psi_2} G(t, u_1, u_2, p_1, p_2)(\hat{\psi}_2) = 0 \qquad \forall \hat{\psi}_2 \in F_2. \end{aligned} \tag{2.2}$$

Note that due to Assumption 2.1, the equations (2.2) are independent of $(p_1, p_2) \in F_1 \times F_2$. We define the set

$$\Lambda(t) := \{(u_1, u_2) \in E_1 \times E_2 | \; u_1 \text{ and } u_2 \text{ solve (2.2)}\}. \tag{2.3}$$

Now we select $u^t = (u_1^t, u_2^t) \in \Lambda(t)$ and $u = (u_1, u_2) \in \Lambda(0)$ and consider a solution $p^t = (p_1^t, p_2^t)$ of

$$\begin{aligned} \int_0^1 \partial_{\varphi_1} G(t, [u_1^t, u_1]_s, u_2^t, p_1^t, p_2^t)(\hat{\varphi}_1) ds = 0, \;\; \text{for all } \hat{\varphi}_1 \in E_1 \\ \int_0^1 \partial_{\varphi_2} G(t, u_1, [u_2^t, u_2]_s, p_1^t, p_2^t)(\hat{\varphi}_2) ds = 0, \;\; \text{for all } \hat{\varphi}_2 \in E_2, \end{aligned} \tag{2.4}$$

where $[u_i^t, u_i]_s := su_i^t + (1 - s)u_i$, $i = 1, 2$. We associate with this system the following solution set

$$\Upsilon(t) := \{p \in F_1 \times F_2 | \; \exists u^t \in \Lambda(t) \text{ and } u \in \Lambda(0) \text{ s.t. } p \text{ solves (2.4)}\}. \tag{2.5}$$

In addition to Assumption 2.1, we require:

Assumption 2.2 *For every* $(p_1, p_2, t) \in F_1 \times F_2 \times [0, \tau]$ *the mappings*

$$\varphi_1 \mapsto G(t, \tilde{u}_1 + s\varphi_1, u_2, p_1, p_2) \text{ and } \varphi_2 \mapsto G(t, u_1, \tilde{u}_2 + s\varphi_2, p_1, p_2)$$

are absolutely continuous and the partial derivatives are such that the integrals in (2.4) *are defined.*

Before proving the main theorem, we recall a simple fact.

Lemma 2.3 *Let* E *be linear space and* $f : E \to \mathbf{R}$ *a real valued function. Assume that for all* $x, y \in E$

$$s \mapsto f(sx + (1 - s)y),$$

is absolutely continuous. Then it holds

$$f(x) - f(y) = \int_0^1 \partial f(sx + (1 - s)y)(x - y) \, ds,$$

where

$$\partial f(sx + (1 - s)y)(x - y) := \lim_{t \to 0} \frac{f(y + (s + h)(x - y)) - f(y + h(x - y))}{t}.$$

Proof: We define the function $\varphi : [0,1] \to E$ by $\varphi(s) := f(sx + (1-s)y)$, where $s \in [0,1]$. Then by the fundamental theorem of calculus

$$f(x) - f(y) = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(s)ds = \int_0^1 \partial f(sx + (1-s)y)(x-y)\, ds.$$

$\square$

We have the following result for computing derivatives of shape functionals without computing the derivative of the state:

Theorem 2.4 *Let E_1, E_2, F_1, F_2 be Banach spaces, $\tau > 0$ a real number, and*

$$\begin{aligned} G : [0,\tau] \times E_1 \times E_2 \times F_1 \times F_2 \to \mathbf{R}, \\ (t, \varphi_1, \varphi_2, \psi_1, \psi_2) \mapsto G(t, \varphi_1, \varphi_2, \psi_1, \psi_2), \end{aligned}$$

be given. Assume the following conditions are satisfied:

- *(B1) Assumption 2.1 and Assumption 2.2 are satisfied.*
- *(B2) For all $t \in [0,\tau]$ the sets $\Upsilon(t)$ and $\Lambda(t)$ are not empty and single valued.*
- *(B3) For any sequence $(t_n)_{n\in\mathbf{N}}$ converging to zero, $t_n \to 0$ as $n \to \infty$, there exists a subsequence $(t_{n_k})_{k\in\mathbf{N}}$, an element $p = (p_1, p_2) \in \Upsilon(0)$ and for every $k \geq 1$ there is a $p^{n_k} = (p_1^{n_k}, p_2^{n_k}) \in \Upsilon(t_{n_k})$ such that for $u = (u_1, u_2) \in \Lambda(0)$*

$$\lim_{\substack{k\to\infty \\ t\searrow 0}} \partial_t G(t, u, p^{n_k}) = \partial_t G(0, u, p).$$

Then for all $(\psi_1, \psi_2) \in F_1 \times F_2$

$$\frac{d}{dt} G(t, u_1^t, u_2^t, \psi_1, \psi_2)|_{t=0} = \partial_t G(0, u_1, u_2, p_1, p_2).$$

Proof: Let $t \in [0,\tau]$ and $\bar{p}^t = (\bar{p}_1^t, \bar{p}_2^t) \in \Upsilon(t)$, $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \Upsilon(0)$, $u^t = (u_1^t, u_2^t) \in \Lambda(t)$, $u = (u_1, u_2) \in \Lambda(0)$ be given. Write

$$\begin{aligned} & G(t, u_1^t, u_2^t, \psi_1, \psi_2) - G(0, u_1, u_2, \psi_1, \psi_2) \\ &\quad = G(t, u_1^t, u_2^t, \bar{p}_1^t, \bar{p}_2^t) - G(0, u_1, u_2, \bar{p}_1, \bar{p}_2) \\ &\quad = G(t, u_1^t, u_2^t, \bar{p}_1^t, \bar{p}_2^t) - G(t, u_1, u_2^t, \bar{p}_1^t, \bar{p}_2^t) \\ &\quad + G(t, u_1, u_2^t, \bar{p}_1^t, \bar{p}_2^t) - G(t, u_1, u_2, \bar{p}_1^t, \bar{p}_2^t) \\ &\quad + G(t, u_1, u_2, \bar{p}_1^t, \bar{p}_2^t) - G(0, u_1, u_2, \bar{p}_1^t, \bar{p}_2^t), \end{aligned} \tag{2.6}$$

for all $\psi = (\psi_1, \psi_2) \in F_1 \times F_2$, where we used Assumption 2.1 to obtain

$$\begin{aligned} G(0, u_1, u_2, \bar{p}_1^t, \bar{p}_2^t) - G(0, u_1, u_2, \bar{p}_1, \bar{p}_2^t) = 0 \\ G(0, u_1, u_2, \bar{p}_1, \bar{p}_2^t) - G(0, u_1, u_2, \bar{p}_1, \bar{p}_2) = 0. \end{aligned}$$

By the mean value theorem, we find for each $t \in [0,\tau]$ an $\eta_t \in (0,1)$ such that

$$G(t, u_1, u_2, \bar{p}_1^t, \bar{p}_2^t) - G(0, u_1, u_2, \bar{p}_1^t, \bar{p}_2^t) = t\partial_t G(\eta_t t, u_1, u_2, \bar{p}_1^t, \bar{p}_2^t),$$

or briefly

$$G(t,u,\bar{p}^t) - G(0,u,\bar{p}^t) = t\partial_t G(\eta_t t, u, \bar{p}^t).$$

This equation, together with Lemma 2.3, yields that (2.6) can be written as

$$\begin{aligned} G(t,u^t,\psi) - G(0,u,\psi) = & \int_0^1 \partial_{\varphi_1} G(t,[u_1^t,u_1]_s, u_2^t, \bar{p}_1^t, \bar{p}_2^t)\left(u_1^t - u_1\right)\, ds \\ & + \int_0^1 \partial_{\varphi_2} G(t,u_1,[u_1^t,u_1]_s, \bar{p}_1^t, \bar{p}_2^t)\left(u_2^t - u_2\right)\, ds \\ & + t\partial_t G(\eta_t t, u, \bar{p}^t), \end{aligned}$$

for all $\psi = (\psi_1,\psi_2) \in F_1 \times F_2$. Using that $\bar{p}^t \in \Upsilon(t)$ and $(u_1^t - u_1) \in E_1,\ (u_2^t - u_2) \in E_2$, we get

$$G(t,u^t,\psi) - G(0,u,\psi) = t\partial_t G(\eta_t t, u, \bar{p}^t), \quad \text{for all } \psi \in F_1 \times F_2.$$

Let $\psi \in F_1 \times F_2$ be arbitrary and set $\delta(t) := G(t,u^t,\psi) - G(0,u,\psi)$. Define $\underline{dg}(0) := \liminf_{t\searrow 0} \delta(t)/t$ and $\overline{dg}(0) := \limsup_{t\searrow 0} \delta(t)/t$. There are sub-sequences $(l_n)_{n\in\mathbf{N}}$ and $(s_n)_{n\in\mathbf{N}}$ of $(t_n)_{n\in\mathbf{N}}$ such that

$$\lim_{n\to\infty} \delta(l_n)/l_n = \underline{dg}(0) \quad \text{and} \quad \lim_{n\to\infty} \delta(s_n)/s_n = \overline{dg}(0).$$

Owing to (B3), we deduce that for every $k \geq 1$ there is $p^{n_k} \in \Upsilon(l_{n_k})$ such that for $u \in \Lambda(0)$

$$\lim_{\substack{k\to\infty \\ t\searrow 0}} \partial_t G(t,u,p^{n_k}) = \partial_t G(0,u,p).$$

This shows that

$$\lim_{n\to\infty} \delta(l_n)/l_n = \lim_{k\to\infty} \delta(l_{n_k})/l_{n_k} = \underline{dg}(0) = \partial_t G(0,u,p),$$

and the same argumentation leads to

$$\lim_{n\to\infty} \delta(s_n)/s_n = \lim_{k\to\infty} \delta(s_{n_k})/s_{n_k} = \overline{dg}(0) = \partial_t G(0,u,p).$$

Finally, we conclude

$$\begin{aligned} \lim_{t\to 0} (G(t,u^t,\psi) - G(0,u,\psi))/t &= \underline{dg}(0) = \overline{dg}(0) \\ &= \lim_{t\searrow 0} \partial_t G(\eta_t t, u, \bar{p}^t) = \partial_t G(0,u,p). \end{aligned}$$

Since $\psi \in F_1 \times F_2$ was arbitrary the proof is finished. □

3 Gradient flow

In this section we give a general setting for computing descent directions in the framework of gradient methods using the domain and boundary representations of the shape derivative according to Theorem 1.2. We show how a descent direction θ with any regularity H^s, $s \geq 1$ can be obtained by solving an appropriate partial differential equation. We also show how to deal with constraints on θ and how to obtain specific displacements such as translations and rotations.

3.1 Smooth descent directions

In order to develop a setting allowing to define general descent directions, we recall sufficient conditions for the solvability of the following operator equation

$$Av = f,$$

where $A : \mathcal{H} \to \mathcal{H}'$ is an operator between a Banach space $\mathcal{H}$ and its dual $\mathcal{H}'$. Sufficient conditions for the bijectivity of A are given by the theorem of Minty-Browder [27, p.364, Theorem 10.49].

Theorem 3.1 (Minty-Browder) *Let $(\mathcal{H}; \|\cdot\|_{\mathcal{H}})$ be a reflexive separable Banach space and $A : \mathcal{H} \to \mathcal{H}'$ a bounded, hemi-continuous, monotone and coercive operator. Then A is surjective, i.e. for each $f \in \mathcal{H}'$ there exists $v \in \mathcal{H}$ such that $Av = f$. Moreover if A is strictly monotone then it is bijective.*

Let $A : \mathcal{H} \to \mathcal{H}'$ be an operator on a reflexive, separable Banach space $\mathcal{H}$ satisfying the assumptions of Theorem 3.1 with $A(0)v \geq 0$ for $v \in \mathcal{H}$. Denote $G := dJ(\Omega) \in \mathcal{H}'$. Introduce the bilinear form

$$b : \mathcal{H} \times \mathcal{H} \to \mathbf{R}, \qquad b(v, w) := \langle Av, w \rangle_{\mathcal{H}',\mathcal{H}}. \tag{3.1}$$

Consider the variational problem:

(VP) Find $v_1 \in \mathcal{H}$ such that $b(v_1, w) = -G[w]$ for all $w \in \mathcal{H}$,

Then the solution v_1 of (VP) is a descent direction since $G[v_1] = -b(v_1, v_1) \leq 0$.

In certain situations it is desirable to have bound constraints such as $v \geq 0$ on the shape perturbation. This may be handled by considering the more general case of a variational inequality. Given a subset $K \subset \mathcal{H}$ with $0 \in K$, consider the variational inequality:

(VI) Find $v_2 \in K$ such that $b(v_2, v_2 - w) \leq G[w - v_2]$ for all $w \in K$.

The solution v_2 of (VI) yields a descent direction for J at Ω since taking $w = 0 \in K$ we get $G[v_2] \leq -b(v_2, v_2) \leq 0$. In view of Theorem 1.2, we choose $\mathcal{H} \subset H^s(\Omega)$ where s is such that $G : H^s(D, \mathbf{R}^d) \to \mathbf{R}^d$ is continuous. When $\mathcal{H}$ is a Hilbert space, one may identify $\mathcal{H}'$ with $\mathcal{H}$. Therefore if b is bilinear, coercive, and continuous, then Lax Milgram's lemma ensures that (VP) has a unique solution. For all other cases we may have to use Theorem 3.1 or similar results.

Example 3.2 (H^1 flow - boundary shape gradient) *Let $D \subset \mathbf{R}^d$ and $\Omega \subset D$, $d = 2, 3$, be of class C^k, $k \geq 3$. Assume $J(\Omega)$ admits a shape derivative in a boundary form as in* (1.9)*:*

$$dJ(\Omega)[\theta] = \int_{\partial\Omega} g\,\theta \cdot n\, ds \tag{3.2}$$

where $g \in H^{k-3/2}(\partial\Omega)$. Take $\mathcal{H} = H^1(\Omega; \mathbf{R}^d)$ and $b(v, w) = (v, w)_{\mathcal{H}}$. Let θ be the solution of the variational problem (VP), i.e. θ solves

$$\begin{aligned} -\Delta\theta + \theta &= 0 \quad \text{in } \Omega, \\ -\partial_n\theta &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

By standard elliptic regularity, we get $\theta \in H^k(\Omega)$. Due to Sobolev imbeddings we get $\theta \in C^{k-2,\alpha}(\overline{\Omega})$, $k \geq 3$ and $0 < \alpha < 1$ for $d = 2$ or $0 < \alpha < 1/2$ for $d = 3$. Therefore the integral in (3.2) is well-defined. Since Ω is of class C^k we may extend θ to a function $\widetilde{\theta} \in C^{k-2,\alpha}(D)$ and $\widetilde{\theta}$ is a descent direction for $J(\Omega)$.

Example 3.3 (Transmission problem - boundary shape gradient) *Let $D \subset \mathbf{R}^d$ be the universe and consider the partition $D = \Omega^+ \cup \Omega^-$ where $\Omega^+ \Subset D$ is open with a C^k boundary, $k \geq 3$, and denote n its unit outward normal vector. Assume $J(\Omega^+)$ yields a boundary expression of the type* (1.9)*:*

$$dJ(\Omega^+)[\theta] = \int_{\partial\Omega^+} g\,\theta \cdot n\,ds \tag{3.4}$$

where $g \in H^{k-3/2}(\partial\Omega^+)$. Take $\mathcal{H} = H_0^1(D;\mathbf{R}^d)$ and $b(v,w) := (v,w)_{\mathcal{H}}$. The solution $\theta \in \mathcal{H}$ of (VP) solves the transmission problem

$$\int_D D\theta : D\xi + \theta\cdot\xi = \int_{\partial\Omega^+} g\xi\cdot n\,ds \quad \text{for all } \xi \in \mathcal{H},$$

which has the following strong form

$$\begin{aligned} -\Delta\theta^+ + \theta^+ &= 0 \quad && \text{in } \Omega^+ \\ -\Delta\theta^- + \theta^- &= 0 \quad && \text{in } \Omega^- \\ [\theta] = 0, \quad [\partial_n\theta] &= g \quad && \text{on } \partial\Omega^+ \\ \theta &= 0 \quad && \text{on } \partial D \end{aligned} \tag{3.5}$$

where $\theta = \theta^+\chi_{\Omega^+} + \theta^-\chi_{\Omega^-}$ and $[\partial_n\theta] := \partial_n\theta^+ - \partial_n\theta^-$. By standard elliptic regularity and since Ω^+ has a C^k boundary we get $\theta^+ \in H^k(\Omega^+)$ and $\theta^- \in H^k(\Omega^-)$. Using Sobolev imbeddings and the transmission conditions on $\partial\Omega^+$ in (3.5) *we get $\theta \in C^{0,1}(\overline{D})$. Therefore the integral in* (3.4) *is well-defined and θ is a descent direction for $J(\Omega^+)$.*

Example 3.4 (H^1 flow - distributed shape derivative) *Let $\Omega \subset D \subset \mathbf{R}^d$ be both of class C^3 and assume J has a shape derivative of the form*

$$dJ(\Omega)[\theta] = \int_\Omega F_\Omega[\theta]dx, \tag{3.6}$$

where $F_\Omega : \mathcal{D}^k(D,\mathbf{R}^d) \to H^1(\Omega,\mathbf{R})$, $k \geq 1$ is linear in θ. Let $\mathcal{H} = H^1(\Omega,\mathbf{R}^d)$ and $b(v,w) = (v,w)_{\mathcal{H}}$. Assume that F_Ω can be extended to a function

$$F_\Omega : H^1(D,\mathbf{R}^d) \to H^1(\Omega,\mathbf{R})$$

A solution of (VP) is defined in the variational sense by: find $\theta \in H^1(\Omega;\mathbf{R}^d)$ such that

$$\int_\Omega D\theta : D\xi + \theta\cdot\xi dx = -\int_\Omega F_\Omega[\xi]dx \quad \text{for all } \xi \in H^1(\Omega;\mathbf{R}^d). \tag{3.7}$$

By standard elliptic regularity, since Ω of class C^3, we get $\theta \in H^3(\Omega)$. Due to Sobolev imbeddings, $\theta \in C^1(\overline{\Omega})$ and θ can be extended to a function in $C^1(D)$.

Choosing $\mathcal{H} = H^1(D,\mathbf{R}^d)$ instead of $H^1(\Omega,\mathbf{R}^d)$ yields the transmission problem: find $\theta \in H^1(D;\mathbf{R}^d)$ such that

$$\int_D D\theta : D\xi + \theta\cdot\xi dx = -\int_\Omega F_\Omega[\xi]dx \quad \text{for all } \xi \in H^1(D;\mathbf{R}^d). \tag{3.8}$$

With such a choice we build an extension of θ to $\overline{D}$. By standard elliptic regularity, due to D of class C^3, we get $\theta|_{D\setminus\overline{\Omega}} \in H^3(D\setminus\overline{\Omega})$ and $\theta|_\Omega \in H^3(\Omega)$. According to Sobolev imbeddings this yields $\theta \in C^{0,1}(\overline{D})$.

Example 3.5 (H^k flow - distributed shape derivative) *Assume that D is bounded and smooth. Let $\Omega \subset D \subset \mathbf{R}^2$ be open, consider the space $\mathcal{H}^k = H^k(D,\mathbf{R}^d) \cap H^1_0(D,\mathbf{R}^d)$ for $k \geq 0$ and the bilinear form*

$$b(\theta,\xi) := (\theta,\xi)_{H^k(\Omega)}. \tag{3.9}$$

Assume J has a shape derivative $dJ(\Omega)[\theta]$ which belongs to the dual of $\mathcal{H}^k$. By the Riesz representation theorem the following equation admits a unique solution $\theta \in \mathcal{H}^k$:

$$(\theta,\xi)_{\mathcal{H}^k} = -dJ(\Omega)[\xi], \quad \text{for all } \xi \in \mathcal{H}^k. \tag{3.10}$$

By Sobolev embeddings, θ can be made arbitrary smooth choosing $k > 0$ large enough, which yields a smooth descent direction.

Example 3.6 (boundary versus domain representation) *Consider the simple example of the volume of a domain Ω:*

$$J(\Omega) = \int_\Omega 1\, dx.$$

When Ω is of class C^1, the normal vector n is $C^0(\partial\Omega)$ and

$$dJ(\Omega)[\theta] = \int_{\partial\Omega} \theta \cdot n\, ds. \tag{3.11}$$

In this case, we are in the framework of Example 3.3 and we can solve (VP) using the Laplacian to get enough regularity for θ.

When Ω is only measurable, the integral representation (3.11) *does not exist but we can still compute the domain representation*

$$dJ(\Omega)[\theta] = \int_\Omega \operatorname{div}(\theta)\, dx. \tag{3.12}$$

In this case, the required regularity $\theta \in W^{1,\infty}(D,\mathbf{R}^d)$ can be obtained by solving (VP) using $\mathcal{H}^k$ with k large enough as in Example 3.5. Using the Laplacian in this case would not provide enough regularity for θ. Obviously, the more irregular Ω is, the higher the order of the operator b has to be taken in (VP).

Using a bilinear form defined in Ω we obtain a vector field θ defined in the domain Ω and not only on the boundary (or the interface) $\partial\Omega$. This is often a desired property for numerical applications. For instance in the level set method, the common practice is to compute the boundary shape gradient, deduce from it a descent direction concentrated on $\partial\Omega$, and extend it on the entire domain Ω or on a narrow band around $\partial\Omega$ by solving a parabolic equation. In many applications, the knowledge of θ on the entire domain Ω is also required to move the mesh points at each iteration in order to avoid an expensive remeshing.

The distributed shape derivative also allows to obtain descent directions for domains with a low regularity such as Lipschitz domains as in example 3.6 unlike using the boundary representation which requires a minimal smoothness of Ω. From a numerical point of view, the domain expression (3.6) is also easier to handle and more accurate than the boundary expression (3.2) as it does not require to determine geometric quantities on the interface such as the normal vector or the curvature, and avoids the interpolation on the boundary $\partial\Omega$ of the derivatives of the state and adjoint states often appearing in the shape derivative (compare for instance the formulae in Proposition 4.2 and Proposition 4.5), which frequently leads to additional approximation errors.

3.2 Translations and rotations

In certain applications it may happen that the shape of an object is known while the location and orientation are to be found. It is then meaningful to use translations and rotations to move a given shape instead of working with a general class of shapes, and to considerably reduce the amount of variables in this way. As mentioned earlier in this section, in the shape optimization literature gradient flows are usually determined by using the boundary shape gradient (1.9) by taking $\theta \cdot n = -g$ and $\theta_\Gamma \equiv 0$. This approach produces a vector field θ which is normal to the boundary, and cannot be a translation in general, although this would be the natural transformation in applications where the shape is known but not the location.

In order to produce a transformation which is locally a translation, one needs to take a non-zero tangential component θ_Γ and an appropriate normal component $\theta \cdot n$. In $\mathbf{R}^d$ a transformation combining translation and rotation is a mapping $\Phi : \mathbf{R}^d \to \mathbf{R}^d$ which is locally of the form

$$\Phi(x) := \mathcal{O}x + b, \tag{3.13}$$

where $b \in \mathbf{R}^d$ and $\mathcal{O} \in \mathbf{R}^{d,d}$ is an orthogonal matrix, i.e. $\mathcal{O}\mathcal{O}^T = \mathcal{O}^T\mathcal{O} = I$. Therefore we can define a rotation requiring $D\Phi(D\Phi)^T = I$. For small t the flow Φ_t of the vector field $\theta \in \mathcal{D}(\mathbf{R}^d; \mathbf{R}^d)$ has the form

$$\Phi_t(x) = x + t\theta(x). \tag{3.14}$$

To obtain a translation one may assume

$$D\theta(D\theta)^T = I. \tag{3.15}$$

For a combination of translations and rotations in $\mathbf{R}^2$ we may choose a vector field θ which locally satisfies with $\beta = (\beta_1\ \beta_2)^T \in \mathbf{R}^2,\ \alpha \in \mathbf{R}$:

$$\theta(x) = \mathcal{O}(\alpha)x + \beta, \quad \mathcal{O}(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}. \tag{3.16}$$

The shape derivative is then determined by three parameters α, β_1, β_2. Note that in this case the vector field θ is not normal to the boundary of Ω. To determine parameters we can solve

$$\min_{(\alpha,\beta_1,\beta_2)\in\mathbf{R}^3} dJ(\Omega)[\mathcal{O}(\alpha)x + \beta]$$

Assume the shape derivative has the form (3.2), i.e.

$$dJ(\Omega)[\theta] = \int_{\partial\Omega} g\,\theta \cdot n\, ds.$$

Then plugging in $\theta(x) = \mathcal{O}x + \beta$ with $x = (x_1, x_2), n = (n_1, n_2)$ one obtains

$$\begin{aligned} dJ(\Omega)[\theta] = \cos(\alpha) \int_{\partial\Omega} g(n_1x_1 + n_2x_2) + \sin(\alpha) \int_{\partial\Omega} g(-n_1x_2 + n_2x_1) \\ + \beta_1 \int_{\partial\Omega} n_1 g + \beta_2 \int_{\partial\Omega} n_2 g. \end{aligned}$$

In view of this formula, one may choose the parameters

$$\alpha = -\arctan \frac{\int_{\partial\Omega} g(n_1x_1 + n_2x_2)}{\int_{\partial\Omega} g(-n_1x_2 + n_2x_1)},$$
$$\beta_1 = -\int_{\partial\Omega} n_1 g, \qquad \beta_2 = -\int_{\partial\Omega} n_2 g$$

to get a descent direction θ which is a translation and a rotation. Note that using the boundary form (3.2), θ is automatically a local transformation and rotation. This is often more meaningful for applications since we consider shapes contained in the universe D which is fixed, which implies that θ must vanish on the boundary of D, in which case we cannot take θ as a translation everywhere.

Using the volume form (3.6), the determination of the parameters α and β is not as straightforward for the reason mentioned above. One may choose θ as a piecewise linear function so that θ is a translation on the interface $\partial\Omega$ and vanishes on ∂D. Assume the shape derivative has the form

$$dJ(\Omega)[\theta] = \int_\Omega F_1[\theta_1] + F_2[\theta_2]dx,$$

where $F_1[\theta_1]$ and $F_2[\theta_2]$ are linear with respect to θ_1 and θ_2 with $\theta = (\theta_1, \theta_2)$. In order to obtain a transformation which is locally a translation, one may choose the following class of vector fields $\theta = (\beta_1\eta, \beta_2\eta)$ where η is a smooth function equal to one in a neighborhood Ω^* of Ω and equal to zero on ∂D. The choice of η depends on Ω and D and we have

$$dJ(\Omega)[\theta] = \beta_1 \int_\Omega F_1[\eta]dx + \beta_2 \int_\Omega F_2[\eta]dx.$$

Therefore a descent direction is easily found as

$$\beta_1 = -\int_\Omega F_1[\eta]dx, \quad \beta_2 = -\int_\Omega F_2[\eta]dx.$$

4 Electrical impedance tomography

We consider an application of the results above to a typical and important interface problem: the inverse problem of electrical impedance tomography (EIT) also known as the inverse conductivity or Calderón's problem [5] in the mathematical literature. It is an active field of research with an extensive literature; for further details we point the reader to the survey papers [4, 6] as well as [20] and the references therein. We consider the particular case where the objective is to reconstruct a piecewise constant conductivity σ which amounts to determine an interface Γ^+ between some inclusions and the background. We refer the reader to [7, 14, 15, 17] for more details on this approach.

The main interest of studying EIT is to apply the approach developed in this paper to a problem which epitomizes general interface problems and simultaneously covers the entire spectrum of difficulties encountered with severely ill-posed inverse problem.

4.1 Problem statement

Let $\Omega \subset \mathbf{R}^d$ be a Lipschitz domain, and $\Omega^+, \Omega^- \subset \Omega$ open sets such that $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$, where $\Gamma^+ = \partial\Omega^+ = \overline{\Omega^+} \cap \overline{\Omega^-}$ and $\Gamma = \partial\Omega$; see Figure 1. In this section n denotes either the outward normal vector to Ω or the outward normal vector to Ω^+. Decompose Γ as $\Gamma = \Gamma_D \cup \Gamma_N$. Let $\sigma = \sigma^+\chi_{\Omega^+} + \sigma^-\chi_{\Omega^-}$ where $\sigma^\pm$ are scalars and $f = f^+\chi_{\Omega^+} + f^-\chi_{\Omega^-}$ where $f^\pm \in H^1(\Omega)$.

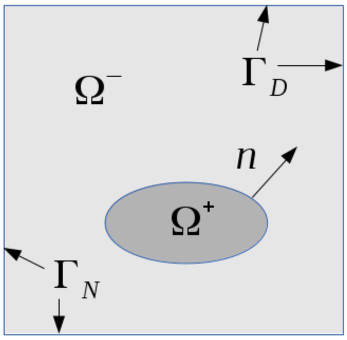


Figure 1: Partition $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$.

Consider the following problems: find $u_N \in H^1_D(\Omega)$

$$\int_\Omega \sigma \nabla u_N \cdot \nabla z = \int_\Omega fz + \int_{\Gamma_N} gz \text{ for all } z \in H^1_D(\Omega) \tag{4.1}$$

and find $u_D \in H^1_{DN}(\Omega)$ such that

$$\int_\Omega \sigma \nabla u_D \cdot \nabla z = \int_\Omega fz \text{ for all } z \in H^1_0(\Omega) \tag{4.2}$$

where

$$\begin{aligned} H^1_D(\Omega) &:= \{v \in H^1(\Omega) \mid \ v = 0 \text{ on } \Gamma_D\}, \\ H^1_{DN}(\Omega) &:= \{v \in H^1(\Omega) \mid \ v = 0 \text{ on } \Gamma_D, v = h \text{ on } \Gamma_N\}, \\ H^1_0(\Omega) &:= \{v \in H^1(\Omega) \mid \ v = 0 \text{ on } \Gamma\} \end{aligned}$$

and $g \in H^{-1/2}(\Gamma_N)$ represents the input, in this case the electric current applied on the boundary and $h \in H^{1/2}(\Gamma_N)$ is the measurement of the potential on Γ_N, or the other way around, i.e. h can be the input and g the measurement. Define also the space

$$PH^k(\Omega) := \{u = u^+\chi_{\Omega^+} + u^-\chi_{\Omega^-} \mid u^+ \in H^k(\Omega^+), \ u^- \in H^k(\Omega^-)\}.$$

Consider the following assumption:

Assumption 4.1 *The domains $\Omega, \Omega^+, \Omega^-$ are of class C^k, $f \in PH^{\max(k-2,1)}(\Omega)$, $g \in H^{k-\frac{3}{2}}(\Omega)$ and $h \in H^{k-\frac{1}{2}}(\Omega)$ for $k \geq 2$.*

Applying Green's formula under assumption 4.1, equations (4.1) and (4.2) are equivalent to the following transmission problems where $u_N = u_N^+\chi_{\Omega^+} + u_N^-\chi_{\Omega^-}$ and $u_D = u_D^+\chi_{\Omega^+} + u_D^-\chi_{\Omega^-}$:

$$-\sigma^+\Delta u_N^+ = f \text{ in } \Omega^+, \quad -\sigma^-\Delta u_N^- = f \text{ in } \Omega^-, \tag{4.3}$$
$$u_N^- = 0 \text{ on } \Gamma_D, \tag{4.4}$$
$$\sigma^-\partial_n u_N^- = g \text{ on } \Gamma_N, \tag{4.5}$$

$$-\sigma^+\Delta u_D^+ = f \text{ in } \Omega^+, \quad -\sigma^-\Delta u_D^- = f \text{ in } \Omega^-, \tag{4.6}$$
$$u_D^- = 0 \text{ on } \Gamma_D, \tag{4.7}$$
$$u_D^- = h \text{ on } \Gamma_N, \tag{4.8}$$

with the transmission conditions

$$\begin{aligned} \sigma^+\partial_n u_N^+ = \sigma^-\partial_n u_N^-, \qquad \sigma^+\partial_n u_D^+ = \sigma^-\partial_n u_D^- \quad \text{on } \Gamma^+ \\ u_N^+ = u_N^-, \qquad u_D^+ = u_D^- \quad \text{on } \Gamma^+ \end{aligned} \tag{4.9}$$

On Γ_D we impose homogeneous Dirichlet conditions, meaning that the voltage is fixed and no measurement is performed. One may take $\Gamma_D = \emptyset$, in which case (4.1) becomes a pure Neumann problem and additional care must be taken for the uniqueness and existence of a solution. The situation $\Gamma_D \neq \emptyset$ corresponds to partial measurements. Alternatively, it is also possible to consider functions u_N and u_D which have both the boundary conditions (4.5) and (4.8) on different parts of the boundary. Several measurements can be made by choosing a set of functions g or h. The result for several measurements can be straightforwardly deduced from the case of one measurement by summing the cost functionals corresponding to each measurement, therefore we stick to the case of one measurement g for simplicity of presentation.

The problem of electrical impedance tomography is:

(EIT): Given $\{g_k\}_{k=1}^K$ and $\{h_k\}_{k=1}^K$, find σ such that $u_D = u_N$ in Ω.

Note that $u_N = u_N(\Omega^+)$ and $u_D = u_D(\Omega^+)$ actually depend on Ω^+ through σ, however we often write u_N and u_D for simplicity.

The notion of well-posedness due to Hadamard requires the existence and uniqueness of a solution and the continuity of the inverse mapping. The severe ill-posedness of EIT is well-known: uniqueness and continuity of the inverse mapping depend on the regularity of σ, the latter being responsible for the instability of the reconstruction process. Additionally, partial measurements often encountered in practice render the inverse problem even more ill-posed. We refer to the reviews [4, 6, 20] and the references therein for more details. A standard cure against the ill-posedness is to regularize the inverse mapping. In this paper the regularization is achieved by considering smooth perturbations of the domains Ω^+.

To solve the EIT problem, we use an optimization approach by considering the shape functionals

$$J_1(\Omega^+) = \frac{1}{2}\int_\Omega (u_D(\Omega^+) - u_N(\Omega^+))^2, \tag{4.10}$$

$$J_2(\Omega^+) = \frac{1}{2}\int_{\Gamma_N} (u_N(\Omega^+) - h)^2. \tag{4.11}$$

Since $u_D, u_N \in H^1(\Omega)$ and $h \in H^{1/2}(\Gamma_N)$, J_1 and J_2 are well-defined. Note that J_1 and J_2 are redundant for the purpose of the reconstruction but our aim is to provide an efficient way of computing the shape derivative of two functions which are often encountered in the literature. To compute these derivatives we follow the new Lagrangian approach from [29]. First of all introduce

$$F_1(\varphi_D, \varphi_N) := \frac{1}{2}\int_\Omega (\varphi_D - \varphi_N)^2 \tag{4.12}$$

$$F_2(\varphi_N) := \frac{1}{2}\int_{\Gamma_N} (\varphi_N - h)^2. \tag{4.13}$$

Note that $J_1(\Omega^+) = F_1(u_D(\Omega^+), u_N(\Omega^+))$ and $J_2(\Omega^+) = F_2(u_N(\Omega^+))$. Next consider $\mathfrak{P}$ a subset of $\mathcal{P}(\Omega)$ and the Lagrangian $\mathcal{L} : \mathfrak{P} \times H_D^1 \times H_D^1 \times H_D^1 \times H_D^1 \times H^{1/2}(\Gamma_N) \to \mathbf{R}$:

$$\begin{aligned} \mathcal{L}(\Omega^+, \boldsymbol{\varphi}, \boldsymbol{\psi}, \lambda) := \alpha_1 F_1(\varphi_D, \varphi_N) + \alpha_2 F_2(\varphi_N) \\ + \int_\Omega \sigma \nabla \varphi_D \cdot \nabla \psi_D - f \psi_D + \int_{\Gamma_N} \lambda(\varphi_D - h) \\ + \int_\Omega \sigma \nabla \varphi_N \cdot \nabla \psi_N - f \psi_N - \int_{\Gamma_N} g \psi_N, \end{aligned} \tag{4.14}$$

where $\boldsymbol{\varphi} := (\varphi_D, \varphi_N)$ and $\boldsymbol{\psi} := (\psi_D, \psi_N)$. The adjoint variable λ is used to enforce the boundary condition (4.8); see (4.20). Introduce the objective functional

$$J(\Omega^+) := \alpha_1 J_1(\Omega^+) + \alpha_2 J_2(\Omega^+).$$

In order to compute the shape derivative for this linear transmission problem we would like to apply the Lagrangian approach based on the theorem of Correa-Seeger introduced in [8]. This application has been done in [3] and [30] for instance. However in these papers, the convexity of the Lagrangian with respect to $\boldsymbol{\varphi}$ is required to obtain the shape derivative. For the transmission problem in this section, the Lagrangian is not convex with respect to $\boldsymbol{\varphi}$. To deal with the Lagrangian depending on (4.3)-(4.8), we may adapt a result from [29] where an extension of the theorem of Correa-Seeger to non-linear partial differential equations is performed.

4.2 State and adjoint equations

Let us denote $\mathbf{u} := (u_D, u_N)$. Since $\mathcal{L}$ is affine linear in ψ_D and ψ_N, respectively, the derivative

$$\partial_{\psi_D} \mathcal{L}(\Omega^+, \mathbf{u}, \boldsymbol{p}, \lambda)(\hat{\psi}_D) = 0 \text{ for all } \hat{\psi}_D \in H_0^1(\Omega)$$

is independent of $\boldsymbol{p}$ and equivalent to

$$\int_\Omega \sigma \nabla u_D \cdot \nabla \hat{\psi}_D = \int_\Omega f \hat{\psi}_D \text{ for all } \hat{\psi}_D \in H_0^1(\Omega),$$

corresponding to (4.2). Under Assumption 4.1, we get $u_D \in PH^k(\Omega)$ and using Green's formula in Ω^+ with $\hat{\psi}_D \in C_c^\infty(\Omega^+)$, we obtain

$$\int_{\Omega^+} -\operatorname{div}(\sigma \nabla \varphi_D) \hat{\psi}_D = \int_{\Omega^+} f \hat{\psi}_D. \tag{4.15}$$

Repeating this operation in Ω^- yields (4.6) for the state u_D. By calculating

$$\partial_{\psi_N} \mathcal{L}(\Omega^+, \mathbf{u}, \boldsymbol{p}, \lambda)(\hat{\psi}_N) = 0, \text{ for all } \hat{\psi}_N \in H_D^1(\Omega),$$

we obtain

$$\int_\Omega \sigma \nabla u_N \cdot \nabla \hat{\psi}_N = \int_\Omega f \hat{\psi}_N dx + \int_{\Gamma_N} g \hat{\psi}_N ds \qquad \text{for all } \hat{\psi}_N \in H_D^1(\Omega),$$

corresponding to the variational equation (4.1). Using Green's formula in (4.15) shows that the solution u_N of the previous variational equation is a solution of (4.3) and (4.4).

Now by solving
$$\partial_\lambda \mathcal{L}(\Omega^+, \mathbf{u}, \boldsymbol{p}, \lambda)(\hat{\lambda}) = 0 \qquad \text{for all } \hat{\lambda} \in H^{1/2}(\Gamma_N),$$
we obtain
$$\int_{\Gamma_N} \hat{\lambda}(u_D - h) = 0, \text{ for all } \hat{\lambda} \in H^{1/2}(\Gamma_N),$$
which gives $u_D = h$ and the boundary condition (4.8) is satisfied.

Solving the equation
$$\partial_{\varphi_D} \mathcal{L}(\Omega^+, \mathbf{u}, \boldsymbol{p}, \lambda)(\hat{\varphi}_D) = 0, \text{ for all } \hat{\varphi}_D \in H^1_D(\Omega),$$
leads to
$$\alpha_1 \int_\Omega (u_D - u_N)\hat{\varphi}_D + \int_\Omega \sigma \nabla p_D \cdot \nabla \hat{\varphi}_D + \int_{\Gamma_N} \lambda \hat{\varphi}_D = 0, \tag{4.16}$$
for all $\hat{\varphi}_D \in H^1_D(\Omega)$, which is the variational formulation for the adjoint state p_D. This yields the following variational formulation when test functions are restricted to $H^1_0(\Omega)$:
$$\alpha_1 \int_\Omega (u_D - u_N)\widetilde{\varphi}_D + \int_\Omega \sigma \nabla p_D \cdot \nabla \widetilde{\varphi}_D = 0, \qquad \forall \widetilde{\varphi}_D \in H^1_0(\Omega). \tag{4.17}$$
Under Assumption 4.1, we get $p_D \in PH^k(\Omega)$ and using Green's formula in Ω^+ with $\hat{\varphi}_D \in C^\infty_c(\Omega^+)$ we obtain
$$\int_{\Omega^+} -\operatorname{div}(\sigma \nabla p_D)\hat{\varphi}_D = -\int_{\Omega^+} \alpha_1 (u_D - u_N)\hat{\varphi}_D \tag{4.18}$$
and a similar procedure provides the equation in Ω^-, which finally yields
$$-\operatorname{div}(\sigma \nabla p_D) = -\alpha_1 (u_D - u_N) \text{ in } \Omega^+ \text{ and } \Omega^-. \tag{4.19}$$
Now using Green's formula in Ω^+ and Ω^- for all $\hat{\varphi}_D \in H^1_D(\Omega)$ yields
$$\begin{aligned}\int_{\Omega^+ \cup \Omega^-} &\alpha_1 (u_D - u_N)\hat{\varphi}_D - \operatorname{div}(\sigma \nabla p_D)\hat{\varphi}_D \\ &+ \int_{\Gamma^+} [\sigma \partial_n p_D]_{\Gamma^+} \hat{\varphi}_D + \int_{\Gamma_N} (\sigma \partial_n p_D + \lambda)\hat{\varphi}_D = 0.\end{aligned}$$
where $[\sigma \partial_n p_D]_{\Gamma^+} = \sigma^+ \partial_n p_D^+ - \sigma^- \partial_n p_D^-$ is the jump of $\sigma \partial_n p_D$ across Γ^+. Using (4.19), we obtain
$$\lambda = -\sigma^- \partial_n p_D \text{ on } \Gamma_N, \tag{4.20}$$
$$p_D = 0 \text{ on } \Gamma, \tag{4.21}$$
$$\sigma^+ \partial_n p_D^+ = \sigma^- \partial_n p_D^- \text{ on } \Gamma^+. \tag{4.22}$$
Having determined λ we consider a new Lagrangian, using the same notation for simplicity:
$$\mathcal{L}(\Omega^+, \boldsymbol{\varphi}, \boldsymbol{\psi}) := \mathcal{L}(\Omega^+, \boldsymbol{\varphi}, \boldsymbol{\psi}, -\sigma^- \partial_n \psi_D),$$
for which we have the fundamental relation
$$J(\Omega^+) = \mathcal{L}(\Omega^+, \mathbf{u}, \boldsymbol{\psi}), \text{ for all } \boldsymbol{\psi} \in H^1_D(\Omega) \times H^1_D(\Omega), \tag{4.23}$$

where $\mathbf{u} = (u_D, u_N)$. Finally solving

$$\partial_{\varphi_N} \mathcal{L}(\Omega^+, \mathbf{u}, \boldsymbol{p})(\hat{\varphi}_N) = 0, \text{ for all } \hat{\varphi}_N \in H^1_D(\Omega),$$

leads to the variational formulation

$$\int_\Omega -\alpha_1(\varphi_D - \varphi_N)\hat{\varphi}_N + \sigma \nabla \psi_N \cdot \nabla \hat{\varphi}_N + \int_{\Gamma_N} \alpha_2(\varphi_N - h)\hat{\varphi}_N = 0 \tag{4.24}$$

for all $\hat{\varphi}_N \in H^1_D(\Omega)$. Under Assumption 4.1 we get $p_N \in PH^k(\Omega)$ and using Green's formula in Ω^+ with $\hat{\varphi}_N \in C_c^\infty(\Omega^+)$ we obtain

$$\int_{\Omega^+} -\operatorname{div}(\sigma \nabla p_N)\hat{\varphi}_N = \int_{\Omega^+} \alpha_1(u_D - u_N)\hat{\varphi}_N$$

and a similar procedure provides the equation in Ω^-, which yields finally

$$-\operatorname{div}(\sigma \nabla p_N) = \alpha_1(u_D - u_N) \text{ in } \Omega^+ \text{ and } \Omega^-. \tag{4.25}$$

Using Green's formula in Ω^+ and Ω^- for all $\hat{\varphi}_N \in H^1_D(\Omega)$ and $\varphi_N = 0$ on Γ_D yields

$$\begin{aligned}&\int_{\Omega^+ \cup \Omega^-} -\alpha_1(u_D - u_N)\hat{\varphi}_N - \operatorname{div}(\sigma \nabla p_N)\hat{\varphi}_N \\ &+ \int_{\Gamma^+} [\sigma \partial_n p_N]_{\Gamma^+} + \int_{\Gamma_N} (\sigma \partial_n p_N + \alpha_2(u_N - h))\hat{\varphi}_N = 0.\end{aligned}$$

This gives the boundary conditions for the adjoint:

$$\sigma \partial_n p_N = -\alpha_2(u_N - h) \text{ on } \Gamma_N, \tag{4.26}$$
$$p_N = 0 \text{ on } \Gamma_D. \tag{4.27}$$

with the transmission conditions

$$\sigma^+ \partial_n p_N^+ = \sigma^- \partial_n p_N^-, \quad p_N^+ = p_N^- \text{ on } \Gamma^+. \tag{4.28}$$

Summarizing, under Assumption 4.1 we obtain the system for p_N:

$$-\sigma^+ \Delta p_N^+ = \alpha_1(u_D^+ - u_N^+) \text{ in } \Omega^+, \quad -\sigma^- \Delta p_N^- = \alpha_1(u_D^- - u_N^-) \text{ in } \Omega^-, \tag{4.29}$$
$$\sigma^- \partial_n p_N^- = -\alpha_2(u_N^- - h^-) \text{ on } \Gamma_N, \tag{4.30}$$
$$p_N^- = 0 \text{ on } \Gamma_D, \tag{4.31}$$

4.3 Shape derivatives

Let us consider a transformation $\Phi_t(\theta)$ defined by (1.3) with $\theta \in \mathcal{D}^1(\Omega, \mathbf{R}^d)$. Note that $\Phi_t(\theta)(\Omega) = \Omega$ but in general $\Phi_t(\theta)(\Omega^+) \neq \Omega^+$. We use the notation $\Omega^+(t) := \Phi_t(\theta)(\Omega^+)$. Our aim is to show the shape differentiability of $J(\Omega^+)$ with the help of Theorem 2.4. For this purpose, introduce

$$G(t, \boldsymbol{\varphi}, \boldsymbol{\psi}) := \mathcal{L}(\Omega^+(t), \boldsymbol{\varphi} \circ \Phi_t^{-1}, \boldsymbol{\psi} \circ \Phi_t^{-1}), \tag{4.32}$$

which reads after the change of variables $\Phi_t(x) = y$

$$
\begin{aligned}
G(t, \boldsymbol{\varphi}, \boldsymbol{\psi}) = \frac{\alpha_1}{2} \int_\Omega (\varphi_D - \varphi_N)^2 j(t) + \frac{\alpha_2}{2} \int_{\Gamma_N} (\varphi_N - h)^2 \\
+ \int_\Omega \sigma A(t) \nabla \varphi_D \cdot \nabla \psi_D - f \circ \Phi_t \psi_D j(t) - \int_{\Gamma_N} \sigma^{-1} \partial_n \psi_D (\varphi_D - h) \\
+ \int_\Omega \sigma A(t) \nabla \varphi_N \cdot \nabla \psi_N - f \circ \Phi_t \psi_N j(t) - \int_{\Gamma_N} g \psi_N,
\end{aligned}
\tag{4.33}
$$

where the Jacobian $j(t)$ and $A(t)$ are defined as

$$
j(t) := \det(D\Phi_t) \tag{4.34}
$$

$$
A(t) := j(t) D\Phi_t^{-1} D\Phi_t^{-T} \tag{4.35}
$$

In the previous expression (4.33), one should note that the integrals on Γ are unchanged since $\Phi_t^{-1} = I$ on Γ. Thus we have $\Phi_t(\theta)(\Omega) = \Omega$, however the terms inside the integrals on Ω are modified by the change of variable since $\Phi_t^{-1} \neq I$ inside Ω. Note that

$$
J(\Omega^+(t)) = G(t, \mathbf{u}^t, \boldsymbol{\psi}), \text{ for all } \boldsymbol{\psi} \in H^1_D(\Omega) \times H^1_D(\Omega),
$$

where $\mathbf{u}^t = (u^t_N, u^t_D) := (u_{N,t} \circ \Phi_t, u_{D,t} \circ \Phi_t)$ and $u_{N,t}, u_{D,t}$ solve (4.1),(4.2), respectively, with the domain Ω^+ replaced by $\Omega^+(t)$. As one can verify by applying a change of variables to (4.1) and (4.2) on the domain $\Omega^+(t)$ the functions u^t_N, u^t_D satisfy

$$
\int_\Omega \sigma A(t) \nabla u^t_N \cdot \nabla \hat{\psi}_N = \int_\Omega f \hat{\psi}_N + \int_{\Gamma_N} g \hat{\psi}_N \text{ for all } \hat{\psi}_N \in H^1_D(\Omega) \tag{4.36}
$$

and

$$
\int_\Omega \sigma A(t) \nabla u^t_D \cdot \nabla \hat{\psi}_D = \int_\Omega f \hat{\psi}_D \text{ for all } \hat{\psi}_D \in H^1_0(\Omega) \tag{4.37}
$$

Testing equation (4.36) with $\hat{\psi}_D = u^t_D$ and equation (4.37) with $\hat{\psi}_N = u^t_N$, we infer the existence of constants $C_1, C_2 > 0$ and $\tau > 0$ such that for all $t \in [0, \tau]$:

$$
\|u^t_D\|_{H^1(\Omega)} \leq C_1, \quad \text{and } \|u^t_N\|_{H^1(\Omega)} \leq C_2. \tag{4.38}
$$

From these estimates, we get

$$
u^t_D \rightharpoonup w_1 \text{ and } u^t_N \rightharpoonup w_2 \text{ in } H^1(\Omega) \text{ as } t \to 0.
$$

Passing to the limit in (4.36) and (4.37) yields $w_1 = u_D$ and $w_2 = u_N$ by uniqueness.

Let us now check the conditions (B1)-(B3) of Theorem 2.4 for the function G given by (4.33) and the Banach spaces $E_1 = F_1 = E_2 = F_2 = H^1_D(\Omega)$. The condition (B1) is automatically satisfied by construction since the function G is affine linear with respect to φ_D and φ_N, and linear with respect to ψ_D and ψ_N. Concerning (B2), note that $\Lambda(t) = \{(u^t_N, u^t_D)\}$; see (2.3). We have $\bar{\boldsymbol{p}}^t = (\bar{p}^t_N, \bar{p}^t_D) \in$

$\Upsilon(t)$ if and only if they solve

$$\begin{aligned}
&\int_\Omega \sigma A(t)\nabla p_D^t\cdot\nabla\hat\varphi_D + \alpha_1\int_0^1\int_\Omega j(t)([u_D^t,u_D]_s - u_N^t)\hat\varphi_D dxds\\
&+\int_{\Gamma_N}\partial_n p_D^t\hat\varphi_D = 0,\\
&\int_\Omega \sigma A(t)\nabla p_N^t\cdot\nabla\hat\varphi_N - \alpha_1\int_0^1\int_\Omega j(t)(u_D - [u_N^t,u_N]_s)\hat\varphi_N dxds\\
&+\alpha_2\int_{\Gamma_N}(u_N-h)\partial_n\hat\varphi_N = 0,
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\int_\Omega \sigma A(t)\nabla p_D^t\cdot\nabla\hat\varphi_D + \alpha_1\int_\Omega j(t)(u_D^t + u_D - u_N^t)\hat\varphi_D\\
&+\int_{\Gamma_N}\partial_n p_D^t\hat\varphi_D = 0, \qquad (4.39)\\
&\int_\Omega \sigma A(t)\nabla p_N^t\cdot\nabla\hat\varphi_N - \alpha_1\int_\Omega j(t)(u_D - (u_N^t + u_N))\hat\varphi_N\\
&+\alpha_2\int_{\Gamma_N}(u_N-h)\partial_n\hat\varphi_N = 0 \qquad (4.40)
\end{aligned}$$

for all $\hat\varphi_D, \hat\varphi_N$ in $H^1_D(\Omega)$. Thanks to the Lax-Milgram's lemma, we check that both equations (4.39) and (4.40) have a unique solution. Testing (4.39) with $\hat\varphi_D = \bar p_D^t$ and (4.40) with $\hat\varphi_N = \bar p_N^t$, we conclude by an application of Hölder's inequality together with (4.38) the existence of constants C_1, C_2 and $\tau > 0$ such that for all $t \in [0,\tau]$

$$\|\bar p_D^t\|_{H^1(\Omega)} \le C_1, \quad \text{and } \|\bar p_N^t\|_{H^1(\Omega)} \le C_2.$$

We get $\bar p_D^t \rightharpoonup q_1$ and $\bar p_N^t \rightharpoonup q_2$ for two elements $q_1, q_2 \in H^1(\Omega)$. Passing to the limit in (4.39) and (4.40) yields $q_1 = p_D$ and $q_2 = p_N$ by uniqueness, where p_D and p_N are solutions of the adjoint equations. Finally, differentiating G with respect to t yields

$$\begin{aligned}
\partial_t G(t,\boldsymbol\varphi,\boldsymbol\psi) &= \frac{\alpha_1}{2}\int_\Omega(\varphi_D-\varphi_N)^2 j(t)\mathrm{tr}(D\theta_t D\Phi_t^{-1})\\
&+\int_\Omega \sigma A'(t)\nabla\varphi_D\cdot\nabla\psi_D - f\circ\Phi_t\psi_D j(t)\mathrm{tr}(D\theta_t D\Phi_t^{-1}) - \psi_D\nabla f\circ\Phi_t\cdot\theta_t j(t)\\
&+\int_\Omega \sigma A'(t)\nabla\varphi_N\cdot\nabla\psi_N - f\circ\Phi_t\psi_N j(t)\mathrm{tr}(D\theta_t D\Phi_t^{-1}) - \psi_N\nabla f\circ\Phi_t\cdot\theta_t j(t).
\end{aligned}$$

where $\theta_t = \theta\circ\Phi_t$, $A'(t) = \mathrm{tr}(\partial\theta^t D\Phi_t^{-1})A(t) - D\Phi_t^{-T}\partial\theta_t A(t) - (D\Phi_t^{-T}\partial\theta_t A(t))^T$ and $D\theta_t$ is the Jacobian matrix of θ_t. In view of $\theta \in \mathcal{D}^1(\Omega, \mathbf{R}^d)$, the functions $t \mapsto D\theta_t$ and $t \mapsto \mathrm{tr}(D\theta_t\Phi_t^{-1}) = \mathrm{div}\,(\theta)\circ\Phi_t$ are continuous on $[0,T]$. Moreover $\varphi_D, \varphi_N, \psi_D, \psi_N$ are in $H^1(\Omega)$, $f \in PH^1(\Omega)$ so that $\partial_t G(t,\boldsymbol\varphi,\boldsymbol\psi)$ is well-defined for all $t \in [0,T]$. Further it follows from the above formula that $(t,\boldsymbol\psi) \mapsto \partial_t G(t,\mathbf{u}^0,\boldsymbol\psi)$ is weakly continuous and therefore

$$\lim_{\substack{k\to\infty\\ t\searrow 0}} \partial_t G(t,\mathbf{u}^0,\bar{\boldsymbol p}^{n_k}) = \partial_t G(0,\mathbf{u}^0,\bar{\boldsymbol p}^0). \qquad (4.41)$$

Using Theorem 2.4 one concludes

$$dJ(\Omega^+)[\theta] = \frac{d}{dt}G(t, \mathbf{u}^t, \boldsymbol{\psi})|_{t=0} = \partial_t G(0, \mathbf{u}^0, \boldsymbol{p}^0) \text{ for all } \boldsymbol{\psi} \in H^1_D(\Omega) \times H^1_D(\Omega),$$

and therefore we have proved the following Proposition.

Proposition 4.2 (distributed shape derivative) *Let $\Omega \subset \mathbf{R}^d$ be a Lipschitz domain, $\theta \in \mathcal{D}^1(\Omega, \mathbf{R}^d)$, $f \in PH^1(\Omega)$, $g \in H^{-1/2}(\Gamma_N)$, $h \in H^{1/2}(\Gamma_N)$, $\Omega^+ \subset \Omega$ is an open set, then the shape derivative of $J(\Omega^+)$ is given by*

$$\begin{aligned} dJ(\Omega^+)[\theta] = &\int_\Omega \Big(\frac{\alpha_1}{2}(u_D - u_N)^2 - f(p_N + p_D)\Big) \operatorname{div}\theta \\ &+ \int_\Omega -(p_D + p_N)\nabla f \cdot \theta + \sigma A'(0)(\nabla u_D \cdot \nabla p_D + \nabla u_N \cdot \nabla p_N), \end{aligned} \tag{4.42}$$

where $A'(0) = (\operatorname{div}\theta)I - D\theta^T - D\theta$, u_N, u_D are solutions of the variational inequalities (4.1),(4.2) *and p_N, p_D of* (4.24),(4.17).

It is remarkable that the volume expression of the shape gradient in Proposition 4.2 corresponding to point (i) of Theorem 1.2 has been obtained without any regularity assumption on Ω^+. In order to obtain a boundary expression on the interface Γ^+ as in Theorem 1.2 (iii) we need more regularity of Ω^+ provided by Assumption 4.1.

Remark 4.3 *Note that* (4.42) *can be rewritten in a canonical form as*

$$dJ(\Omega^+)[\theta] = \int_\Omega \mathbb{S} : D\theta + \mathfrak{S} \cdot \theta, \tag{4.43}$$

where

$$\begin{aligned} \mathbb{S} = &-\sigma(\nabla u_D \otimes \nabla p_D + \nabla p_D \otimes \nabla u_D + \nabla u_N \otimes \nabla p_N + \nabla p_N \otimes \nabla u_N) \\ &+ \sigma(\nabla u_D \cdot \nabla p_D + \nabla u_N \cdot \nabla p_N)I + \Big(\frac{\alpha_1}{2}(u_D - u_N)^2 - f(p_N + p_D)\Big) I, \\ \mathfrak{S} = &-(p_D + p_N)\nabla f. \end{aligned}$$

The tensor $\mathbb{S}$ can be seen as a generalization of the Eshelby energy momentum tensor in continuum mechanics introduced in [9]; see also [22].

From now on we assume that Assumption 4.1 is satisfied. Then one may differentiate (4.32) directly using the following result for the differentiation of domain integrals; see [13] for instance.

Theorem 4.4 *Let $\Phi : [0, T[\to W^{1,\infty}(\mathbf{R}^d)$ differentiable at $t = 0$ with $\Phi(0) = I$ and $\Phi'(0) = \theta$. Assume $t \in [0, T[\to F(t, \cdot) \in L^1(\mathbf{R}^d)$ is differentiable at 0 and $F(0, \cdot) \in W^{1,1}(\mathbf{R}^d)$. Then*

$$\int_{\Omega^+(t)} F(t, x)dx$$

is differentiable at 0 and we have

$$\frac{d}{dt}\int_{\Omega^+(t)} F(t, x)dx\bigg|_{t=0} = \int_{\partial\Omega^+} F(0, x)\theta \cdot n\, ds + \int_{\Omega^+} \frac{\partial F}{\partial t}(0, x)dx \tag{4.44}$$

when Ω^+ is Lipschitz.

It is important to split the integrals using $\Omega = \Omega^+(t) \cup \Omega^-(t)$ in order to apply the above formula. Writing $n = n_{\Omega^+} = -n_{\Omega^-}$ we obtain

$$\begin{aligned}\partial_t G(0,\boldsymbol{\varphi},\boldsymbol{\psi}) = {} & \frac{\alpha_1}{2}\int_{\Gamma^+} \left[(u_D-u_N)^2\right]_{\Gamma^+} \theta\cdot n \\ & + \alpha_1 \int_\Omega (u_D-u_N)(\dot u_D - \dot u_N) \qquad (4.45)\\ & + \int_{\Gamma^+} \left[\sigma\nabla u_D\cdot\nabla p_D - f p_D + \sigma\nabla u_N\cdot\nabla p_N - f p_N\right]_{\Gamma^+} \theta\cdot n \\ & + \int_\Omega \sigma(\nabla u_D\cdot\nabla\dot p_D + \nabla\dot u_D\cdot\nabla p_D) + f\dot p_D \qquad (4.46)\\ & + \int_\Omega \sigma(\nabla u_N\cdot\nabla\dot p_N + \nabla\dot u_N\cdot\nabla p_N) + f\dot p_N \qquad (4.47)\end{aligned}$$

where

$$\dot\phi := \frac{d(\phi\circ\Phi_t^{-1})}{dt}|_{t=0} = -\nabla\phi\cdot\theta. \qquad (4.48)$$

and

$$[\phi]_{\Gamma^+}(x) := \lim_{z\to x, z\in\Omega^+}\phi(x) - \lim_{z\to x, z\in\Omega^-}\phi(x)$$

denotes the jump of a function ϕ across Γ^+. Thanks to Assumption 4.1 we have that u_N, u_D, p_D, p_N are in $PH^k(\Omega)$ for $k\geq 2$, thus $\dot u_N, \dot u_D, \dot p_D, \dot p_N$ are in $PH^{k-1}(\Omega)$ for $k\geq 2$. Using Green's formula in (4.46) and the fact that $\dot p_D\in PH^1(\Omega)$ has compact support in Ω we get

$$\begin{aligned}\int_\Omega \sigma\nabla u_D\cdot\nabla\dot p_D + \int_\Omega f\dot p_D &= \int_{\Omega^+}\sigma\nabla u_D\cdot\nabla\dot p_D + \int_{\Omega^-}\sigma\nabla u_D\cdot\nabla\dot p_D + \int_\Omega f\dot p_D \\ &= -\int_{\Omega^+}\operatorname{div}(\sigma\nabla u_D)\dot p_D - \int_{\Omega^-}\operatorname{div}(\sigma\nabla u_D)\dot p_D + \int_\Omega f\dot p_D \\ &\quad + \int_{\partial\Omega^+}\sigma\partial_n u_D\dot p_D - \int_{\partial\Omega^-}\sigma\partial_n u_D \dot p_D \\ &= -\int_{\Gamma^+}[\sigma\partial_n u_D\partial_n p_D]_{\Gamma^+}\theta\cdot n\end{aligned}$$

where we have used (4.6) with the test function $\dot p_D$. Using a similar calculation for the other terms in (4.46) and (4.47) using (4.3), (4.19), (4.25)-(4.26) and the test functions $\dot p_N, \dot u_D, \dot u_N \in PH^1(\Omega)$, we get

$$\begin{aligned}\partial_t G(0,\boldsymbol{\varphi},\boldsymbol{\psi}) = {} & \frac{\alpha_1}{2}\int_{\Gamma^+}\left[(u_D-u_N)^2\right]_{\Gamma^+}\theta\cdot n \\ & + \int_{\Gamma^+}\left[\sigma\nabla u_D\cdot\nabla p_D + \sigma\nabla u_N\cdot\nabla p_N\right]_{\Gamma^+}\theta\cdot n \\ & - 2\int_{\Gamma^+}[\sigma\partial_n u_D\partial_n p_D + \sigma\partial_n u_N\partial_n p_N]_{\Gamma^+}\theta\cdot n.\end{aligned}$$

Due to $u_D, u_N\in H^1(\Omega)$ we also have

$$\left[(u_D-u_N)^2\right]_{\Gamma^+} = 0.$$

For given $\phi\in C^1(\Gamma^+)$ and an arbitrary $C^1(U(\Gamma^+))$ extension $\tilde\phi : U(\Gamma^+)\to\mathbf{R}$ in a neighbourhood $U(\Gamma^+)$ of Γ^+, we define the tangential part of the gradient (or tangential gradient) as $\nabla_{\Gamma^+}\phi :=$

$\nabla\tilde{\phi}|_{\Gamma^+} - \partial_n\tilde{\phi}\, n$, which is independent of the extension, see [8, pp.492-493]. We may then consider the decomposition

$$\begin{aligned}[\sigma\nabla u_D \cdot \nabla p_D]_{\Gamma^+} &= [\sigma\partial_n u_D \cdot \partial_n p_D + \sigma\nabla_{\Gamma^+} u_N \cdot \nabla_{\Gamma^+} p_N]_{\Gamma^+} \\ &= [\sigma\partial_n u_D \cdot \partial_n p_D]_{\Gamma^+} + [\sigma]_{\Gamma^+}\, \nabla_{\Gamma^+} u_N \cdot \nabla_{\Gamma^+} p_N\end{aligned}$$

where we have used

$$[\nabla_{\Gamma^+} u_N \cdot \nabla_{\Gamma^+} p_N]_{\Gamma^+} = 0$$

since $u_N^+ = u_N^-$ and $p_N^+ = p_N^-$ on Γ^+. We have obtained the following result:

Proposition 4.5 (boundary expression) *Under Assumption 4.1 and $\theta \in \mathcal{D}^1(\Omega, \mathbf{R}^d)$ the shape derivative of $J(\Omega^+)$ is given by*

$$\begin{aligned}dJ(\Omega^+)[\theta] = &\int_{\Gamma^+} [\sigma(-\partial_n u_D \partial_n p_D - \partial_n u_N \partial_n p_N)]_{\Gamma^+}\, \theta \cdot n. \\ &+ \int_{\Gamma^+} [\sigma]_{\Gamma^+} (\nabla_{\Gamma^+} u_D \cdot \nabla_{\Gamma^+} p_D + \nabla_{\Gamma^+} u_N \cdot \nabla_{\Gamma^+} p_N)\theta \cdot n.\end{aligned}$$

Note that our results cover and generalize several results that can be found in the literature of shape optimization approaches for EIT, including [2, 14]. For instance when taking $\alpha_2 = 1$, $\alpha_1 = 0$ we get $p_D \equiv 0$ and consequently

$$dJ(\Omega^+)[\theta] = \int_{\Gamma^+} ([-\sigma\partial_n u_N \partial_n p_N]_{\Gamma^+} + [\sigma]_{\Gamma^+} \nabla_{\Gamma^+} u_N \cdot \nabla_{\Gamma^+} p_N)\theta \cdot n. \tag{4.49}$$

Formula (4.49) is the same as the one obtained in [2, pp. 533] (under the name $DJ_{DLS}(\omega).V$) by computing the shape derivative of u_N and u_D. The adjoint is given by

$$-\operatorname{div}(\sigma\nabla p_N) = 0, \tag{4.50}$$

$$\sigma\partial_n p_N = -(u_N - h). \tag{4.51}$$

According to Proposition 4.2 we have obtained the following more general domain expression which is valid for any open set Ω^+:

$$dJ(\Omega^+)[\theta] = \int_\Omega \sigma A'(0)\nabla u_N \cdot \nabla p_N - f p_N \operatorname{div}\theta - p_N \nabla f \cdot \theta. \tag{4.52}$$

The two formulas (4.49) and (4.52) are equal when Assumption 4.1 is satisfied.

Note also that from a numerical point of view, the boundary expression in Proposition 4.5 is delicate to compute compared to the domain expression in Proposition 4.2 for which the gradients of the state and adjoint states can be straightforwardly computed at grid points when using the finite element method for instance. The boundary expression, on the other hand, needs here the computation of the normal vector and the interpolation of the gradients on the interface Γ^+ which requires a precise description of the boundary and introduces an additional error.

5 Level set method

The level set method, originally introduced in [24], gives a general framework for the computation of evolving interfaces using an implicit representation of these interfaces. The core idea of this method is to represent the boundary of $\Omega \subset D \in \mathbf{R}^N$ as the level set of a continuous function $\phi : D \to \mathbf{R}$.

Let us consider the family of domains $\Omega_t \subset D$ as defined in (1.4). Each domain Ω_t can be defined as

$$\Omega_t := \{x \in D,\ \phi(x,t) < 0\} \tag{5.1}$$

where $\phi : D \times \mathbf{R}^+ \to \mathbf{R}$ is the so-called level set function. Indeed, we have

$$\partial\Omega_t = \{x \in D,\ \phi(x,t) = 0\}, \tag{5.2}$$

i.e. the boundary $\partial\Omega_t$ is the zero level set of $\phi(\cdot, t)$.

Let $x(t)$ be the position of a particle on the boundary $\partial\Omega_t$ moving with velocity $\dot{x}(t) = \theta(x(t))$ according to (1.3). Differentiating the relation $\phi(x(t), t) = 0$ with respect to t yields the Hamilton-Jacobi equation:

$$\partial_t\phi + \theta \cdot \nabla\phi = 0 \quad \text{in } \partial\Omega(t) \times \mathbf{R}^+, \tag{5.3}$$

which can be extended to D. Traditionally, the level set method has been designed to track smooth interfaces moving along the normal direction to the boundary. Theoretically, this is supported by Theorem 1.2 (ii) and (iii), i.e. if the domain Ω_t and the shape gradient are smooth enough then the shape derivative only depends on $\theta \cdot n$ on $\partial\Omega$. In this case, we may choose a vector field $\theta = \vartheta_n n$ for the optimization and by noting that the outward normal vector n to Ω_t is given by $n = \nabla\phi/|\nabla\phi|$ one obtains the level set equation

$$\partial_t\phi + \vartheta_n|\nabla\phi| = 0 \quad \text{in } D \times \mathbf{R}^+. \tag{5.4}$$

The initial data $\phi(x, 0) = \phi_0(x)$ accompanying the Hamilton-Jacobi equation (5.3) or (5.4) is chosen as the signed distance function to the initial boundary $\partial\Omega_0 = \partial\Omega$ i.e.

$$\phi_0(x) = \begin{cases} d(x, \partial\Omega_0), & \text{if } x \in (\Omega_0)^c, \\ -d(x, \partial\Omega_0), & \text{if } x \in \Omega_0. \end{cases} \tag{5.5}$$

5.1 Level set method and domain expression

In the case of the distributed shape derivative (3.6), ϕ is not governed by (5.4) but rather by the Hamilton-Jacobi equation (5.3) since we are given θ and not ϑ_n. Numerically it is actually more straightforward to use (5.3) instead of (5.4). Indeed, when using (5.4), ϑ_n is initially only given on $\partial\Omega_t$ (through the computation of the boundary shape derivative for instance) and must be extended to the entire domain D or at least to a narrow band around $\partial\Omega_t$. Therefore it is more natural to use (5.3) with θ already defined in D as is the case of the distributed shape derivative, providing a natural extension to D or to a narrow band around $\partial\Omega_t$.

In shape optimization, ϑ_n usually depends on the solution of several partial differential equations and their gradient. Since the boundary $\partial\Omega_t$ does not usually match the grid nodes where ϕ and the solutions of the partial differential equations are defined in the numerical application, the computation of ϑ_n requires the interpolation on $\partial\Omega_t$ of functions defined at the grid points only, complicating the numerical implementation and introducing an additional interpolation error. This is an issue in particular for interface problems where ϑ_n is the jump of a function across the interface, as in Proposition 4.5, which requires multiple interpolations and is error-prone. In the distributed shape derivative framework θ only needs to be defined at grid nodes.

5.2 Discretization of the level set equation

Let D be the unit square $D = (0,1) \times (0,1)$ to fix ideas. For the discretization of the Hamilton-Jacobi equation (5.3), we first define the mesh grid corresponding to D. We introduce the nodes P_{ij} whose coordinates are given by $(i\Delta x, j\Delta y)$, $1 \le i, j \le N$ where Δx and Δy are the steps discretization in the x and y directions respectively. Let us also write $t^k = k\Delta t$ the discrete time for $k \in \mathbb{N}$, where Δt is the time step. We are seeking for an approximation $\phi_{ij}^k \simeq \phi(P_{ij}, t^k)$.

In the usual level set method, the level set equation (5.4) is discretized using an explicit upwind scheme proposed by Osher and Sethian [23, 24, 28]. This scheme applies to the specific form (5.4) but is not suited to discretize (5.3) required for our application. Equation (5.3) is of the form

$$\partial_t \phi + H(\nabla\phi) = 0 \quad \text{in } D \times \mathbf{R}^+. \tag{5.6}$$

where $H(\nabla\phi) := \theta \cdot \nabla\phi$ is the so-called Hamiltonian. We use the Local Lax-Friedrichs flux originally conceived in [25] and which reduces in our case to:

$$\hat{H}^{LLF}(p^-, p^+, q^-, q^+) = H\left(\frac{p^- + p^+}{2}, \frac{q^- + q^+}{2}\right) - \frac{1}{2}(p^+ - p^-)\alpha^x - \frac{1}{2}(p^+ - p^-)\alpha^y$$

where $\alpha^x = |\theta_x|$, $\alpha^y = |\theta_y|$, $\theta = (\theta_x, \theta_y)$ and

$$\begin{aligned} p^- &= D_x^- \phi_{ij} = \frac{\phi_{ij} - \phi_{i-1,j}}{\Delta x}, & p^+ &= D_x^+ \phi_{ij} = \frac{\phi_{i+1,j} - \phi_{ij}}{\Delta x}, \\ q^- &= D_y^- \phi_{ij} = \frac{\phi_{ij} - \phi_{i,j-1}}{\Delta y}, & q^+ &= D_y^+ \phi_{ij} = \frac{\phi_{i,j+1} - \phi_{ij}}{\Delta y} \end{aligned}$$

are the backward and forward approximations of the x-derivative and y-derivative of ϕ at P_{ij}, respectively. Using a forward Euler time discretization, the numerical scheme corresponding to (5.3) is

$$\phi_{ij}^{k+1} = \phi_{ij}^k - \Delta t\, \hat{H}^{LLF}(p^-, p^+, q^-, q^+) \tag{5.7}$$

For numerical accuracy, the solution of the level set equation (5.3) should not be too flat or too steep. This is fulfilled for instance if ϕ is the distance function i.e. $|\nabla\phi| = 1$. Even if one initializes ϕ using a signed distance function, the solution ϕ of the level set equation (5.3) does not generally remain close to a distance function. We may occasionally perform a reinitialization of ϕ by solving a parabolic equation up to the stationary state; see [10, 11, 26]. Although in the level set method this reinitialization is standard, in the case of the distributed shape gradient, due to the regularization, the level set function ϕ stays close to a distance function during the iterations and we do not need to reinitialize.

The computational efficiency of the level set method can be improved by using the so-called "narrow band" approach introduced in [1], which consists in computing and updating the level set function only on a thin stripe around the interface. This allows to reduce the complexity of the problem to $N\log(N)$ instead of N^2 in two dimensions. For simplicity we do not implement this approach here but we mention that it can be easily generalized to the distributed shape derivative approach and equation (5.3) by taking θ with a support in a narrow band around the moving interface, which can be achieved by choosing the appropriate space $\mathcal{H}$ in (3.1).

6 Application and numerical results

6.1 Volume integral

Let $f \in W^{1,1}_{loc}(\mathbf{R}^2)$ and $\Omega \subset \mathbf{R}^2$ an open set. Assume that $\theta \in \text{Lip}(D;\mathbf{R}^2)$ Consider the cost functional

$$J(\Omega) = \int_\Omega f(x)dx. \tag{6.1}$$

According to [8], the distributed shape derivative of J exists and is given by

$$dJ(\Omega)[\theta] = \int_\Omega \operatorname{div}(\theta) f + \nabla f \cdot \theta dx. \tag{6.2}$$

We make use of the general framework of Section 3.1 and solve

$$(\theta, \xi)_{H^1(\Omega)} = -dJ(\Omega)[\xi], \text{ for all } \xi \in H^1(\Omega), \tag{6.3}$$

which is discretized using the finite element method. Two choices of f are used:

(i) $f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \in C^\infty(\mathbf{R}^2)$,

(ii) $f(x,y) = |x| + |y| - 1 \in W^{1,\infty}_{loc}(\mathbf{R}^2)$.

The optimal shape is an ellipse in case (i) and is the unit disk in the Mannathan norm, i.e. a square, in case (ii). In Figure 2 an example for case (i) with the choices $a = 2$ and $b = 3.5$ is given. The initial shape (heart-shaped) is not convex and not smooth, whereas the final shape (an ellipse) is convex and smooth. Case (ii) is illustrated in Figure 3. One observes that the method is able to create the corners of the square starting from a smooth boundary.

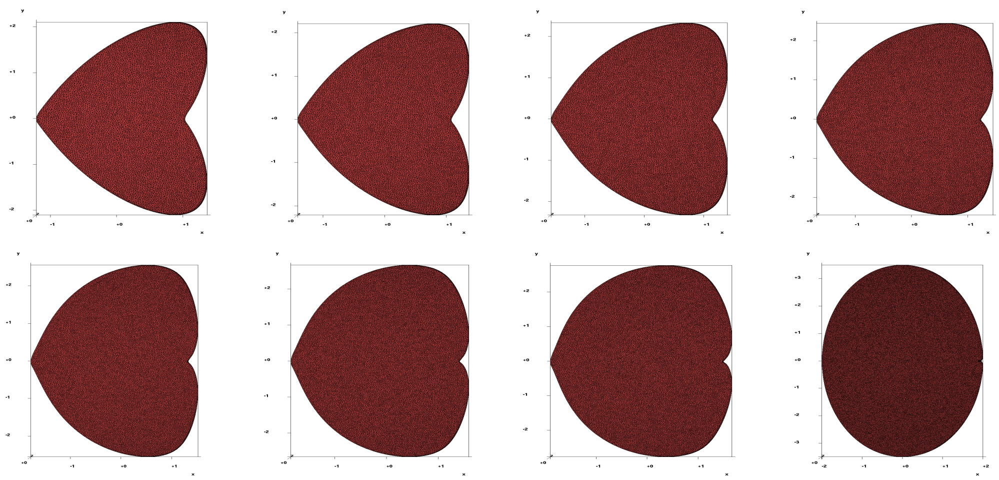


Figure 2: From left to right and top to bottom: iterations $0, 1, 2, 3, 4, 5, 7, 36$

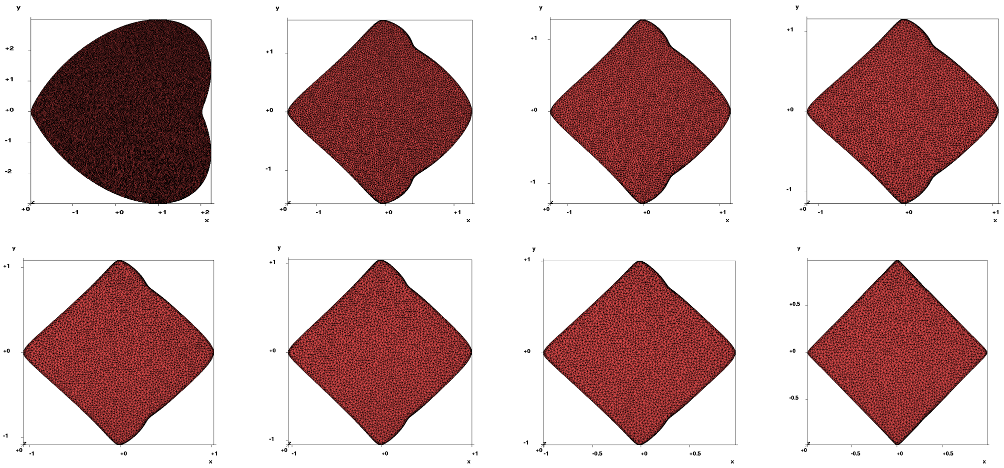


Figure 3: From left to right and top to bottom: iterations $1, 2, 3, 4, 5, 6, 7, 30$

6.2 The Dirichlet problem

The purpose of this example is twofold. On one hand we show that using the distributed shape derivative one may start with an irregular initial domain Ω, which is more difficult with the boundary shape gradient. On the other hand, we observe that an irregular domain can be more accurately approximated when the initial domain is irregular.

Let $\Omega \subset \mathbf{R}^d$ be open, bounded with Lipschitz boundary. Consider the cost functional

$$J(\Omega) = \frac{1}{2} \int_\Omega |u - u_d|^2 dx, \tag{6.4}$$

where $u_d \in H^1(\mathbf{R})$ is a given target and $u \in H^1_0(\Omega)$ denotes the weak solution of the Dirichlet problem

$$\begin{aligned} -\Delta u + u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

The shape derivative of (6.4) at $\Omega \subset \mathbf{R}^d$ in direction $\theta \in \mathcal{D}^1(\mathbf{R}^d, \mathbf{R}^d)$ reads (cf.[8, p.560, formula 5.62])

$$\begin{aligned} dJ(\Omega)[\theta] = &\int_\Omega A'(0)\nabla u \cdot \nabla p + \operatorname{div}(\theta)(up - fp) + \nabla f \cdot \theta\, p\, dx \\ &+ \int_\Omega \frac{1}{2}(u - u_d)^2 \operatorname{div}(\theta) - (u - u_d)\nabla u_d \cdot \theta\, dx. \end{aligned}$$

where $A'(0) = \operatorname{div}(\theta) I - D\theta^T - D\theta$. Here $u, p \in H^1_0(\Omega)$ solve

$$\int_\Omega \nabla u \cdot \nabla \varphi + u\varphi\, dx = \int_\Omega f\varphi \quad \forall \varphi \in H^1_0(\Omega) \tag{6.5}$$

$$\int_\Omega \nabla p \cdot \nabla \psi + p\psi\, dx = -\int_\Omega (u - u_d)\psi\, dx \quad \forall \psi \in H^1_0(\Omega) \tag{6.6}$$

Assuming Ω is of class C^3 and using the same bilinear form as in example 3.4 one obtains the regularity $\theta \in C^{0,1}(D; \mathbf{R}^2)$. Indeed $A'(0)$ and $\operatorname{div}(\theta)$ are then continuous and $u, p \in H^1(\Omega)$ so that $dJ(\Omega)[\theta]$ is well-defined.

We construct an analytical example for a precise comparison with the numerical reconstruction. Introduce the square $\Omega^* := (0,1) \times (0,1)$ and consider the problem

$$\begin{aligned} -\Delta u + u &= f \text{ in } \Omega^*, \\ u &= 0 \text{ on } \partial\Omega^*. \end{aligned} \tag{6.7}$$

where the right hand side is given by

$$f(x,y) := -2x(x-1) - 2y(y-1) + (x-1)(y-1)xy.$$

It may be checked that $u^*(x,y) = x(x-1)y(y-1)$ is the unique solution to (6.7). Set $u_d := u^*$, then Ω^* is a solution of

$$\min_{\Omega \subset \mathbf{R}^d} J(\Omega) = \frac{1}{2} \int_\Omega |u(\Omega) - u_d|^2 dx. \tag{6.8}$$

We start with an initial set Ω different from Ω^* and use the solution θ of the following problem as a descent direction:

$$\theta \in H^1(\Omega) : \quad b(\theta,\xi) = -dJ(\Omega)[\xi], \text{ for all } \xi \in H^1(\Omega), \tag{6.9}$$

where

$$b(\theta,\xi) := (D\theta, D\xi)_{L^2(\Omega)} + (\theta,\xi)_{L^2(\Omega)} + (\theta,\xi)_{L^2(\partial\Omega)}.$$

The evaluation of $dJ(\Omega)[\xi]$ requires the calculation of the state u and the adjoint state p solutions of (6.5)-(6.6) on Ω. The set Ω_n is updated by displacing the grid points according to

$$\Omega_{n+1} = \{x + t\theta(x) |\ x \in \Omega_n\}.$$

The results are displayed in Figure 4 where we have used P1 finite elements with $6 \cdot 10^4$ degrees of freedom. The simulations were performed using the toolbox PDElib from the WIAS. Two different initializations of the algorithm are compared. In Figure 4(a), the initial domain is a polygon, i.e. only a Lipschitz domain. One observes that it allows to reconstruct the square Ω^* accurately. Such a domain could not be used with the boundary shape gradient, since the normal is not defined at the corner points. In Figure 4 (b), the initial domain is a disk. We observe that this initialization leads to a rounded square instead of Ω^*, meaning that the algorithm is sensitive to the initial guess and a nonsmooth initialization seems more appropriate in order to reconstruct a nonsmooth shape.

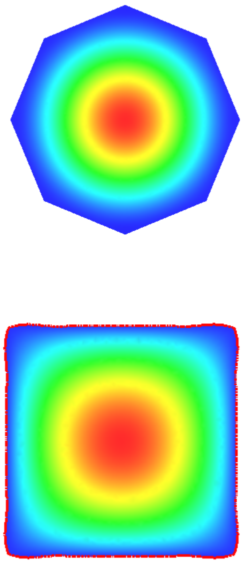
The results from Figure 6 were obtained as in (6.9), with the bilinear form

$$b(\theta,\xi) = (D\theta, D\xi)_{L^2(D)} + (\theta,\xi)_{L^2(D)} + (\theta,\xi)_{L^2(\partial D)} \tag{6.10}$$

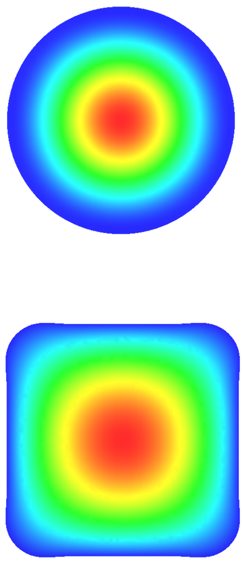
i.e. the variable sets Ω are all embedded in $D = (-3.5, 3.5) \times (-3.5, 3.5)$. In view of the structure theorem, the distributed shape derivative is concentrated on $\partial\Omega$, therefore a finer discretization is used in this area.

6.3 Electrical impedance tomography

In this section we give numerical results for the problem of electrical impedance tomography presented in Section 4.1. Using the notations of Section 4.1 we take $\Omega = (0,1) \times (0,1)$ and $\Gamma_D = \emptyset$, i.e. we have measurements on the entire boundary Γ. For easeness of implementation, we consider a slightly different problem than the one in Section 4.1. Denote Γ_t, Γ_b, Γ_l and Γ_r the four sides of the square,



(a) Initialization with polygon.



(b) Initialization with disk.

Figure 4: Two results using the distributed shape derivative and two different initial guess. First row: initial guess, second row: optimal solution after convergence.

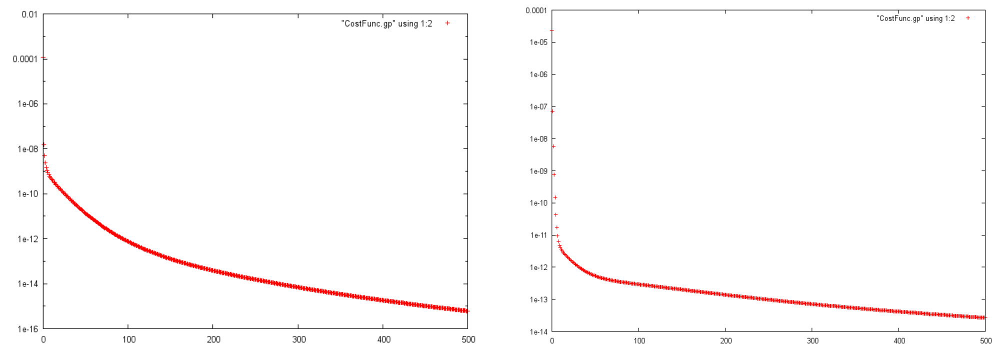


Figure 5: Convergence history in log scale corresponding to Figure 4.

where the indices t, b, l, r stands for top, bottom, left and right, respectively. We consider the following problems: find $u_N \in H^1_{tb}(\Omega)$

$$\int_\Omega \sigma \nabla u_N \cdot \nabla \varphi = \int_\Omega f\varphi + \int_{\Gamma_l \cup \Gamma_r} g\varphi \text{ for all } \varphi \in H^1_{0,tb}(\Omega) \tag{6.11}$$

and find $u_D \in H^1_{lr}(\Omega)$ such that

$$\int_\Omega \sigma \nabla u_D \cdot \nabla \varphi = \int_\Omega f\varphi + \int_{\Gamma_t \cup \Gamma_b} g\varphi \text{ for all } \varphi \in H^1_{0,lr}(\Omega) \tag{6.12}$$

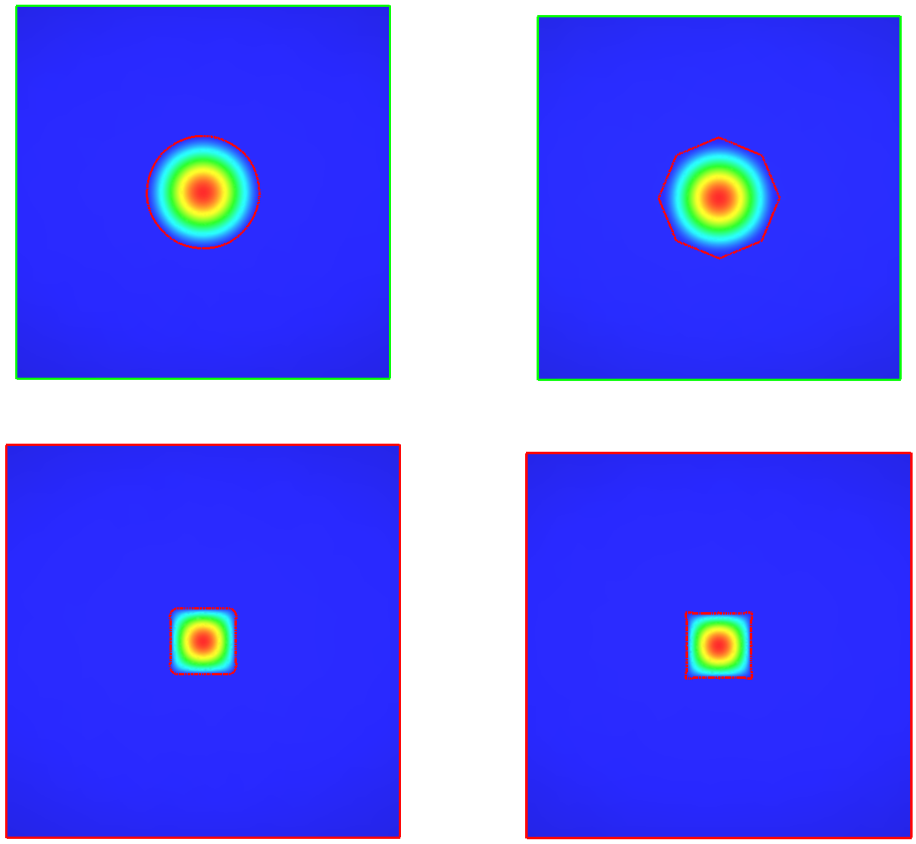


Figure 6: The domain Ω (enclosed by the red line) is embedded in D. Left column: initial shape is a circle (top). Right column: initial shape is an octagon (top). Colors indicate the values of u on its domain of definition. Blue is zero and red is a high value of u.

where

$$\begin{aligned} H^1_{tb}(\Omega) &:= \{v \in H^1(\Omega) \mid \ v = h \text{ on } \Gamma_t \cup \Gamma_b\}, \\ H^1_{lr}(\Omega) &:= \{v \in H^1(\Omega) \mid \ v = h \text{ on } \Gamma_l \cup \Gamma_r\}, \\ H^1_{0,tb}(\Omega) &:= \{v \in H^1(\Omega) \mid \ v = 0 \text{ on } \Gamma_t \cup \Gamma_b\}, \\ H^1_{0,lr}(\Omega) &:= \{v \in H^1(\Omega) \mid \ v = 0 \text{ on } \Gamma_l \cup \Gamma_r\}. \end{aligned}$$

The results of Section 4.1 can be straightforwardly extended to equations (6.11), (6.12) and using functional (4.10) leads to the same optimization problem.

We use the software package FEniCS for the implementation; see [19]. The domain Ω is meshed using a regular grid of 128×128 elements. The conductivity values are set to $\sigma_0 = 1$ and $\sigma_1 = 10$. We employ the regularization (VP) with the bilinear form

$$b(v, w) = \int_\Omega Dv : Dw.$$

We obtain measurements h_k corresponding to fluxes g_k, $k = 1, .., K$, by taking the trace on Γ of the solution of a Neumann problem where the fluxes are equal to g_k. To simulate real noisy EIT data, the measurements h_k are corrupted by adding a normal Gaussian noise with mean zero and standard deviation $\delta * \|h_k\|_\infty$, where δ is a parameter. The noise level is computed as

$$noise = \frac{\sum_{k=1}^{K} \|h_k - \tilde{h}_k\|_{L^2(\Gamma)}}{\sum_{k=1}^{K} \|h_k\|_{L^2(\Gamma)}} \tag{6.13}$$

where $\tilde{h}_k$ is the noisy measurement and h_k the synthetic measurement without noise on Γ.

We use the functional (4.10), i.e. in our context:

$$J(\Omega^+) = \frac{1}{2} \int_\Omega \sum_{k=1}^{K} |u_{D,k}(\Omega^+) - u_{N,k}(\Omega^+)|^2, \tag{6.14}$$

where $u_{D,k}$ and $u_{N,k}$ correspond to the different fluxes g_k.

Since we use a gradient-based method we implement an Armijo line search to adjust the time-stepping. The algorithm is stopped when the decrease of the functional becomes unsignificant, practically when the following stopping criterion is repeatedly satisfied:

$$J(\Omega_n^+) - J(\Omega_{n+1}^+) < \gamma(J(\Omega_0^+) - J(\Omega_1^+))$$

where Ω_n^+ denotes the n-th iterate of Ω^+. We take $\gamma = 5.10^{-5}$ in our tests.

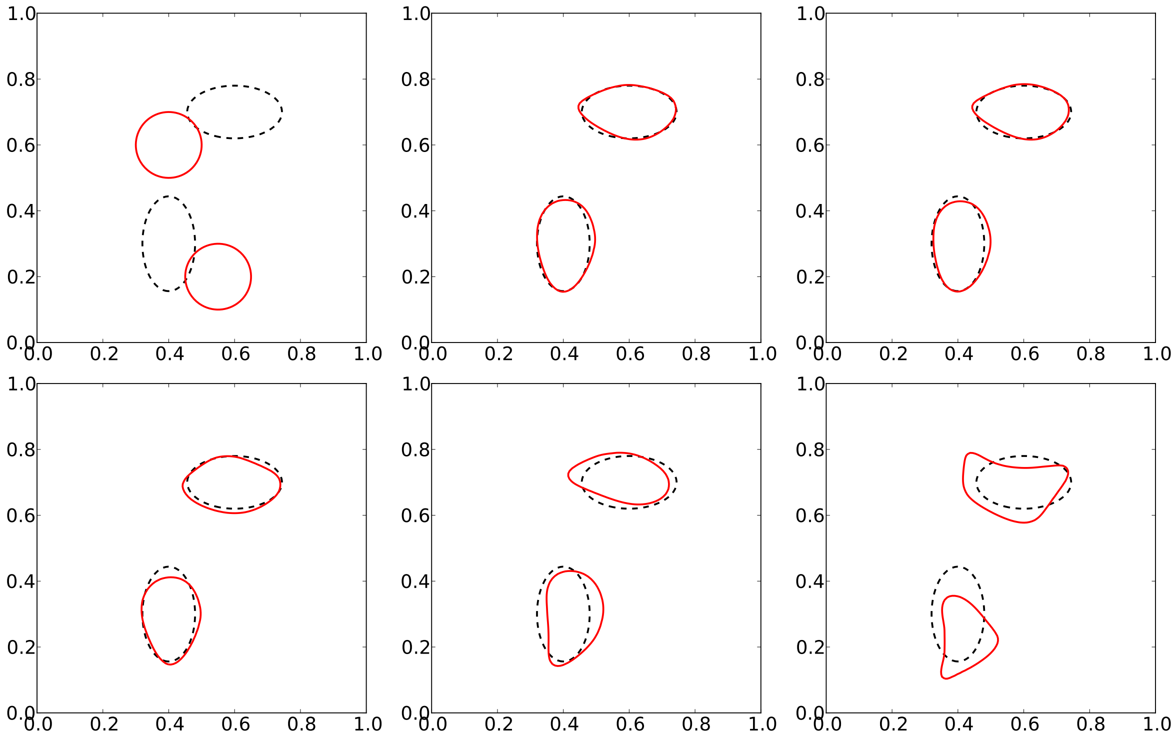


Figure 7: Reconstruction (continuous contours) of two ellipses (dashed contours) with different noise levels and using three measurements. From left to right and top to bottom: initialization (continuous contours - top left), 0% noise (367 iterations), 0.43% noise (338 iterations), 1.44% noise (334 iterations), 2.83% noise (310 iterations), 7% noise (356 iterations).

In Figure 7 we compare the reconstruction for different noise levels computed using 6.13. We take in this example $K = 3$, i.e. we use three fluxes g_k, $k = 1, 2, 3$, defined as follows:

$$g_1 = 1 \text{ on } \Gamma_l \cup \Gamma_r \text{ and } g_1 = -1 \text{ on } \Gamma_t \cup \Gamma_b,$$
$$g_2 = 1 \text{ on } \Gamma_l \cup \Gamma_t \text{ and } g_2 = -1 \text{ on } \Gamma_r \cup \Gamma_b,$$
$$g_3 = 1 \text{ on } \Gamma_l \cup \Gamma_b \text{ and } g_3 = -1 \text{ on } \Gamma_r \cup \Gamma_t.$$

Without noise, the reconstruction is very close to the true object and degrades as the measurements become increasingly noisy, as is usually the case in EIT. However, the reconstruction is quite robust with respect to noise considering that the problem is severely ill-posed. We reconstruct two ellipses and initialize with two balls placed at the wrong location. The average number of iterations until convergence is around 340 iterations.

In Figure 8 we reconstruct three inclusions this time using $K = 7$ different measurements, with 1.55% noise. The reconstruction is close to the true inclusion and is a bit degraded due to the noise. Figure 9 shows the convergence history of the cost functional in log scale for this example.

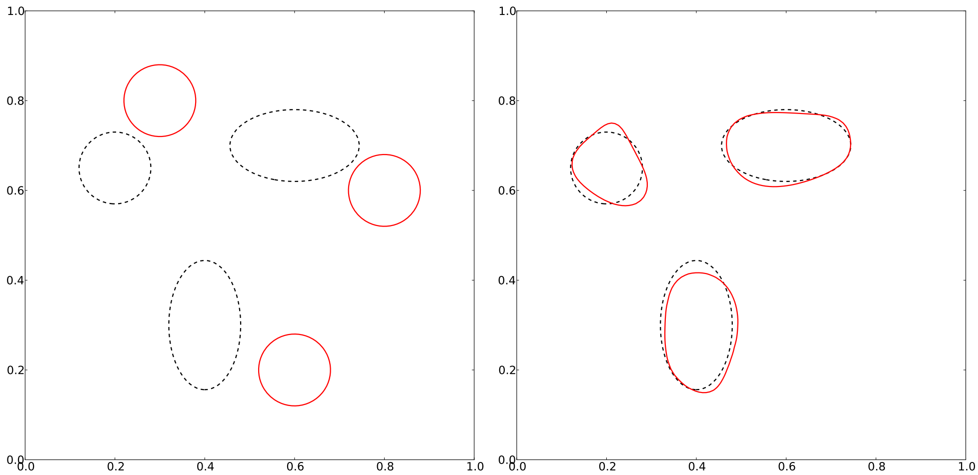


Figure 8: Initialization (continuous contours - left) and reconstruction (continuous contours - right) of two ellipses and a ball (dashed contours) with 1.55% noise (371 iterations) and using seven measurements.

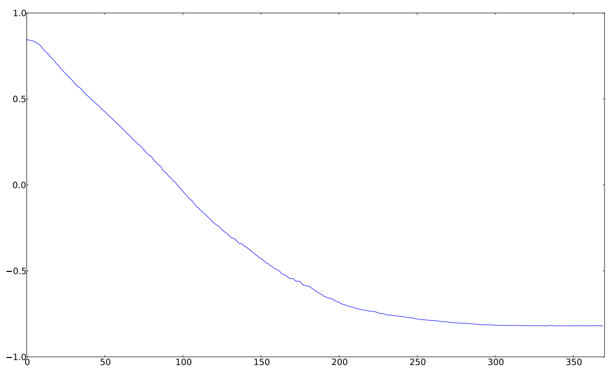


Figure 9: History of cost functional corresponding to Figure 8

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