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Gevrey regularity for integro-differential operators

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ABSTRACT. We prove for a wide class of kernels $K(x,y)$ that viscosity solutions of the integro-differential equation

$$\int_{\mathbb{R}^n} \delta u(x,y)\, K(x,y) dy = f(x)$$

locally belong to some Gevrey class if so does f. The fractional Laplacian equation is included in this framework as a special case.

1. INTRODUCTION

Recently, a great attention has been devoted to equations driven by nonlocal operators of fractional type. From the physical point of view, these equations take into account long-range particle interactions with a power-law decay. When the decay at infinity is sufficiently weak, the long-range phenomena may prevail and the nonlocal effects persist even on large scales (see e.g. [5, 1]).

The probabilistic counterpart of these fractional equation is that the underlying diffusion is run by a stochastic process with power-law tail probability distribution (the so-called Pareto or Lévy distribution), see for instance [1, 2]. Since long relocations are allowed by the process, the diffusion obtained is sometimes referred to with the name of anomalous (in contrast with the classical one coming from Poisson distributions). Physical realizations of these models occur in different fields, such as fluid dynamics (and especially quasi-geostrophic and water wave equations), dynamical systems, elasticity and micelles, see, among the others [7, 8, 1, 1]. Also, the scale invariance of the nonlocal probability distribution may combine with the intermittency and renormalization properties of other nonlinear dynamics and produce complex patterns with fractional features. For instance, there are indications that the distribution of food on the ocean surface has scale invariant properties (see see e.g. [2] and references therein) and it is possible that optimal searches of predators reflect these patterns in the effort of locating abundant food in sparse environments, also considering that power-law distribution of movements allow the individuals to visit more sites than the classical Brownian situation (see e.g. [2, 1]).

The regularity theory of integro-differential equations has been extensively studied in continuous and smooth spaces, see e.g. [3, 1, 1]. The purpose of this paper is to deal with the regularity theory in a Gevrey framework. The proof combines a quantitative bootstrap argument of [1,] and the iteration scheme of [1, 1]. Here the bootstrap argument is more delicate than in the classical case due to the nonlocality of the operator, since the value of the function in a small ball is affected by the values of the function everywhere, not only in a slightly bigger ball; in particular the derivatives of the function cannot be controlled in the whole space and a suitable truncation argument is needed to bound derivatives of any order.

We recall that solutions of fractional equations are in general not better than Hölder continuous up to the boundary. This provides additional conceptual difficulties even for the interior regularity since a nonlocal term coming from far may complicate or invalidate the local estimates for the higher order derivatives.

Before stating the main results of the paper, we recall the definition of Gevrey function. For a detailed treatment of the theory of Gevrey functions and their relation with analytic functions we refer to [1, 1]. Let $\Omega \in \mathbb{R}^n$ be an open set, we define for any fixed real number $\sigma \geqslant 1$ the class $\mathcal{G}^\sigma(\Omega)$ of Gevrey functions of order σ in Ω. This is the set of functions f in $f \in \mathcal{C}^\infty(\Omega)$ such that for every compact subset C of Ω there exist positive constants M and K such that for all $i \in \mathbb{N}$

$$\left\|D^i f\right\|_C \leqslant M\, K^i\, (i!)^\sigma .$$

We remark that the spaces $\mathcal{G}^\sigma(\Omega)$ form a nested family, in the sense that $\mathcal{G}^\sigma(\Omega) \subseteq \mathcal{G}^\tau(\Omega)$ whenever $\sigma \leqslant \tau$ and furthemore the inclusion is strict whenever the inequality is. Clearly the class $\mathcal{G}^1(\Omega)$

coincides with $\mathcal{C}^\omega(\Omega)$, that of analytical functions. It should be stressed that both the inclusions

$$\mathcal{C}^\omega(\Omega) \subset \bigcap_{\sigma>1} \mathcal{G}^\sigma(\Omega) \qquad\qquad \bigcup_{\sigma\geqslant 1} \mathcal{G}^\sigma(\Omega) \subset \mathcal{C}^\infty(\Omega)$$

are strict, see [1]. The notion Gevrey class of functions is quite useful in applications. For instance, it possess a nice characterization in Fourier spaces. Moreover, cut-off functions are never analytic, but they may be chosen to belong to a Gevrey space. Roughly speaking, for a smooth function f the notion of Gevrey order measures "how much"the Taylor series of f diverges.

As in [1,] we consider a quite general kernel $K = K(x,y) : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to (0,+\infty)$ satisfying some structural assumptions. From now we assume that $s \in (1/2,1)$.

We suppose that K is close to the kernel of the fractional Laplacian in the sense that

$$\text{(1.1)} \qquad \begin{cases} \text{there exist } a_0, r_0 \text{ and } \eta \in (0, a_0/4) \text{ such that} \\ \left| \dfrac{|y|^{n+2s} K(x,y)}{2-2s} - a_0 \right| \leqslant \eta \qquad \text{for all } x \in B_1, y \in B_{r_0} \setminus \{0\}. \end{cases}$$

Since we are interested in the Gevrey regularity, in order to ensure that our solution are $\mathcal{C}^\infty$ we assume that $K \in \mathcal{C}^\infty(B_1 \times (\mathbb{R}^n \setminus \{0\}))$ and moreover

$$\text{(1.2)} \qquad \begin{cases} \text{for all } k \in \mathbb{N} \cup \{0\} \text{ there exist } \mathrm{H}_k > 0 \text{ such that} \\ \left\| D_x^\mu D_y^\vartheta K(\cdot, y) \right\|_{B_1} \leqslant \dfrac{\mathrm{H}_k}{|y|^{n+2s+|\vartheta|}} \\ \qquad \text{for all } \mu, \vartheta \in \mathbb{N}^n, |\mu| + |\vartheta| = k, y \in B_{r_0} \setminus \{0\}. \end{cases}$$

Furthermore, since we need a quantitative asymptotic control on the tails of the derivatives of K, we assume that

$$\text{(1.3)} \qquad \begin{cases} \text{there exist } \nu \geqslant 0 \text{ and } \Lambda > 0 \text{ such that} \\ \mathrm{H}_k \leqslant \Lambda^k (k!)^\nu \qquad \text{for all } k \in \mathbb{N} \cup \{0\}. \end{cases}$$

We adopt the following notation for the second increment

$$\delta u(x,y) = u(x+y) + u(x-y) - 2u(x).$$

Our main result is about the Gevrey regularity of the integro-differential operators with a general kernel K.

Theorem 1.1. *Suppose f is in $\mathcal{G}^\tau(B_6)$ and suppose that $u \in L^\infty(\mathbb{R}^n)$ is a viscosity solution of*

$$\text{(1.4)} \qquad \int_{\mathbb{R}^n} \delta u(x,y)\, K(x,y) dy = f(x) \qquad \textit{inside } B_6$$

with $s \in (1/2,1)$. Assume that $K : B_1 \times (\mathbb{R}^n \setminus \{0\}) \to (0,+\infty)$ satisfies assumptions (1.1), (1.2), and (1.3) for η sufficiently small.
Then there exists a $0 < \overline{R} < 5$ wich depends only on n, f, and $\|u\|_{\mathbb{R}^n}$ such that $u \in \mathcal{G}^\sigma(B_R)$ for each $\sigma \geqslant \max\{1+\nu, \tau\}$ and $R \leqslant \overline{R}$.

Furthermore we specialize the analysis to the case of the fractional Laplacian kernel and obtain the following

Corollary 1.2. *Suppose f is in $\mathcal{G}^\tau(B_6)$ and suppose that $u \in L^\infty(\mathbb{R}^n)$ is a viscosity solution of*

$$\text{(1.5)} \qquad (-\Delta)^s u(x) = f(x) \qquad \textit{inside } B_6$$

with $s \in (1/2,1)$.
Then there exists a $0 < \overline{R} < 5$ wich depends only on n, f, and $\|u\|_{\mathbb{R}^n}$ such that $u \in \mathcal{G}^\sigma(B_R)$ for each $\sigma \geqslant \max\{2, \tau\}$ and $R \leqslant \overline{R}$.

We remark that most of the difficulties in our problems come from having the equation in a domain rather than in the whole of the space and from the fact that we look for local, rather than global, estimates. For instance, if a summable u satisfies

$$(-\Delta)^s u(x) + f(x) = 0$$

for every $x \in \mathbb{R}^n$ and f belongs to some Gevrey class, then so is u, and this can be proved directly via Fourier methods, see e.g Theorem 1.6.1 of [1].

Concerning the Gevrey exponent $\sigma = \max\{1+\nu, \tau\}$ in Theorem 1.1, we think that it is an interesting open problem to establish whether or not such exponent is optimal or it can be lowered, for instance, to the value $\max\{\nu, \tau\}$ (in our case, the exponent value σ is due to a nonlocal boundary effect and the worst factor comes from the higher derivatives of the kernel).

Remark 1.3. *In the sequel we will adopt some notations in order to keep the esposition clear. For $\Omega \in \mathbb{R}^n$, u a measurable function defined in Ω or $\mathbb{R}^n$, and K a kernel, then*

$$\|u\|_\Omega = \|u\|_{L^\infty(\Omega)}$$

and

$$\mathcal{I}_K(u) = \int_{\mathbb{R}^n} \delta u(x,y)\, K(x,y) dy\,.$$

2. Incremental quotients

In this section we recall some basic facts and results about incremental quotients.

Given $k \in \mathbb{N}$, we observe that, for any $a \in \mathbb{R}$, there are exactly $k+1$ integers belonging to the interval $(a, a+k+1]$ (and, moreover, they are consecutive, this can be easily proved by induction over k). In particular, taking $a := -(k+1)/2$, we have that there are exactly $k+1$ integers belonging to the interval $(-(k+1)/2, (k+1)/2]$. We call these integers $j_1, \dots, j_{k+1}$.

Now, we consider $V \in \operatorname{Mat}((k+1)\times(k+1))$ to be the matrix

$$V_{m,i} := {j_i}^{m-1} \text{ for any } i, m = 1, \dots, k+1.$$

Then, we consider the vector $c^{(k)} = (c_1^{(k)}, \dots, c_{k+1}^{(k)}) \in \mathbb{R}^{k+1}$ as the unique solution of

$$Vc = (0, \dots, 0, k!). \tag{2.1}$$

We remark that V is invertible, being a Vandermonde matrix, and therefore the definition of $c^{(k)}$ is well posed.

With this notation, given $h > 0$, $v \in \mathbb{S}^{n-1}$, and a function u, we consider the h-incremental quotient of u of order k in direction v, that is

$$T_h^v u(x) := \sum_{i=1}^{k+1} c_i^{(k)} u(x + j_i h v). \tag{2.2}$$

We remark that $T_h^v u(x)/h^k$ behaves like $D_v^k u(x)$, as next result shows:

Lemma 2.1. *Let $r > (k+1)h > 0$ and $u \in C^k(B_r(x))$. Then*

$$T_h^v u(x) = h^k D_v^k u(x) + o(h^k).$$

Proof. By Taylor's Theorem

$$u(x + j_i h v) = \sum_{\ell=0}^{k-1} \frac{D_v^\ell u(x)}{\ell!} j_i^\ell h^\ell + \frac{D_v^k u(x+\xi_i)}{k!} j_i^k h^k$$

for some $\xi_i \in \mathbb{R}^n$ with $|\xi_i| \leqslant (k+1)h$. Then we have

$$\begin{aligned} T_h^v u(x) &= \sum_{i=1}^{k+1}\sum_{\ell=0}^{k-1} \frac{D_v^\ell u(x)}{\ell!} c_i^{(k)} j_i^\ell h^\ell + \sum_{i=1}^{k+1} c_i^{(k)} \frac{D_v^k u(x+\xi_i)}{k!} j_i^k h^k \\ &= \sum_{i=1}^{k+1}\sum_{m=1}^{k} \frac{D_v^\ell u(x)}{\ell!} c_i^{(k)} V_{m,i} h^{m-1} + \sum_{i=1}^{k+1} c_i^{(k)} \frac{D_v^k u(x+\xi_i)}{k!} V_{k+1,i} h^k \\ &= \sum_{m=1}^{k} \frac{D_v^\ell u(x)}{\ell!} (Vc^{(k)})_m h^{m-1} + (Vc^{(k)})_{k+1} \frac{D_v^k u(x)}{k!} h^k \\ &\quad + \sum_{i=1}^{k+1} c_i^{(k)} \frac{D_v^k u(x+\xi_i) - D_v^k u(x)}{k!} V_{k+1,i} h^k \end{aligned}$$

This and (2.1) imply

$$T_h^v u(x) = h^k D_v^k u(x) + h^k \sum_{i=1}^{k+1} c_i^{(k)} \frac{D_v^k u(x+\xi_i) - D_v^k u(x)}{k!} V_{k+1,i} \tag{2.3}$$

and this gives the desired results. ☐

We observe that $T_h^v(T_h^{\tilde v}u) = T_h^{\tilde v}(T_h^v u)$, hence we can extend the notation in (2.2) to the multi-index case. Namely, if $k = (k_1, \dots, k_\ell) \in \mathbb{N}^\ell$ and $v = (v_1, \dots, v_\ell) \in (\mathbb{S}^{n-1})^\ell$, we define

$$\begin{aligned} T_h^v u(x) &:= T_h^{v_1}(\dots(T_h^{v_\ell} u(x))\dots) \\ &= \sum_{i_1=1}^{k_1+1} \cdots \sum_{i_\ell=1}^{k_\ell+1} c_{i_1}^{(k_1)} \dots c_{i_\ell}^{(k_\ell)} u(x + j_{i_1} h v_1 + \dots + j_{i_\ell} h v_\ell) \end{aligned}$$

and this definition is, in fact, independent of the order. In this way, we have the following multi-index version of Lemma 2.1:

Lemma 2.2. *Let $r > (k+1)h > 0$ and $u \in \mathcal{C}^k(B_r(x))$. Then*

$$T_h^v u(x) = h^{|k|} D_v^k u(x) + o(h^{|k|}).$$

To prove Theorem 1.1 we will need the following integral analogue of Lemma 2.2. From now on, we will adopt the convention that for any kernel $K(x,y)$

$$T_h^\gamma K(x,y)$$

denotes the incremental quotient with respect to y (i.e. letting fixed x).

Proposition 2.3. *Let $0 < \varepsilon < r$ and $K \in \mathcal{C}^\infty(B_1 \times \mathbb{R}^n \backslash B_{r-\varepsilon})$ satisfying condition (1.2). Then for each $\gamma \in \mathbb{N}^n$*

$$\lim_{h\to 0} \int_{r<|y|} \frac{1}{h^{|\gamma|}} |T_h^\gamma K(x,y)|\, dy \;=\; \int_{r<|y|} \left|D_y^\gamma K(x,y)\right| dy\,.$$

Proof. For $\varepsilon > h(|\gamma|+1) > 0$ we set

$$f_h(x,y) = \frac{1}{h^{|\gamma|}} |T_h^\gamma K(x,y)|\;,$$

it follows from Lemma 2.1 that $f_h(x,y)$ converges pointwise in Ω to

$$f(x,y) = \left|D_y^\gamma K(x,y)\right|\;.$$

The idea is to show that there exists a $g \in L^1(\Omega)$ such that $f_h(y) \leqslant g(y)$. From (2.3) it follows that there exists $\xi_i \in \mathbb{R}^n$ with $|\xi_i| \leqslant h(|\gamma|+1)$ and positive constants A and A_i such that

$$\begin{aligned} f_h(x,y) &\leqslant |D_y^\gamma K(x,y)| + \sum_{i=1}^{|\gamma|+1} c_i^{(|\gamma|)} \frac{\left|D_y^\gamma K(x,y+\xi_i) - D_y^\gamma K(x,y)\right|}{|\gamma|!} V_{|\gamma|+1,i} \\ &\leqslant A \left|D_y^\gamma K(x,y)\right| + \sum_{i=1}^{|\gamma|+1} A_i \left|D_y^\gamma K(x,y+\xi_i)\right| \,. \end{aligned}$$

From condition (1.2) and $\varepsilon < r$ we have

$$\begin{aligned} \left|D_y^\gamma K(x,y+\xi_i)\right| &\leqslant \frac{\mathrm{H}_{|\gamma|}}{|y+\xi_i|^{n+2s+|\gamma|}} \\ &\leqslant \frac{\mathrm{H}_{|\gamma|}}{(|y|-|\xi_i|)^{n+2s+|\gamma|}} \\ &\leqslant \frac{\mathrm{H}_{|\gamma|}}{(|y|-\varepsilon)^{n+2s+|\gamma|}} \,. \end{aligned}$$

Since this last estimate is independent of h, the proposition follows from the Lebesgue dominated convergence theorem. $\square$

3. An a priori estimate

In this section we deduce the following *a priori* estimate of Friedrichs type for solutions of (1.4). It will be the key tool in proving the Gevrey regularity of the solutions of the equation.

Lemma 3.1. *Let u be a solution of (1.4) with $s \in \left(\frac{1}{2}, 1\right)$ and $f \in \mathcal{C}^\infty(B_6)$. Then for any $0 < r < r + \delta < 5$ and $p \geqslant 0$ the following estimate holds*

$$\begin{aligned} \left\|\nabla^{p+2} u\right\|_{B_r} \leqslant C \Bigg[&\frac{1}{\delta} \left\|\nabla^{p+1} u\right\|_{B_{r+\delta}} + \frac{1}{\delta^2} \left\|\nabla^p u\right\|_{B_{r+\delta}} \\ &+ \delta^{2s-1} \left\|\nabla^{p+1} f\right\|_{B_{r+\delta}} + \frac{\mathrm{H}_{p+1} 2^p}{\delta^{p+2}} \|u\|_{\mathbb{R}^n} \Bigg] , \end{aligned}$$

where C is a constant depending only on n and s.

We note that the estimate above corresponds to Lemma 5.7.1 of [1]. It should be stressed that while in the local case the estimate for a generic p is easily deduced from the estimate for $p = 0$ by differentiating the equation, in the nonlocal case we have to be more accurate in order to take account of the long range interactions. In particular to obtain this estimate we will exploit the bootstrap machinery developed in the [1,] to prove the $\mathcal{C}^\infty$ regularity of solutions.

Proof. First of all we recall that by Theorem 5 of [1,] viscosity solutions of (1.4) are of class $\mathcal{C}^\infty$. Now we choose a $\mathcal{C}^\infty$ cutoff function $\eta(x) \geqslant 0$ such that

$$\eta(x) = \begin{cases} 1 & B_4 \\ 0 & \mathbb{R}^n \backslash B_5 \end{cases}$$

and

$$\|\nabla \eta(x)\|_{B_5} \leqslant 2 \,. \tag{3.1}$$

Now for $\gamma \in \mathbb{N}^n$ with $|\gamma| = p$ and e_i a unit vector we set $\gamma_* = \gamma + e_i$ and define for each $0 < h < \frac{1}{p+2}$

$$w(x) = \frac{1}{h^{p+1}} T_h^{\gamma_*} u(x)\,.$$

By linearity $w(x) = w_1(x) + w_2(x)$ where

$$w_1(x) = \frac{1}{h^{p+1}} T_h^{e_i} \left(\eta\, T_h^{\gamma} u(x)\right)\,,$$

and

$$w_2(x) = \frac{1}{h^{p+1}} T_h^{e_i} \left((1-\eta)\, T_h^{\gamma} u(x)\right).$$

The strategy is to get a good estimate of $|\mathcal{I}_K(w_1)|$ inside B_1 using the technique of [1,]. We stress the fact that $w_2(x) \equiv 0$ for $x \in B_1$ since $\eta \equiv 1$ there.
From the triangle inequality we have that

$$\begin{aligned} |\mathcal{I}_K(w_1)| &= |\mathcal{I}_K(w) - \mathcal{I}_K(w_2)| \\ &\leqslant |\mathcal{I}_K(w)| + |\mathcal{I}_K(w_2)| \\ &= \left| \frac{1}{h^{p+1}} T_h^{\gamma_*} f(x) \right| + |\mathcal{I}_K(w_2)|\,. \end{aligned} \tag{3.2}$$

Using the discrete integration by parts and recalling that $x \in B_1$ we get the inequality

$$\begin{aligned} |\mathcal{I}_K(w_2)| &= \left| \int_{\mathbb{R}^n} \left[w_2(x+y) + w_2(x-y) - 2w_2(x) \right]\, K(x,y)\, dy \right| \\ &= \left| \int_{\mathbb{R}^n} \left[w_2(x+y) + w_2(x-y) \right]\, K(x,y)\, dy \right| \\ &\leqslant \left| \int_{\mathbb{R}^n} w_2(x+y)\, K(x,y)\, dy \right| + \left| \int_{\mathbb{R}^n} w_2(x-y)\, K(x,y)\, dy \right| \\ &= \left| \int_{\mathbb{R}^n} \left[\frac{1}{h^{p+1}} T_h^{e_i} \left((1-\eta)\, T_h^{\gamma} u(x+y)\right) \right]\, K(x,y)\, dy \right| \\ &\qquad + \left| \int_{\mathbb{R}^n} \left[\frac{1}{h^{p+1}} T_h^{e_i} \left((1-\eta)\, T_h^{\gamma} u(x-y)\right) \right]\, K(x,y)\, dy \right| \\ &= \left| \int_{\mathbb{R}^n} \left[\frac{1}{h^{p+1}} \left((1-\eta)\, T_h^{\gamma} u(x+y)\right) \right]\, T_{-h}^{e_i} K(x,y)\, dy \right| \\ &\qquad + \left| \int_{\mathbb{R}^n} \left[\frac{1}{h^{p+1}} \left((1-\eta)\, T_h^{\gamma} u(x-y)\right) \right]\, T_{-h}^{e_i} K(x,y)\, dy \right|\,. \end{aligned}$$

Now, exploiting the properties of the cutoff function η we get the following estimate

$$
\begin{aligned}
|\mathcal{I}_K(w_2)| \leqslant & \left| \int\limits_{4<|x+y|<5} \left[\frac{1}{h^{p+1}}\left((1-\eta)\,T_h^{\gamma}u(x+y)\right)\right]\ T_{-h}^{e_i}K(x,y)\,dy \right| \\
& + \left| \int\limits_{4<|x-y|<5} \left[\frac{1}{h^{p+1}}\left((1-\eta)\,T_h^{\gamma}u(x-y)\right)\right]\ T_{-h}^{e_i}K(x,y)\,dy \right| \\
& + \left| \int\limits_{5<|x+y|} \frac{1}{h^{p+1}} u(x+y)\,T_{-h}^{\gamma_*}K(x,y)\,dy \right| \\
& + \left| \int\limits_{5<|x-y|} \frac{1}{h^{p+1}} u(x-y)\,T_{-h}^{\gamma_*}K(x,y)\,dy \right| .
\end{aligned}
$$

Using the triangle inequality we conclude that

$$
\begin{aligned}
|\mathcal{I}_K(w_2)| \leqslant & \left| \int\limits_{\substack{2<|y|\\|x+y|<5}} \left[\frac{1}{h^{p+1}}\left((1-\eta)\,T_h^{\gamma}u(x+y)\right)\right]\ T_{-h}^{e_i}K(x,y)\,dy \right| \\
& + \left| \int\limits_{\substack{2<|y|\\|x-y|<5}} \left[\frac{1}{h^{p+1}}\left((1-\eta)\,T_h^{\gamma}u(x-y)\right)\right]\ T_{-h}^{e_i}K(x,y)\,dy \right| \\
& + \left| \int\limits_{3<|y|} \frac{1}{h^{p+1}} u(x+y)\,T_{-h}^{\gamma_*}K(x,y)\,dy \right| \\
& + \left| \int\limits_{3<|y|} \frac{1}{h^{p+1}} u(x-y)\,T_{-h}^{\gamma_*}K(x,y)\,dy \right|
\end{aligned}
$$

and the RHS is majorized by

$$
2\left\|\frac{1}{h^p}T_h^{\gamma}u\right\|_{B_5} \int\limits_{2<|y|} \frac{1}{h}\ \left|T_{-h}^{e_i}K(x,y)\right|\ dy + 2\,\|u\|_{\mathbb{R}^n} \int\limits_{3<|y|} \frac{1}{h^{p+1}}\ \left|T_{-h}^{\gamma_*}K(x,y)\right|\ dy\,.
$$

Inserting the inequality above into (3.2) we obtain the desired estimate inside the ball of radius 1

$$
\begin{aligned}
|\mathcal{I}_K(w_1)| \leqslant & \left|\frac{1}{h^{p+1}}T_h^{\gamma_*}f(x)\right| + 2\,\|u\|_{\mathbb{R}^n} \int\limits_{3<|y|} \frac{1}{h^{p+1}}\ \left|T_{-h}^{\gamma_*}K(x,y)\right|\ dy \\
& + 2\left\|\frac{1}{h^p}T_h^{\gamma}u\right\|_{B_5} \int\limits_{2<|y|} \frac{1}{h}\ \left|T_{-h}^{e_i}K(x,y)\right|\ dy\,.
\end{aligned}
\tag{3.3}
$$

Furthermore, thanks to the discrete Leibnitz rule we have that

$$w_1(x) = \frac{1}{h^{p+1}}\left[\eta(x+he_i)T_h^{\gamma_*}u(x) + T_h^{e_i}\eta(x)T_h^{\gamma}u(x)\right]$$

and this implies

$$\|w_1\|_{\mathbb{R}^n} \leqslant \left\|\frac{1}{h^{p+1}}T_h^{\gamma_*}u\right\|_{B_5(he_i)} + \left\|\frac{1}{h}T_h^{e_i}\eta\right\|_{B_5}\left\|\frac{1}{h^p}T_h^{\gamma}u\right\|_{B_5} \tag{3.4}$$

where $B_5(he_i)$ is the ball of radius 5 centered at he_i.
Thanks to (3.3), (3.4), and Theorem 61 of [4,] we get the following estimate for w_1

$$\begin{aligned}
\|w_1\|_{\mathcal{C}^{1,\alpha}(B_1)} &\leqslant C\left(\|w_1\|_{\mathbb{R}^n} + \|\mathcal{I}_K(w_1)\|_{B_2}\right)\\
&\leqslant C\Bigg(\|w\|_{B_5(he_i)} + \left\|\frac{1}{h}T_h^{e_i}\eta\right\|_{B_5}\left\|\frac{1}{h^p}T_h^{\gamma}u\right\|_{B_5} + \left\|\frac{1}{h^{p+1}}T_h^{\gamma_*}f\right\|_{B_2}\\
&\qquad +2\left\|\frac{1}{h^p}T_h^{\gamma}u\right\|_{B_5}\int\limits_{2<|y|}\frac{1}{h}\left|T_{-h}^{e_i}K(x,y)\right|\,dy\\
&\qquad +2\|u\|_{\mathbb{R}^n}\int\limits_{3<|y|}\frac{1}{h^{p+1}}\left|T_{-h}^{\gamma_*}K(x,y)\right|\,dy\Bigg),
\end{aligned}$$

where, from now on, C will denote a positive constant depending only on n and s. This means that in calculations we will reabsorb constant factors inside C. Letting $h \to 0$, from (1.2), Proposition 2.3, and (3.1) we conclude that

$$\begin{aligned}
\|D^{\gamma_*}u\|_{\mathcal{C}^{1,\alpha}(B_1)} \leqslant C\Big(&\|D^{\gamma_*}u\|_{B_5} + 2\|D^{\gamma}u\|_{B_5} + \|D^{\gamma_*}f\|_{L^\infty(B_2)}\\
&+\mathrm{H}_1\|D^{\gamma}u\|_{B_5} + \frac{\mathrm{H}_{p+1}}{2^{2s+p+1}}\|u\|_{\mathbb{R}^n}\Big).
\end{aligned} \tag{3.5}$$

By a scaling argument, from (3.5) we get the following estimate valid for any $\sigma \in (0,1]$

$$\begin{aligned}
\left\|\nabla^{p+2}u\right\|_{B_\sigma} \leqslant C\Bigg[&\frac{1}{\sigma}\left\|\nabla^{p+1}u\right\|_{B_{5\sigma}} + \frac{1}{\sigma^2}\|\nabla^p u\|_{B_{5\sigma}}\\
&+\sigma^{2s-1}\left\|\nabla^{p+1}f\right\|_{B_{2\sigma}} + \frac{\mathrm{H}_{p+1}}{2^p}\frac{1}{\sigma^{p+2}}\|u\|_{\mathbb{R}^n}\Bigg]
\end{aligned}$$

this implies, by covering, the following global estimate, valid for $0 < r < r+\delta \leqslant 5$

$$\begin{aligned}
\left\|\nabla^{p+2}u\right\|_{B_r} \leqslant C\Bigg[&\frac{1}{\delta}\left\|\nabla^{p+1}u\right\|_{B_{r+\delta}} + \frac{1}{\delta^2}\|\nabla^p u\|_{B_{r+\delta}}\\
&+\delta^{2s-1}\left\|\nabla^{p+1}f\right\|_{B_{r+\delta}} + \frac{\mathrm{H}_{p+1}\,2^p}{\delta^{p+2}}\|u\|_{\mathbb{R}^n}\Bigg].
\end{aligned}$$

which is the desired estimate. □

4. Proof of Theorem 1.1

In this section we will adapt the iteration scheme of [1, 1] to obtain Gevrey regularity of solutions of (1.4). The main idea here is to introduce some rescaled companions of the L^∞ norms in order to exploit effectively the estimate of Lemma 3.1.

As in [1, 1] we introduce the following

Definition 4.1. *For f and u in $\mathcal{C}^\infty(B_R)$, we define the quantities*

$$\mathrm{M}^s_{R,p}(f) = \sup_{R/2<r<R} (R-r)^{2s+p+1} \left\|\nabla^{p+1} f\right\|_{B_r}, \quad p \in \mathbb{N} \cup \{0\}$$

$$\mathrm{N}^*_{R,p}(u) = \sup_{R/2<r<R} (R-r)^{p+2} \left\|\nabla^{p+2} u\right\|_{B_r}, \qquad p \in \{-2,-1\}.$$

Since there is no ambiguity, in the sequel will be suppressed the explicit dependence from f and u in $\mathrm{M}^s_{R,p}$ and $\mathrm{N}^*_{R,p}$. It is apparent from this definition that a function $u \in \mathcal{C}^\infty(B_R)$ will be also in $\mathcal{G}^\sigma(B_R)$ if and only if there exist positive constants M and K such that the following inequality holds for any $p \geqslant -2$

$$\mathrm{N}^*_{R,p} \leqslant M \cdot K^p \cdot [p!]^\sigma, \tag{4.1}$$

where $[p!]$ is defined as

$$\begin{cases} p! & \text{if } p \geqslant 0 \\ 1 & \text{if } p < 0. \end{cases}$$

Thus the strategy for proving the Theorem 1.1 will consist in showing the validity of (4.1) for solutions of (1.4). The following lemma will be the induction step of the proof.

Lemma 4.2. *Let u be a solution of (1.4) with $f \in \mathcal{C}^\infty(B_6)$. Then there exist positive constants E and F depending only on s and n such that the estimate*

$$\mathrm{N}^*_{R,p} \leqslant E\left[p\,\mathrm{N}^*_{R,p-1} + p(p-1)\,\mathrm{N}^*_{R,p-2} + \mathrm{M}^s_{R,p} + F^p\,\mathrm{H}_{p+1}\,p!\,\|u\|_{\mathbb{R}^n}\right]$$

holds for any $p \geqslant 0$.

Proof. By Theorem 5 of [1,] we have that $u \in \mathcal{C}^\infty(B_R)$. Plugging the estimate of Lemma 3.1 into the definition of $\mathrm{N}^*_{R,p}$ we obtain the following

$$\begin{aligned} \mathrm{N}^*_{R,p} \leqslant C \sup_{R/2<r<R} (R-r)^{p+2} \Bigg[& \frac{1}{\delta}\left\|\nabla^{p+1} u\right\|_{B_{r+\delta}} + \frac{1}{\delta^2}\left\|\nabla^p u\right\|_{B_{r+\delta}} \\ & + \delta^{2s-1}\left\|\nabla^{p+1} f\right\|_{B_{r+\delta}} + \frac{\mathrm{H}_{p+1} 2^p}{\delta^{p+2}}\|u\|_{\mathbb{R}^n}\Bigg]. \end{aligned} \tag{4.2}$$

By Definition 4.1 we get the inequalities

$$\begin{aligned} \left\|\nabla^{p+1} u\right\|_{B_{r+\delta}} &\leqslant \frac{1}{(R-r-\delta)^{p+1}} \mathrm{N}^*_{R,p-1} \\ \left\|\nabla^{p} u\right\|_{B_{r+\delta}} &\leqslant \frac{1}{(R-r-\delta)^{p}} \mathrm{N}^*_{R,p-2} \\ \left\|\nabla^{p+1} f\right\|_{B_{r+\delta}} &\leqslant \frac{1}{(R-r-\delta)^{2s+p+1}} \mathrm{M}^s_{R,p} \end{aligned} \tag{4.3}$$

and inserting them into (4.2) we have

$$\begin{aligned} \mathrm{N}^*_{R,p} \leqslant C \sup_{r\in[R/2,R]} \Bigg[& \frac{(R-r)^{p+2}}{\delta(R-r-\delta)^{p+1}} \mathrm{N}^*_{R,p-1} + \frac{(R-r)^{p+2}}{\delta^2(R-r-\delta)^{p}} \mathrm{N}^*_{R,p-2} \\ & + \frac{\delta^{2s-1}(R-r)^{p+2}}{(R-r-\delta)^{2s+p+1}} \mathrm{M}^s_{R,p} + \mathrm{H}_{p+1} 2^p \frac{(R-r)^{p+2}}{\delta^{p+2}} \|u\|_{\mathbb{R}^n} \Bigg]. \end{aligned}$$

Now we eliminate the dependence on r by setting

$$\delta = \frac{R-r}{p},$$

in this way we get the following estimate

$$
\begin{aligned}
\mathrm{N}^*_{R,p} &\leqslant C\left[\frac{p^{p+2}}{(p-1)^{p+1}}\mathrm{N}^*_{R,p-1}+\frac{p^{p+2}}{(p-1)^{p}}\mathrm{N}^*_{R,p-2}\right.\\
&\qquad\qquad\left.+\frac{p^{p+2}}{(p-1)^{2s+p+1}}\mathrm{M}^s_{R,p}+\mathrm{H}_{p+1}2^p p^{p+2}\left\|u\right\|_{\mathbb{R}^n}\right]\\
&\leqslant C\left[p\left(\frac{p}{p-1}\right)^{p+1}\mathrm{N}^*_{R,p-1}+p(p-1)\left(\frac{p}{p-1}\right)^{p+1}\mathrm{N}^*_{R,p-2}\right.\\
&\qquad\qquad\left.+p^{1-2s}\left(\frac{p}{p-1}\right)^{p+1+2s}\mathrm{M}^s_{R,p}+\mathrm{H}_{p+1}2^p p^{p+2}\left\|u\right\|_{\mathbb{R}^n}\right].
\end{aligned}
$$

The proof follows from the fact that $2s>1$ and $\left(\frac{p}{p-1}\right)^p$ is bounded. □

By Theorem 5 of [1,] we know that u is of class $\mathcal{C}^\infty$ inside the ball B_6, this implies that the quantities $\mathrm{N}^*_{R,p}$ are well defined and finite for any $p\geqslant -2$ and $R<6$. Since $\mathcal{G}^\tau(B_6)\subset\mathcal{C}^\infty(B_6)$ also $\mathrm{M}^s_{R,p}$ is well defined for all p, moreover there exist positive numbers L and A such that for any $R\leqslant 6$ the derivatives of f satisfy

$$
\left\|\nabla^p f\right\|_{B_R}\leqslant L\left(\frac{A}{R}\right)^p (p!)^\tau .
$$

This implies that

$$
\begin{aligned}
\mathrm{M}^s_{R,p} &= \sup_{R/2<r<R}(R-r)^{2s+p+1}\left\|\nabla^{p+1}f\right\|_{B_r}\\
&\leqslant L\left(\frac{R}{2}\right)^{2s}\left(\frac{A}{2}\right)^{p+1}((p+1)!)^\tau .
\end{aligned}
$$

From Lemma 4.2 it follows that

$$
\begin{aligned}
\mathrm{N}^*_{R,p} &\leqslant E\left[p\,\mathrm{N}^*_{R,p-1}+p(p-1)\,\mathrm{N}^*_{R,p-2}+F^p(p!)^2\left\|u\right\|_{\mathbb{R}^n}\right.\\
&\qquad\left.+L\left(\frac{R}{2}\right)^{2s}\left(\frac{A}{2}\right)^{p+1}((p+1)!)^\tau\right].
\end{aligned}
\tag{4.4}
$$

Theorem 1.1 will proved by showing that we can choose K and M so that (4.1) holds. For this, we are going to proceed by induction on p. Clearly we can choose K and M so that (4.1) holds for $p=-2$ and $p=-1$, then we suppose that this choice holds up to $p-1$ with $p>0$ and we prove it for p. From (4.4) we have that

$$
\begin{aligned}
\mathrm{N}^*_{R,p} &\leqslant E\left[M\cdot K^{p-1}\cdot p[(p-1)!]^\sigma+M\cdot K^{p-2}\cdot p(p-1)[(p-2)!]^\sigma\right.\\
&\qquad\left.+F^p(p!)^2\left\|u\right\|_{\mathbb{R}^n}+L\left(\frac{R}{2}\right)^{2s}\left(\frac{A}{2}\right)^{p+1}((p+1)!)^\tau\right]\\
&= M\cdot K^p\cdot[p!]^\sigma\,E\left[\frac{1}{K}\,p^{1-\sigma}+\frac{1}{K^2}(p(p-1))^{1-\sigma}+\frac{1}{M}\left(\frac{F}{K}\right)^p(p!)^{2-\sigma}\left\|u\right\|_{\mathbb{R}^n}\right.\\
&\qquad\left.+\frac{L}{M\,K^p}\left(\frac{R}{2}\right)^{2s}\left(\frac{A}{2}\right)^{p+1}(p+1)^\tau(p!)^{\tau-\sigma}\right].
\end{aligned}
$$

If we choose $\sigma \geqslant \max\{2, \tau\}$ the inequality above implies that

$$\mathrm{N}^*_{R,p} = M \cdot K^p \cdot [p!]^\sigma \, E \left[\frac{1}{K} + \frac{1}{K^2} + \frac{1}{M} \left(\frac{F}{K} \right)^p \|u\|_{\mathbb{R}^n} + \frac{L}{M\,K^p} \left(\frac{R}{2} \right)^{2s} \left(\frac{A}{2} \right)^{p+1} (p+1)^\tau \right].$$

At this point we are left to show that it is possible to choose M and K in such a way that

$$E \left[\frac{1}{K} + \frac{1}{K^2} + \frac{1}{M} \left(\frac{F}{K} \right)^p \|u\|_{\mathbb{R}^n} + \frac{L}{M\,K^p} \left(\frac{R}{2} \right)^{2s} \left(\frac{A}{2} \right)^{p+1} (p+1)^\tau \right] \leqslant 1, \tag{4.5}$$

for all p. It is clear that this is always the case, more precisely, since K appears always at the denominator with the highest exponent in p, it is possible to choose K (depending on E, F, $\|u\|_{\mathbb{R}^n}$, and A) so that (4.5) holds for all $M \geqslant 1$.

It is well known that for $s \in (0,1)$ the fractional Laplacian $(-\Delta)^s$ is the translation invariant integro-differential operator whose kernel, denoted by K_0, is defined as

$$K_0(y) = -\frac{1}{2} c_{n,s} \frac{1}{|y|^{n+2s}}$$

here $c_{n,s}$ denotes a normalization constant, see for instance [9,] for a survey on the topic.

We observe that the kernel K_0 is analytic out of the origin since it is a composition of analytic functions (see for instance [1] Proposition 1.6.7). Furthermore it is a homogeneous function of degree $-(n+2s)$ and this implies that for each multi-index $\alpha \in \mathbb{N}^n$, the partial derivative $D^\alpha K_0$ is a homogeneous function of degree $-(|\alpha| + n + 2s)$.
Analiticity of K_0 implies that there exist positive constants R and C such that for any y with $|y| = 1$ and for any $\alpha \in \mathbb{N}^n$ we have that

$$|D^\alpha K_0(y)| \leqslant C \frac{j!}{R^j},$$

where $j = |\alpha|$, (see, e.g. [1]).
From the homogeneity of the kernel we can conclude that conditions (1.1), (1.2), and (1.3) hold with $\nu = 1$ and Corollary 1.2 follows.

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