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The fundamental solution of the unidirectional pulse propagation equation

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Abstract

In the article the fundamental solution of a variant of the three-dimensional wave equation known as "unidirectional pulse propagation equation" (UPPE) and its paraxial approximation is obtained. It is shown that the fundamental solution can be presented as a projection of a fundamental solution of the wave equation to some functional subspace. We discuss the degree of equivalence of the UPPE and the wave equation in this respect. In particular, we show that the UPPE, in contrast to the widespread belief, describes the wave propagation in both directions simultaneously, and remark non-causal character of its solutions.

1 Introduction

In many cases, the problem of light propagation in a nonlinear homogeneous isotropic medium requires solving the nonlinear wave equation (WE)

$$\Box E(t,\mathbf{r}) \equiv \Delta E(t,\mathbf{r}) - \frac{1}{c^2}\partial_{tt}E(t,\mathbf{r}) = Q[E], \tag{1}$$

where $\mathbf{r} = \{x, y, z\}$ are the spatial coordinates, t is time, $\Delta = \partial_{zz} + \partial_{yy} + \partial_{xx}$, $E(t,\mathbf{r}) \in \mathbb{R}$ represents the electric field (we assume in the following the scalar field, that is, linear polarization and nonlinearity which does not alter the polarization state). $Q[E]$ is, in general, a nonlinear operator describing response of the medium. For instance, for the case of an electromagnetic wave propagating in a plasma we have $Q = \mu_0\partial_t J$, where J is the plasma current, μ_0 is the vacuum permeability. In the case of strong optical fields the plasma current J depends itself on E in rather complicated way [1–4], making the equation nonlinear.

Independently on the nature of inhomogeneity Q, Eq. (1) is typically accomplished with initial and boundary conditions. Unfortunately, this problem is, in large number of practically important cases, is extremely difficult to treat numerically. A typical situation is a propagation of a few-cycle pulse through a waveguide [5–10] or in a long filament [2, 11], which assumes large extension in one spatial dimensions (say, z), making the amount of data required for solving the initial value problem intractably large. To deal with such cases, so-called unidirectional pulse propagation equation (UPPE) were formulated [2, 12], which is the best written for the Fourier-transformed electric field $E(z, \omega, \mathbf{k}_\perp) \in \mathbb{C}$.

$$\partial_z E(\omega, z, \mathbf{k}_\perp) - i\beta_z E(\omega, z, \mathbf{k}_\perp) = \frac{1}{2i\beta_z}\tilde{Q}[E], \tag{2}$$

Here, $E(\omega, z, \mathbf{k}_\perp) = \mathcal{F}_{\{t,x,y\}}[E(t,\mathbf{r})]$, $\tilde{Q} = \mathcal{F}_{\{t,x,y\}}[Q]$, $\mathcal{F}_{\{t,x,y\}}$ is the Fourier transform from coordinates $\{t, x, y\}$ to $\{\omega, k_x, k_y\}$, $\beta_z = \sqrt{\omega^2/c^2 - k_\perp^2}$, $\mathbf{k}_\perp = \{k_x, k_y\}$. Eq. (9), if we consider it in the $\{t, \mathbf{r}\}$ space, is now not a PDE anymore, but belongs to more general class of pseudo-differential equations; The problem is completed by providing the field $E(t, x, y)$ at $z = 0$ and boundary conditions, and is solved along z-direction rather in time. Note that the problem remains in the same extended class if we include nontrivial dispersion of the medium, that is, assume $c = c(\omega)$.

Eq. (2) is formally a result of a kind of factorization of the original WE Eq. (1), typically [2, 12, 13] (but not necessary [14]) neglecting the waves propagating backward in z-direction. Very schematically, the factorization can be maid in the following way: the WE, after the partial Fourier transform $\mathcal{F}_{\{t,x,y\}}$, can be written as $(\partial_z - i\beta_z)(\partial_z + i\beta_z)E = Q$. The dispersion relation of the homogeneous WE contains two branches $k_x = \pm\beta_z$. If we have a short optical pulse, that is, a wave-packet, that propagates in certain direction in z in a medium with relatively weak nonlinearity and dispersion, it is located, in the Fourier domain, near one of those two branches. Therefore, we can approximate, for instance, $\partial_z + i\beta_z$ by $2i\beta_z$, which gives us Eq. (2).

Eq. (2), its 1-dimensional analogs and other modifications are nowadays heavily used in optics to describe ultrashort pulses and ultra-broad spectra (see [2, 5–11, 14–25] and references therein), because they include minimal assumptions about the spectral width of $E(t, \mathbf{r})$.

Both Eq. (1) and Eq. (2) can be solved analytically only in exceptional cases. Nevertheless, if we consider the wave equation Eq. (1), a lot can be said about the general behavior of solutions, considering the right hand side Q as a pre-known quantity and thus Eq. (1) as a linear inhomogeneous PDE. In particular, in the plasma example given above, with $Q = \mu_0 J$, it is sometimes useful to consider an approximation when the current J does not depend on E.

It is well known, that a solution of the linear inhomogeneous variant of Eq. (1) with "well enough defined" inhomogeneity $Q(\mathbf{r}, t)$ can be obtained using a fundamental solution approach, which is the solution (in the sense of generalized functions) with the inhomogeneity in the form of Dirac δ-function $Q = \delta(\mathbf{r}, t)$. The two most useful linearly-independent fundamental solutions of $\Box$ are known to be:

$$\mathcal{E}_{\Box\pm} = \frac{-1}{4\pi r}\delta(t \mp r/c), \tag{3}$$

where $r = |\mathbf{r}|$. They describe spherical waves propagating forward ($\mathcal{E}_{\Box+}$) or backward ($\mathcal{E}_{\Box-}$) in time. In particular, physically meaningful solution of Eq. (1) for an arbitrary "good enough" function Q is given by a convolution $\mathcal{E}_{\Box+} \star Q$, whereas $\mathcal{E}_{\Box-} \star Q$ does not fulfill the causality principle.

In the similar way, one can think about the fundamental solution of Eq. (2) (formulated in $\{\mathbf{r}, t\}$-space, see Eq. (9)), that is, its generalized solution with the inhomogeneity $Q = \delta(\mathbf{r}, t)$. Nevertheless, up to now neither the fundamental solution, nor its basic properties are known for the UPPE Eq. (2) [Eq. (9)].

In the present article, we construct a fundamental solution of Eq. (2) [Eq. (9)] and show that it is a projection of the fundamental solution of Eq. (1) to some functional subspace, formed by waves, propagating either "forward-" or "backward-" in z-direction (see Theorem 4.1). We explore some consequences of this result such as the intrinsic non-causality of Eq. (2) [Eq. (9)]. We also consider a variation of Eq. (2) [Eq. (9)] where the right-hand side is taken in the paraxial approximation.

2 Assumptions and Denotations

In the present article we use the following assumptions: we consider the solution in the sense of distributions, that is we assume that all objects $u(\mathbf{r},t)$ we work with are tempered distributions $u \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$ belonging to the space dual to Schwartz space $\mathscr{S}(\mathbb{R}^3 \times \mathbb{R})$, with the scalar product defined as $\langle u, \phi \rangle = \int u\phi d^3 r dt$. With this assumption, we ensure that the Fourier transform of all distributions considered here exists. In the following we define the the Fourier transform of distributions $\mathcal{F}[u]$ as $\langle \mathcal{F}u, \phi \rangle \equiv \langle u, \mathcal{F}\phi \rangle$ [26], and the inverse one as $\langle \mathcal{F}^{-1}u, \phi \rangle \equiv \langle u, \mathcal{F}^{-1}\phi \rangle$. The Fourier transform of $\phi \in \mathscr{S}(\mathbb{R}^3 \times \mathbb{R})$ is defined as:

$$\mathcal{F}[\phi(\mathbf{r},t)](\mathbf{k},\omega) = \int \phi(\mathbf{r},t) \exp\left(-i\mathbf{k}\mathbf{r} + i\omega t\right) dxdydzdt, \tag{4}$$

where $\mathbf{k} = \{k_x, k_y, k_z\}$. The inverse transform is then defined as:

$$\mathcal{F}^{-1}[\phi(\mathbf{k},\omega)](\mathbf{r},t) = 1/(2\pi)^4 \int \phi(\mathbf{k},\omega) \exp\left(i\mathbf{k}\mathbf{r} - i\omega t\right) dk_x dk_y dk_z d\omega. \tag{5}$$

Note that the signs in spatial and temporal parts of the Fourier transform are different, following the convention often used in electrodynamics [27].

We will use also partial Fourier transformations in respect to subsets of coordinates. For example, the Fourier transform $\mathcal{F}_{(\mathbf{r}_\perp,t)}$ in transverse coordinates $\mathbf{r}_\perp = \{x, y\}$ and time is defined as:

$$\mathcal{F}_{(\mathbf{r}_\perp,t)}[\phi(\mathbf{r},t)](\mathbf{k}_\perp, z, \omega) = \int \phi(\mathbf{r},t) \exp\left(-i\mathbf{k}_\perp \mathbf{r}_\perp + i\omega t\right) dxdydt, \tag{6}$$

where $\mathbf{k}_\perp = \{k_x, k_y\}$. We note that in this formulation the partial Fourier transform of the Dirac delta-function is $\mathcal{F}_{(\mathbf{r}_\perp,t)}[\delta(\mathbf{r},t)] = \delta(z)$. The partial Fourier transforms and their combinations are always possible for the test functions (and hence for tempered distributions) because every of particular Fourier transforms leave the test function $\phi \in \mathscr{S}(\mathbb{R}^3 \times \mathbb{R})$ in $\mathscr{S}(\mathbb{R}^3 \times \mathbb{R})$. An important relation we will use is the known Plancherel's formula, which allows to define the scalar product in the Fourier domain:

$$\langle u, \phi \rangle = \frac{1}{(2\pi)^4} \langle\, \mathcal{F}u, \mathcal{F}\phi \rangle, \tag{7}$$

Similar expressions also valid for all partial Fourier transforms.

We also use the following definition of the Heaviside step-function:

$$\Theta(x) = \frac{1}{2}\left(1 + \mathrm{sign}(x)\right). \tag{8}$$

Using the definitions above we can now reformulate Eq. (2) in the $\{\mathbf{r}, t\}$-space as:

$$\partial_z E(\mathbf{r},t) - i\hat{\beta}_z E(\mathbf{r},t) = \frac{1}{2i}\hat{\beta}_i Q(\mathbf{r},t), \tag{9}$$

where $E, Q \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$ and where we explicitly assume that Q does not depend on E, thus making the equation linear. Here, by definition,

$$\hat{\beta}_z E(\mathbf{r}, t) = \mathcal{F}^{-1}\left[\beta_z \mathcal{F}[E(\mathbf{r}, t)]\right], \tag{10}$$

$$\hat{\beta}_i E(\mathbf{r}, t) = \mathcal{F}^{-1}\left[\frac{1}{\beta_z} \mathcal{F}[E(\mathbf{r}, t)]\right], \tag{11}$$

With β_z defined below Eq. (2). We remark that, if $u \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$, then $\hat{\beta}_i u \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$. In contrast, $\hat{\beta}_z u$ may not be in $\mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$ anymore. However, for us it is important that the operator $e^{i\hat{\beta}_z z}$ is always in $\mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$ for any $z \in \mathbb{R}$.

We can now define the fundamental solution of Eq. (9) as following:

Definition 2.1. *Fundamental solution of Eq. (9) is a tempered distribution $\mathcal{E} \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$ giving a solution (in the sense of distributions) of Eq. (9) with $Q = \delta(\mathbf{r}, t)$.*

We remark that if the fundamental solution exists, the solution of Eq. (9) for an arbitrary $Q \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$ is given by a convolution $\mathcal{E} \star Q$, if it exists. A convolution $\star$ is defined as a multiplication in Fourier domain:

$$[\mathcal{E} \star Q](\mathbf{r}, t) = \mathcal{F}^{-1}\left[\mathcal{F}[\mathcal{E}](\mathbf{k}, t)\mathcal{F}[Q](\mathbf{k}, t)\right](\mathbf{r}, t), \tag{12}$$

where $\mathbf{k} = \{k_x, k_y, k_z\}$. Of course, the fundamental solution is defined up to a solution of a homogeneous problem. We will aim to find the fundamental solution which is ”most similar” to the one of the WE defined by Eq. (3).

3 Forward- and Backward- Propagating Waves

Eq. (9) which we are going to consider is in some sense anisotropic. Namely, the direction z, in contrast to the wave equation Eq. (1), is not on the same footing with the other spatial coordinates. Thus, before we proceed further, we must define the notion of “forward-” and “backward-” propagating waves having in mind z- direction. First, we do this for the functions, representing plane waves in the form $f(\mathbf{r}, t) = e^{i\mathbf{kr} - i\omega t}$. The function $f(\mathbf{r}, t)$ for arbitrary $\mathbf{k} = \{k_x, k_y, k_z\} \in \mathbb{R}^3$, $\omega \in \mathbb{R}$, $\omega \neq 0$, $k_z \neq 0$ is a eigenfunction of both operator $\square$ and the operators $\hat{\beta}_z$, $\hat{\beta}_i$ in Eq. (9). Solving Eq. (1) or Eq. (9) with $E|_{t=0} = f(\mathbf{r}, 0)$, we see that f “propagates” in z as t increases, either backward or forward, that is, $E(t - \tau, \mathbf{r}) = E(t, \mathbf{r} + \mathbf{R})$ for arbitrary $\tau \in \mathbb{R}$ and $\mathbf{R} = \omega\tau\mathbf{k}/k^2$, where $k = |\mathbf{k}|$. That is, if we change the sign of the product ωk_z, the direction of propagation along z changes. Thus, we can define the plane wave as “forward-” or “backward- propagating” in the direction z using the condition

$$\operatorname{sign} k_z = \pm \operatorname{sign} \omega, \tag{13}$$

with $+$ for forward and $-$ for backward case (see Fig. 1). The values on the axes ($k_z = 0$ or $\omega = 0$) we deliberately ascribe to both forward- and backward waves.

Being able to define the propagation direction along z-axis for a single wave, we can track it down to an arbitrary combination of such waves using the Fourier transform;

Definition 3.1. *We now define the projecting operators* $\hat{\mathcal{P}}_{z+}$,$\hat{\mathcal{P}}_{z-}$, $\hat{\mathcal{P}}_{+}$,$\hat{\mathcal{P}}_{-}$,$\hat{\mathcal{P}}_{ij}, i, j = 0, 1$ *acting from* $\mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$ *to* $\mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$ *as:*

$$\hat{\mathcal{P}}_{ij} u = \mathcal{F}^{-1} \left[P_{ij}(k_z, \omega) \mathcal{F}[u] \right], \tag{14}$$

$$\hat{\mathcal{P}}_{+} = \hat{\mathcal{P}}_{00} + \hat{\mathcal{P}}_{11}, \; \hat{\mathcal{P}}_{-} = \hat{\mathcal{P}}_{01} + \hat{\mathcal{P}}_{10}, \tag{15}$$

$$\hat{\mathcal{P}}_{z+} = \hat{\mathcal{P}}_{00} + \hat{\mathcal{P}}_{01}, \; \hat{\mathcal{P}}_{z-} = \hat{\mathcal{P}}_{10} + \hat{\mathcal{P}}_{11}, \tag{16}$$

where $P_{ij}(k_z, \omega) = \Theta\left((-1)^i \omega\right) \Theta\left((-1)^j k_z\right) \; i, j = 0, 1$, u *is an arbitrary tempered distribution* $u \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$, Θ *is defined by Eq. (8).*

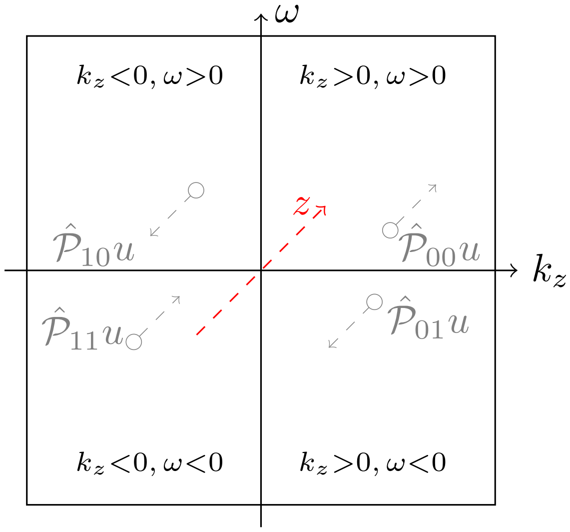

Figure 1: The forward- and backward propagating waves in $\{k_z, \omega\}$-plane, defined by projectors $\hat{\mathcal{P}}_{ij}$ from Eq. (14). The arrows parallel to z-axis show the direction of propagation of waves of every type.

That is, the projector $\hat{\mathcal{P}}_{00}$ cuts off the part of the spectrum of u, which does not belongs to the quadrant with $k_z > 0$, $\omega > 0$ (that is, it keeps only the upper right quadrant in the $\{k_z, \omega\}$-plane, see Fig. 1). In the same way, $\hat{\mathcal{P}}_{01}$ keeps only the right lower quadrant, $\hat{\mathcal{P}}_{01}$ — the left upper quadrant and finally $\hat{\mathcal{P}}_{11}$ — the left lower quadrant in the $\{k_z, \omega\}$-plane. It is easy to see that according to Eq. (8) the border values $k_z = 0$ or $\omega = 0$ make an equal impact to both forward- and backward-propagating waves. Obviously, all the operators defined above are continuous and linear, and possess the property

$$\hat{\mathcal{P}}_{i}^2 = \hat{\mathcal{P}}_{i}, \; i = \pm, z\pm; \; \hat{\mathcal{P}}_{ij}^2 = \hat{\mathcal{P}}_{ij}, \; i, j = 0, 1, \tag{17}$$

which is common for projecting operators.

Definition 3.2. *A tempered distribution* $u \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$, $u \neq 0$, *is called forward- (backward-) propagating in* z*-direction iff* $\hat{\mathcal{P}}_{+} u = u$ *(*$\hat{\mathcal{P}}_{-} u = u$*).*

Equipped with these definitions we can formulate the following result:

Lemma 3.3. *An arbitrary $u(\mathbf{r}, t) \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$ can be decomposed into the sum forward- and backward- propagating in z-direction functions.*

From Def. 3.1 we easily see that $\hat{P}_+ + \hat{P}_- = \hat{I}$, where $\hat{I}$ is a unit operator. Thus, $u = \hat{P}_+ u + \hat{P}_- u$.

It should be noted that the decomposition defined by lemma 3.3 is not unique, since we can prescribe both forward- and backward- direction to the parts of u with $\mathcal{F}\left[u\right](\mathbf{k}, \omega)\big|_{k_z=0}$, $\mathcal{F}\left[u\right](\mathbf{k}, \omega)\big|_{\omega=0}$. Nevertheless, for the subspace $(u \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R}) : \mathcal{F}[u]_{k_z=0} = 0,\ \mathcal{F}[u]_{\omega=0} = 0)$ the decomposition is, indeed, unique.

Definition 3.4. *We can also define the action of $\hat{\mathcal{P}}_{ij}$, $\hat{\mathcal{P}}_{\pm}$, $\hat{\mathcal{P}}_{z\pm}$ to the functions $\phi \in \mathscr{S}(\mathbb{R}^3 \times \mathbb{R})$ in the following way: we postulate, that for an arbitrary $u \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$:*

$$\langle u, \hat{\mathcal{P}}_i \phi\rangle = \langle \hat{\mathcal{P}}_{z\pm} u, \phi\rangle,\ i = \pm, z\pm,\ \langle u, \hat{\mathcal{P}}_{ij}\phi\rangle = \langle \hat{\mathcal{P}}_{ij} u, \phi\rangle,\ i, j = 0, 1. \tag{18}$$

Comparing this definition with Eq. (7) we see, that $\hat{\mathcal{P}}_{\pm} u$,$\hat{\mathcal{P}}_{z\pm} u$, $\hat{\mathcal{P}}_{ij}$ in application to the test functions do the same job as in the case of tempered distributions, that is, cut off the corresponding parts in the Fourier domain.

Remark 3.5. *The fundamental solutions of the WE $\mathcal{E}_{\Box\pm}$ [Eq. (3)] contain parts propagating both forward and backward in z.*

This can be seen by making the temporal Fourier transform of Eq. (3), which gives $-e^{\pm i\omega r/c}/(4\pi r)$. This expression does not change upon the change of the sign of z, therefore its z-Fourier transform must contain components both with $z > 0$ and $z < 0$ for every ω.

4 The Fundamental Solution for the UPPE Eq. (9)

Now, equipped with the definitions in Sec. 2, 3, we can formulate our main result:

Theorem 4.1. *The fundamental solution $\mathcal{E} \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$ of Eq. (9) exists and can be represented as:*

$$\mathcal{E} = \Theta(z)\hat{\mathcal{P}}_{z+}\left\{\mathcal{E}_{\Box+} + \mathcal{E}_{\Box-}\right\}, \tag{19}$$

where $\mathcal{E}_{\Box\pm}$, $\mathcal{P}_{z+}$ and $\Theta(z)$ are given by Eq. (3), Eq. (16) and Eq. (8).

Before we proceed with a prove of this theorem, few remarks are needed. The fundemental solution given by Eq. (19) is visualized in Fig. 2. For $z > 0, t > 0$, it coincides with $\mathcal{E}_{\Box+}$ and for $z > 0, t < 0$ with $\mathcal{E}_{\Box-}$. Nevertheless, two features making it different from the fundamental solution of the wave equation can be immediately seen:

Remark 4.2. *In contrast to every of the two fundamental solutions of the wave equation $\mathcal{E}_{\Box\pm}$, the fundamental solution $\mathcal{E}$ of the UPPE is extended in both directions in time.*

We also note, that the projection operator $\hat{\mathcal{P}}_{z+}$ does not project to the set of only forward- or only backward- propagating functions. Thus, taking into account the Remark 3.5 we can conclude that:

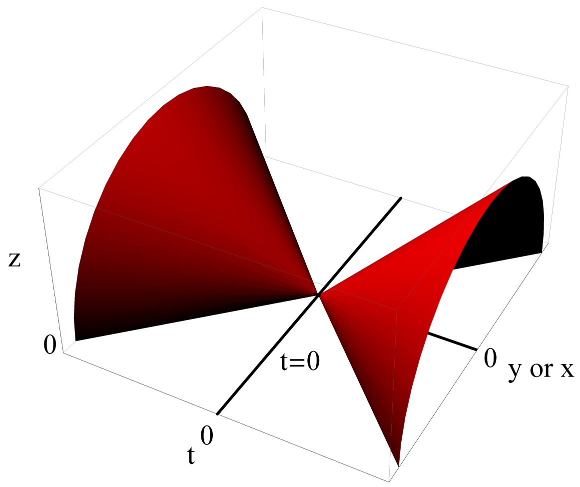

Figure 2: Visual representaion of the fundamental solution $\mathcal{E}$ defined by Eq. (19) projected to $\{t, x, z\}$- (or, equivalently, to $\{t, x, z\}$-) plane.

Remark 4.3. *The fundamental solution of the UPPE $\mathcal{E}$ [Eq. (19)] contains parts propagating both forward and backward in z.*

The rest of the section is a technical proof of the Theorem 4.1. We proceed by solving Eq. (2), the Fourier-transformed analog of Eq. (9) for $Q = \delta(z)$ (recall that Eq. (9) is written in $\{z, k_x, k_y, \omega\}$ space so that $\mathcal{F}[\delta(\mathbf{r}, t)] = \delta(z)$). As it is known (see for example [26]) the fundamental solution of the operator $\frac{d}{dz} + a$ for any $a \in \mathbb{C}$ exists and is given by the expression $\Theta(z)e^{-az}$.

This gives us immediately fundamental solution $\mathcal{E}$ of Eq. (9), which also obviously belongs to $\mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$:

$$\mathcal{E}(\mathbf{r}, t) = \Theta(z)\mathcal{F}^{-1}_{\mathbf{r}_\perp, t}\left[\frac{\exp(i\beta_z z)}{2i\beta_z}\right], \tag{20}$$

We will use also the following relation:

$$\exp(i\beta_z z) = -i\mathcal{F}_z^{-1}\left[\left\{\frac{1}{k_z - \beta_z - i0} - \frac{1}{k_z - \beta_z + i0}\right\}\right], \tag{21}$$

where we define $\lim_{\epsilon\to 0} f(x \pm i\epsilon)$ as $f(x \pm i0)$. Of course, Eq. (21) should be understood in the sense of distributions. Eq. (21) follows directly from the Sokhotsky's formulas $1/(x \pm i0) = \mp i\pi\delta(0) + \mathscr{P}[1/x]$. In addition, taking into account that $k_z^2 + \beta_z^2 = k^2 + \beta^2$ (where $\mathbf{k} = \{k_z, \mathbf{k}_\perp\}$, $k^2 = k_z^2 + k_\perp^2$, $\beta = \omega/c$), the expressions $1/(k_z - \beta_z \pm i0)$ can be rewritten as:

$$\frac{\pm 1}{k_z - \beta_z \mp i0} = \pm\frac{k_z + \beta_z}{k^2 - \beta^2 \mp i0} = \pm\frac{k_z + \beta_z}{k^2 - (\beta \pm i0\,\mathrm{sign}\,\beta)^2}. \tag{22}$$

Substituting Eq. (21), Eq. (22) into Eq. (20) and dividing the resulting expression into parts corresponding to the summands in Eq. (22), we will have:

$$\mathcal{E} \equiv \Theta(z)(\mathcal{E}_+ + \mathcal{E}_-);\; \mathcal{E}_\pm = \pm\mathcal{F}^{-1}\left[\frac{k_z + \beta_z}{2\beta_z}\frac{1}{k^2 - (\beta \mp i0\,\mathrm{sign}\,\beta)^2}\right]. \tag{23}$$

We can explicitly perform the Fourier transform $\mathcal{F}_t^{-1}$, that is, make the integration over ω, thus obtaining:

$$\mathcal{E}_\pm = \pm\frac{ic}{2}\mathcal{F}_{x,y,z}^{-1}\left[\Theta(k_z)\frac{e^{\mp ickt}}{k}\right]. \tag{24}$$

$\Theta(k_z)$ appears because the integration over ω gives $\beta_z \to |k_z|$ and thus $k_z+\beta_z \to k_z+|k_z| = 2\Theta(k_z)|k_z|$. Taking into account that $\mathcal{E}_{\Box\pm}$ can be rewritten as [27]

$$\mathcal{E}_{\Box\pm} = -c\Theta(\pm t)\mathcal{F}_{x,y,z}^{-1}\left[\frac{\sin(ckt)}{k}\right], \tag{25}$$

and using Eq. (23) we obtain finally Eq. (19).

5 the UPPE with the Paraxial Nonlinearity

The UPPE in Eq. (9) is completely free from the paraxial approximation, that is, no assumptions is taken about the ratio k_z/k allowed in the solution. On the other hand, sometimes [11, 20] a simplified (but computationally more effective) variant of the UPPE is considered:

$$\partial_z E - i\hat{\beta}_z E = \frac{1}{2i}\hat{\beta}_{i0}Q, \tag{26}$$

where

$$\hat{\beta}_{i0}E(\mathbf{r},t) = \mathcal{F}^{-1}\left[\frac{\mathcal{F}[E(\mathbf{r},t)]}{\beta}\right], \tag{27}$$

that is, β_z is replaced by β in denominator of right hand side of Eq. (26) [cf. Eq. (9), Eq. (11)]. This equation can be obtained using the approximation $k_z/k \ll 1$ for the right hand side of Eq. (9).

Definition 5.1. *We will call the Eq. (26) the paraxial UPPE.*

Theorem 5.2. *The fundamental solution $\mathcal{E}_p \in \mathscr{S}'(\mathbb{R}^3 \times \mathbb{R})$ of Eq. (26) exists and is given by:*

$$\mathcal{E}_p = c\Theta(z)\int_{-\infty}^{t} \partial_z\left\{\mathcal{E}_+ - \mathcal{E}_-\right\} d\tau, \tag{28}$$

where $\mathcal{E}_\pm$ are given by Eq. (23).

Proceeding as in the previous section, that is, introducing $\mathcal{E}_{p\pm}$ similar to Eq. (23) so that $\mathcal{E}_p = \Theta(z)(\mathcal{E}_{p+} + \mathcal{E}_{p-})$, we obtain, that the analogous formula for $\mathcal{E}_{p\pm}$ contains only additional factor β_z/β in the Fourier-image of the analog of Eq. (23), and thus the factor k_z/k in the analog of Eq. (24), in other respects it is the same. This fact can be in $\{\mathbf{r}, t\}$-space expressed as:

$$\mathcal{E}_{p\pm} = \pm c\int_{-\infty}^{t} \partial_z\mathcal{E}_\pm \, d\tau, \tag{29}$$

which gives finally Eq. (28).

6 Discussions and Conclusions

We see now that, although in the 3-dimensional UPPE Eq. (2) [Eq. (9)] the "selected axis" z is pre-imposed, it in many respects remains similar to the 3-dimensional wave equation. Nevertheless, in contrast to the WE, an inhomogeneity in the form of δ-function in the UPPE produces waves propagating also both forward and backward in z-direction, and in both directions in time, as formulated in Eq. (19), Remarks 4.2 and 4.3. This is in some contradiction with the commonly-used name "unidirectional" given to this equation. On the other hand, if the inhomogeneity Q belongs to the class of forward- or backward-propagating in z-direction waves, the solution $\mathcal{E} \star Q$ also "almost" belongs to the same class with the only deviation resulting from the presence of Θ-function in Eq. (19), which extends the spectrum of the solution to both $k_z < 0$ and $k_z > 0$ even it is not so for the spectrum of Q.

Strictly speaking, the fact that the inhomogeneity Q excites the waves which propagate in both directions in time breaks the causality principle; that is, the solution of the UPPE at some distance from the excitation Q is determined not only by the past but also by the future of the excitation history. Nevertheless, when the UPPE is applied to the short pulses and the inhomogeneity is created by nonlinearity, the "waves from the future" interact with the pulse only very limited time, and quickly travel away, remaining very weak. In this situation, using of the UPPE is, of course, acceptable.

The results above allow to see immediately also some important qualitative features of the UPPE response in some particular cases. For instance, assuming the source is created by a plasma current J near some spatial point $\mathbf{r}_0$, that is, $Q \propto J$, the solution will be given by $E \propto \partial_t J$, as in the case of the 3D wave equation, but unlike 1D wave equation, where we would have $E \propto J$ instead [26, 28].

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