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Uniform Poincaré-Sobolev and relative isoperimetric inequalities for classes of domains

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Abstract

The aim of this paper is to prove an isoperimetric inequality relative to a convex domain $\Omega \subset \mathbb{R}^d$ intersected with balls with a uniform relative isoperimetric constant, independent of the size of the radius $r > 0$ and the position $y \in \overline{\Omega}$ of the center of the ball. For this, uniform Sobolev, Poincaré and Poincaré-Sobolev inequalities are deduced for classes of (not necessarily convex) domains that satisfy a uniform cone property. It is shown that the constants in all of these inequalities solely depend on the dimensions of the cone, space dimension d, the diameter of the domain and the integrability exponent $p \in [1, d)$.

1 Introduction

Main goal of this paper is to prove a uniform relative isoperimetric inequality for a suitable class of bounded domains $\Omega \subset \mathbb{R}^d$ intersected with balls. More precisely, we shall verify the following statement: *Let $\Omega \subset \mathbb{R}^d$ be a suitable, bounded domain. Then, there exists a constant $C_\Omega > 0$ such that for all $y \in \overline{\Omega}$, for all $\rho > 0$ and for every $\mathcal{L}^d$-measurable set $A \subset \Omega$ with finite perimeter in Ω, i.e. $P(A, \Omega) < \infty$, it holds:*

$$\min\left\{\mathcal{L}^d(A \cap (\Omega \cap B_\rho(y))), \mathcal{L}^d((\Omega \cap B_\rho(y))\backslash A)\right\}^{\frac{d-1}{d}} \leq C_\Omega P(A, \Omega \cap B_\rho(y))\,. \tag{1.1}$$

Here and in the following, $d \in \mathbb{N}$ denotes the space dimension, $\mathcal{L}^d$ the d-dimensional Lebesgue-measure and $B_\rho(y) \subset \mathbb{R}^d$ the open ball of radius ρ with center in $y \in \mathbb{R}^d$. Moreover, $\partial\Omega$ stands for the topological boundary of Ω.

As it is pointed out in e.g. [BZ80, pp. 132] the relative isoperimetric constant (in an isoperimetric inequality relative to a domain Ω) can be understood as a domain characteristic in the sense that its value is essentially determined by the geometrical properties of the domain. Thus, in other words, the domains Ω suited for (1.1) shall be such that a change of $\rho > 0$ or $y \in \overline{\Omega}$ does not exorbitantly worsen the geometrical properties of $\Omega \cap B_\rho(y)$, so that the respective relative isoperimetric constants $C_{\Omega\cap B_\rho(y)}$ can be bounded uniformly from above. In particular, if $\Omega \cap B_\rho(y) = \Omega$ for ρ sufficiently large, the respective relative isoperimetric constant $C_{\Omega\cap B_\rho(y)}$ should coincide with the one for Ω.

For a set A such that $\min\left\{\mathcal{L}^d(A \cap (\Omega \cap B_\rho(y))), \mathcal{L}^d((\Omega \cap B_\rho(y))\backslash A)\right\} = \mathcal{L}^d(A \cap (\Omega \cap B_\rho(y)))$ the relative isoperimetric inequality (1.1) is equivalent to the Poincaré-Sobolev inequality (for functions with zero mean value) between $\mathrm{BV}(\Omega \cap B_\rho(y))$ and $L^{d/(d-1)}(\Omega \cap B_\rho(y))$, applied to the characteristic function of the finite-perimeter set A. This is exactly the strategy of our proof: We will first verify that the Poincaré-Sobolev inequality in domains Ω holds with a constant that depends solely on certain geometric properties of the domain. All the domains with the same geometric properties will be gathered in a class and it will be shown that, for $\Omega \cap B_\rho(y)$, changes of $\rho > 0$ and $y \in \overline{\Omega}$ do not lead out of this class.

Also for $\Omega = \mathbb{R}^d$ the Poincaré-Sobolev inequality and the isoperimetric inequality (on $\mathbb{R}^d$) are equivalent. The isoperimetric inequality in $\mathbb{R}^d$ states that the perimeter $P(A)$ in $\mathbb{R}^d$ of an $\mathcal{L}^d$-measurable set A is necessarily larger than the perimeter of a ball with the same volume, i.e.

$$d\omega_d^{1/d}\mathcal{L}^d(A)^{(d-1)/d} \leq P(A)\,,$$

where ω_d is the volume of the ball in $\mathbb{R}^d$ with radius $r = 1$. In order to quantify the deviation of a finite-perimeter set A from being a ball, so-called *quantitative* isoperimetric inequalities have been studied in e.g. [Fug89, FMP10, FMP08, CN10]:

$$d\omega_d^{1/d}\mathcal{L}^d(A)^{(d-1)/d}(1 + \lambda(A)) \leq P(A)\,,$$

where the function λ measures how far A is from being a ball; in particular $\lambda(A) = 0$ iff $A \subset \mathbb{R}^d$ is a ball. Analogously, in [FMP07], a quantitative (Poincaré-)Sobolev inequality has been derived which expresses how far a $\mathrm{BV}(\mathbb{R}^d)$-function is from being the characteristic function of a ball.

The above mentioned works, however, are concerned with estimates in $\mathbb{R}^d$ and it is not obvious at all whether and how the constants obtained in $\mathbb{R}^d$ relate to the constants of the respective inequalities on bounded domains, i.e. *relative* to a bounded domain Ω. In the latter context, it is elaborated in [GG01, GG03, Gaj01] that the *optimal* relative isoperimetric constant is given by the first non-trivial eigenvalue of the 1-Laplacian equation in $\Omega \subset \mathbb{R}^d$ subject to homogeneous Neumann boundary conditions. Furthermore, the relation of this optimal constant to the *optimal* constant of the Poincaré-Sobolev inequality for convex $\Omega \subset \mathbb{R}^2$ is described in [EFK$^+$12] and it is proved that the respective optimal constants for Ω are always bounded from above by the ones for a ball of volume $\mathcal{L}^d(\Omega)$. In the present work, however, we are rather interested in the *uniform* applicability of such constants for a collection of domains. Hence, to obtain (1.1) we rather rely on the "classical" methods to derive embedding inequalities on bounded domains, like they are elaborated in e.g. [Agm65, Ada75, Maz11, Zie89]. In particular, for the derivation of uniform estimates we will confine the analysis to bounded domains $\Omega \subset \mathbb{R}^d$ with diameter $\operatorname{diam}\Omega \leq b$, which satisfy the *uniform cone property.* Hereby, we will use the definition of the uniform cone property given by Chenais in [Che75, Che77], see Definition 1.2 here below, and in order to distinguish it from other definitions, see Definition 2.2, we will from now on refer to it as *Chenais' uniform cone property.* Let us mention that in subsequent works related to shape optimization, this type of cone property is also called ε-cone property, see e.g. [Pir87, LS04, BC07]. The advantage of Chenais' uniform cone property is that this definition is tuned according to the relevant parameters that enter into the constants in inequalities on such type of bounded domains. Therefore it is possible to gather domains satisfying Chenais' uniform cone property with the same parameters into a class, see Definition 1.3 and to thus conclude *uniform* constants within this class.

Definition 1.1 (Convex cone $K(\xi,\theta,h)$) *Let $h>0$, $\theta \in (0,\pi/2)$ and $\xi \in \mathbb{R}^d$ with $|\xi|=1$. The set*

$$K(\xi,\theta,h) := \{x \in \mathbb{R}^d;\, x\cdot\xi > |x|\cos\theta,\ |x|<h\} \tag{1.2}$$

is the cone of angle θ, height h and axis ξ.

The restriction $\theta \in (0,\pi/2)$ is for the reason of convexity of the cone and because of $\cos\pi/2 = 0$. Since $x\cdot\xi = |x||\xi|\cos\alpha(x,\xi)$ with $\alpha(x,\xi)$, the intersection angle between the lines tx and $t\xi$, $t\in\mathbb{R}$, and since $\cos(\cdot)$ is monotonously decreasing on $(0,\pi/2)$, the cone $K(\xi,\theta,h)$ indeed consists of the vectors $x \in \mathbb{R}^d$ with $|x|<h$ and $\alpha(x,\xi)<\theta$.

Definition 1.2 (Chenais' uniform cone property [Che77]) *Let $h>0$, $\theta\in(0,\pi/2)$ and $r>0$ with $2r\leq h$ fixed. A set $\Omega\subset\mathbb{R}^d$ is said to satisfy the uniform cone property iff for all $x\in\partial\Omega$ there exists a cone $K_x = K(\xi_x,\theta,h)$ of angle θ, height h and axis ξ_x with vertex in x such that*

$$\text{for all} \quad y\in B_r(x)\cap\Omega \text{ it holds } y+K_x\subset\Omega. \tag{1.3}$$

All the sets $\Omega\subset\mathbb{R}^d$ with diameter $\operatorname{diam}\Omega\leq b$, which satisfy the cone property from Definition 1.2 with cones congruent to the same cone $K(\xi,\theta,h)$ with fixed parameters θ,h,r are now gathered in the class $\Pi(\theta,h,r,b)$. In particular, note that $\Pi(\theta_1,h_1,r_1,b)\subset\Pi(\theta_2,h_2,r_2,b)$ for all $\theta_2\leq\theta_1$, $h_2\leq h_1$ and $r_2\leq r_1$.

Definition 1.3 ($\Pi(\theta,h,r,b)$) *Let $b>0$ fixed. The class $\Pi(\theta,h,r,b)$ is the set of all the domains $\Omega\subset\mathbb{R}^d$ with diameter $\operatorname{diam}\Omega\leq b$, satisfying the cone property of Definition 1.2 with the parameters θ,h,r.*

Since Chenais' uniform cone property looks quite restrictive compared to other definitions of cone properties, we give an overview on geometrical properties of domains and discuss their relations in Section 2.1. Moreover, in Section 2.2, we state and explain existing results developed in [Che75, Che77, BC07] on the class $\Pi(\theta,h,r,b)$, such as a uniform extension theorem and Poincaré inequalities as well as compactness results with respect to different types of convergence of sets. By refining the arguments of [Ada75, L. 5.10, p. 103] we will derive a uniform Sobolev inequality in Section 3.1. Combined with the uniform Poincaré inequality this results in a uniform Poincaré-Sobolev inequality and in a uniform relative isoperimetric inequality for domains $\Omega\in\Pi(\theta,h,r,b)$. This will finally allow us in Section 3.2 to deduce the uniform relative isoperimetric inequality (1.1) for convex domains.

2 Qualities of $\Pi(\theta,r,h,b)$ & comparison of geometrical properties

2.1 Comparison of geometrical properties of domains

In the following we give an (incomplete) list of geometrical properties of domains. Moreover, we compare Chenais' uniform cone property with other definitions of the uniform cone property. We will observe that all of them coincide for bounded domains, while, for unbounded domains there is a hierarchy with Chenais' uniform cone property as the most restrictive one.

Definition 2.1 (Cone property [Ada75, 4.3, p. 66]) *A domain $\Omega \subset \mathbb{R}^d$ has the cone property iff there exists a finite cone K such that each point $x \in \Omega$ is the vertex of a finite cone $K_x \subset \Omega$ congruent to K.*

Following the lines of [Ada75, p. 66] a countable collection $\mathcal{O}$ of open subsets of $\mathbb{R}^d$ is said to be a *locally finite open cover* of a set $S \subset \mathbb{R}^d$ iff any compact set $C \subset \mathbb{R}^d$ intersects at most finitely many elements of $\mathcal{O}$. Involving this notion of covering more restrictive cone properties can be formulated.

Definition 2.2 (Uniform/restricted cone property) *A domain $\Omega \subset \mathbb{R}^d$ has the*

1) (Adams') uniform cone property, [Ada75, 4.4, p. 66], *iff there exists a locally finite open cover $\{U_j\}$ of $\partial\Omega$ and a corresponding sequence $\{K_j\}$ of finite cones, each congruent to a fixed finite cone K, such that:*
   (i) For some finite $M \in \mathbb{R}$ fixed, every U_j has diameter less than M.
   (ii) For some $\delta > 0$, it is $\Omega_\delta \subset \cup_{j=1}^{\infty} U_j$, where $\Omega_\delta := \{x \in \Omega,\ \mathrm{dist}(x, \partial\Omega) < \delta\}$.
   (iii) For every j there holds $Q_j := \cup_{x\in\Omega\cap U_j}(x + K_j) \subset \Omega$.
   (iv) For some finite $R \in \mathbb{N}$, every collection of $R+1$ of the sets Q_j has empty intersection.

2) (Agmon's) restricted cone property, [Agm65, Def. 2.1, p. 11], *iff there exists a locally finite open cover $\{O_i\}$ of $\partial\Omega$ and a corresponding sequence $\{K_i\}$ of finite cones, each congruent to a fixed finite cone K, such that:*

$$\text{for all } x \in O_i \cap \Omega : \quad x + K_i \subset \Omega \,. \tag{2.1}$$

On the first glance, Chenais' uniform cone property introduced in [Che77] seems to be more restrictive than the uniform cone property stated in [Ada75] or the restrictive cone property used in [Agm65]. Indeed, in Proposition 2.3 we will verify the hierarchy that Chenais' uniform cone property implies Adams' uniform cone property, which, in turn implies Agmon's restricted cone property, while the converse implications are in general false in the case of unbounded domains, as can be seen from Examples 2.5 and 2.6. For bounded domains, however, we will observe in Proposition 2.4 that the three properties are equivalent. For this, we first verify the following

Proposition 2.3 *Let $\Omega \subset \mathbb{R}^d$ be a domain.*

1.) Assume that Ω satisfies Chenais' uniform cone property (Def. 1.2). Then the collection of open balls $\{B_r(x),\ x \in \partial\Omega\}$ contains a locally finite subcover $\{B_r(x_j),\ j \in \mathbb{N}\}$ with the properties:

- *For every x_j and any of its nearest neighbours x_k the distance is less than r,* (2.2a)
- $\Omega_{r/2} \subset \cup_{j\in\mathbb{N}} B_r(x_j)$, (2.2b)
- *for every j there holds $Q_j := \cup_{x\in\Omega\cap B_r(x_j)}(x + K_{x_j}) \subset \Omega$.* (2.2c)
- *For some finite $R \in \mathbb{N}$, every collection of $R+1$ of the sets Q_j has empty intersection,* (2.2d)

i.e. the domain Ω satisfies Adams' uniform cone property (Def. 2.2)

2.) Assume that Ω has Adams' uniform cone property. Then Ω also satisfies Agmon's restricted cone property.

Proof: **Ad 1.):** Let $\Omega \subset \mathbb{R}^d$ satisfy Chenais' uniform cone property. Firstly, we fill $\mathbb{R}^d$ with the countable collection of closed cubes $\{C_i\}_{i\in\mathbb{N}}$ of uniform diagonal length $\mathrm{diag}\, C_i < r/2$ and $\mathrm{int}\, C_i \cap \mathrm{int}\, C_j = \emptyset$ for all $i \neq j$. We consider all the cubes C_i that have nonempty intersection with $\partial\Omega$ and in each of the C_i we pick one point $x_i \in \partial\Omega \cap C_i$. In particular, both $\{B_{r/2}(x_i),\ i \in \mathbb{N}\}$ and $\{B_r(x_i),\ i \in \mathbb{N}\}$ are locally finite open covers of $\partial\Omega$. Moreover, as $\mathrm{diag}\, C_i < r/2$, we have that $\mathrm{dist}(x_i, x_j) < r$ for x_i x_j in neighboring cubes C_i, C_j. This implies that $\Omega_{r/2} \subset \cup_{i\in\mathbb{N}} B_r(x_i)$.

By Chenais' cone property, for any $y \in B_r(x_i) \cap \Omega$ we have that $y + K_{x_i} \subset B_{r+h}(x_i) \cap \Omega$. Moreover, for any $z \in B_{r+h}(x_i)$ there is a cube C_j such that $z \in C_j$. Clearly, $B_{r+h}(y)$ intersects with at most R cubes and R is a finite number $R \in \mathbb{N}$, depending on d, r, and h, only. Hence, z is an element of at most R of the sets Q_i.

Ad 2.): Clearly, an locally finite open cover corresponding to Adam's uniform cone property also serves as a suitable open cover for Agmon's restricted cone property. ∎

As a direct consequence of of Proposition 2.3 we deduce the equivalence of the three uniform cone properties in the case of bounded domains.

Proposition 2.4 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Then, the uniform cone properties of Chenais (Def. 1.2), Adams (Def. 2.2, 1.)), and Agmon (Def. 2.2, 2.)) are equivalent.*

Proof: In view of Proposition 2.3 it remains to verify that Agmon's restricted cone property implies Chenais' uniform cone property for a bounded domain $\Omega \subset \mathbb{R}^d$. Thus, let $\mathcal{O} = \{O_i, i \in \mathbb{N}\}$ be a locally finite open cover serving for Agmon's restricted cone property. As Ω is bounded, we have that $\partial\Omega = \overline{\Omega}\backslash\Omega$ is compact. By Heine-Borel's covering theorem there exists a finite subcover $\{O_i, i = 1, \ldots N\} \subset \mathcal{O}$. We first show that there is $\delta > 0$ such that $\Omega_\delta \subset \cup_{i=1}^N O_i$ as claimed in *(ii)* of Adams' uniform cone property. In other words, this ensures that $\cup_{x\in\partial\Omega} B_r(x) \cap \Omega \subset \Omega$ for every $r < \delta$, which is needed for Chenais' uniform cone property.

For every $x \in \partial\Omega$ we introduce $\delta_x := \sup\{\rho > 0 : B_\rho(x) \subset \cup_{i\in\mathbb{N}} O_i\}$. As $\cup_{i\in\mathbb{N}} O_i$ is open we have that $\delta_x > 0$ for all $x \in \partial\Omega$. Proceeding by contradiction we assume that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \partial\Omega$ and a sequence of radii $(\delta_n)_{n\in\mathbb{N}}$ with $\delta_n = \delta_{x_n}$ such that $\delta_n \to 0$ as $n \to \infty$. But by compactness there is a subsequence $x_{n'} \to x \in \partial\Omega$. Hence $\delta_x > 0$ in contradiction to the assumption $\delta_{n'} \to 0$ and we conclude that indeed $\delta := \min\{\delta_x, x \in \partial\Omega\} > 0$.

Let now $r := \delta/8$. Then $B_r(x)\cap\Omega \subset \Omega_\delta$ for all $x \in \partial\Omega$. Moreover, consider $\widetilde{K}$, the cone from Agmon's cone property of, say, opening angle θ and height $\tilde{h}$. We shorten $\widetilde{K}$ to the cone K of opening angle θ and height $h = \delta/2$. Then, with this choice of r and h we have ensured that $2r \leq h$ and in particular that $y + K_x \subset \Omega_\delta \subset \Omega$ for every $y \in B_r(x)$ and every $x \in \partial\Omega$, independently of the vertex orientation ξ_x. Thus, the bounded domain Ω has Chenais' uniform cone property. ∎

In the following we give two examples confirming that the three cone properties are not necessarily equivalent in the case of unbounded domains. We start with an example of a domain with Adams' uniform cone property but lacking Chenais' uniform cone property.

Example 2.5 (Adams' uniform cone property $\not\Rightarrow$ Chenais' uniform cone property) As in Fig. 2.1 we consider a semi-infinite strip $\Omega \subset \mathbb{R}^2$ perforated by a sequence of holes of degenerating width. For a clearer presentation of the arguments we fix $\Omega := [0, \infty) \times [-5, 5] \backslash \cup_{n\in\mathbb{N}} H_n$, where $H_n = [5n - 1/n, 5n + 1/n] \times [-1, 1] \cup B_{\frac{1}{n}}((5n, 1)) \cup B_{\frac{1}{n}}((5n, -1))$ is the hole centered in $(5n, 0)$. Clearly, the collection of open sets $\mathcal{O} = \{U_1, U_2, U_3, U_{1_n}, U_{2_n}, U_{3_n}, U_{4_n}, U_{5_n}, U_{6_n}, n \in \mathbb{N} \cup \{0\}\}$ where

$$\begin{aligned}
&U_1 := (-1, 1) \times (-5, 5), \quad U_2 := (-1, 1) \times (4, 6), \quad U_3 := (-1, 1) \times (-6, -4),\\
&U_{1_n} := (n - 1/3, n + 1/3) \times (4, 6), \quad U_{2_n} := (n - 1/3, n + 1/3) \times (-6, -4),\\
&U_{3_n} := (5n - 1, 5n + 1/n + 1/(3n) \times (-2, 2), \quad U_{4_n} := (5n + 1/n - 1/(3n), 5n + 1) \times (-2, 2),\\
&U_{5_n} := (5n - 1, 5n + 1) \times (1, 1 + 1/n), \quad U_{6_n} := (5n - 1, 5n + 1) \times (-1 - 1/n, -1),
\end{aligned}$$

provides an open cover of $\partial\Omega$ with the properties *(i)–(iv)* of Adams' cone property. More precisely, the open cover is chosen in correspondence to the cones sketched in Fig. 2.1 and, in particular *(i)* holds true for all $M > 2$, *(ii)* for all $\delta \in (0, 1)$ and at most 3 of the open sets have nonempty intersection, i.e. $R = 3$ in *(iv)*. However, as $n \to \infty$, the width of the holes H_n tends to 0 and hence there is no uniform radius $r > 0$ such that the cone K_x can be used in the ball $B_r(x)$ for $x \in \cup_{n\in\mathbb{N}} \partial H_n$. Hence, Ω does not satisfy Chenais' uniform cone property.

Now we state an example for an unbounded domain enjoying Agmon's uniform cone property but lacking Adams' uniform cone property.

Example 2.6 (Agmon's uniform cone property $\not\Rightarrow$ Adams' uniform cone property) As indicated in Fig. 2.2 we consider a semi-infinite strip $\Omega \subset \mathbb{R}^2$ perforated by a sequence of non-convex holes of degenerating width and distance such that condition *(iii)* of Adams' uniform cone property is violated. To simplify our arguments we fix $\Sigma := ((0, \infty) \times (-5, 5))$ the semi-infinite strip and the domain

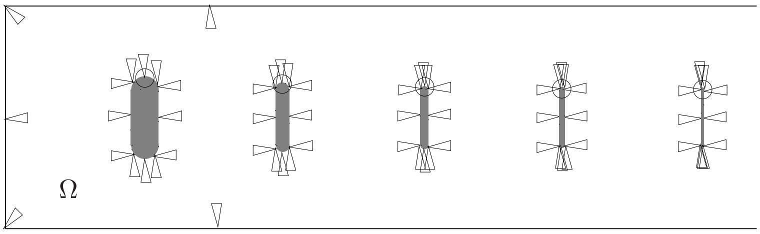

Figure 2.1: Unbounded domain with holes of degenerating width and cones corresponding to $\mathcal{O}$.

$\Omega := S \backslash \cup_{i,n\in\mathbb{N}} H_n^i$, where H_n^i are the holes constructed and positioned as follows: For every $n \in \mathbb{N}$ we put a rectangle $(5n-1/(4n), 5n+1/(4n)) \times (-1,1)$. Inside of these rectangles we cut out squares of edge length $1/(2n)$ and distance $1/(2n)$, , i.e. at each horizontal position $5n$ the semi-infinite strip S is perforated by n squares S_n^i, $i \in \{1,\dots,n\}$. Now we fix a cone K of opening angle $\theta \in (0,\pi/2)$ and height $h < 3$. For each $n \in \mathbb{N}$ fixed, for all $i \in \{1,\dots,n\}$ we choose a point $x_n^i \in S_n^i$ with coordinate component in horizontal direction larger than $5n$. In addition, for all $i \in \{1,\dots,n\}$ we choose an orientation ξ_n^i with positive horizontal component such that $x_n^i + K(\theta,h,\xi_n^i) \in \Sigma$ and such that $\xi_n^i \neq \xi_n^j$ for all $i,j \in \{1,\dots,n\}$. For all $n \in \mathbb{N}$, $i \in \{1,\dots,n\}$ the hole H_n^i is then defined by $H_n^i := S_n^i \backslash (x_n^i + K(\theta,h,\xi_n^i))$, i.e. optically, each square S_n^i has a cone-shaped notch N_n^i formed by $x_n^i + K(\theta,h,\xi_n^i)$. For every H_n^i there exists a finite open cover of ∂H_n^i and corresponding cones congruent to K such that for Ω Agmon's restricted cone property (2.1) holds true. In particular, we note that we can find elements U_n^i of the open cover such that $S_n^i \cap (x_n^i + K(\theta,h,\xi_n^i)) \subset U_n^i$ and such that $Q_n^i := \cup_{y\in U_n^i}(y + K(\theta,h,\xi_n^i)) \in \Omega$. But since the distance between the holes H_n^i and their edge length is $1/(2n)$ we conclude that the number of sets Q_n^i with nonempty intersection tends to ∞ as $n \to \infty$.

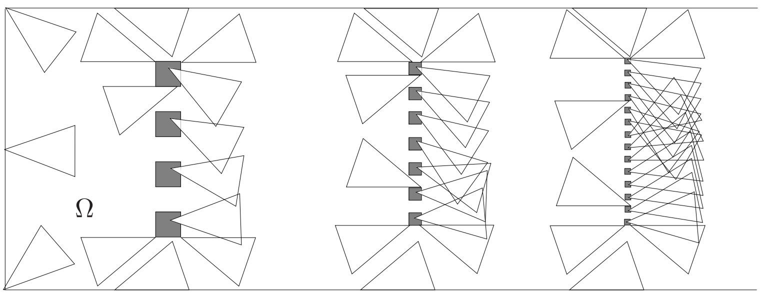

Figure 2.2: Unbounded domain with non-convex holes of degenerating size and distance and cones corresponding to $\mathcal{O}$.

In the following we state several types of Lipschitz properties and compare them with Chenais' uniform cone property.

Definition 2.7 (Lipschitz properties of bounded domains) *Let* (X, d) *and* $(\widetilde{X}, \tilde{\mathrm{d}})$ *be two metric spaces. A mapping* $F : X \to \widetilde{X}$ *is called* Lipschitzian *if there exists a constant* $L > 0$ *such that for all* $x, y \in X$ *we have* $\tilde{\mathrm{d}}(F(x), F(y)) \leq L\mathrm{d}(x,y)$. *If* F^{-1} *is injective and also Lipschitzian, the mapping* F *is called* bi-Lipschitzian, *see* [Reh12].

1. Domain with strong local Lipschitz property [Ada75, 4.5, p. 67] ($\equiv$ Domains of the class $C^{0,1}$ [Maz11, Def. 2, p. 15, Rem. 2, p. 16]): *A bounded domain* $\Omega \subset \mathbb{R}^d$ *belongs to the class* $C^{0,1}$ *if each point* $x \in \partial\Omega$ *has a neighborhood* U_x *such that the set* $U_x \cap \Omega$ *is represented by the inequality* $x_d < F_x(x_1,\dots,x_{d-1})$ *in some Cartesian coordinate system and the function* F_x *is Lipschitzian. In other words,* [Ada75, p. 67], Ω *is a domain with (locally) Lipschitz boundary.*
2. Lipschitz domain [Maz11, Def. 3, p. 16]: *A bounded domain* Ω *is a Lipschitz domain (Lipschitzian) iff each point of its boundary has a neighborhood* $U_x \subset \mathbb{R}^d$ *such that a quasi-isometric transformation maps* $U_x \cap \Omega$ *onto a cube.*

Clearly, Ω being of class $C^{0,1}$ implies that Ω being a Lipschitz domain.

Compared to the strong local Lipschitz property in Definition 2.7 above [Che75, Def. III.1, p. 201] gives a refined definition, which allows it to gather all the bounded domains Ω with neighborhoods U_x being balls $B_\delta(x)$ and Lipschitz constants of F_x of the same size $L > 0$ in the class $\mathrm{Lip}(L, \delta)$. For this, we will make use of the following notation: *Let $\delta, \delta' > 0$ and $x = (\hat{x}, x_d)^\top \in \mathbb{R}^d$ with $\hat{x} = (x_1, \dots, x_{d-1})^\top \in \mathbb{R}^{d-1}$. We introduce the sets*

$$\begin{aligned} P_{\delta\delta'}(x) &:= \{y \in \mathbb{R}^d,\ |y_i - x_i| < \delta \text{ for } i = 1, \dots, d-1 \text{ and } |y_d - x_d| < \delta'\}, \\ P_\delta(\hat{x}) &:= \{\hat{y} \in \mathbb{R}^{d-1},\ |x_i - y_i| < \delta \text{ for } i = 1, \dots, d-1\}. \end{aligned}$$

Definition 2.8 (Chenais' strong Lipschitz property) *Let $L > 0$ and $\delta > 0$ be given and $\delta' := L\delta(d-1)^{1/2}$. We denote by $\mathrm{Lip}(L, \delta)$ the set of all open sets $\Omega \subset \mathbb{R}^d$ with $\mathrm{diam}\,\Omega \le b$ such that for all $x \in \partial\Omega$ there exists a local coordinate system and a function $\Phi_x : P(\hat{x}) \to \mathbb{R}$, which is Lipschitzian with Lipschitz constant L such that*

$$y \in P_{\delta\delta'}(x) \cap \Omega \ \Leftrightarrow \ \begin{cases} y \in P_{\delta\delta'}(x), \\ y_d > \Phi_x(\hat{y}). \end{cases} \tag{2.3}$$

We first convince ourselves that Chenais' strong Lipschitz property is equivalent to the one stated in Definition 2.7, item 1..

Lemma 2.9 *A bounded domain Ω of class $C^{0,1}$ with diameter $\mathrm{diam}\,\Omega \le b$ satisfies Chenais' strong Lipschitz property and vice versa.*

Proof: Consider an open set $\Omega \subset \mathbb{R}^d$ with diameter $\mathrm{diam}\,\Omega \le b$ of class $C^{0,1}$. For all $x \in \partial\Omega$ there exists an open neighborhood U_x with $x \in U_x$. Hence, each U_x contains an open ball $B_{r_x}(x)$ with center in x. Assume that there is a sequence of points $(x_j)_j \subset \partial\Omega$ such that $r_{x_j} \to 0$ for the respective radii. Since Ω is bounded there exists a subsequence $(x_{j'})_{j'} \subset (x_j)_j$ such that $x_{j'} \to x$ and by compactness of $\partial\Omega$ we have that $x \in \partial\Omega$ with $r_x = 0$. This states a contradiction to the fact that U_x is open. Thus there is a lower bound $\tilde{\delta} > 0$ such that $r_x > \tilde{\delta}$ for all $x \in \partial\Omega$. We fix $\tilde{\delta} > 0$ such that $\overline{B_\delta(x)} \subset U_x$ for all $x \in \partial\Omega$.

With the same arguments we can conclude that the Lipschitz constants L_x of the Lipschitz mappings F_x are uniformly bounded for all $x \in \partial\Omega$, both from below and from above. Hence, there exists a Lipschitz constant L such that $|F_x(\hat{x}) - F_x(\hat{y})| \le L|\hat{x} - \hat{y}|$ for all $y \in B_{\tilde{\delta}}(x)$, uniformly for all $x \in \partial\Omega$.

Moreover, by Def. 2.7, Item 1., we find that, for all $x \in \partial\Omega$, the set $B_\delta(x) \cap \Omega$ is represented by the relation $y_d < F_x(\hat{y})$. Since F_x is Lipschitz continuous on $U_x \supset \overline{B_{\tilde{\delta}}(x)}$ we conclude that $\max_{y \in B_{\tilde{\delta}}(x)} |F_x(\hat{y})| =: M_x$ is attained. Additionally, we have that $m_x < y_d < \tilde{m}_x$ for all $y \in B_{\tilde{\delta}}(x)$. Therefore, $\Phi_x := F_x - M_x + m_x$ is Lipschitzian with Lipschitz constant L and satisfies $y_d > \Phi_x(\hat{y})$ for all $y \in B_{\tilde{\delta}}(x)$.

It remains to determine δ in a suitable relation to $\tilde{\delta}$ such that (2.3) holds true. For this, consider $y \in P_{\delta\delta'}(x)$. We have to ensure that $|y - x|^2 < (d-1)(L^2+1)\delta^2 =: \tilde{\delta}^2$, because then $y \in P_{\delta\delta'}(x)$ implies that $y \in B_{\tilde{\delta}}(x)$ and thus, (2.3) is guaranteed.

In order to verify that a domain $\Omega \in \mathrm{Lip}(L, \delta)$ is of class $C^{0,1}$, just observe that $P_{\delta\delta'}(x)$ is a particular choice of neighbourhood U_x. With analogous calculations as above one can turn the Lipschitzian Φ_x into a Lipschitzian F_x satisfying $F_x(\hat{y}) > y_d$ for all $y \in U_x \cap \Omega$. ∎

Chenais' refined definition of the strong local Lipschitz property allows it to establish the equivalence between the classes $\mathrm{Lip}(L, \delta)$ and $\Pi(\theta, h, r, b)$, see [Che75, Prop. III.1, p. 203 and Prop. III.2, p. 204].

Proposition 2.10 (Chenais' uniform cone property $\Leftrightarrow$ strong Lipschitz property) *For all θ, h, r as in Def. 1.3 there exist $L, \delta > 0$ as in Def. 2.8 such that $\Pi(\theta, h, r, b) \subset \mathrm{Lip}(L, \delta)$ and, vice versa, for all $L, \delta > 0$ as in Def. 2.8 there exist θ, h, r as in Def. 1.3 such that $\mathrm{Lip}(L, \delta) \subset \Pi(\theta, h, r, b)$.*

A weaker property of domains is the segment property, which will be exploited later in Section 3.1 for the proof of the uniform Sobolev inequality.

Definition 2.11 (Segment property [Ada75, 4.2, p.66]) *An open domain $\Omega \subset \mathbb{R}^d$ has the segment property if there exists a locally finite open cover $\{U_j\}$ of $\partial\Omega$ and a corresponding sequence $\{y_j\}$ of nonzero vectors such that if $x \in \overline{\Omega} \cap U_j$ for some j, then $x + ty_j \in \Omega$ for all $t \in (0, 1)$.*

Clearly, the segment property is implied by the uniform cone property, since, by compactness of $\partial\Omega$ there is a finite open cover with balls $B_r(x_i)$, $x_i \in \partial\Omega$ and the direction y_i is given by the axis ξ_{x_i} of the cone K_{x_i}. The existence of line segments inside Ω is crucial for the proof of embedding theorems in Sobolev spaces and we will exploit this in the proof of the Sobolev inequality. Nevertheless, the existence of segments in Ω is already guaranteed by the cone property, see Definition 1.2, since, due to a theorem by Gagliardo, see [Ada75, Thm. 4.8, p. 68], a bounded domain Ω with the cone property can be composed by a finite collection of $C^{0,1}$-domains, which means that each of them has the uniform cone property by Proposition 2.10.

Remark 2.12 (Further properties of domains) In [AF77] it was established that for many of the embedding and interpolation theorems in Sobolev spaces the so-called *weak cone condition* of a domain is sufficient: *A domain $\Omega \subset \mathbb{R}^d$ satisfies the* weak cone condition *iff there exists $\delta > 0$ such that*

$$\mathcal{L}^d(\Gamma(x)) \geq \delta \quad \text{for all } x \in \Omega\,. \tag{2.4}$$

Here, $\Gamma(x) := \{y \in R(x),\ |y - x| < 1\}$ and, given $x \in \Omega$, the set $R(x)$ consists of all points $y \in \Omega$ such that the line segment joining x to y lies entirely in Ω; thus $R(x)$ is a union of rays and line segments emanating from x (see [AF77, p. 714] *or* [AF03, 4.7, p. 82]*).*

When treating elliptic or parabolic PDEs with mixed boundary conditions, it is necessary that the underlying domain Ω is of better regularity. A geometrical property which has been established exactly for this setting is the notion of *regular domains in the sense of Gröger*. This geometrical condition enhances the notion of Lipschitz domains, see Def. 2.7, Item 2., with the refinement that the Dirichlet and Neumann parts of the boundary are mapped suitably by the bi-Lipschitz functions, i.e. the Dirichlet and the Neumann parts of the boundary $\partial\Omega$ are separated by a Lipschitzian hypersurface of the boundary. See [Grö89, Def. 2, p. 680 and Rem. 1, p. 681] for the definition and e.g. [GGKR02, GR01] for further applications.

2.2 Properties of $\Pi(\theta, h, r, b)$

In this section we give an overview over the uniform properties of the class $\Pi(\theta, h, r, b)$, which were mainly established in [Che75, Che77, BC07]. Since these results will be relevant in Section 3, we will explain their relations and outline the methods used for the proofs.

The following way to define a uniform covering for sets $\Omega \in \Pi(\theta, h, r, b)$ is a slight modification of the one given in [Che75, Prop. II.2, p. 198].

Proposition 2.13 (Uniform covering & partition of unity for $\Omega \in \Pi(\theta, h, r, b)$) *Let $\theta \in (0, \pi/2)$, $h > 0$, $r > 0$ with $2r \leq h$ fixed. There exists an integer $N(r, b, d)$ and a constant $M(r, b) > 0$ such that for each $\Omega \subset \Pi(\theta, h, r, b)$, the closure $\overline{\Omega}$ has an open covering $\mathcal{O}_\Omega := \mathcal{O}_\Omega^I \cup \mathcal{O}_\Omega^B$ with the following properties:*

- $\mathcal{O}_\Omega^B := \{B_i' := B(x_i(r/2), x_i \in \partial\Omega,\ i = 1, \ldots, \nu_B(\Omega)\}$,
- $\mathcal{O}_\Omega^I := \{B_i' := B(x_i(r/2), x_i \in \Omega,\ i = \nu_B(\Omega)+1, \ldots, \nu(\Omega)\}$,
- $\nu(\Omega) \leq N(r, b, d)$,
- $B_0' := \cup_{i=\nu_B(\Omega)+1}^{\nu(\Omega)} B_i' \subset \Omega$.

Moreover, there exist $\nu_B(\Omega) + 1$ functions $\zeta_i \in C_0^\infty(\mathbb{R}^d)$, $i = 0, 1, \ldots, \nu(\Omega)$ such that:

- $\operatorname{supp} \zeta_i \subset B_i'$, $\zeta_i(x) \in [0, 1]$ *for all* $x \in \mathbb{R}^d$ *and* $\sum_{i=0}^{\nu(\Omega)} \zeta_i(x) = 1$ *for all* $x \in \Omega$,
- $\sup_{x \in \mathbb{R}^d} |\nabla \zeta_i(x)| \leq M(r, b)$.

Proof: The second part of the proposition is proved in detail in [Che75, Prop. II.2, p. 198]. Here, we clarify the existence of a uniform upper bound $N(r, b, d)$ on the number of open sets. Let $\Omega \in \Pi(\theta, h, r, b)$ arbitrarily but fixed. Since $\operatorname{diam} \Omega \leq b$, the set Ω is contained in a closed cube $Q_{\tilde{b}}$ of edge length $\tilde{b} := ([2b/r] + 1)r$. Here $[\cdot]$ denotes the Gauß bracket, i.e. $[a] = n \in \mathbb{N}$ for $a = n + \lambda$ with $\lambda \in [0, 1)$.

The cube $Q_{\tilde{b}}$ can be covered with open balls $B_{r/2}$ of radius $r/2$ as follows: Starting in each corner we put a ball $B_{r/2}$ with center in the corner. We cover the edges of $Q_{\tilde{b}}$ with balls $B_{r/2}$ with their centers on the edges and with the distance $r/2$ of their centers. We fill the faces of the cube by translating the balls from the edges, normal to the edges such that the center of every ball to its neighbor has distance $r/2$.

We fill the interior of the cube in a similar way. We observe that every edge is covered with $2([2b/r]+1)$ balls, which implies that $Q_{\tilde{b}}$ is covered by $N(r,b,d) = 2^d([2b/r]+1)^d$ balls $B_{r/2}$.

In what follows, the collection of balls $B_{r/2}$ constructed above is denoted by $\mathcal{O}$ and the corresponding collection of their center points by $C_{Q_{\tilde{b}}}$.

The open covering $\mathcal{O}_\Omega \subset \mathcal{O}$ for $\Omega \subset Q_{\tilde{b}}$ can now be picked as follows: Those balls in $\mathcal{O}$ which do not intersect with Ω do not contribute to the cover $\mathcal{O}_\Omega$. Furthermore, we introduce the set of center points $C_I := \{x_i \in \mathbb{R}^d,\, B_{r/2}(x_i) \subset \Omega,\, B_{r/2}(x_i) \in \mathcal{O}\}$ and we find $B_0' = \cup_{x_i \in C_I} B_{r/2}(x_i)$. For balls $B_{r/2}(y_i) \in \mathcal{O}$ with $B_{r/2}(y_i) \cap \Omega \neq \emptyset$ but $B_{r/2}(y_i) \not\subset \Omega$ we choose a point $x_i \in \partial\Omega$ such that $B_{r/2}(y_i) \cap \Omega \subset B_{r/2}(x_i)$. The collection of these center points is denoted by C_B. Hence, the collection of open balls $\mathcal{O}_\Omega := \{B_{r/2}(x_i),\, x_i \in C_I \cup C_B\}$ is an open cover of Ω. It consists of $\nu(\Omega)$ balls with $\nu(\Omega) \leq N(r,b,d)$ by construction. ■

The uniform partition of unity is the crucial tool to construct linear, continuous extension operators $E_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$, $p \in [1,\infty)$ with their operator norms uniformly bounded for all $\Omega \in \Pi(\theta,h,r,b)$.

Theorem 2.1 (Uniform extension) *Let $\theta \in (0,\pi/2)$, $0 < 2r \leq h$ and $p \in [1,\infty)$. Then there exists a constant $K(\theta,h,r,b,d,p) > 0$ and for every $\Omega \in \Pi(\theta,h,r,b)$ there is a linear, continuous extension operator $E_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ such that:*

$$\|E_\Omega\| \leq K(\theta,h,r,b,d,p)\,. \tag{2.5}$$

The above statement was proved in [Che75, Thm. II.1, p. 199] for $p=2$ using Calderon-Zygmund kernels for the construction of the extension operators; see also [Agm65, Chap. 11] for more details related to extensions via Calderon-Zygmund kernels. This method, however, cannot be applied for $p=1$. This case is covered by [Che77, Thm. II.1, p. 213] using reflection techniques to construct the extension operators. In that work the uniform statements are developed for the class $\mathrm{Lip}(L,\delta)$. Nevertheless, Proposition 2.10 directly translates this result into the $\Pi(\theta,h,r,b)$-setting.

Definition 2.14 (Different notions of set convergence) *Consider a sequence sets $(A_k)_k \subset \mathbb{R}^d$.*

1. Convergence in the sense of characteristic functions: *For all $k \in \mathbb{N}$ assume that A_k is $\mathcal{L}^d$-measurable. The sequence $(A_k)_k$ is said to converge to an $\mathcal{L}^d$-measurable set A, i.e. $A_k \overset{\mathrm{ch}}{\to} A$, iff the sequence of their characteristic functions $(\mathcal{X}_{A_k})_k$ converge strongly in $L^1(\mathbb{R}^d)$, i.e. $\mathcal{X}_{A_k} \to \mathcal{X}_A$ in $L^1(\mathbb{R}^d)$.*
2. Convergence of compact sets in Hausdorff-sense: *For two compact sets K_1 and K_2 the Hausdorff distance can be defined by, see* [Hau62, p. 167]*, or e.g.* [RW98, p. 117],
$$d_{\mathrm{H}}(K_1,K_2) := \inf\{\eta \in [0,\infty),\, K_1 \subset K_2 + B_\eta(0) \text{ and } K_2 \subset K_1 + B_\eta(0)\}\,. \tag{2.6}$$
For all $k \in \mathbb{N}$, assume that A_k are compact. We say that $(A_k)_k$ converges in Hausdorff-sense to a compact set $A \subset \mathbb{R}^d$, i.e.
$$A_k \overset{\mathrm{H}}{\to} A \quad \text{iff} \quad d_{\mathrm{H}}(A_k,A) \to 0\,. \tag{2.7}$$
3. Convergence of open sets in Hausdorff-complement-sense: *Let $D \subset \mathbb{R}^d$ be open. For all $k \in \mathbb{N}$, assume that A_k is open and $A_k \subset D$. We say that $(A_k)_k$ converges in Hausdorff-complement-sense to an open set $A \subset D$, i.e.*
$$A_k \overset{\mathrm{H^c}}{\to} A \quad \text{iff} \quad d_{\mathrm{H}}(\overline{D}\backslash A_k, \overline{D}\backslash A) \to 0\,. \tag{2.8}$$

Remark 2.15 (Translations of sets $\Omega \in \Pi(\theta,h,r,b)$) Since $\operatorname{diam}\Omega \leq b$ for any set $\Omega \in \Pi(\theta,h,r,b)$, there are points $x_\Omega \in \Omega$ such that $\Omega \subset Q_{\tilde{b}}(x_\Omega)$, where $Q_{\tilde{b}}(x_\Omega)$ is the cube with center $x_\Omega \in \mathbb{R}^d$ and edge length $\tilde{b} := ([2b/r]+1)r$ with edges parallel to the planes spanned by the coordinate axes. Here, $[\cdot]$ denotes the Gauß bracket, i.e. $[a] = n \in \mathbb{N}$ for $a = n + \lambda$ with $\lambda \in [0,1)$. The collection of sets $\Pi(\theta,h,r,b)$ therefore can be composed by translating $\Pi(\theta,h,r,Q_{\tilde{b}}(0)) := \{\Omega \subset Q_{\tilde{b}}(0) \text{ and } \Omega \in \Pi(\theta,h,r,b)\}$, i.e.

$$\Pi(\theta,h,r,b) = \bigcup_{x \in \mathbb{R}^d} x + \Pi(\theta,h,r,Q_{\tilde{b}}(0))\,. \tag{2.9}$$

In addition, for all $u \in W^{1,p}(\Omega)$ we have that $\|u\|_{W^{1,p}(\Omega)} = \|u \circ \tau_\Omega^{-1}\|_{W^{1,p}(\tau_\Omega \Omega)}$, where $\tau_{x_\Omega} : \mathcal{Q}_{\tilde{b}}(x_\Omega) \to \mathcal{Q}_{\tilde{b}}(0)$ is the translation that centers the cube $\mathcal{Q}_{\tilde{b}}(x_\Omega)$ in the origin.

For the collection of sets $\Pi(\theta,h,r,Q_{\tilde b}(0))$ introduced in Remark 2.15 the results in [LS04, Lemma 3.3, p. 4] state compactness with respect to the above convergences.

Lemma 2.16 (Compactness of $\Pi(\theta,h,r,Q_{\tilde b}(0))$) *Let $(\Omega_k)_k \subset \Pi(\theta,h,r,Q_{\tilde b}(0))$. Then, there exists a subsequence $(\Omega_{k'})_{k'} \subset (\Omega_k)_k$ and a set $\Omega \in \Pi(\theta,h,r,Q_{\tilde b}(0))$ such that*

$$\Omega_{k'} \overset{\mathrm{H^c}}{\to} \Omega\,, \quad \Omega_{k'} \overset{\mathrm{ch}}{\to} \Omega \quad \textit{and} \quad \overline{\Omega_{k'}} \overset{\mathrm{H}}{\to} \overline{\Omega}\,. \tag{2.10}$$

With the aid of Remark 2.15 one can therefrom conclude the closedness of the class $\Pi(\theta,h,r,b)$ with respect to the above convergences.

Theorem 2.2 (Closedness of $\Pi(\theta,h,r,b)$) *The set $\Pi(\theta,h,r,b)$ is closed with respect to both the convergence in the sense of characteristic functions and the convergence in Hausdorff-complement-sense.*

The compactness result in combination with the uniform extension is used in [BC07, Thm. 1, p. 1442] to derive a uniform Poincaré inequality; in the following,

$$[u]_\Omega := \tfrac{1}{\mathcal{L}^d(\Omega)} \int_\Omega u \,\mathrm{d}x \tag{2.11}$$

denotes the mean value of u in Ω.

Theorem 2.3 (Uniform Poincaré inequality for $\Omega \in \Pi(\theta,h,r,b)$) *Let $p \in [1,\infty)$. There exists a constant $C_{\mathrm{P}} = C_{\mathrm{P}}(\theta,h,r,d,p)$ such that for every $\Omega \in \Pi(\theta,h,r,b)$ and for all $u \in W^{1,p}(\Omega)$ it is*

$$\|u - [u]_\Omega\|_{L^p(\Omega)} \le C_{\mathrm{P}} \|\mathrm{D}u\|_{L^p(\Omega)}\,. \tag{2.12}$$

The way to prove the above uniform result is indirect, by contradiction. It reveals its dependence on the quantities θ,h,r,d,b and p but it does not render the constant C_{P} in detail. For convex domains in arbitrary space dimension, however, optimal constants in Poincaré inequalities for functions with zero-mean value could be derived in [PW60] for $p=2$ and in [AD03] for $p=1$. In these cases it turns out that the optimal constant solely depends on the diameter of the domain. In particular, for $p=1$, [AD03, Thm. 1, p. 199] states that

$$C_{\mathrm{P}} = b/2 \qquad \text{for every convex domain } \Omega \subset \mathbb{R}^d \text{ with diameter } b\,. \tag{2.13}$$

3 Uniform inequalities for $\Pi(\theta,h,r,b)$

In this section we derive uniform Poincaré-Sobolev inequalities and a relative isoperimetric inequality for the class $\Pi(\theta,h,r,b)$ as well as the uniform relative isoperimetric inequality for convex domains $\Omega \in \Pi(\theta,h,r,b)$ intersected with balls. Similar to the works [Che75, Che77, BC07], where uniform extension operators and Poincaré inequalities are deduced, we will obtain that the constant in the Poincaré-Sobolev inequality for $\Omega \in \Pi(\theta,h,r,b)$ solely depends on the the exponent p, space dimension d, the bound on the diameter b and on the parameters θ,h and r of the cone that defines the cone property for the class $\Pi(\theta,h,r,b)$. This uniform dependence carries over both to the relative isoperimetric inequality in $\Pi(\theta,h,r,b)$ in Section 3.1 and to the uniform relative isoperimetric inequality for convex domains $\Omega \in \Pi(\theta,h,r,b)$ intersected with balls in Section 3.2.

3.1 Uniform Poincaré-Sobolev and isoperimetric inequalities for $\Pi(\theta,h,r,b)$

It was elaborated in [AF77, Thm. 1, p. 715 and pp. 726] that the Sobolev embedding $W^{1,1}(\Omega) \to L^{d/(d-1)}(\Omega)$, i.e. the respective Sobolev inequality, holds also for domains with the weak cone property, only, see Remark 2.12. It is even pointed out there, that the respective embedding constants exhibit the dependence on the previously mentioned parameters, also for Ω having the weak cone property, only. However, the deduction of the Sobolev-Poincaré inequality from this Sobolev inequality requires the use of the uniform Poincaré inequality from Theorem 2.3, which in turn is proven via Theorem 2.1 on the existence of a uniform extension operator. Thus, to have these uniform results at hand, we will confine ourselves to the strengthened assumption that Ω even has the strong local Lipschitz property and prove the Sobolev inequality for domains $\Omega \in \Pi(\theta, h, r, b)$, only.

The sole dependence of the Sobolev constant on the previously mentioned parameters for domains with strong local Lipschitz property, is also pointed out in [Ada75, L. 5.10, p. 103], but the proof therein does not reveal how exactly these parameters enter the constant. Therefore, we will here give a modified proof of [Ada75, L. 5.10, p. 103] where the influence of the parameters in the constant is displayed more clearly. In the proof, we will apply the uniform, finite covering given in Proposition 2.13. In each of the subdomains we will derive the uniform Sobolev inequality by exploiting the segment property of the domain, as suggested in the proof of [Ada75, L. 5.10, p. 103]. The uniform boundedness of the number of elements in the covering will then allow it to find a global inequality for $\Omega \in \Pi(\theta, h, r, b)$.

Theorem 3.1 (Sobolev inequality in $\Omega \in \Pi(\theta, h, r, b)$) *Let $p < d$. There is a constant $C_{\mathrm{S}} = C_{\mathrm{S}}(\theta, h, r, d, p)$ such that for all $\Omega \in \Pi(\theta, h, r, b)$ and for all $u \in W^{1,p}(\Omega)$ it holds*

$$\|u\|_{L^{dp/(d-p)}(\Omega)} \leq C_{\mathrm{S}} N(r, b, d)^{1-\frac{1}{p}} \|u\|_{W^{1,p}(\Omega)} \,. \tag{3.1}$$

Proof: Let $\Omega \in \Pi(\theta, h, r, b)$. Recall that Ω satisfies Chenais' uniform cone property and that the sets $B_i' \in \mathcal{O}_\Omega$ from Proposition 2.13 have the radius $r/2$. Hence, for all $x \in B_i' \cap \Omega$ we have that a cone $K_i = K(\theta, h, \xi_i)$ with opening angle θ, height h and vertex orientation ξ_i is contained in Ω. In particular, $2r \leq h$. Therefore, K_i contains a parallelepiped P_i with opening angle θ and edge length $l = h/(2\cos(\theta/2))$. The parallelepiped can be transformed into a cube $Q_{l/2}i$ of edge length $l = h/(2\cos(\theta/2))$ by a suitable transformation T.

In a first step we assume that the parallel epiped indeed is the cube $Q^i_{l/2}$ and we derive a local Sobolev estimate for Q^i_l. This can be done in analogy to the proof of [Ada75, L. 5.10, p. 103], where we exploit that the domain $\Omega_i := (\Omega \cap B_i') + Q^i_{l/2}$ can be regarded as the parallel translate of the cube $Q^i_{l/2}$. Secondly we treat the general case of a parallel epiped $P_i \neq Q^i_{l/2}$ by applying the above mentioned transformation T that allows it to lead this case back to the setting of the cube $Q^i_{l/2}$. Here, for $\Omega_i := (\Omega \cap B_i') + P_i$, the transformed domain $T\Omega_i$ is then again given as the parallel translate of the cube $Q^i_{l/2}$, so that the results of Step 1 apply. With the aid of [Ada75, Thm. 3.35, p. 63] the Sobolev estimate obtained for the transformed domain can be carried over to Ω_i.

In a third step, the Poincaré-Sobolev estimate for Ω is obtained by summing up the local contributions.

Step 1 (Estimate for $P_i = Q^i_{l/2}$): We choose a local coordinate system with axes in parallel to the faces of the cube $Q^i_{l/2}$. We introduce the set $\Omega_i := (\Omega \cap B_i') + Q^i_{l/2}$, which is the parallel translate of the cube $Q^i_{l/2}$. For $x \in \Omega_i$ let $w_j(x)$ denote the intersection of Ω_i with the straight line through x in parallel to the x_j-axis. Hence, $w_j(x)$ contains the line segment $\{x + te_j\,, 0 \leq t < 1\}$. Here, e_j is a vector in parallel to the x_j-axis with $|e_j| = l$ which points either in the positive or in the negative x_j-direction in dependence of the position of $x \in \Omega_i$.

Let $\gamma = (dp - p)/(d - p)$ and consider $u \in C^\infty(\overline{\Omega_i})$. Then, integration by parts yields

$$\int_0^1 |u(x + (1-t)e_j|^\gamma \,\mathrm{d}t = |u(x)|^\gamma - \gamma \int_0^1 t|u(x + (1-t)e_j|^{\gamma-1}\, \partial_t |u(x + (1-t)e_j)| \mathrm{d}t \,, \tag{3.2}$$

where $|u(x + (1-t)e_j|^{\gamma-1}\, \partial_t |u(x + (1-t)e_j)|$ is well-defined for every $u \in C^\infty(\overline{\Omega_i})$ although the absolute value function is non-differentiable in 0. To see this, assume that $u(x_0) = 0$, but $u(y) \neq 0$ for all y in a neighborhood $B(x_0)$ of x_0. Otherwise, if $u \equiv 0$ in $B(x_0)$, the part of the line segment intersecting with $B(x_0)$ can be neglected on the left-hand side of (3.2). Then $\lim_{h\to 0}(|u(x_0 \pm he_j)|^{\gamma-1}\partial_{x_j}|u(x_0 \pm he_j)|) = \lim_{h\to 0}(|u(x_0 \pm he_j)|^{\gamma-1}\,\mathrm{sign}(u(x_0 \pm he_j))\partial_{x_j} u(x_0 \pm he_j)) = 0$. In particular, this observation implies the following estimate, which will be used later:

$$\text{For all } z \in \overline{\Omega_i}: \qquad |u(z)|^{\gamma-1}\partial_{z_j}|u(z)| \leq |u(z)|^{\gamma-1}|\partial_{z_j} u(z)| \,. \tag{3.3}$$

Rearranging the terms in (3.2) leads to

$$|u(\tilde{x})|^\gamma = \int_0^1 |u(\tilde{x} + (1-t)e_j)|^\gamma \,\mathrm{d}t + \gamma \int_0^1 t|u(\tilde{x} + (1-t)e_j)|^{\gamma-1}\, \partial_t |u(\tilde{x} + (1-t)e_j)| \mathrm{d}t \,, \tag{3.4}$$

which holds true for every $\tilde{x} \in w_j(x)$ with e_j pointing either in the positive or in the negative x_j-direction. Let $\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$ and set

$$F_j(\hat{x}_j) = \sup_{\tilde{x} \in w_j(x)} |u(\tilde{x})|^{p/(d-p)} . \tag{3.5}$$

Assume that $\tilde{x}_s$ realizes the supremum in (3.5). Since $\gamma = p(d-1)/(d-p)$ we deduce from (3.4) with $\tilde{x} = \tilde{x}_s$ that

$$\begin{aligned} |F_j(\hat{x}_j)|^{d-1} = |u(\tilde{x}_s)|^\gamma &= \int_0^1 |u(\tilde{x}_s + (1-t)e_j)|^\gamma \,\mathrm{d}t + \gamma \int_0^1 t|u(\tilde{x}_s + (1-t)e_j|^{\gamma-1}\, \partial_t |u(\tilde{x}_s + (1-t)e_j)| \mathrm{d}t \\ &\le \int_{w_j(x)} |u(\tilde{x})|^\gamma \,\mathrm{d}\tilde{x}_j + \gamma \int_{w_j(x)} |u(\tilde{x})|^{\gamma-1} |\mathrm{D}_j u(\tilde{x})| \,\mathrm{d}\tilde{x}_j , \end{aligned}$$

where the second integral is estimated with the aid of (3.3).

Until now we have just integrated in x_j-direction. Integration with respect to the remaining coordinates can be done by integration over O_j, the projection of Ω_i onto the plane with $x_j = 0$. This leads to

$$\int_{O_j} |F_j(\hat{x}_j)|^{d-1} \,\mathrm{d}\hat{x}_j \le \int_{Q_i'} |u(x)|\, |u(x)|^{\gamma-1} \,\mathrm{d}x + \gamma \int_{\Omega_i} |u(x)|^{\gamma-1} |\mathrm{D}_j u(x)| \,\mathrm{d}x . \tag{3.6}$$

If $p > 1$, then $\gamma > 1$ and we apply Hölder's inequality with exponent p to the right-hand side. Hence, $p'(\gamma - 1) = dp/(d-p) = q$ and we obtain

$$\|F_j\|_{L^{d-1}(O_j)}^{d-1} \le \gamma \left(\int_{\Omega_i} \big(|u| + |\nabla u|\big)^p \,\mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{\Omega_i} |u(x)|^{(\gamma-1)p'} \,\mathrm{d}x \right)^{\frac{1}{p'}} \le 2^{(p-1)/p} \gamma \|u\|_{W^{1,p}(\Omega_i)} \|u\|_{L^q(\Omega_i)}^{q/p'} . \tag{3.7}$$

Now, [Ada75, L. 5.9, p. 101] guarantees that $F(\hat{x}) = \Pi_{i=1}^d F_j(\hat{x}_j) \in L^1(\Omega_i)$ and $\|F\|_{L^1(\Omega_i)} \le \Pi_{j=1}^d \|F_j\|_{L^\lambda(O_j)}$ with $\lambda = d-1$. Hence,

$$\begin{aligned} \|u\|_{L^q(\Omega_i)}^q = \int_{\Omega_i} |u|^{dp/(d-p)} \,\mathrm{d}x \le \int_{\Omega_i} |\Pi_{j=1}^d F_j(\hat{x}_j)| \,\mathrm{d}x \le \Pi_{j=1}^d \|F_j\|_{L^{d-1}(O_j)} \\ \le \big(2^{(p-1)/p} \gamma \|u\|_{W^{1,p}(\Omega_i)} \|u\|_{L^q(\Omega_i)}^{q/p'} \big)^{d/(d-1)} . \end{aligned}$$

Since $q(d-1)/d - q/p' = 1$ we find

$$\|u\|_{L^q(\Omega_i)} \le 2^{(p-1)/p} \gamma \|u\|_{W^{1,p}(\Omega_i)} , \tag{3.8}$$

i.e. we have first taken the root $(d-1)/d$ and then divided by $\|u\|_{L^q(\Omega_i)}^{q/p'} > 0$. For $\|u\|_{L^q(\Omega_i)}^{q/p'}$ the inequality clearly holds. We put $2^{(p-1)/p}\gamma = K$.

Step 2 (General case $P_i \neq Q_{l/2}^i$): Recall that Ω satisfies Chenais' uniform cone property and that the sets $B_i' \in \mathcal{O}_\Omega$ have the radius $r/2$. Hence, for all $x \in B_i' \cap \Omega$ we have that a cone $K_i = K(\theta, h, \xi_i)$ with opening angle θ, height h and vertex orientation ξ_i is contained in Ω. In particular, $2r \le h$. Therefore, K_i contains a parallelepiped P_i with opening angle θ and edge length $l = h/(2\cos(\theta/2))$. We introduce the set $\Omega_i := (\Omega \cap B_i') + P_i$. The parallel epiped P_i can be obtained by a suitable transformation of a cube with edge length l, having one face in common with P_i, by the angle $\pi/2 - \theta$. We denote this cube by $Q_{l/2}^i$. In particular, we can choose the shear transformation $T : P_i \to Q_{l/2}^i$ uniformly for all the sets $B_i' \in \mathcal{O}_\Omega$. For all $Tx \in T(\Omega_i \cap B_i')$ we have that $Tx + Q_{l/2}^i \subset T\Omega_i$. In other words, the domain $T\Omega_i$ is the parallel translate of the cube $Q_{l/2}^i$, as in Step 1. For $u \in C^\infty(\overline{\Omega_i})$ we set $\tilde{u} := u \circ T^{-1} \in W^{1,p}(T\Omega_i)$. By [Ada75, Thm. 3.35, p. 63] there exists constants $c^0(T), C^0(T), c^1(T)$ and $C^1(T)$ such that

$$c^0(T)\|u\|_{L^\gamma(\Omega_i)} \le \|\tilde{u}\|_{L^\gamma(T\Omega_i)} \le C^0(T)\|u\|_{L^\gamma(\Omega_i)} , \tag{3.9}$$

$$c^1(T)\|u\|_{W^{1,p}(\Omega_i)} \le \|\tilde{u}\|_{W^{1,p}(T\Omega_i)} \le C^1(T)\|u\|_{W^{1,p}(\Omega_i)} . \tag{3.10}$$

Since the parallelepiped for $T\Omega_i$ is a cube, we can treat the domain $T\Omega_i$ as described in Step 1. From this we obtain that $\|\tilde{u}\|_{L^\gamma(T\Omega_i)} \le K\|\tilde{u}\|_{W^{1,p}(T\Omega_i)}$. Applying the estimates (3.9) and (3.10) results in

$$\|u\|_{L^\gamma(\Omega_i)} \le c^0(T)^{-1}\|\tilde{u}\|_{L^\gamma(T\Omega_i)} \le c^0(T)^{-1} K \|\tilde{u}\|_{W^{1,p}(T\Omega_i)} \le c^0(T)^{-1} C^1(T) K \|u\|_{W^{1,p}(\Omega_i)} .$$

Step 3 (Sobolev estimate for Ω): Let $u \in C^\infty(\overline{\Omega})$. Recall that the covering $\mathcal{O}_\Omega$ consists of $\nu(\Omega)$ sets with $\nu(\Omega) \leq N(r,b,d)$. Hence we conclude that

$$\begin{aligned}\|u\|_{L^\gamma(\Omega)} &\leq \sum_{i=1}^{\nu(\Omega)} \|u\|_{L^\gamma(\Omega_i)} \leq (c^0(T)^{-1}C^1(T))K \sum_{i=1}^{\nu(\Omega)} \|u\|_{W^{1,p}(\Omega_i)} \\ &\leq (c^0(T)^{-1}C^1(T))KN(r,b,d)^{1-\frac{1}{p}} \|u\|_{W^{1,p}(\Omega)}\,.\end{aligned}$$

We set $C_\mathrm{S} = (c^0(T)^{-1}C^1(T))K$.

By density of $C^\infty(\overline{\Omega})$ in $W^{1,p}(\Omega)$ we carry the estimate over to functions $f \in W^{1,p}(\Omega)$. ■

As a direct consequence of Theorem 3.1 we obtain the Poincaré-Sobolev inequality, which involves the mean value of f, see (2.11). It can be deduced by setting $u = f - [f]_\Omega$ in (3.1) and by applying the uniform Poincaré inequality (2.12).

Corollary 3.1 (Poincaré-Sobolev inequality in $\Omega \in \Pi(\theta,h,r,b)$) *Let $p < d$. There is a constant $C_\mathrm{PS} = C_\mathrm{PS}(\theta,h,r,d,p)$ such that for all $\Omega \in \Pi(\theta,h,r,b)$ and for all $f \in W^{1,p}(\Omega)$ it holds*

$$\|f - [f]_\Omega\|_{L^{dp/(d-p)}(\Omega)} \leq C_\mathrm{PS} N(r,b,d)^{1-\frac{1}{p}} \|\mathrm{D}f\|_{L^p(\Omega)}\,. \tag{3.11}$$

Proof: By Theorem 3.1 it follows that there is a constant C_S such that for all $\Omega \in \Pi(\theta,h,r,b)$ and all $u \in W^{1,p}(\Omega)$ the uniform Sobolev inequality holds true. Consider $u = f - [f]_\Omega$ with $f \in W^{1,p}(\Omega)$. Then, the uniform Poincaré inequality (2.12) can be applied and we find

$$\|u\|_{L^{dp/(d-p)}(\Omega)} \leq C_\mathrm{S} N(r,b,d)^{1-\frac{1}{p}} \|u\|_{W^{1,p}(\Omega)} \leq C_\mathrm{S} N(r,b,d)^{1-\frac{1}{p}} (C_\mathrm{P}+1)^{1/p} \|\nabla u\|_{L^p(\Omega)}\,,$$

Setting $C_\mathrm{PS} = C_\mathrm{S}(C_\mathrm{P}+1)^{1/p}$ yields (3.11). ■

Exploiting that characteristic functions $\mathcal{X}_A \in \mathrm{BV}(\Omega)$ of sets A of finite perimeter in Ω can be approximated by a sequence of mollifiers $(f_k)_k \subset C^\infty(\Omega)$ such that $f_k \to \mathcal{X}_A$ in $L^1(\Omega)$ and their total variations $\|\nabla f_k\|_{L^1(\Omega)} = |\mathrm{D}f_k|(\Omega) \to |\mathrm{D}\mathcal{X}_A|(\Omega) = P(A,\Omega)$ the Poincaré-Sobolev inequality (3.11) can be carried over to sets of finite perimeter. By carrying out the classical steps of the proof of the relative isoperimetric inequality in balls, see e.g. [Zie89, Thm. 5.4.3, p. 230] or [EG92, Thm. 2, p. 190], one obtains the uniform isoperimetric inequality relative to $\Omega \in \Pi(\theta,h,r,b)$.

Corollary 3.2 (Uniform relative isoperimetric inequality for $\Omega \in \Pi(\theta,h,r,b)$) *Let $\Omega \in \Pi(\theta,h,r,b)$. There exists a constant $C_\mathrm{I} = 2C_\mathrm{PS}(\theta,h,r,b,d)$ such that for all sets $A \subset \Omega$ with finite perimeter in Ω, i.e. $P(A,\Omega) < \infty$, it holds*

$$\min\left\{\mathcal{L}^d(A\cap\Omega), \mathcal{L}^d(\Omega\backslash A)\right\}^{\frac{d-1}{d}} \leq C_\mathrm{I} P(A,\Omega)\,. \tag{3.12}$$

Proof: See Steps 2 and 3 in the proof of Theorem 3.2 for details. ■

Let us mention the works [MV05, MV08], where optimal Poincaré-Sobolev inequalities with trace terms and related inequalities are deduced using transportation techniques. This includes the isoperimetric inequality in $\mathbb{R}^d$. Clearly, since our proof of the uniform Sobolev inequality involves the uniform covering from Proposition 2.13, which does not use the minimal number of sets needed to cover a set $\Omega \in \Pi(\theta,h,r,b)$, the uniform constants obtained with our method are not the optimal ones.

3.2 Uniform isoperimetric inequality in convex domains intersected with balls

It is well known for balls, see e.g. [Zie89, Thm. 5.4.3, p. 230] or [EG92, Thm. 2, p. 190], that the relative isoperimetric inequality in balls is scaling invariant, i.e. that the isoperimetric constant does not depend on the radius of the ball. An analogous result holds when replacing the ball by an arbitrary Lipschitz domain, see [Pfe01, Thm. 1.8.7, p. 36]. In this section we aim at a slightly different situation, which cannot be concluded solely from scaling arguments: We deduce a relative isoperimetric inequality for a fixed convex domain Ω intersected with a ball $B_\rho(y)$ of radius $\rho > 0$ and center $y \in \overline{\Omega}$. We obtain that the isoperimetric constant is independent of both the radius $\rho > 0$ and the choice of the center $y \in \overline{\Omega}$. In particular the constant will solely depend on space dimension d, the bound b on the diameter of Ω and on the angle θ, the height h and the radius r, i.e. the three parameters governing the uniform cone property of Ω.

By [Gri85, Cor. 1.2.2.3] it is ensured that every convex domain is of class $C^{0,1}$. Hence it has Chenais' uniform cone property. Furthermore, clearly, the intersection of two convex domains results in a convex domain, and hence, again in a $C^{0,1}$-domain with Chenais' uniform cone property. This is why the result is established for convex domains Ω, since, on the one hand the assumption of convexity yields the uniform cone property, and on the other hand it ensures that $\Omega \cap B_\rho(y)$ is connected for every choice of $\rho > 0$ and $y \in \overline{\Omega}$. The latter property is crucial to apply the Poincaré-Sobolev inequality with the mean value (3.11) and it must not hold if Ω is non-convex. A further crucial reason to rule out non-convex domains Ω is the fact that the intersection angle of a ball with $\partial\Omega$ may degenerate to 0 as the center of the ball is moved along $\partial\Omega$, see Fig. 3.1. More precisely, in the proof of the uniform relative isoperimetric inequality it is exploited that every domain $\Omega \cap B_\rho(y)$ for $y \in \overline{\Omega}$ satisfies the cone property with a cone of the same opening angle as the one of Ω. This is due to the fact that the intersection angle $\alpha(y)$ of the boundary $\partial\Omega$ and a ball $B_\rho(y)$ with center $y \in \overline{\Omega}$ is at least $90°$ for a convex domain Ω. Hence, the cone defining the cone property for $\Omega \cap B_\rho(y)$ may have a smaller height than the one for Ω, but the opening angles of the cones are the same. In this case the cones can be scaled to the same size by a suitable scaling of $\Omega \cap B_\rho(y)$. In contrast, for a non-convex domain Ω, the intersection angle $\alpha(y)$ can degenerate to zero as the center y moves along the boundary $\partial\Omega$ away from a re-entrant corner, indicated in Fig. 3.1. Therefore, the opening angle of the cone differs for every domain $\Omega \cap B_\rho(y)$ in dependence of the location of $y \in \overline{\Omega}$. Thus, in the non-convex case, the cones of Ω and $\Omega \cap B_\rho(y)$ cannot be transformed into each other simply by scaling.

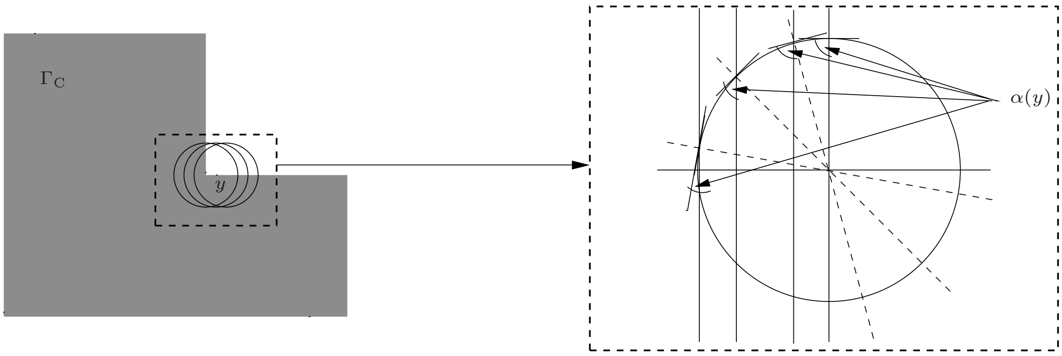

Figure 3.1: The intersection angle $\alpha(y) \to 0$ as y moves from the re-entrant corner to the right.

In the following we consider a convex domain $\Omega \subset \Pi(\theta_*, h_*, r_*, b)$ for θ_*, h_*, r_* fixed. Then also $\Omega \subset \Pi(\theta_*, 2a_*, a_*, b)$ with $a_* = \min\{h_*/2, r_*\}$. As Ω is convex we observe that its intersection $\Omega \cap B_\rho(y)$ with any ball $B_\rho(y)$ with center $y \in \overline{\Omega}$ and arbitrary radius $\rho > 0$ is again a convex domain. In particular we conclude that the opening angle of the cone, which constitutes the cone property for $\Omega \cap B_\rho(y)$, is again θ_*, the opening angle of the cone for Ω. This is due to the fact that the boundary $\partial\Omega$ intersects with $\partial B_\rho(y)$ in an angle larger or equal than $\pi/2$, because the center $y \in \overline{\Omega}$. Moreover, there are $h_\circ, r_\circ > 0$, which depend on $y \in \overline{\Omega}$ and $\rho > 0$, such that $\Omega \cap B_\rho(y) \in \Pi(\theta_*, h_\circ, r_\circ, b)$. Again, we set $a_\circ := \min\{h_\circ/2, r_\circ\}$ and find that $\Omega \cap B_\rho(y) \in \Pi(\theta_*, 2a_\circ, a_\circ, b)$.

Assume that $a_\circ \neq a_*$. Then we may rescale domain $\Omega \cap B_\rho(y)$ to the size such that the corresponding cone has the height $2a_*$ and the balls of radius a_*. More precisely, the rescaled domain is given by

$$\tfrac{a_*}{a_\circ}(\Omega \cap B_\rho(y)) = \tfrac{a_*}{a_\circ}\Omega \cap B_{a_*\rho/a_\circ}(a_* y/a_\circ) \tag{3.13}$$

and satisfies $\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y)) \in \Pi(\theta_*, 2a_*, a_*, b)$. By Corollary 3.1 there is a constant $C_{\mathrm{PS}} = C_{\mathrm{PS}}(\theta, 2a_*, a_*, d, p)$ such that the uniform Poincaré-Sobolev inequality (3.11) for $p = 1$ holds in $\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y))$ for all $f \in C^\infty(\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y)))$ independently of the upper bound $N(r, b, d)$ on the order of the covering:

$$\|f - [f]_{\frac{a_*}{a_\circ}(\Omega\cap B_\rho(y))}\|_{L^{d/(d-1)}(\frac{a_*}{a_\circ}(\Omega\cap B_\rho(y)))} \leq C_{\mathrm{PS}}\|\mathrm{D}f\|_{L^1(\frac{a_*}{a_\circ}(\Omega\cap B_\rho(y)))} \,. \tag{3.14}$$

By a change of variables it can be proved that (3.14) holds true independently of the fraction $a_*/a_\circ$, i.e. that C_{PS} is independent of the radius ρ and the center y of the ball $B_\rho(y)$ that was used for the intersection with Ω. Then, by density arguments, (3.14) can be carried over to BV and the following isoperimetric inequality relative to $\Omega \cap B_\rho(y)$ can be proved with a constant $C_\Omega = C_{\mathrm{PS}}(\theta, 2a_*, a_*, d)$, uniformly for all $y \in \overline{\Omega}$ and all $\rho > 0$.

Theorem 3.2 (Uniform relative isoperimetric inequality for convex Ω intersected with balls) *Let $\Omega \in \Pi(\theta, 2a_*, a_*, b)$ be a convex domain and $B_\rho(y)$ the open ball with radius $\rho > 0$ and center $y \in \overline{\Omega}$. There exists a constant $C_\Omega = 2C_{\mathrm{PS}}(\theta, 2a_*, a_*, d)$ such that for all $y \in \overline{\Omega}$, all $\rho > 0$ and every set $A \subset \Omega$ of finite perimeter in Ω, i.e. $P(A, \Omega) < \infty$, it holds*

$$\min\left\{\mathcal{L}^d(A \cap (\Omega \cap B_\rho(y))), \mathcal{L}^d((\Omega \cap B_\rho(y))\backslash A)\right\}^{\frac{d-1}{d}} \le C_\Omega P(A, \Omega \cap B_\rho(y)) . \tag{3.15}$$

Proof: As a first step we perform a change of variables in (3.14). This will reveal that rescaling the domain $\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y))$ to $\Omega \cap B_\rho(y)$ does not change the constants in (3.14). As a second step we carry (3.14) over to BV-functions via density arguments. Finally, in a third step, we deduce the relative isoperimetric inequality by applying (3.14) to the characteristic function of $A \cap (\Omega \cap B_\rho(y))$ for an arbitrary but fixed set A with finite perimeter.

Step 1 (Change of variables in (3.14)): Let $f \in C^\infty(\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y)))$. For $\sigma \in \Omega \cap B_\rho(y)$ we set $x = \frac{a_*}{a_\circ}\sigma \in \frac{a_*}{a_\circ}(\Omega \cap B_\rho(y))$. Then $\sigma = \frac{a_\circ}{a_*}x$ and $\mathrm{d}x = (\frac{a_*}{a_\circ})^d \mathrm{d}\sigma$. Thus, it is

$$[f]_{\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y))} = (\tfrac{a_*}{a_\circ})^{-d}\mathcal{L}^d(\Omega \cap B_\rho(y))^{-d}\int_{\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y))} f(x)\,\mathrm{d}x = \fint_{\Omega \cap B_\rho(y)} f(a_*\sigma/a_\circ)\,\mathrm{d}\sigma = [f_\circ]_{\Omega \cap B_\rho(y)} , \tag{3.16}$$

where we introduced the notation $f_\circ(\sigma) = f(a_*\sigma/a_\circ)$. With the same ideas we can transform the full norm on the left-hand side of (3.14), i.e. we find

$$\begin{aligned} \|f - [f]_{\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y))}\|_{L^{d/(d-1)}(\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y)))} &= \Big((\tfrac{a_*}{a_\circ})^d \int_{\Omega \cap B_\rho(y)} f_\circ - [f_\circ]_{\Omega \cap B_\rho(y)}\,\mathrm{d}\sigma\Big)^{(d-1)/d} \\ = (\tfrac{a_*}{a_\circ})^{d-1}\|f_\circ - [f_\circ]_{\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y))}\|_{L^{d/(d-1)}(\Omega \cap B_\rho(y)))} &\le C_{\mathrm{PS}}(\theta, 2a_*, a_*, d)\|\mathrm{D}f\|_{L^1(\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y)))} . \end{aligned} \tag{3.17}$$

It remains to transform $\|\mathrm{D}f\|_{L^1(\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y)))}$ in (3.17) to the domain $\Omega \cap B_\rho(y)$. Using that $\mathrm{D}_x f(x) = \mathrm{D}_x f(\frac{a_*}{a_\circ}\sigma) = \frac{a_\circ}{a_*}\mathrm{D}_\sigma f(\frac{a_*}{a_\circ}\sigma) = \frac{a_\circ}{a_*}\mathrm{D}_\sigma f_\circ$ we find

$$\begin{aligned} \|\mathrm{D}f\|_{L^1(\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y)))} &= \int_{\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y))} |\tfrac{a_*}{a_\circ}\mathrm{D}_x f(x)|\tfrac{a_\circ}{a_*}\,\mathrm{d}x = \int_{\frac{a_*}{a_\circ}(\Omega \cap B_\rho(y))} |\tfrac{a_*}{a_\circ}\mathrm{D}_x f(x)|(\tfrac{a_\circ}{a_*})^d(\tfrac{a_*}{a_\circ})^{d-1}\,\mathrm{d}x \\ &= (\tfrac{a_*}{a_\circ})^{d-1}\int_{\Omega \cap B_\rho(y)} |\mathrm{D}_\sigma f_\circ(\sigma)|\,\mathrm{d}\sigma = (\tfrac{a_*}{a_\circ})^{d-1}\|\mathrm{D}f_\circ\|_{L^1(\Omega \cap B_\rho(y))} . \end{aligned} \tag{3.18}$$

Comparing (3.17) and (3.18) we see that the transformation factor $(\frac{a_*}{a_\circ})^{d-1}$ cancels out and hence we have for all $f_\circ \in C^\infty(\Omega \cap B_\rho(y))$

$$\|f_\circ - [f_\circ]_{(\Omega \cap B_\rho(y))}\|_{L^{d/(d-1)}(\Omega \cap B_\rho(y))} \le C_{\mathrm{PS}}\|\mathrm{D}f_\circ\|_{L^1(\Omega \cap B_\rho(y))} \tag{3.19}$$

with the constant $C_{\mathrm{PS}} = C_{\mathrm{PS}}(\theta, a_*, d)$ depending solely on the parameters of the cone ensuring the cone property for Ω but being independent of $B_\rho(y)$.

Step 2 ((3.19) for characterstic functions by density): Let $A \subset \Omega \cap B_\rho(y)$ be a set of finite perimeter in $\Omega \cap B_\rho(y)$. For $f = \mathcal{X}_A$ we consider a sequence of mollifiers $(f_k)_k \subset C^\infty(\Omega \cap B_\rho(y))$ such that $f_k \to f$ in $L^1(\Omega \cap B_\rho(y))$ and $|\mathrm{D}f_k|(\Omega \cap B_\rho(y)) = \|\mathrm{D}f_k\|_{L^1(\Omega \cap B_\rho(y))} \to |\mathrm{D}f|(\Omega \cap B_\rho(y)) = P(A, \Omega \cap B_\rho(y))$. Since $f_k \to f$ in $L^1(\Omega \cap B_\rho(y))$ there is a subsequence converging pointwise a.e.. Hence, we can conclude by lower semicontinuity of $\|\cdot\|_{L^{d/(d-1)(\Omega \cap B_\rho(y))}}$ that

$$\begin{aligned} \|f - [f]_{\Omega \cap B_\rho(y)}\|_{L^{d/(d-1)}(\Omega \cap B_\rho(y))} &\le \liminf_{k\to\infty} \|f_k - [f_k]_{\Omega \cap B_\rho(y)}\|_{L^{d/(d-1)}(\Omega \cap B_\rho(y))} \\ &\le C_{\mathrm{PS}} \lim_{k\to\infty} \|\mathrm{D}f_k\|_{L^1(\Omega \cap B_\rho(y))} = C_{\mathrm{PS}}P(A, \Omega \cap B_\rho(y)) . \end{aligned} \tag{3.20}$$

Step 3 (Deduction of the relative isoperimetric inequality from (3.20)**):** This step is the same as the proof of the relative isoperimetric inequality in balls, see e.g. [Zie89, Thm. 5.4.3, p. 230] or [EG92, Thm. 2, p. 190]. Consider $f = \mathcal{X}_{A\cap(\Omega\cap B_\rho(y))}$ as the characteristic function of the set $A\cap(\Omega\cap B_\rho(y))$ with $A \subset \Omega$ and $P(A,\Omega) < \infty$. Immediate calculation yields

$$\begin{aligned} &\|f - [f]_{\Omega\cap B_\rho(y)}\|_{L^{d/(d-1)}(\Omega\cap B_\rho(y))} \\ &= \left(\tfrac{\mathcal{L}^d((\Omega\cap B_\rho(y))\setminus A)}{\mathcal{L}^d(\Omega\cap B_\rho(y))}\right)\mathcal{L}^d((\Omega\cap B_\rho(y))\cap A)^{(d-1)/d} + \left(\tfrac{\mathcal{L}^d((\Omega\cap B_\rho(y))\cap A)}{\mathcal{L}^d(\Omega\cap B_\rho(y))}\right)\mathcal{L}^d((\Omega\cap B_\rho(y))\setminus A)^{(d-1)/d} \end{aligned} \tag{3.21}$$

Assume that $\mathcal{L}^d((\Omega\cap B_\rho(y))\setminus A) \geq \mathcal{L}^d((\Omega\cap B_\rho(y))\cap A)$. Then $\mathcal{L}^d((\Omega\cap B_\rho(y))\setminus A)/\mathcal{L}^d(\Omega\cap B_\rho(y)) \geq 1/2$ and hence $\|f - [f]_{\Omega\cap B_\rho(y)}\|_{L^{d/(d-1)}(\Omega\cap B_\rho(y))} \geq \frac{1}{2}\mathcal{L}^d((\Omega\cap B_\rho(y))\cap A)$. Similarly, if $\mathcal{L}^d((\Omega\cap B_\rho(y))\cap A) \geq \mathcal{L}^d((\Omega\cap B_\rho(y))\setminus A)$, we find $\|f - [f]_{\Omega\cap B_\rho(y)}\|_{L^{d/(d-1)}(\Omega\cap B_\rho(y))} \geq \frac{1}{2}\mathcal{L}^d((\Omega\cap B_\rho(y))\setminus A)$. Putting the two cases together results in the desired relative isoperimetric inequality. ■

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