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Simulation based policy iteration for American style derivatives – A multilevel approach

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Abstract

This paper presents a novel approach to reduce the complexity of simulation based policy iteration methods for pricing American options. Typically, Monte Carlo construction of an improved policy gives rise to a nested simulation algorithm for the price of the American product. In this respect our new approach uses the multilevel idea in the context of the inner simulations required, where each level corresponds to a specific number of inner simulations. A thorough analysis of the crucial convergence rates in the respective multilevel policy improvement algorithm is presented. A detailed complexity analysis shows that a significant reduction in computational effort can be achieved in comparison to standard Monte Carlo based policy iteration.

1 Introduction

Pricing high-dimensional American options in an efficient way has been a challenge for decades. For low or moderate dimensions, deterministic (PDE) based methods may be applicable, but for higher dimensions Monte Carlo based methods are practically the only way out. Besides the dimension independent convergence rates, Monte Carlo methods are also popular because of their generic applicability. In the late nineties several regression methods for constructing "good" exercise policies yielding price lower bounds were introduced in the literature (see Carriere (1996), Longstaff and Schwartz (2001), and Tsistsiklis and Van Roy (1999), for a detailed description see also Glasserman (2003)). Among many other approaches we mention that Broadie and Glasserman (2004) developed a stochastic mesh method, Bally and Pages (2003) introduced quantization methods, and Kolodko and Schoenmakers (2006) considered a class of policy iterations. In Bender et al. (2008) it is demonstrated that the latter approach can be combined effectively with the Longstaff-Schwartz approach.

The methods mentioned above commonly provide a (generally suboptimal) exercise policy, hence a lower price for an American product. They are therefore called primal methods. As a next breakthrough in Monte Carlo simulation of American options, a dual approach was developed by Rogers (2002) and independently by Haugh and Kogan (2004), related to earlier ideas in Davis and Karatzas (1994). Due to the dual formulation one considers "good" martingales rather than "good" stopping times. In fact, based on a "good" martingale the price of an American derivative can be bounded from above by a "look-back" option due to the difference of the cash-flow and this martingale. Probably one of the most popular numerical methods for computing dual upper bounds is the method of Andersen and Broadie (2004). However, this method has a drawback, namely a high computational complexity due to the need of nested Monte Carlo simulations. In a recent paper, Belomestny and Schoenmakers (2011) mend this problem by considering a multilevel version of the Andersen and Broadie (2004) algorithm.

In this paper we consider a new multilevel primal approach due to Monte Carlo based policy iteration. The basic concept of policy iteration goes back to Howard (1960) in fact (see also Puterman (1994)). A detailed probabilistic treatment of a class of policy iterations (that includes Howard's one as a special case) as well as the description of corresponding Monte Carlo algorithms is provided in Kolodko and Schoenmakers (2006). In the spirit of Belomestny and Schoenmakers (2011) we here develop a multilevel estimator, where the multilevel concept is applied to the number of inner Monte Carlo simulations needed for constructing a new policy, rather than the discretization step size of a particular SDE as in Giles (2008). In this context we give a detailed analysis of the bias rates and the related variance rates that are crucial for the performance of the multilevel algorithm. For transparency we restrict ourself in this respect to the case of Howard's policy iteration, but, with no doubt the results carry over to the more refined policy iteration procedure in Kolodko and Schoenmakers (2006) as well.

The contents of the paper is as follows. In Section 2 we recap the iterative construction of optimal exercise policies in Kolodko and Schoenmakers (2006). A description of the Monte Carlo based policy iteration algorithm, and a detailed convergence analysis is presented in Section 3. After a concise assessment of the complexity of the standard Monte Carlo approach in Section 4, we then introduce its multilevel version in Section 5 and provide a detailed analysis of the multilevel complexity and the corresponding computational gain with respect to the standard approach.

2 Policy iteration for optimal stopping

In this section we review the (probabilistic) policy iteration method for the optimal stopping problem in discrete time, hence pricing of Bermudan derivatives. We will work in a stylized setup where $(\Omega, \mathbb{F}, \mathbb{P})$ is a filtered probability space with discrete filtration $\mathbb{F} = (\mathcal{F}_j)_{j=0,...,T}$ for $T \in \mathbb{N}_+$. A Bermudan derivative on a nonnegative adapted cash-flow process $(Z_j)_{j\geq 0}$ entitles the holder to exercise or receive cash Z_j at an exercise time $j \in \{0, ..., T\}$ that he may choose once. It is assumed that Z_j is expressed in units of some specific pricing numeraire N with $N_0 := 1$ (w.l.o.g. we may take $N \equiv 1$). Then the value of the Bermudan option at time $j \in \{0, ..., T\}$ (in units of the numeraire) is given by the optimal stopping problem

$$Y_j^* = \underset{\tau\in\mathcal{T}[j,...,T]}{\text{ess.sup}} \mathbb{E}_{\mathcal{F}_j}[Z_\tau], \tag{1}$$

provided that the option is not exercised before j. In (1), $\mathcal{T}[j, \ldots, T]$ is the set of $\mathbb{F}$-stopping times taking values in $\{j, \ldots, T\}$ and the process $\left(Y_j^*\right)_{j\geq 0}$ is called the Snell envelope. It is well known that Y^* is a supermartingale satisfying the Bellman principle

$$Y_j^* = \max\left(Z_j, \mathbb{E}_{\mathcal{F}_j}[Y_{j+1}^*]\right), \quad 0 \leq j < T, \quad Y_T^* = Z_T.$$

An exercise policy is a family of stopping times $(\tau_j)_{j=0,...,T}$ such that $\tau_j \in \mathcal{T}[j, \ldots, T]$.

Definition 1 *An exercise policy $(\tau_j)_{j=0,...,T}$ is said to be* ***consistent*** *if*

$$\tau_j > j \Longrightarrow \tau_j = \tau_{j+1}, \quad 0 \leq j < T, \quad \text{and} \quad \tau_T = T.$$

Definition 2 ***(standard) policy iteration***
Given a consistent stopping family $(\tau_j)_{j=0,...,T}$ *we consider a new family* $(\widehat{\tau}_j)_{j=0,...,T}$ *defined by*

$$\widehat{\tau}_j = \inf\left\{k : j \leq k < T,\ Z_k > \mathbb{E}_{\mathcal{F}_k}[Z_{\tau_{k+1}}]\right\} \wedge T, \quad j = 0, ..., T \tag{2}$$

with $\wedge$ *denoting the minimum operator and* $\inf \varnothing := +\infty$. *The new family* $(\widehat{\tau}_j)$ *is termed a policy iteration of* (τ_j) .

The basic idea behind (2) goes back to Howard (1960) (see also Puterman (1994)). The key issue is that (2) is actually a *policy improvement* by the following theorem.

Theorem 3 *It holds (i):*

$$Y_j^* \geq \widehat{Y}_j := \mathbb{E}_{\mathcal{F}_j}[Z_{\widehat{\tau}_j}] \geq \mathbb{E}_{\mathcal{F}_j}[Z_{\tau_j}] =: Y_j, \quad j = 0, ..., T,$$

and moreover (ii): If $\tau_j^{(0)} := \tau_j, \tau_j^{(m+1)} = \widehat{\tau}_j^{(m)}, Y_j^{(0)} = Y_j, Y_j^{(m)} := \mathbb{E}_{\mathcal{F}_j}[Z_{\tau_j^{(m)}}], j = 0, ..., T,$ $m = 0, 1, 2, ...,$ *then,*

$$Y_k^{(T-j)} = Y_k^*, \quad k = j, ..., T.$$

Theorem 3 is in fact a corollary of Th. 3.1 and Prop. 4.3 in Kolodko and Schoenmakers (2006), where a detailed convergence analysis is provided for a whole class of policy iterations of which (2) is a special case. See also Bender and Schoenmakers (2006) for a further analysis regarding stability issues, and extensions to policy iteration methods for multiple stopping. Due to Theorem 3, one may iterate any consistent policy in finitely many steps to an optimal one. Moreover, the respective (lower) approximations to the Snell envelope converge in a nondecreasing manner.

3 Simulation based policy iteration

In order to apply the policy iteration method in practice, we henceforth assume that the cash-flow Z_j is of the form (while slightly abusing of notation) $Z_j = Z_j(X_j)$ for some underlying (possibly high-dimensional) Markovian process X. As a consequence, the Snell envelope then has the form $Y_j^* = Y_j^*(X_j), \quad j = 0, ..., T$, also. Furthermore, it is assumed that a consistent stopping family (τ_j) depends on ω only through the path $X_.$ in the following way: For each j the event $\{\tau_j = j\}$ is measurable w.r.t. X_j, and τ_j is measurable w.r.t. $(X_k)_{j \leq k \leq T}$, i.e.

$$\tau_j(\omega) = h_j(X_j(\omega), ..., X_T(\omega)) \tag{3}$$

for some Borel measurable function h_j. A typical example of such a stopping family is

$$\tau_j = \inf\{k : j \leq k \leq T, \quad Z_k(X_k) \geq f_k(X_k)\}$$

for a set of real valued functions $f_k(x)$. The next issue is the estimation of the conditional expectations in (2). A canonical approach is the use of sub simulations. In this respect we consider an enlarged probability space $(\Omega, \mathbb{F}', \mathbb{P})$, where $\mathbb{F}' = (\mathcal{F}'_j)_{j=0,...,T}$ and $\mathcal{F}_j \subset \mathcal{F}'_j$ for each j. By assumption, $\mathcal{F}'_j$ specified as

$$\mathcal{F}'_j = \mathcal{F}_j \vee \sigma\left\{X_{\cdot}^{i,X_i},\ i \leq j,\right\} \text{ with } \mathcal{F}_j = \sigma\left\{X_i,\ i \leq j\right\},$$

where for a generic $(\omega, \omega_{in}) \in \Omega$, $X_{\cdot}^{i,X_i} := X_k^{i,X_i(\omega)}(\omega_{in})$, $k \geq i$ denotes a sub trajectory starting at time i in the state $X_i(\omega) = X_i^{i,X_i(\omega)}$ of the outer trajectory $X(\omega)$. In particular, the random variables $X_{\cdot}^{i,X_i}$ and $X_{\cdot}^{i',X_{i'}}$ are by assumption independent, conditionally $\{X_i, X_{i'}\}$, for $i \neq i'$. On the enlarged space we consider $\mathcal{F}'_j$ measurable estimations $\mathcal{C}_{j,M}$ of $C_j := \mathbb{E}_{\mathcal{F}_j}\left[Z_{\tau_{j+1}}\right]$ as being standard Monte Carlo estimates based on M sub simulations. More precisely, if

$$\begin{aligned} C_j(X_j) &:= \mathbb{E}_{X_j}\left[Z_{\tau_{j+1}}\right], \quad \text{then consider} \\ \mathcal{C}_{j,M} &:= \frac{1}{M}\sum_{m=1}^{M} Z_{\tau_{j+1}^{(m)}}(X_{\tau_{j+1}^{(m)}}^{j,X_j}), \quad \text{where the} \\ \tau_{j+1}^{(m)} &:= h_{j+1}(X_{j+1}^{j,X_j,(m)}, ..., X_T^{j,X_j,(m)}), \quad 0 \leq j < T, \end{aligned} \tag{4}$$

(cf. (3)) are evaluated on M sub trajectories all starting at time j in X_j. Obviously, $\mathcal{C}_{j,M}$ is an unbiased estimator for C_j with respect to $\mathbb{E}_{\mathcal{F}_j}[\cdot]$. We thus end up with a simulation based version of (2),

$$\begin{aligned} \widehat{\tau}_{j,M} &= \inf\left\{k : j \leq k < T,\ Z_k > \mathcal{C}_{k,M}\right\} \wedge T, \quad \text{while (2) reads} \\ \widehat{\tau}_j &= \inf\left\{k : j \leq k < T,\ Z_k > C_k\right\} \wedge T, \quad j = 0, ..., T. \end{aligned}$$

Let us define

$$\widehat{Y}_j := \mathbb{E}_{\mathcal{F}_j}[Z_{\widehat{\tau}_j}], \quad \widehat{C}_j = \mathbb{E}_{\mathcal{F}_j}[Z_{\widehat{\tau}_{j+1}}], \quad j = 0, \ldots, T.$$

Lemma 4 *In a model with time horizon $T+1$ it holds*

$$\left|\mathbb{E}_{\mathcal{F}_T}\left(1_{\widehat{\tau}_M = T+1} - 1_{\widehat{\tau}=T+1}\right)\right| \leq \sum_{j=0}^{T}\left|\mathbb{P}_{\mathcal{F}_j}\left(Z_j \leq \langle Z_{\widehat{\tau}_{j+1}}^{(\cdot)}\rangle_M\right) - 1\left(Z_j \leq \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}}\right)\right|.$$

Proof. It holds

$$\begin{aligned} \mathbb{E}_{\mathcal{F}_T}\left(1_{\widehat{\tau}_M = T+1} - 1_{\widehat{\tau}=T+1}\right) &= \mathbb{E}_{\mathcal{F}_T}\Big(1_{Z_0 \leq \langle Z_{\widehat{\tau}_1}^{(\cdot)}\rangle_M} \cdots 1_{Z_{T-1} \leq \langle Z_{\widehat{\tau}_T}^{(\cdot)}\rangle_M} 1_{Z_T \leq \langle Z_{\widehat{\tau}_{T+1}}^{(\cdot)}\rangle_M} \\ &\qquad -1_{Z_0 \leq \mathbb{E}_{\mathcal{F}_0} Z_{\widehat{\tau}_1}} \cdots 1_{Z_{T-1} \leq \mathbb{E}_{\mathcal{F}_{T-1}} Z_{\widehat{\tau}_T}} 1_{Z_T \leq \mathbb{E}_{\mathcal{F}_T} Z_{\widehat{\tau}_{T+1}}}\Big) \\ &= \mathbb{E}_{\mathcal{F}_T}\left(\prod_{i=0}^{T} 1_{Z_i \leq \langle Z_{\widehat{\tau}_{i+1}}^{(\cdot)}\rangle_M} - \prod_{i=0}^{T} 1_{Z_i \leq \mathbb{E}_{\mathcal{F}_i} Z_{\widehat{\tau}_{i+1}}}\right) \end{aligned}$$

and

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{F}_T}\left(\prod_{i=0}^{T} 1_{Z_i \le \langle Z^{(\cdot)}_{\widehat{\tau}_{i+1}}\rangle_M} - \prod_{i=0}^{T} 1_{Z_i \le \mathbb{E}_{\mathcal{F}_i} Z_{\widehat{\tau}_{i+1}}}\right) \\
&= \mathbb{E}_{\mathcal{F}_T}\left(\left(\prod_{i=0}^{T-1} 1_{Z_i \le \mathbb{E}_{\mathcal{F}_i} Z_{\widehat{\tau}_{i+1}}}\right)\left(1_{Z_T \le \langle Z^{(\cdot)}_{\widehat{\tau}_{T+1}}\rangle_M} - 1_{Z_T \le \mathbb{E}_{\mathcal{F}_T} Z_{\widehat{\tau}_{T+1}}}\right)\right) \\
&+ \mathbb{E}_{\mathcal{F}_T}\left(\left(\prod_{i=0}^{T-1} 1_{Z_i \le \langle Z^{(\cdot)}_{\widehat{\tau}_{i+1}}\rangle_M} - \prod_{i=0}^{T-1} 1_{Z_i \le \mathbb{E}_{\mathcal{F}_i} Z_{\widehat{\tau}_{i+1}}}\right) 1_{Z_T \le \langle Z^{(\cdot)}_{\widehat{\tau}_{T+1}}\rangle_M}\right) \\
&= \left(\prod_{i=0}^{T-1} 1_{Z_i \le \mathbb{E}_{\mathcal{F}_i} Z_{\widehat{\tau}_{i+1}}}\right) \mathbb{E}_{\mathcal{F}_T}\left(1_{Z_T \le \langle Z^{(\cdot)}_{\widehat{\tau}_{T+1}}\rangle_M} - 1_{Z_T \le \mathbb{E}_{\mathcal{F}_T} Z_{\widehat{\tau}_{T+1}}}\right) \\
&+ \mathbb{E}_{\mathcal{F}_{T-1}}\left(\prod_{i=0}^{T-1} 1_{Z_i \le \langle Z^{(\cdot)}_{\widehat{\tau}_{i+1}}\rangle_M} - \prod_{i=0}^{T-1} 1_{Z_i \le \mathbb{E}_{\mathcal{F}_i} Z_{\widehat{\tau}_{i+1}}}\right) \mathbb{P}_{\mathcal{F}_T}\left(Z_T \le \langle Z^{(\cdot)}_{\widehat{\tau}_{T+1}}\rangle_M\right)
\end{aligned}
$$

where the independence of inner simulations for different time steps is used.

Hence

$$
\left|\mathbb{E}_{\mathcal{F}_T}\left(1_{\widehat{\tau}_M = T+1} - 1_{\widehat{\tau} = T+1}\right)\right| \le \sum_{j=0}^{T} \left|\mathbb{P}_{\mathcal{F}_j}\left(Z_j \le \langle Z^{(\cdot)}_{\widehat{\tau}_{j+1}}\rangle_M\right) - 1\left(Z_j \le \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}}\right)\right|.
$$

■

Let us define

$$
\eta_j(x) := \theta_j x - \log \phi(\theta_j), \quad \theta_j := \theta_j(x)
$$

where θ_j solves the equation:

$$
\phi_j'(\theta_j)/\phi_j(\theta_j) = x
$$

with

$$
\phi_j(\theta_j) = E \exp\left[\theta_j \left(Z_{\widehat{\tau}_{j+1}}(X^{j,X_j}_{\widehat{\tau}_{j+1}}) - \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}})\right)\right]
$$

and

$$
\sigma_j(x)^2 = \frac{\phi''(\theta_j)}{\phi(\theta_j)} - x^2 = \frac{d^2}{d\theta_j^2} \ln(\phi(\theta_j)).
$$

Theorem 5 *Suppose there exists a constant B_1such that $|Z_j| < B_1$. Let p_j be the conditional density of the r.v. $x = Z_{\widehat{\tau}_{j+1}}(X^{j,X_j}_{\widehat{\tau}_{j+1}}) - \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}}$ given that $Z_j(X_j) - \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}} > 0$ and $\bar{p}_j$ the conditional density of the same r.v. given $Z_j(X_j) - \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}} \le 0$. Assume that all the functions $\eta_j(x)$ are nonegative, attain their minimum at $x = 0$ and $\eta_j''(0) > 0$. Moreover suppose*

$$
\frac{p_j(x)}{\sigma_j(x)\theta_j(x)} = x^{\beta_j - 1} f_j(x), \quad \frac{\bar{p}_j(x)}{\sigma_j(x)\theta_j(x)} = x^{\beta_j - 1} \bar{f}_j(x), \quad x \in (0, a)
$$

for some $a > 0,$ where functions f_j and $\bar{f}_j$ are smooth at $0.$ Then it holds

$$\left|\widehat{Y}_{0,M} - \widehat{Y}_0\right| \leq BM^{-(1+\alpha)/2}$$

for some $\alpha > 0$ where B only depends on α, B_1.

Proof. Define $\widehat{\tau}_M := \widehat{\tau}_{0,M}$, $\widehat{\tau} := \widehat{\tau}_0$, and use induction to the number of exercise dates T. We will show the induction basis at the end of the proof cause it requires similar calculations as the induction step. Suppose it is shown that

$$\mathbb{E}\left(Z_{\widehat{\tau}_M} - Z_{\widehat{\tau}}\right) = O(\frac{1}{M})$$

for T exercise dates. Now consider the cashflow process $Z_0, ..., Z_{T+1}$. Then

$$\begin{aligned}
\left|\mathbb{E}\left(Z_{\widehat{\tau}_M} - Z_{\widehat{\tau}}\right)\right| &= \left|\mathbb{E}\left(Z_{\widehat{\tau}_M} 1_{\widehat{\tau}_M \leq T} + Z_{T+1} 1_{\widehat{\tau}_M = T+1} - Z_{\widehat{\tau}} 1_{\widehat{\tau} \leq T} - Z_{T+1} 1_{\widehat{\tau} = T+1}\right)\right| \\
&= \left|\mathbb{E}\left(Z_{\widehat{\tau}_M} 1_{\widehat{\tau}_M \leq T} - Z_{\widehat{\tau}} 1_{\widehat{\tau} \leq T}\right) + \mathbb{E}\left(Z_{T+1}\left(1_{\widehat{\tau}_M = T+1} - 1_{\widehat{\tau} = T+1}\right)\right)\right| \\
&\leq O(\frac{1}{M}) + \mathbb{E}\left(Z_{T+1}\left|\mathbb{E}_{\mathcal{F}_T}\left(1_{\widehat{\tau}_M = T+1} - 1_{\widehat{\tau} = T+1}\right)\right|\right) \qquad (5)
\end{aligned}$$

by applying the induction hypothesis. Applying Lemma 4 we have

$$\begin{aligned}
&\mathbb{E}\left[Z_{T+1}\left|\mathbb{E}_{\mathcal{F}_T}\left(1_{\widehat{\tau}_M = T+1} - 1_{\widehat{\tau} = T+1}\right)\right|\right] \\
&\leq \mathbb{E}\left[Z_{T+1} \sum_{j=0}^{T} \left|\mathbb{P}_{\mathcal{F}_j}\left(Z_j \leq \langle Z^{(\cdot)}_{\widehat{\tau}_{j+1}} \rangle_M\right) - 1\left(Z_j \leq \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}}\right)\right|\right] \\
&\leq B \sum_{j=0}^{T} \mathbb{E}\left[\left|\mathbb{P}_{\mathcal{F}_j}\left(Z_j \leq \langle Z^{(\cdot)}_{\widehat{\tau}_{j+1}} \rangle_M\right) - 1\left(Z_j \leq \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}}\right)\right| \middle| Z_j \leq \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}}\right] \\
&+ \mathbb{E}\left[\left|\mathbb{P}_{\mathcal{F}_j}\left(Z_j \leq \langle Z^{(\cdot)}_{\widehat{\tau}_{j+1}} \rangle_M\right) - 1\left(Z_j \leq \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}}\right)\right| \middle| Z_j > \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}}\right] \\
&= B \sum_{j=0}^{T} \mathbb{E}\left[\left(1 - \mathbb{P}_{\mathcal{F}_j}\left(Z_j \leq \langle Z^{(\cdot)}_{\widehat{\tau}_{j+1}} \rangle_M\right)\right) \middle| Z_j \leq \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}}\right] \\
&+ \mathbb{E}\left[\mathbb{P}_{\mathcal{F}_j}\left(Z_j \leq \langle Z^{(\cdot)}_{\widehat{\tau}_{j+1}} \rangle_M\right) \middle| Z_j - \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}} > 0\right] = S_1 + S_2
\end{aligned}$$

It holds (see, e.g., Lee and Glynn (2003))

$$P_{\mathcal{F}_j}\left(\frac{1}{M} \sum_{m=1}^{M}\left[Z_{\widehat{\tau}^{(m)}_{j+1}}(X^{j,X_j}_{\widehat{\tau}^{(m)}_{j+1}}) - \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}}\right] > \xi_j > 0\right) \asymp \frac{\exp(-M\eta_j(\xi_j))}{\sqrt{2\pi}\sigma_j(\xi_j)\theta_j(\xi_j)\sqrt{M}},$$

where $\xi_j = Z_j(X_j) - \mathbb{E}_{\mathcal{F}_j} Z_{\widehat{\tau}_{j+1}}$. Under our assumptions we have

$$S_2 \leq \frac{B}{\sqrt{M}} \sum_{j=0}^{T} \int_0^\infty \frac{\exp(-M\eta_j(x))}{\sqrt{2\pi}\sigma_j(x)\theta_j(x)} p_j(x) dx$$

Then the Laplace method yields

$$\int_0^\infty \frac{\exp(-M\eta_j(x))}{\sqrt{2\pi}\sigma_j(x)\theta_j(x)} p_j(x)dx \lesssim M^{-\beta_j/2}$$

as $M \to \infty$. Hence

$$S_2 \leq B_2 M^{-(1+\min(\alpha_0,\ldots\alpha_T))/2}.$$

Analogously, under similar assumptions on the conditional density $\bar{p}_j$, we get

$$S_1 \leq B_1 M^{-(1+\min(\alpha_0,\ldots\alpha_T))/2}.$$

For the induction basis $T = 0$ we have with $Z_{\widehat{\tau}_M} 1_{\widehat{\tau}_M \leq 0} - Z_{\widehat{\tau}} 1_{\widehat{\tau} \leq 0} = 0$

$$\begin{aligned}
|\mathbb{E}\ (Z_{\widehat{\tau}_M} - Z_{\widehat{\tau}})| &= |\mathbb{E}\ (Z_{\widehat{\tau}_M} 1_{\widehat{\tau}_M \leq T} + Z_{T+1} 1_{\widehat{\tau}_M = T+1} - Z_{\widehat{\tau}} 1_{\widehat{\tau} \leq T} - Z_{T+1} 1_{\widehat{\tau} = T+1})| \\
&= |\mathbb{E}\ (Z_{\widehat{\tau}_M} 1_{\widehat{\tau}_M \leq T} - Z_{\widehat{\tau}} 1_{\widehat{\tau} \leq T}) + \mathbb{E}\ (Z_{T+1} (1_{\widehat{\tau}_M = T+1} - 1_{\widehat{\tau} = T+1}))| \\
&\leq \mathbb{E}\ (Z_{T+1} |\mathbb{E}_{\mathcal{F}_T} (1_{\widehat{\tau}_M = T+1} - 1_{\widehat{\tau} = T+1})|) \\
&= O\left(\frac{1}{M}\right)
\end{aligned}$$

where the last equality follows with the arguments from above. ■

Theorem 5 controls the bias of the estimator $\widehat{Y}_{0,M}$ for the lower approximation $\widehat{Y}_0$ to the Snell envelope due to the improved policy $(\widehat{\tau}_j)$. Concerning the improvement we infer from Kolodko and Schoenmakers (2006), Lemma 4.5, that

$$0 \leq \widehat{Y}_0 - Y_0 \leq \mathbb{E} \sum_{k=\tau_0}^{\widehat{\tau}_0 - 1} [\mathbb{E}_{\mathcal{F}_k} Y_{k+1} - Y_k].$$

Hence, for a bounded cash-flow process with $|Z_j| < B_1$we get

$$0 \leq \widehat{Y}_0 - Y_0 \leq T B_1 \mathbb{P}(\tau_j \neq \widehat{\tau}_j) \leq T B_1 \mathbb{P}(\tau_j \neq \tau_j^*),$$

as $\tau_j = \tau_j^*$ implies $\tau_j = \widehat{\tau}_j = \tau_j^*$. The following proposition, is important in the context of multilevel Monte Carlo estimation of $\widehat{Y}_0$, introduced in the next section.

Proposition 6 *Suppose that there are constants B_1 and B_2, such that $|Z_j| < B_1$ and*

$$\mathit{Var}_{\mathcal{F}_j}\left[Z_{\tau_{j+1}}\right] := \mathbb{E}_{\mathcal{F}_j}[(Z_{\tau_{j+1}} - C_j)^2] < B_2, \quad j = 0, \ldots, T-1 \quad \textit{a.s.} \tag{6}$$

Let us further assume that there exist constants $\overline{B}_{0,j} > 0$, $j = 0, \ldots, T-1$, and $\overline{\alpha} > 0$, such that for any $\delta > 0$ and $j = 0, \ldots, T-1$,

$$\mathbb{P}(|C_j - Z_j| \leq \delta) \leq \overline{B}_{0,j} \delta^{\overline{\alpha}}. \tag{7}$$

It then holds,

$$\mathbb{E}[(Z_{\widehat{\tau}_{0,M}} - Z_{\widehat{\tau}_0})^2] \leq M^{-\overline{\alpha}/2} 2 B_1^2 \overline{B} \sum_{j=0}^{T-1} \overline{B}_{0,j}. \tag{8}$$

Proof. We have

$$\mathbb{E}[\left(Z_{\widehat{\tau}_{0,M}}-Z_{\widehat{\tau}_0}\right)^2]=\mathbb{E}[\left(Z_{\widehat{\tau}_{0,M}}-Z_{\widehat{\tau}_0}\right)^2\mathbf{1}_{\left\{\widehat{\tau}_{0,M}\neq\widehat{\tau}_0\right\}}]\leq B_1^2\mathbb{P}\left(\widehat{\tau}_{0,M}\neq\widehat{\tau}_0\right). \tag{9}$$

Let us write $\{\widehat{\tau}_{0,M}\neq\widehat{\tau}_0\}=\{\widehat{\tau}_{0,M}>\widehat{\tau}_0\}\cup\{\widehat{\tau}_{0,M}<\widehat{\tau}_0\}$. It then holds (see definitions (4))

$$\begin{aligned}\{\widehat{\tau}_{0,M}>\widehat{\tau}_0\}&\subset\bigcup_{j=0}^{T-1}\{C_j<Z_j\leq\mathcal{C}_{j,M}\}\cap\{\widehat{\tau}_0=j\}\\&=:\bigcup_{j=0}^{T-1}A_j^{M+}\cap\{\widehat{\tau}_0=j\},\end{aligned}$$

and similarly,

$$\begin{aligned}\{\widehat{\tau}_{0,M}<\widehat{\tau}_0\}&\subset\bigcup_{j=0}^{T-1}\{C_j\geq Z_j>\mathcal{C}_{j,M}\}\cap\{\widehat{\tau}_0=j\}\\&=:\bigcup_{j=0}^{T-1}A_j^{M-}\cap\{\widehat{\tau}_{0,M}=j\}.\end{aligned}$$

So we have

$$\mathbb{P}\left(\widehat{\tau}_{0,M}\neq\widehat{\tau}_0\right)\leq\sum_{j=0}^{T-1}\mathbb{P}\left(A_j^{M+}\cup A_j^{M-}\right).$$

By the conditional version of the Bernstein inequality we have,

$$\begin{aligned}\mathbb{P}_{\mathcal{F}_T}\left(A_j^{M+}\right)&=\mathbb{P}_{X_j}\left(0<Z_j-C_j\leq\frac{1}{M}\sum_{m=1}^{M}\left(Z_{\tau_{j+1}^{(m)}}(X^{j,X_j}_{\tau_{j+1}^{(m)}})-C_j\right)\right)\\&\leq\mathbf{1}_{\left\{|Z_j-C_j|\leq M^{-1/2}\right\}}+\sum_{k=1}^{\infty}\mathbf{1}_{\left\{2^{k-1}M^{-1/2}<|Z_j-C_j|\leq 2^kM^{-1/2}\right\}}\cdot\\&\cdot\mathbb{P}_{X_j}\left(2^{k-1}M^{-1/2}<\frac{1}{M}\sum_{m=1}^{M}\left(Z_{\tau_{j+1}^{(m)}}(X^{j,X_j}_{\tau_{j+1}^{(m)}})-C_j\right)\right)\\&\leq\mathbf{1}_{\left\{|Z_j-C_j|\leq M^{-1/2}\right\}}+\sum_{k=1}^{\infty}\mathbf{1}_{\left\{2^{k-1}M^{-1/2}<|Z_j-C_j|\leq 2^kM^{-1/2}\right\}}\cdot\\&\cdot\exp\left[-\frac{2^{2k-3}M}{M\overline{B}_2+\overline{B}_1 2^{k-1}M^{1/2}/3}\right]\\&\leq\mathbf{1}_{\left\{|Z_j-C_j|\leq M^{-1/2}\right\}}+\sum_{k=1}^{\infty}\mathbf{1}_{\left\{|Z_j-C_j|\leq 2^kM^{-1/2}\right\}}\cdot\\&\cdot\exp\left[-\frac{2^{2k-2}}{\overline{B}_2+\overline{B}_1 2^{k-1}/3}\right].\end{aligned}$$

So by assumption (7),

$$\mathbb{P}\left(A_j^{M+}\right) \leq \overline{B}_{0,j}M^{-\overline{\alpha}/2} + \overline{B}_{0,j}\sum_{k=1}^{\infty} 2^{\overline{\alpha}k}M^{-\overline{\alpha}/2}\exp\left[-\frac{2^{2k-2}}{B_2 + B_1 2^{k-1}M^{-1/2}/3}\right]$$
$$\leq \overline{B}\overline{B}_{0,j}M^{-\overline{\alpha}/2}$$

for $\overline{B}$ depending on $\overline{B}_1$, $\overline{B}_2$ and α. After obtaining a similar estimate for $\mathbb{P}\left(A_j^{M-}\right)$, we finally conclude that

$$\mathbb{P}\left(\widehat{\tau}_{0,M} \neq \widehat{\tau}_0\right) \leq M^{-\overline{\alpha}/2} 2\overline{B}\sum_{j=0}^{T-1}\overline{B}_{0,j},$$

and then (8) follows from (9). ■

4 Standard Monte Carlo approach

Within a Markovian setup as introduced in Section 3, we consider for some fixed natural numbers N and M, the estimator

$$\widehat{\mathcal{Y}}_{N,M} := \frac{1}{N}\sum_{n=1}^{N} Z_{\widehat{\tau}_M^{(n)}} \tag{10}$$

for $\widehat{Y}_M := \widehat{Y}_{0,M}$ with $\widehat{\tau}_M := \widehat{\tau}_{0,M}$, based on a set of samples

$$\left\{Z_{\widehat{\tau}_M^{(n)}}, \quad n = 1, ..., N\right\}$$

of the cash-flow $Z_{\widehat{\tau}_M}$. Let us investigate the complexity, i.e. the required computational costs, in order to compute $\widehat{Y} := \widehat{Y}_0$ with a prescribed (root-mean-square) accuracy ϵ, by using the estimator (10). Under the assumptions of Theorem 5 we have with $\gamma := (1+\alpha)/2 \geq 1/2$, for the squared accuracy,

$$\mathbb{E}\left[\widehat{\mathcal{Y}}_{N,M} - \widehat{Y}\right]^2 \leq N^{-1}\text{Var}\left[\widehat{Y}_M\right] + \left|\widehat{Y} - \widehat{Y}_M\right|^2 \tag{11}$$
$$\leq N^{-1}\sigma_\infty^2 + \mu_\infty M^{-2\gamma}, \quad M \geq M_0,$$

for some constants μ_∞ and $\sigma_\infty^2 := \sup_{M\geq M_0}\text{Var}\left[\widehat{Y}_M\right]$, where M_0 denotes some fixed minimum number of sub trajectories used for computing the stopping time $\widehat{\tau}_M$. In order to bound (11) by ϵ^2, we set

$$M = \left\lceil \frac{(2\mu_\infty)^{1/2\gamma}}{\epsilon^{1/\gamma}} \right\rceil, \quad N = \left\lceil \frac{2\sigma_\infty^2}{\epsilon^2} \right\rceil$$

with $\lceil x \rceil$ denoting the smallest integer bigger or equal than x. For notational simplicity we will henceforth disregard the Entier brackets and carry out calculations with generally non-integer valued M, N. This will neither affect complexity rates nor asymptotic proportionality constants. Thus the computational complexity for reaching accuracy ϵ when $\epsilon \downarrow 0$ is given by

$$\mathcal{C}_{\text{stan}}^{N,M}(\epsilon) := NM = \frac{2\sigma_\infty^2 (2\mu_\infty)^{1/2\gamma}}{\epsilon^{2+1/\gamma}}, \tag{12}$$

where, again for simplicity, it is assumed that both the cost of one outer trajectory and one sub trajectory is equal to one unit. In virtually all applications the random variable $\widehat{C}_j - Z_j$ has a positive but non-exploding density in zero, therefore we have $\alpha = \gamma = 1$. Hence the complexity of the standard Monte Carlo method is of order $O(\epsilon^{-3})$.

5 Multilevel Monte Carlo approach

For a fixed natural number L and a set of natural numbers $\mathbf{m} := (m_0, ..., m_L)$ satisfying $1 \leq m_0 < ... < m_L$, we consider in the spirit of Giles (2008) the telescoping sum,

$$\widehat{Y}_{m_L} = \widehat{Y}_{m_0} + \sum_{l=1}^{L} \left(\widehat{Y}_{m_l} - \widehat{Y}_{m_{l-1}} \right). \tag{13}$$

Next we take a set of natural numbers $\mathbf{n} := (n_0, ..., n_L)$ satisfying $n_0 > ... > n_L \geq 1$, and simulate an initial set of cash-flows

$$\left\{ Z^{(j)}_{\widehat{\tau}_{m_0}}, \quad j = 1, ..., n_0 \right\},$$

due to an initial set of trajectories $X^{0,x,(j)}_{\cdot}$, $j = 1, ..., n_0$, where

$$Z^{(j)}_{\widehat{\tau}_{m_0}} := Z_{\widehat{\tau}^{(j)}_{0,m_0}} (X^{0,x,(j)}_{\widehat{\tau}^{(j)}_{0,m_0}}).$$

Next we simulate *independently* for each level $l = 1, ..., L$, a set of pairs

$$\left\{ Z^{(j)}_{\widehat{\tau}_{m_l}}, Z^{(j)}_{\widehat{\tau}_{m_{l-1}}}, \quad j = 1, ..., n_l \right\}$$

due to a set of trajectories $X^{0,x,(j)}_{\cdot}$, $j = 1, ..., n_l$ to obtain a multilevel estimator

$$\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}} := \frac{1}{n_0} \sum_{j=1}^{n_0} Z^{(j)}_{\widehat{\tau}_{m_0}} + \sum_{l=1}^{L} \frac{1}{n_l} \sum_{j=1}^{n_l} \left(Z^{(j)}_{\widehat{\tau}_{m_l}} - Z^{(j)}_{\widehat{\tau}_{m_{l-1}}} \right) \tag{14}$$

as an approximation to $\widehat{Y}$. (cf. Belomestny and Schoenmakers (2011)). Henceforth we always take $\mathbf{m}$ to be a geometric sequence $m_l = m_0 \kappa^l$, for some m_0, $\kappa \in \mathbb{N}$, $\kappa \geq 2$.

Complexity analysis

Let us now consider the complexity of the multilevel estimator (14) under the assumption that the conditions of Theorem 5 and Proposition 6 are fulfilled. For the bias we have due to Theorem 5 with $\gamma = (1 + \alpha)/2 \geq 1/2$,

$$\left| \mathbb{E} \left[\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}} \right] - \widehat{Y} \right| = \left| \mathbb{E} \left[Z_{\widehat{\tau}_{m_L}} - Z_{\widehat{\tau}} \right] \right| \leq \mu_\infty m_L^{-\gamma},$$

and for the variance it holds

$$\mathsf{Var}\left[\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}}\right]=\frac{1}{n_0}\mathsf{Var}\left[Z_{\widehat{\tau}_{m_0}}\right]+\sum_{l=1}^{L}\frac{1}{n_l}\mathsf{Var}\left[Z_{\widehat{\tau}_{m_l}}-Z_{\widehat{\tau}_{m_{l-1}}}\right],$$

where due to Proposition 6, the terms with $l>0$ may be estimated by

$$\begin{aligned}\mathsf{Var}\left[Z_{\widehat{\tau}_{m_l}}-Z_{\widehat{\tau}_{m_{l-1}}}\right]&\leq\mathbb{E}\left[\left(Z_{\widehat{\tau}_{m_l}}-Z_{\widehat{\tau}_{m_{l-1}}}\right)^2\right]\\&\leq 2\mathbb{E}\left[\left(Z_{\widehat{\tau}_{m_l}}-Z_{\widehat{\tau}}\right)^2\right]+2\mathbb{E}\left[\left(Z_{\widehat{\tau}_{m_{l-1}}}-Z_{\widehat{\tau}}\right)^2\right]\\&\leq C\left(m_l^{-\beta}+m_{l-1}^{-\beta}\right)\leq\mathcal{V}_\infty m_l^{-\beta},\end{aligned}$$

with $\beta:=\overline{\alpha}/2,$ and suitable constants C, $\mathcal{V}_\infty$. In most applications, also C_j-Z_j in (7) has a positive but non-exploding density in zero which implies $\overline{\alpha}=1,$ hence $\beta=1/2$. This rate is confirmed by numerical experiments.

Theorem 7 *Let us assume $0<\beta\leq 1$ and $\gamma\geq\frac{1}{2}$ and $m_l=m_0\kappa^l$ for some fixed κ and suitably fixed m_0. Fix some $0<\epsilon<1$. Let L be (the integer part of)*

$$L=L(\epsilon)=\gamma^{-1}\ln^{-1}\kappa\ln\left[\frac{\sqrt{2}\mu_\infty}{m_0^\gamma\epsilon}\right],$$

and

Then the computational complexity of the estimator (14) is bounded by

$$\begin{aligned}\mathcal{C}_{ML}^{\mathbf{n},\mathbf{m}}(\epsilon)&:=n_0m_0+\sum_{l=1}^{L}n_l\left(m_l+m_{l-1}\right)\\&=\frac{2\mathcal{V}_\infty\kappa^{1-\beta}m_0^{1-\beta}\left(1+\kappa^{-1}\right)}{\epsilon^2}\left(\frac{\left(\frac{\left(\sqrt{2}\mu_\infty\right)^{\frac{1}{\gamma}}}{m_0\epsilon^{\frac{1}{\gamma}}}\right)^{(1-\beta)/2}-1}{\kappa^{(1-\beta)/2}-1}+\frac{\sigma_\infty^2m_0^\beta}{\mathcal{V}_\infty\kappa^{(1-\beta)/2}}\right)\cdot\\&\cdot\left(\frac{\left(\frac{\left(\sqrt{2}\mu_\infty\right)^{\frac{1}{\gamma}}}{m_0\epsilon^{\frac{1}{\gamma}}}\right)^{(1-\beta)/2}-1}{\kappa^{(1-\beta)/2}-1}+\frac{1}{\kappa^{(1-\beta)/2}\left(1+\kappa^{-1}\right)}\right),\qquad(15)\\&=O(\epsilon^{-2-\frac{1-\beta}{\gamma}}),\quad\epsilon\downarrow 0,\quad\beta<1.\end{aligned}$$

Proof. of Theorem 7

First we have

$$C_{\mathsf{ML}}^{\mathbf{n},\mathbf{m}}(\epsilon) = n_0 m_0 + \sum_{l=1}^{L} n_0 \kappa^{-l(1+\beta)/2} \left(m_0 \kappa^l + m_0 \kappa^{l-1} \right)$$
$$= n_0 m_0 \left(1 + \frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1} \kappa^{(1-\beta)/2} \left(1 + \kappa^{-1} \right) \right). \qquad (16)$$

In view of

$$\mathbb{E}\left[\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}} - \widehat{Y} \right]^2 = \mathsf{Var}\left[\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}} \right] + \left(\mathbb{E}\left[\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}} \right] - \widehat{Y} \right)^2$$

we set

$$\frac{\mu_\infty}{m_0^\gamma \kappa^{\gamma L}} = \frac{\epsilon}{\sqrt{2}}$$
$$\frac{\sigma_\infty^2}{n_0} + \frac{\mathcal{V}_\infty}{n_0 m_0^\beta} \frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1} \kappa^{(1-\beta)/2} = \frac{\epsilon^2}{2}.$$

This yields, respectively

$$L = \gamma^{-1} \ln^{-1} \kappa \ln \left[\frac{\sqrt{2} \mu_\infty}{m_0^\gamma \epsilon} \right], \qquad (17)$$

and

$$n_0 = \frac{2\sigma_\infty^2}{\epsilon^2} + \frac{2\mathcal{V}_\infty}{\epsilon^2 m_0^\beta} \frac{\frac{\left(\sqrt{2}\mu_\infty\right)^{\frac{1-\beta}{2\gamma}}}{m_0^{(1-\beta)/2} \epsilon^{\frac{1-\beta}{2\gamma}}} - 1}{\kappa^{(1-\beta)/2} - 1} \kappa^{(1-\beta)/2}. \qquad (18)$$

We then obtain from (16), (17), and (18) the desired result after a little algebra. ■

Remark 8 *Note that for $\beta \uparrow 1$, Theorem 7 reads*

$$n_l = n_0 \kappa^{-l} \quad \textit{with}$$
$$n_0 = n_0(\epsilon) = \frac{2\sigma_\infty^2}{\epsilon^2} + \frac{2\mathcal{V}_\infty}{\epsilon^2 m_0} L,$$

where the computational complexity of the estimator (14) is bounded by

$$\mathcal{C}_{ML}^{\mathbf{n},\mathbf{m}}(\epsilon) = \frac{2\mathcal{V}_\infty \left(1 + \kappa^{-1}\right)}{\epsilon^2} \left(\frac{\ln \left[\frac{\left(\sqrt{2}\mu_\infty\right)^{\frac{1}{\gamma}}}{m_0 \epsilon^{\frac{1}{\gamma}}} \right]}{\ln \kappa} + \frac{\sigma_\infty^2 m_0}{\mathcal{V}_\infty} \right) \cdot$$
$$\cdot \left(\frac{\ln \left[\frac{\left(\sqrt{2}\mu_\infty\right)^{\frac{1}{\gamma}}}{m_0 \epsilon^{\frac{1}{\gamma}}} \right]}{\ln \kappa} + \frac{1}{1 + \kappa^{-1}} \right),$$
$$= O(\epsilon^{-2} \ln^2 \epsilon), \quad \epsilon \downarrow 0, \quad \beta = 1.$$

(cf. the situation in Belomestny and Schoenmakers (2011)).

Complexity comparison of multilevel versus standard MC

Let us consider the ratio of the multilevel costs over the standard MC costs. From (15) and (12) we obtain for $\beta < 1$,

$$\frac{\mathcal{C}_{\mathsf{ML}}^{\mathbf{n},\mathbf{m}}(\epsilon)}{\mathcal{C}_{\mathsf{stan}}^{N,M}(\epsilon)} = \frac{\mathcal{V}_\infty \kappa^{1-\beta} m_0^{1-\beta}\left(1+\kappa^{-1}\right)}{\sigma_\infty^2 \left(2\mu_\infty\right)^{1/2\gamma}} \epsilon^{1/\gamma} \left(\frac{\left(\frac{\left(\sqrt{2}\mu_\infty\right)^{\frac{1}{\gamma}}}{m_0 \epsilon^{\frac{1}{\gamma}}} \right)^{(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1} + \frac{\sigma_\infty^2 m_0^\beta}{\mathcal{V}_\infty \kappa^{(1-\beta)/2}} \right) \cdot$$

$$\cdot \left(\frac{\left(\frac{\left(\sqrt{2}\mu_\infty\right)^{\frac{1}{\gamma}}}{m_0 \epsilon^{\frac{1}{\gamma}}} \right)^{(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1} + \frac{1}{\kappa^{(1-\beta)/2}\left(1+\kappa^{-1}\right)} \right)$$

$$= O(\epsilon^{\beta/\gamma}), \quad \epsilon \downarrow 0,$$

and for $\beta = 1$,

$$\frac{\mathcal{C}_{\mathsf{ML}}^{\mathbf{n},\mathbf{m}}(\epsilon)}{\mathcal{C}_{\mathsf{stan}}^{N,M}(\epsilon)} = \frac{\mathcal{V}_\infty \left(1+\kappa^{-1}\right)}{\sigma_\infty^2 \left(2\mu_\infty\right)^{1/2\gamma}} \epsilon^{1/\gamma} \left(\frac{\ln\left[\frac{\left(\sqrt{2}\mu_\infty\right)^{\frac{1}{\gamma}}}{m_0 \epsilon^{\frac{1}{\gamma}}} \right]}{\ln \kappa} + \frac{\sigma_\infty^2 m_0}{\mathcal{V}_\infty} \right) \cdot$$

$$\cdot \left(\frac{\ln\left[\frac{\left(\sqrt{2}\mu_\infty\right)^{\frac{1}{\gamma}}}{m_0 \epsilon^{\frac{1}{\gamma}}} \right]}{\ln \kappa} + \frac{1}{1+\kappa^{-1}} \right)$$

$$= O(\epsilon^{1/\gamma} \ln^2 \epsilon), \quad \epsilon \downarrow 0.$$

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