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A criterion for a two-dimensional domain to be Lipschitzian

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ABSTRACT. We prove that a two-dimensional domain is already Lipschitzian if only its boundary admits locally a one-dimensional, bi-Lipschitzian parametrization.

1. INTRODUCTION

In the last decades it has been anticipated in applied analysis that many problems originating from science, engineering, and technology lead to elliptic/parabolic problems on nonsmooth domains. Aiming at a class of domains where a lot of 'classical' instruments still work, Lipschitz domains have proved to be an adequate setting. Having in mind elliptic, second order divergence operators, this concerns optimal elliptic Sobolev regularity [8], [9], maximal parabolic regularity on a huge scale of spaces [13], [10], [11], Hölder conituity for the solution of the elliptic/parabolic problems (even in case of mixed boundary conditions) [12], [3], and also interpolation [6]. This is in particular true if one considers two-dimensional problems - either as the 'original' or as an artefact of a corresponding three-dimensional one. Note that many of these (often nonlinear) two-dimensional models are at present still indispensable because one is still unable to treat the three-dimensional model in full mathematically or/and the computer ressources do not suffice for doing so [1], [2], [5], [14], [15], [16], [18].
On the other hand, it is known that the class of Lipschitz domains contains rather strange representatives as Grisvard's flash [7, Ch.1.2] for which it is not at all obvious that it indeed belongs to this class. Thus, it seems desirable to obtain criteria for the Lipschitz property of a domain which are simpler to handle as the definition itself. We present such a criterion in the case of two space dimensions, basing on a deep theorem of Tukia [19, Thm. B]. Roughly spoken, a two-dimensional domain is already Lipschitzian, if only the boundary itself admits one-dimensional, bi-Lipschitzian charts.
Unfortunately, there is no higher dimensional analogon, as is already pointed out in [19].

2. THE CRITERION

Let us briefly introduce some notations and definitions. Let K_d be the open unit cube in $\mathbb{R}^d$ with center $0 \in \mathbb{R}^d$, K_d^- its lower half $K_d^- := K_d \cap \{\mathrm{x} : x_d < 0\}$ and P_d the midplate $P_d := K_d \cap \{\mathrm{x} : x_d = 0\}$ of K_d.

Definition 2.1. Let (X, ρ) and (Y, ϱ) be two metric spaces. Then we call a mapping $F : X \to Y$ Lipschitzian if there is constant γ such that $\varrho(F(\mathrm{x}_1), F(\mathrm{x}_2)) \le \gamma\rho(\mathrm{x}_1, \mathrm{x}_2)$ for all $\mathrm{x}_1, \mathrm{x}_2 \in X$. If F^{-1} is injective and also Lipschitzian, we call F bi-Lipschitzian.

Definition 2.2. A bounded domain $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain (or Lipschitzian), if for any point $\mathrm{x} \in \partial\Omega$ there is an open neighbourhood $V_\mathrm{x} \ni \mathrm{x}$ and a bi-Lipschitzian mapping Φ_x from V_x onto K_d, such that $\Phi_\mathrm{x}(V_\mathrm{x} \cap \Omega) = K_d^-$, $\Phi_\mathrm{x}(V_\mathrm{x} \cap \partial\Omega) = P_d$, $\Phi_\mathrm{x}(\mathrm{x}) = 0 \in \mathbb{R}^d$.

Let us quote, for the convenience of the reader, the pioneering central result from [19].

Proposition 2.3. *Let $L \subset \mathbb{R}^2$ be a bounded line segment and f a mapping from L into $\mathbb{R}^2$, which is bi-Lipschitzian. Then there is a bi-Lipschitzian extension F of f which maps $\mathbb{R}^2$ onto $\mathbb{R}^2$.*

We formulate now our criterion for the Lipschitz property of a two-dimensional, bounded domain.

Theorem 2.4. *A bounded domain $\Omega \subset \mathbb{R}^2$ is a Lipschitz domain if and only if for any $\mathrm{x} \in \partial\Omega$ there is an open neighbourhood $U_\mathrm{x} \ni \mathrm{x}$ and a bi-Lipschitzian mapping ϕ_x from $U_\mathrm{x} \cap \partial\Omega$ onto the interval $]-\frac{1}{2}, \frac{1}{2}[$.*

Proof. During the proof we identify the interval $]-\frac{1}{2}, \frac{1}{2}[$ with the line segment $P_2 =]-\frac{1}{2}, \frac{1}{2}[\times\{0\}$ in $\mathbb{R}^2$. The condition is clearly necessary. In the sequel we show that it is also sufficient. Let x be any element from $\partial\Omega$, U_x and ϕ_x the neighbourhood and the bi-Lipschitzian mapping from the supposition. Modulo a bi-Lipschitz mapping from $]-\frac{1}{2}, \frac{1}{2}[$ onto itself, we may assume that $\phi_\mathrm{x}(\mathrm{x}) = 0$. The Tukia theorem, applied to the mapping $f := \phi_\mathrm{x}^{-1}$, yields a bi-Lipschitz extension $\Psi_\mathrm{x} := F^{-1}$ of ϕ_x which maps $\mathbb{R}^2$ onto itself. Let $\epsilon \in]0, 1]$ be a number such that $\Psi_\mathrm{x}^{-1}(\epsilon K_2) \subset U_\mathrm{x}$. We define $V_\mathrm{x} := \Psi_\mathrm{x}^{-1}(\epsilon K_2)$. Since $U_\mathrm{x} \cap \partial\Omega$ is mapped by ϕ_x onto $]-\frac{1}{2}, \frac{1}{2}[$, Ψ_x maps $V_\mathrm{x} \cap \partial\Omega$ necessarily onto the interval $]-\epsilon, \epsilon[$. This, together with the definition of V_x, leads to the equality

$$\epsilon K_2^- = \left(\epsilon K_2^- \cap \Psi_\mathrm{x}(V_\mathrm{x} \cap \Omega)\right) \cup \left(\epsilon K_2^- \cap \Psi_\mathrm{x}(V_\mathrm{x} \setminus \overline{\Omega})\right). \tag{2.1}$$

$V_\mathrm{x} \cap \Omega$ and $V_\mathrm{x} \setminus \overline{\Omega}$ are open, thus $\epsilon K_2^- \cap \Psi_\mathrm{x}(V_\mathrm{x} \cap \Omega)$ and $\epsilon K_2^- \cap \Psi_\mathrm{x}(V_\mathrm{x} \setminus \overline{\Omega})$ are both open in ϵK_2^-. Since ϵK_2^- is connected, either $\epsilon K_2^- \cap \Psi_\mathrm{x}(V_\mathrm{x} \cap \Omega)$ or $\epsilon K_2^- \cap \Psi_\mathrm{x}(V_\mathrm{x} \setminus \overline{\Omega})$ must, hence, be empty, due to 2.1. Thus, we are in one of the following two cases

$$\epsilon K_2^- \cap \Psi_\mathrm{x}(V_\mathrm{x} \setminus \overline{\Omega}) = \emptyset, \quad \text{or, equivalently,} \quad \Psi_\mathrm{x}(V_\mathrm{x} \cap \Omega) = \epsilon K_2^- \tag{2.2}$$

$$\epsilon K_2^- \cap \Psi_\mathrm{x}(V_\mathrm{x} \cap \Omega) = \emptyset, \quad \text{or, equivalently,} \quad \Psi_\mathrm{x}(V_\mathrm{x} \cap \Omega) = \epsilon K_2^+. \tag{2.3}$$

In the first case we define $\Phi_\mathrm{x} := \frac{1}{\epsilon}\Psi_\mathrm{x}$ and are done. In the second we define Φ_x as the composition of Ψ_x with the transformation $\mathbb{R}^2 \ni (y_1, y_2) \mapsto \frac{1}{\epsilon}(y_1, -y_2)$. □

Remark 2.5. The bi-Lipschitzian parametrization of the boundary also provides the boundary measure on $\partial\Omega$ (which is identical with the restriction of the $(d-1)$-dimensional Hausdorff measure to $\partial\Omega$) see [4, Section 3.3.4 C].

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