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Some remarks on stability of generalized equations

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Abstract

The paper concerns the computation of the graphical derivative and the regular (Fréchet) coderivative of the solution map to a class of generalized equations, where the multi-valued term amounts to the regular normal cone to a (possibly nonconvex) set given by C^2 inequalities. Instead of the Linear Independence qualification condition, standardly used in this context, one assumes a combination of the Mangasarian-Fromovitz and the Constant Rank qualification conditions. On the basis of the obtained generalized derivatives, new optimality conditions for a class of mathematical programs with equilibrium constrains are derived, and a workable characterization of the isolated calmness of the considered solution map is provided.

1 Introduction

In [4], the authors investigated regular coderivatives of solution maps to perturbed *generalized equations* (GEs)

$$0 \in F(x,y) + \widehat{N}_\Gamma(y), \tag{1}$$

where $x \in \mathbb{R}^n$ is a *parameter*, $y \in \mathbb{R}^m$ is the *(decision) variable*, $F[\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m]$ is a continuously differentiable mapping,

$$\Gamma = \{y \in \mathbb{R}^m | \, q_i(y) \le 0, i = 1,2,\ldots,s\}, \tag{2}$$

and $\widehat{N}_\Gamma(y)$ stands for the *regular normal cone* to Γ at y (see Section 2 for the definition). In (2), the functions q_i $[\mathbb{R}^m \to \mathbb{R}]$ were assumed to be twice continuously differentiable. In the first part of [4], it was assumed that, at the reference pair $(\bar{x},\bar{y})$, a strong second-order sufficient condition (SSOSC) held and Γ fulfilled both the Mangasarian-Fromovitz and the Constant Rank qualification conditions (MFCQ and CRCQ) at $\bar{y}$. In the second part, the SSOSC was dropped, but the results from the first part were used for the computation of the regular normal cone to the graph of $\widehat{N}_\Gamma$, which then again enabled the authors to compute the regular coderivative of the solution map. Thereby they utilized the well-known relationship between the projection and the normal-cone operators in the convex case, and so they had to assume that Γ was convex.

The main aim of this note is to show that these results from [4] remain valid *without any additional assumptions* on the functions q_i. This improvement is based on the theory of prox-regular sets and, in particular, on a more subtle relationship between the projection and the (limiting) normal-cone operator, valid in this context. This relationship can be found e.g. in [8, Excercise 13.38] and it is the *key* reference for our development. For the reader's convenience, we state a (for our purposes relevant) part of this result together with some other important auxiliary results at the end of the next section.

The structure of the paper is as follows. Section 2 contains besides the mentioned auxiliary statements also definitions of some basic notions from variational analysis which are used in the sequel. In Section 3, we compute both the regular normal cone and the contingent (Bouligand, tangent) cone to the graph of $\widehat{N}_\Gamma$. On the basis of these results, we obtain then easily the regular coderivative and the graphical derivative of the solution map to (1) at the reference pair. The final Section 4 is devoted to applications. Specifically, similarly to [4], we derive two types of sharp optimality conditions to an optimization problem, where the GE (1) arises as a constraint, and state a characterization of the so-called isolated calmness of the solution maps to (1).

Our notation is standard. All spaces in use are assumed Euclidean. $\mathbb{B}$ is the closed unit ball, P_C denotes the projection map onto a set C and, for a multifunction Φ, $\mathrm{Gr}\,\Phi$ stands for its graph. If K is a cone, then its negative polar $\{v|\ \langle v, x\rangle \leq 0$ for all $x \in K\}$ is denoted K^0.

2 Preliminaries

Let $C \subset \mathbb{R}^m$ be a closed set and $\bar{x} \in C$. Then the *contingent (Bouligand, tangent) cone* to C at $\bar{x}$ is the set

$$T_C(\bar{x}) := \operatorname*{Lim\,sup}_{t \downarrow 0} \frac{C - \bar{x}}{t} = \{d \in \mathbb{R}^m |\ \exists t_k \downarrow 0,\ d_k \to d : \bar{x} + t_k d_k \in C\ \forall\, k\}.$$

“Limsup” stands here for the Painlevé-Kuratowski upper (outer) limit, cf. [8, Definition 4.1], [6, Definition 1.1].

The *regular (Fréchet) normal cone* to C at $\bar{x}$ can now be defined by

$$\widehat{N}_C(\bar{x}) := (T_C(\bar{x}))^0.$$

Consequently, $\widehat{N}_C(\bar{x})$ is a closed convex cone.

The *limiting (Mordukhovich) normal cone* to C at $\bar{x}$, is defined by

$$N_C(\bar{x}) := \operatorname*{Lim\,sup}_{x \to \bar{x},\ x \in C} \widehat{N}_C(x).$$

C is called *regular* at $\bar{x}$ provided $N_C(\bar{x}) = \widehat{N}_C(\bar{x})$.

Consider now a multifunction $\Phi[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ with a closed graph $\mathrm{Gr}\,\Phi$ and a point $(\bar{u}, \bar{v}) \in \mathrm{Gr}\,\Phi$. On the basis of the contingent and the regular normal cones to $\mathrm{Gr}\,\Phi$, one can define the following notions.

The multifunctions $D\Phi(\bar{u}, \bar{v})(\cdot)[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ and $\widehat{D}^*\Phi(\bar{u}, \bar{v})(\cdot)[\mathbb{R}^m \rightrightarrows \mathbb{R}^n]$ defined by

$$D\Phi(\bar{u}, \bar{v})(h) := \{k \in \mathbb{R}^m |\ (h, k) \in T_{\mathrm{Gr}\Phi}(\bar{u}, \bar{v})\},\ h \in \mathbb{R}^n,$$

and

$$\widehat{D}^*\Phi(\bar{u}, \bar{v})(y^*) := \{x^* \in \mathbb{R}^n |\ (x^*, -y^*) \in \widehat{N}_{\mathrm{Gr}\,\Phi}(\bar{u}, \bar{v})\},\ y^* \in \mathbb{R}^m,$$

are called the *graphical derivative* and the *regular coderivative* of Φ at $(\bar{x}, \bar{y})$, respectively.

Both these notions are well suited for description of the local behavior of Φ around $(\bar{x}, \bar{y})$ and will be extensively used in the sequel.

Let $S[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ be the solution map to (1), i.e.,

$$S(x) := \{y \in \mathbb{R}^m |\ 0 \in F(x, y) + \widehat{N}_\Gamma(y)\},$$

and consider a reference point $(\bar{x}, \bar{y}) \in \mathrm{Gr}\, S$. If the partial Jacobian matrix $\nabla_x F(\bar{x}, \bar{y})$ is surjective (i.e., we are dealing with the so-called ample perturbations, cf. [2]), then by virtue of [8, Exercise 6.7], for all $h \in \mathbb{R}^n$

$$DS(\bar{x}, \bar{y})(h) = \{k \in \mathbb{R}^m |\ 0 \in \nabla_x F(\bar{x}, \bar{y})h + \nabla_y F(\bar{x}, \bar{y})k + D\widehat{N}_\Gamma(\bar{y}, -F(\bar{x}, \bar{y}))(k)\}, \quad (3)$$

and for all $y^* \in \mathbb{R}^m$

$$\widehat{D}^* S(\bar{x}, \bar{y})(y^*) = \{(\nabla_x F(\bar{x}, \bar{y}))^T b |\, 0 \in y^* + (\nabla_y F(\bar{x}, \bar{y}))^T b + \widehat{D}^* \widehat{N}_\Gamma(\bar{y}, -F(\bar{x}, \bar{y}))^T(b)\}. \quad (4)$$

To apply these formulas, one has thus to compute $D\widehat{N}_\Gamma(\bar{y}, -F(\bar{x}, \bar{y}))$ and $\widehat{D}^* \widehat{N}_\Gamma(\bar{y}, -F(\bar{x}, \bar{y}))$, which will be done in the next section.

Let us now return to the set Γ given by (2). With each $y \in \Gamma$ and arbitrary $v \in \mathbb{R}^m$ we can associate the *critical cone* to Γ at u with respect to v, given by $K(y, v) := T_\Gamma(y) \cap \{v\}^\perp$. The next proposition collects some simple properties of the critical cone which will be used in the sequel.

Proposition 1. (i) *If* $v = 0$*, then* $K(y, v) = T_\Gamma(y)$.

(ii) *If* $v \in \widehat{N}_\Gamma(y) \setminus \{0\}$*, then* $K(y, v) \subset \operatorname{bd} T_\Gamma(y)$.

(iii) *If* $v \in \operatorname{int} \widehat{N}_\Gamma(y)$*, then* $K(y, v) = \{0\}$.

(iv) $K(y, v) = K(y, \nu v)$ *for any* $\nu \neq 0$.

Proof. Assertions (i) and (iv) are evident.

(ii) Let $v \in \widehat{N}_\Gamma(y) \setminus \{0\}$. Since $K(y, v) \subset T_\Gamma(y)$, it is sufficient to show that $K(y, v) \cap \operatorname{int} T_\Gamma(y) = \emptyset$. Suppose $u \in K(y, v) \cap \operatorname{int} T_\Gamma(y)$. Then $\langle u, v\rangle = 0$ and $u + \varepsilon\mathbb{B} \subset T_\Gamma(y)$ for some $\varepsilon > 0$. Hence, by assumption, for any $b \in \mathbb{B}$, it holds $0 \geq \langle v, u + \varepsilon b\rangle = \varepsilon\langle v, b\rangle$, and consequently $v = 0$. A contradiction.

(iii) Let $v \in \operatorname{int} \widehat{N}_\Gamma(y)$ and $u \in K(y, v)$. Then $v + \varepsilon\mathbb{B} \subset \widehat{N}_\Gamma(y)$ for some $\varepsilon > 0$, $u \in T_\Gamma(y)$, and $\langle u, v\rangle = 0$. Hence, for any $b \in \mathbb{B}$, it holds $0 \geq \langle v + \varepsilon b, u\rangle = \varepsilon\langle u, b\rangle$, and consequently $u = 0$. □

We say that Γ fulfills the MFCQ at y provided

$$\left.\begin{aligned} (\nabla q(y))^T \lambda = 0 \\ \lambda \geq 0 \\ \langle q(y), \lambda\rangle = 0 \end{aligned}\right\} \Rightarrow \lambda = 0.$$

To introduce the second needed constraint qualification, namely the CRCQ, we associate with each $y \in \Gamma$ the index set $I(y)$ of *active* inequalities, i.e.,

$$I(y) := \{i \in \{1, 2, \ldots, s\}|\, q_i(y) = 0\}.$$

One says that Γ fulfills the CRCQ at $\bar{y}$ provided there exists a neighborhood $\mathcal{M}$ of $\bar{y}$ such that for any subsets I of $I(\bar{y})$, the family of gradients $\{\nabla q_i(y)|\, i \in I\}$ has the same rank for all $y \in \mathcal{M}$.

If Γ fulfills MFCQ at y, then it is easy to see that Γ is fully amenable at this point.

Recall from [8, Definition 10.23(b)] that a set $C \subset \mathbb{R}^m$ is *fully amenable* at $y \in C$ if there is an open neighborhood $\mathcal{U}$ of y along with a $\mathcal{C}^2$ mapping g from $\mathcal{U}$ into a space $\mathbb{R}^s$ and a polyhedral convex set $D \subset \mathbb{R}^s$ such that

$$C \cap \mathcal{U} = \{u \in \mathcal{U}|\, g(u) \in D\},$$

and the only vector $\lambda \in N_D(g(y))$ with $\nabla g(y)^T \lambda = 0$ is $\lambda = 0$.

We finish this section with three important auxiliary statements sorted out from our basic references [7], [4] and [8]. To avoid any confusion, the first two statements are formulated in terms of another GE of the type (1), namely

$$0 \in G(p, y) + \widehat{N}_\Gamma(y), \quad (5)$$

where $p \in \mathbb{R}^l, y \in \mathbb{R}^m, G[\mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}^m]$ is continuously differentiable, and Γ is given by (2). Moreover, we assume that G amounts to the partial Jacobian of a twice continuously differentiable function $\varphi[\mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}]$ with respect to the second variable.

Let $\Xi[\mathbb{R}^l \rightrightarrows \mathbb{R}^m]$ be the solution map to (5) and suppose that $(p, y) \in \operatorname{Gr} \Xi$. Under MFCQ at y there exists a multiplier $\lambda \in \mathbb{R}^m$ such that

$$0 = \mathcal{L}(p, y, \lambda), \quad \lambda \geq 0, \quad \langle \lambda, q(y) \rangle = 0, \tag{6}$$

where

$$\mathcal{L}(p, y, \lambda) := G(p, y) + \sum_{i=1}^{s} \lambda_i \nabla q_i(y)$$

is the *Lagrangian* associated with the GE (5). In the next statement we make use of the index set

$$I_+(y, \lambda) := \{i \in I(y) |\ \lambda_i > 0\}$$

of *strongly active* inequalities.

Theorem 2 ([7]). *Consider the GE (5) around the reference point $(\bar{p}, \bar{y}) \in \operatorname{Gr} \Xi$. Further suppose that*

(i) MFCQ and CRCQ hold at $\bar{y}$;

(ii) For each λ satisfying (6) with $(p, y) = (\bar{p}, \bar{y})$ and each $v \neq 0$ such that $\langle \nabla q_i(y), v \rangle = 0$ if $i \in I_+(\bar{y}, \lambda)$ one has

$$\langle v, \nabla_y \mathcal{L}(\bar{p}, \bar{y}, \lambda) v \rangle > 0.$$

Then the following statements hold:

1) There exist neighborhoods $\mathcal{V}$ of $\bar{p}$, $\mathcal{U}$ of $\bar{y}$ and a Lipschitz function $\sigma[\mathcal{V} \to \mathcal{U}]$ such that $\sigma(\bar{p}) = \bar{y}$ and

$$\Xi(p) \cap \mathcal{U} = \sigma(p) \text{ for } p \in \mathcal{V}.$$

2) For each $p \in \mathcal{V}$ and $d \in \mathbb{R}^l$, σ is directionally differentiable at p in the direction d and $\sigma'(p; d) = v$, the unique solution of the GE

$$0 \in \nabla_p G(p, y) d + \nabla_y \mathcal{L}(p, y, \lambda) v + N_{\mathcal{K}}(v), \tag{7}$$

where $y = \sigma(p)$, $\lambda \in \mathbb{R}^s$ fulfills the relations (6) and

$$\mathcal{K} := K(y, G(p, y)) = T_\Gamma(y) \cap \{G(p, y)\}^\perp$$

is the critical cone to Γ at y with respect to $G(p, y)$.

Remark 1. *Assumption (ii) in the above theorem is the SSOSC mentioned in the Introduction.*

Next we state a slight modification of [4, Corollary 3.2] which, however, by virtue of the preceding statement does not require any changes in the proof, cf. [4, Theorem 3.1].

Theorem 3. *Consider the setting of Theorem 2 and let $p \in \mathcal{V},\ y = \sigma(p)$ and $\lambda \in \mathbb{R}^s$ satisfy the relations (6). If $\nabla_p G(p, y)$ is surjective, then one has for all $y^* \in \mathbb{R}^m$ that*

$$\widehat{N}_{\mathrm{Gr}\,\Xi}(p, y) = \{(p^*, y^*) \in \mathbb{R}^l \times \mathbb{R}^m |\ p^* \in \mathcal{K},\ y^* + (\nabla_y \mathcal{L}(p, y, \mu))p^* \in \mathcal{K}^0\}. \tag{8}$$

In the last auxiliary statement, we collect those parts of [8, Exercise 13.38] which play an essential role in the proof of Theorem 6 in the next section.

Theorem 4. *Let $C \subset \mathbb{R}^m$ be fully amenable at $\bar{y}$. Then there exists a neighborhood of $\bar{y}$ on which P_C is single-valued and Lipschitz with*

$$P_C = (I + T)^{-1}$$

for some localization T of N_C around $(\bar{y}, 0)$.

3 Main results

Lemma 5. *Consider a cone-valued multifunction $\Phi[\mathbb{R}^n \rightrightarrows \mathbb{R}^n]$, a pair $(\bar{u}, \bar{v}) \in \mathrm{Gr}\,\Phi$ and a number $\nu > 0$. Then*

(i) $(h, k) \in T_{\mathrm{Gr}\,\Phi}(\bar{u}, \bar{v})$ *if and only if* $(h, \nu k) \in T_{\mathrm{Gr}\,\Phi}(\bar{u}, \nu\bar{v})$.

(ii) $(u^*, v^*) \in \widehat{N}_{\mathrm{Gr}\,\Phi}(\bar{u}, \bar{v})$ *if and only if* $(u^*, v^*/\nu) \in \widehat{N}_{\mathrm{Gr}\,\Phi}(\bar{u}, \nu\bar{v})$.

Proof. (i) Let $(h, k) \in T_{\mathrm{Gr}\,\Phi}(\bar{u}, \bar{v})$. Then there exist sequences $\lambda_i \downarrow 0, h_i \to h, k_i \to k$ such that $(\bar{u} + \lambda_i h_i, \bar{v} + \lambda_i k_i) \in \mathrm{Gr}\,\Phi$. By virtue of the assumption, $(\bar{u}, \nu\bar{v}) \in \mathrm{Gr}\,\Phi$ and $(\bar{u} + \lambda_i h_i, \nu\bar{v} + \lambda_i(\nu k_i)) \in \mathrm{Gr}\,\Phi$. Hence, $(h, \nu k) \in T_{\mathrm{Gr}\,\Phi}(\bar{u}, \nu\bar{v})$.

The converse implication is a consequence of the just proved one applied with the positive constant ν^{-1}.

(ii) The second assertion follows from the first one by using the relationship between polar cones. □

Theorem 6. *Suppose both MFCQ and CRCQ hold at $\bar{y}$. Let $\bar{v} \in N_\Gamma(\bar{y})$ and λ be an arbitrary multiplier satisfying the conditions*

$$0 = \sum_{i=1}^m \lambda_i \nabla q_i(\bar{y}) - \bar{v}, \quad \lambda \geq 0, \quad \langle \lambda, q(\bar{y}) \rangle = 0. \tag{9}$$

Then

$$\begin{aligned} T_{\mathrm{Gr}\,\widehat{N}_\Gamma}(\bar{y}, \bar{v}) &= \left\{ (a, b) \in \mathbb{R}^m \times \mathbb{R}^m \,\middle|\, b \in \left(\sum_{i=1}^m \lambda_i \nabla^2 q_i(\bar{y}) \right) a + N_{K(\bar{y},\bar{v})}(a) \right\} \\ &= \left\{ (a, b) \,\middle|\, a \in K(\bar{y}, \bar{v}),\ b - \left(\sum_{i=1}^m \lambda_i \nabla^2 q_i(\bar{y}) \right) a \in (K(\bar{y}, \bar{v}))^0, \right. \\ &\qquad \left. \left\langle a, b - \left(\sum_{i=1}^m \lambda_i \nabla^2 q_i(\bar{y}) \right) a \right\rangle = 0 \right\}, \end{aligned} \tag{10}$$

$$\widehat{N}_{\mathrm{Gr}\,\widehat{N}_\Gamma}(\bar{y}, \bar{v}) = \left\{ (y^*, v^*) \,\middle|\, y^* + \left(\sum_{i=1}^m \lambda_i \nabla^2 q_i(\bar{y}) \right) v^* \in (K(\bar{y}, \bar{v}))^0,\ v^* \in K(\bar{y}, \bar{v}) \right\}. \tag{11}$$

Proof. We start with proving (11). For that we need to analyze the projection P_Γ. Take p as a parameter and consider the GE of the type (5) in the variable y

$$p \in y + \widehat{N}_\Gamma(y). \tag{12}$$

It corresponds to taking $G(p,y) = y - p$ in (5) and can be associated with the optimization problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|p-y\|^2 \\ \text{subject to} & y \in \Gamma. \end{array}$$

Let Ξ be the solution map to (12) and consider the point $(\bar{y},\bar{y}) \in \mathrm{Gr}\,\Xi$. Due to the imposed MFCQ at $\bar{y}$, to this point we can assign only the multiplier $\lambda = 0_{\mathbb{R}^m}$ satisfying the conditions

$$0 = G(\bar{y},\bar{y}) + (\nabla q(\bar{y}))^T\lambda, \quad \lambda \geq 0, \quad \langle q(\bar{y}), \lambda\rangle = 0.$$

We observe that all assumptions of Theorem 2 are fulfilled and, consequently, there are neighborhoods $\mathcal{V}$ and $\mathcal{U}$ of $\bar{y}$ and a single-valued Lipschitz function $\sigma[\mathcal{V} \to \mathcal{U}]$ such that $\sigma(\bar{y}) = \bar{y}$ and

$$\Xi(p) \cap \mathcal{U} = \{\sigma(p)\} \text{ for } p \in \mathcal{V}.$$

Since P_Γ is nonempty-valued, $P_\Gamma(p) \subset \Xi(p)$ for all p, and

$$P_\Gamma(\bar{y}) = \{\bar{y}\} = \sigma(\bar{y}),$$

the neighborhood $\mathcal{V}$ and $\mathcal{U}$ can be shrunk if necessary (without changing the notation) so that

$$\mathrm{Gr}\,P_\Gamma \cap (\mathcal{V} \times \mathcal{U}) = \mathrm{Gr}\,\sigma.$$

We apply now Theorem 3 to a pair $(u,\bar{y})$ with $u \in \mathcal{V}$ and $P_\Gamma(u) = \{\bar{y}\}$. Denoting $\bar{v} = u - \bar{y}$ and taking into account that $K(\bar{y},-\bar{v}) = K(\bar{y},\bar{v})$ (Proposition 1 (iv)), we arrive at

$$\widehat{N}_{\mathrm{Gr}P_\Gamma}(u,\bar{y}) = \left\{(u^*,y^*) \in \mathbb{R}^m \times \mathbb{R}^m \,\middle|\, u^* \in K(\bar{y},\bar{v}),\ y^* + \left(I + \sum_{i=1}^m \lambda_i \nabla^2 q_i(\bar{y})\right) u^* \in (K(\bar{y},\bar{v}))^0\right\}, \tag{13}$$

where $\lambda \in \mathbb{R}^m$ is any multiplier satisfying conditions (9).

Next we make use of the full amenability of Γ at $\bar{y}$, mentioned in Section 2. By Theorem 4, there exist neighborhoods $\mathcal{Z}$ of $\bar{y}$ and $\mathcal{W}$ of 0 and a single-valued mapping $T[\mathcal{Z} \to \mathcal{W}]$ such that for $p \in \mathcal{Z}$

$$P_\Gamma(p) = (I+T)^{-1}(p)$$

and

$$\mathrm{Gr}\,T = \mathrm{Gr}\,N_\Gamma \cap (\mathcal{Z} \times \mathcal{W}). \tag{14}$$

Moreover, since Γ is regular on a neighborhood of $\bar{y}$, the limiting normal cone on the right-hand side of (14) can be replaced by the regular normal cone $\widehat{N}_\Gamma$. Hence,

$$(b,c) \in \mathrm{Gr}\,\widehat{N}_\Gamma \cap (\mathcal{Z} \times \mathcal{W}) \quad \text{if and only if} \quad (b+c,b) \in \mathrm{Gr}\,P_\Gamma \text{ and } (b,c) \in \mathcal{Z} \times \mathcal{W}. \tag{15}$$

If we now shrink $\mathcal{Z}$ and $\mathcal{W}$ in such a way that $\mathcal{Z} \subset \mathcal{U}$, $\mathcal{Z} + \mathcal{W} \subset \mathcal{V}$, we can invoke (13) and [8, Exercise 6.7] and obtain that for $\bar{v} \in \mathcal{W}$ with $P_\Gamma(\bar{y}+\bar{v}) = \{\bar{y}\}$

$$
\widehat{N}_{\mathrm{Gr}\,\widehat{N}_\Gamma}(\bar{y},\bar{v}) = \begin{bmatrix} I & I \\ I & 0 \end{bmatrix} \widehat{N}_{\mathrm{Gr}\,P_\Gamma}(u,\bar{y}) =
$$

$$
\left\{ (y^*,v^*) \in \mathbb{R}^m \times \mathbb{R}^m \,\middle|\, v^* \in K(\bar{y},\bar{v}),\ y^* + \left(\sum_{i=1}^m \lambda_i \nabla^2 q_i(\bar{y}) \right) v^* \in (K(\bar{y},\bar{v}))^0 \right\}. \quad (16)
$$

It remains now to analyze a general pair $(\bar{y},\bar{v}) \in \mathrm{Gr}\,\widehat{N}_\Gamma$, where $\bar{v}$ does not necessarily belong to $\mathcal{W}$. Let λ be an arbitrary multiplier satisfying conditions (9). We readily see that there is a positive real $\nu \in (0,1]$ such that $\nu\bar{v} \in \mathcal{W}$ and $\nu\lambda$ is a multiplier satisfying conditions (9) with $\nu\bar{v}$ instead of $\bar{v}$. Moreover, by Proposition 1 (iv), $K(\bar{y},\bar{v}) = K(\bar{y},\nu\bar{v})$. By virtue of (16),

$$
\begin{aligned}
\widehat{N}_{\mathrm{Gr}\,\widehat{N}_\Gamma}(\bar{y},\nu\bar{v}) &= \left\{ (y^*,b) \,\middle|\, b \in K(\bar{y},\bar{v}),\ y^* + \left(\sum_{i=1}^m \nu\lambda_i \nabla^2 q_i(\bar{y}) \right) b \in (K(\bar{y},\bar{v}))^0 \right\} \\
&= \left\{ (y^*,b) \,\middle|\, \nu b \in K(\bar{y},\bar{v}),\ y^* + \left(\sum_{i=1}^m \lambda_i \nabla^2 q_i(\bar{y}) \right) \nu b \in (K(\bar{y},\bar{v}))^0 \right\}.
\end{aligned}
$$

It follows by Lemma 5 (ii) that

$$
\begin{aligned}
\widehat{N}_{\mathrm{Gr}\,\widehat{N}_\Gamma}(\bar{y},\bar{v}) &= \left\{ (y^*,v^*) |\, v^* = \nu b,\ (y^*,b) \in \widehat{N}_{\mathrm{Gr}\,\widehat{N}_\Gamma}(\bar{y},\nu\bar{v}) \right\} \\
&= \left\{ (y^*,v^*) \,\middle|\, v^* \in K(\bar{y},\bar{v}),\ y^* + \left(\sum_{i=1}^m \lambda_i \nabla^2 q_i(\bar{y}) \right) v^* \in (K(\bar{y},\bar{v}))^0 \right\},
\end{aligned}
$$

and so (11) has been established.

Let $\mathcal{V}$ be the neighborhood specified above and let $u \in \mathcal{V}$ be such that $P_\Gamma(u) = \{\bar{y}\}$. By virtue of Theorem 2,

$$
T_{\mathrm{Gr}P_\Gamma}(u,\bar{y}) = \left\{ (h,k) \in \mathbb{R}^m \times \mathbb{R}^m \,\middle|\, h \in \left(I + \sum_{i=1}^s \lambda_i \nabla^2 q_i(\bar{y}) \right) k + N_{K(\bar{y},\bar{v})}(k) \right\},
$$

where $\bar{v} = u - \bar{y}$ and $\lambda \in \mathbb{R}^m$ is any multiplier satisfying conditions (9). If $\bar{v}$ lies even in the neighborhood $\mathcal{W}$ specified above, one can make use of (15) and conclude (with the help of [8, Exercise 6.7]) that

$$
\begin{aligned}
T_{\mathrm{Gr}\,\widehat{N}_\Gamma}(\bar{y},\bar{v}) &= \left\{ (a,b) \,\middle|\, \begin{bmatrix} I & I \\ I & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \in T_{\mathrm{Gr}P_\Gamma}(u,\bar{y}) \right\} = \\
&= \left\{ (a,b) \,\middle|\, b \in \left(\sum_{i=1}^m \lambda_i \nabla^2 q_i(\bar{y}) \right) a + N_{K(\bar{y},\bar{v})}(a) \right\}.
\end{aligned}
$$

It remains to analyze a general pair $(\bar{y},\bar{v})$, where $\bar{v}$ does not necessarily belong to $\mathcal{W}$. By using the same reasoning as in the proof of (11), based this time on Lemma 5 (i), we conclude that $T_{\mathrm{Gr}\,\widehat{N}_\Gamma}(\bar{y},\bar{v})$ is given by (10), where $\lambda \in \mathbb{R}^s$ is an arbitrary multiplier satisfying conditions (9). So, the statement has been proved. □

On the basis of formulas (3), (4) and the preceding theorem, we can now immediately compute the desired graphical derivative and regular coderivative of S at $(\bar{x},\bar{y})$.

Theorem 7. *Let $\nabla_x F(\bar{x}, \bar{y})$ be surjective, both MFCQ and CRCQ hold at $\bar{y}$, and λ be an arbitrary multiplier satisfying conditions (9). Then for all $h \in \mathbb{R}^n$,*

$$DS(\bar{x}, \bar{y})(h) = \{k \in \mathbb{R}^m |\ 0 \in \nabla_x F(\bar{x}, \bar{y})h + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \lambda)k + N_{\mathcal{K}}(k)\}, \tag{17}$$

and for all $y^* \in \mathbb{R}^m$

$$\widehat{D}^* S(\bar{x}, \bar{y})(y^*) = \{(\nabla_x F(\bar{x}, \bar{y}))^T b |\ 0 \in y^* + (\nabla_y \mathcal{L}(\bar{x}, \bar{y}, \lambda))^T b + \mathcal{K}^0, -b \in \mathcal{K}\}, \tag{18}$$

where

$$\mathcal{L}(x, y, \lambda) := F(x, y) + \sum_{i=1}^{s} \lambda_i \nabla q_i(y)$$

is the Lagrangian associated with the GE (1) and

$$\mathcal{K} := K(\bar{y}, F(\bar{x}, \bar{y})). \tag{19}$$

Formulas (17), (18) have multiple applications, some of which will be discussed in the next section.

4 Applications

A multifunction $\Phi[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ is said to have the *isolated calmness* property at $(\bar{u}, \bar{v}) \in \operatorname{Gr} \Phi$, provided there exist neighborhoods $\mathcal{U}$ of $\bar{u}$ and $\mathcal{V}$ of $\bar{v}$ and a constant $\kappa \geq 0$ such that

$$\Phi(u) \cap \mathcal{V} \subset \{\bar{v}\} + \kappa \|u - \bar{u}\| \mathbb{B} \text{ when } u \in \mathcal{U}.$$

In [5], it has been proved that Φ possesses the isolated calmness property at $(\bar{u}, \bar{v})$ if and only if

$$D\Phi(\bar{u}, \bar{v})(0) = \{0\}, \tag{20}$$

cf. also [3, Theorem 4C.1]. In that monograph, this characterization has been applied to variational inequalities with polyhedral constraint sets [3, Theorem 4E.1]. Our Theorem 4 yields a substantial generalization of this result to the GE (1).

Theorem 8. *Consider the GE (1), $(\bar{x}, \bar{y}) \in \operatorname{Gr} S$. Let λ be an arbitrary multiplier satisfying conditions (9) and assume that MFCQ and CRCQ hold at $\bar{y}$. Then S has the isolated calmness property at $(\bar{x}, \bar{y})$, provided the GE*

$$0 \in \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \lambda)k + N_{\mathcal{K}}(k) \tag{21}$$

(with $\mathcal{K}$ given in (19)) possesses only the trivial solution $k = 0$.

Moreover, if $\nabla_x F(\bar{x}, \bar{y})$ is surjective, then the above condition is not just sufficient but also necessary for S to have the isolated calmness property at $(\bar{x}, \bar{y})$.

Proof. The full characterization under the surjectivity of $\nabla_x F(\bar{x}, \bar{y})$ follows directly from condition (20) combined with formula (17). The "sufficiency" part follows from the fact that, in absence of the surjectivity of $\nabla_x F(\bar{x}, \bar{y})$, equation (3) becomes inclusion of the type "$\subset$", cf. [8, Theorem 6.31]. $\square$

Example 1. *Consider the GE*

$$0 \in x + \widehat{N}_\Gamma(y), \tag{22}$$

where

$$\Gamma = \left\{ y \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} -\frac{1}{2}y_1^2 + y_1 - y_3 \leq 0, \\ -\frac{1}{2}y_1^2 - y_1 - y_3 \leq 0, \\ -\frac{1}{2}y_2^2 + y_2 - y_3 \leq 0, \\ -\frac{1}{2}y_2^2 - y_2 - y_3 \leq 0, \end{array} \right\}. \tag{23}$$

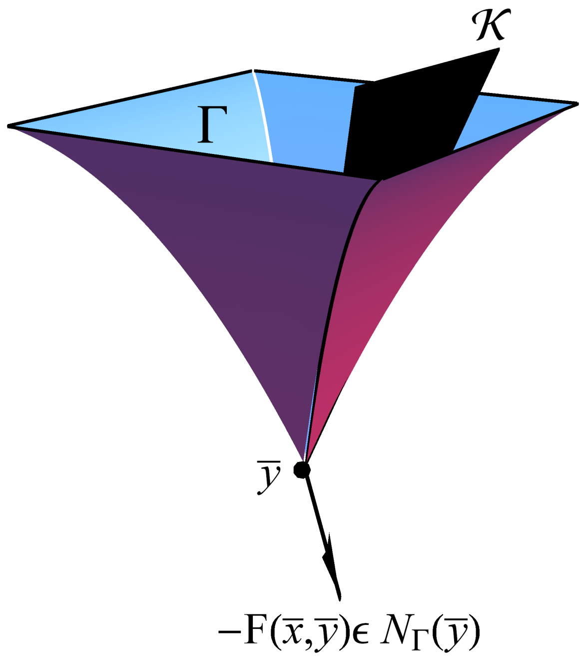

Figure 1: Illustration of the feasible set Γ defined by (23) and of the critical cone $\mathcal{K}$.

Observe that the corresponding solution map S assigns x the stationary points of the nonconvex program

$$\begin{array}{ll} \textit{minimize} & \langle x, y \rangle \\ \textit{subject to} & y \in \Gamma. \end{array}$$

As the reference point, take the pair $(\bar{x}, \bar{y})$ with $\bar{x} = (-1, 0, 1)$ and $\bar{y} = 0_{\mathbb{R}^3}$. All assumptions of Theorem 8 are fulfilled and the equality

$$0 = \bar{x} + \sum_{i=1}^{4} \lambda_i \nabla q_i(\bar{y})$$

holds, for instance, with $\bar{\lambda}_1 = 1$, $\bar{\lambda}_2 = \bar{\lambda}_3 = \bar{\lambda}_4 = 0$. Clearly, the GE (21) can be written down in the form

$$k \in \mathcal{K}, \; - \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} k \in \mathcal{K}^0, \; k_1^2 = 0,$$

where $\mathcal{K} = \{(y_1, y_2, y_3) |\; y_1 = y_3 \geq |y_2|\}$ and $\mathcal{K}^0 = \{(v_1, v_2, v_3) |\; v_1 + v_3 + |v_2| \leq 0\}$. It follows that $k_2 = k_3 = 0$ as well and, consequently, the respective S has the isolated calmness property at $(\bar{x}, \bar{y})$. $\triangle$

Theorem 7 will now be used in a generalization of [4, Theorem 4.2], where we remove the requirement of convexity imposed on the functions q_i. Consider the *mathematical program with equilibrium constraints* (MPEC)

$$\begin{array}{ll} \text{minimize} & f(x,y) \\ \text{subject to} & 0 \in F(x,y) + \widehat{N}_\Gamma(y), \end{array} \tag{24}$$

where $f[\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}]$ is continuously differentiable and, apart from the GE (1), we do not have any constraints.

Theorem 9. *Let $(\bar{x}, \bar{y})$ be a (local) solution of the MPEC (24), and assume that $\nabla_x F(\bar{x}, \bar{y})$ is surjective, and MFCQ and CRCQ hold at $\bar{y}$. Then, there is an MPEC multiplier $\bar{b} \in -\mathcal{K}$ such that*

$$0 = \nabla_x f(\bar{x}, \bar{y}) + (\nabla_x F(\bar{x}, \bar{y}))^T \bar{b}, \tag{25}$$

$$0 = \nabla_y f(\bar{x}, \bar{y}) + (\nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda}))^T \bar{b} + \mathcal{K}^0, \tag{26}$$

where $\bar{\lambda}$ is an arbitrary multiplier satisfying conditions (9) with $\bar{v} = -F(\bar{x}, \bar{y})$ and $\mathcal{K}$ is given by (19).

Proof. The statement follows immediately from the standard optimality condition

$$0 \in \nabla f(\bar{x}, \bar{y}) + \widehat{N}_{\mathrm{Gr} S}(\bar{x}, \bar{y})$$

by virtue of Theorem 7. $\square$

Example 2. *Consider the following modification of the MPEC from [4, Example 4.1]:*

$$\begin{array}{ll} \text{minimize} & y_3 + \frac{1}{2}\|x - a\|^2 \\ \text{subject to} & 0 \in x + \widehat{N}_\Gamma(y), \quad x, y \in \mathbb{R}^3 \end{array} \tag{27}$$

where $a = (-1 - \varepsilon, 0, 1)$ with some $\varepsilon \geq 0$, Γ is given by (23). It can easily be checked that $\bar{x} = (-1 - \frac{\varepsilon}{2}, 0, 1 + \frac{\varepsilon}{2})$, $\bar{y} = 0_{\mathbb{R}^3}$ is a local solution to (27). All assumptions of Theorem 9 are fulfilled and the equality

$$0 = \bar{x} + \sum_{i=1}^{4} \lambda_i \nabla q_i(\bar{y})$$

holds, for instance, with $\bar{\lambda}_1 = 1 + \frac{\varepsilon}{2}$, $\bar{\lambda}_2 = \bar{\lambda}_3 = \bar{\lambda}_4 = 0$. Conditions (25), (26) take the form

$$0 = \begin{pmatrix} \frac{\varepsilon}{2} \\ 0 \\ \frac{\varepsilon}{2} \end{pmatrix} + \bar{b}, \quad 0 \in \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \bar{\lambda}_1 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \bar{b} + \mathcal{K}^0,$$

where $\mathcal{K}$ and $\mathcal{K}^0$ are the same as in Example 1. Hence,

$$\bar{b} = (-\frac{\varepsilon}{2}, 0, -\frac{\varepsilon}{2}) \in -\mathcal{K} \quad \textit{and} \quad -\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - (1 + \frac{\varepsilon}{2}) \begin{pmatrix} \frac{\varepsilon}{2} \\ 0 \\ 0 \end{pmatrix} \in \mathcal{K}^0.$$

The optimality conditions in Theorem 9 are fulfilled. $\triangle$

The theory developed in Section 3 enables us also to derive the so-called fuzzy optimality conditions for a general MPEC, where one has, apart from the equilibrium constraint, also additional constraints, for instance in the form of equalities and inequalities.

Consider the following MPEC:

$$\begin{array}{ll} \text{minimize} & f_0(x,y) \\ \text{subject to} & f_i(x,y) \le 0,\ i=1,\dots,l, \\ & f_i(x,y) = 0,\ i=l+1,\dots,k, \\ & 0 \in F(x,y) + \widehat{N}_\Gamma(y), \end{array} \tag{28}$$

where $0 \le l \le k$ and the functions $f_i[\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}]$ are lower semicontinuous for $i = 0,\dots,l$ and continuous for $i = l+1,\dots,k$ near $(\bar{x},\bar{y}) \in \operatorname{Gr} S$ (recall that S denotes the solution map of the equilibrium constraint). Observe that under the imposed assumptions the graph $\operatorname{Gr} S$ is locally closed near $(\bar{x},\bar{y})$. Indeed, Γ is regular on a neighborhood of $\bar{y}$ and so, on this neighborhood, the regular normal cone in (1) can be replaced by the limiting one. The local closedness of S follows then immediately from the outer semicontinuity of the map $N_\Gamma(\cdot)$.

Theorem 10. *Let $(\bar{x},\bar{y})$ be a (local) solution of problem (28). Suppose that $\nabla_x F(\bar{x},\bar{y})$ is surjective and both MFCQ and CRCQ hold at $\bar{y}$. Then for any $\varepsilon > 0$, there exist points $(x_i,y_i) \in (\bar{x},\bar{y}) + \varepsilon\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}^m}$, $i = 0,\dots,k+1$; numbers $\mu_i \ge 0$, $i = 0,\dots,k$; a point $(u^*,v^*) \in \mathbb{R}^n \times \mathbb{R}^m$, and an MPEC multiplier $\bar{b} \in \mathbb{R}^m$ such that*

$$f_i(x_i,y_i) \le f_i(\bar{x},\bar{y}) + \varepsilon,\ i=1,\dots,l, \tag{29}$$

$$(u^*,v^*) \in \sum_{i=0}^{l} \mu_i \widehat{\partial} f_i(x_i,y_i) + \sum_{i=l+1}^{k} \mu_i \left(\widehat{\partial} f_i(x_i,y_i) \cup \widehat{\partial}(-f_i)(x_i,y_i)\right) + \varepsilon\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}^m}, \tag{30}$$

$$-\bar{b} \in K(y_{k+1}, F(x_{k+1},y_{k+1})), \tag{31}$$

$$0 = u^* + (\nabla_x F(x_{k+1},y_{k+1}))^T \bar{b}, \tag{32}$$

$$0 = v^* + (\nabla_y \mathcal{L}(x_{k+1},y_{k+1},\bar{\lambda}))^T \bar{b} + [K(y_{k+1},F(x_{k+1},y_{k+1}))]^0, \tag{33}$$

$$\mu_i f_i(x_i,y_i) = 0,\ i=1,\dots,l, \tag{34}$$

$$\sum_{i=0}^{k} \mu_i = 1, \tag{35}$$

where $\bar{\lambda} = (\lambda_1,\dots,\lambda_m) \ge 0$ is an arbitrary multiplier satisfying

$$\begin{aligned} 0 \ = \ & \sum_{i=1}^{m} \lambda_i \nabla q_i(y_{k+1}) + F(x_{k+1},y_{k+1}), \\ & \lambda_i q_i(y_{k+1}) = 0,\ i=1,\dots,m. \end{aligned}$$

Proof. Applying to problem (28) the fuzzy/approximate multiplier rule (see e.g. [1, Theorem 3.3.8]), we get for any $\varepsilon > 0$, the existence of points $(x_i,y_i) \in (\bar{x},\bar{y}) + \varepsilon\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}^m}$, $i = 0,\dots,k+1$; numbers $\mu_i \ge 0$, $i = 0,\dots,k$, and a point $(u^*,v^*) \in \mathbb{R}^n \times \mathbb{R}^m$ such that conditions (29), (30), (34), and (35) hold true and $-(u^*,v^*) \in \widehat{N}_{\operatorname{Gr} S}(x_{k+1},y_{k+1})$. The rest follows from Theorem 7 due to the fact that the surjectivity of $\nabla_x F$ and MFCQ and CRCQ possess a nice property: once they are satisfied at a point, they also hold in its neighborhood. $\square$

Note that in Theorem 10, the case $\mu_0 = 0$ is not excluded. This is because the constraint qualifications in its statement are for the equilibrium constraint only. To guarantee $\mu_0 > 0$ some additional qualification conditions are needed.

5 Conclusion

In most finite dimensional applications of variational analysis we use nowadays various limiting derivative-like objects, because they typically admit a much richer calculus than the basic constructions (like regular normal cones, subdifferentials and coderivatives). On the other hand, basic notions yield mostly sharper optimality / stationarity conditions than their limiting counterparts and in some stability considerations (related e.g. to the isolated calmness), just these basic notions are needed.

In this note we continue the research started in [4] and investigate a situation which seems to be especially suitable for the computation of graphical derivatives and regular coderivatives of solution maps to a class of perturbed GEs. Thereby we employ, as a main tool, a deep result from the theory of prox-regular sets relating a local notion (normal cone mapping) with a global one (projection map)). It is, however, not clear, to what extent this approach could be applied to not fully amenable sets Γ arising e.g. in the context of conical programming.

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