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Global spatial regularity for elasticity models with cracks, contact and other nonsmooth constraints

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Abstract

A global higher differentiability result in Besov spaces is proved for the displacement fields of linear elastic models with self contact. Domains with cracks are studied, where non-penetration conditions/Signorini conditions are imposed on the crack faces. It is shown that in a neighborhood of crack tips (in 2D) or crack fronts (3D) the displacement fields are $B^{3/2}_{2,\infty}$ regular. The proof relies on a difference quotient argument for the directions tangential to the crack. In order to obtain the regularity estimates also in the normal direction, an argument due to Ebmeyer/Frehse/Kassmann [EFK02] is modified. The methods are then applied to further examples like contact problems with nonsmooth rigid foundations, to a model with Tresca friction and to minimization problems with nonsmooth energies and constraints as they occur for instance in the modeling of shape memory alloys. Based on Falk's approximation Theorem for variational inequalities, convergence rates for FE-discretizations of contact problems are derived relying on the proven regularity properties. Several numerical examples illustrate the theoretical results.

1 Introduction

In this note, a global higher differentiability result in Besov spaces for the elastic fields of a model with self contact is proved. In particular, domains with cracks are studied, where non-penetration conditions/Signorini conditions are imposed on the crack faces.

Local regularity results for the Signorini or contact problem are well-known, and we refer to [Kin81, BGK87, Sch89] for instance. There, Hölder continuity properties as well as higher differentiability in the scale of Sobolev spaces is shown for displacement fields u. For example, it is shown that $u \in H^2_{\text{loc}}(\Omega \cup \Gamma_C)$, where $\Omega \subset \mathbb{R}^d$ is the body and $\Gamma_C \subset \partial\Omega$ the open set where the body might be in contact with an obstacle. However, not so much is known about the regularity of u in the neighborhood of the interface separating Γ_C from the rest of $\partial\Omega$.

Neglecting contact conditions, it is well known that in a neighborhood of crack tips (in 2D) or of smooth crack fronts (in 3D) the displacement fields have a behavior of the type $r^{\frac{1}{2}}$, where r stands for the distance to the crack tip or front. Hence, the displacements belong to the Besov space $B^{\frac{3}{2}}_{2,\infty}$. Such results are derived using the method of singular expansions and we refer to [CD93, CDY04], for example.

In this paper we extend these results to the case, where contact conditions are imposed on the crack faces. Moreover, the crack front need not to be smooth. Instead, we assume that the crack front is a Lipschitz curve (e.g. in 3D). We show that also in these cases the displacement fields are $B^{\frac{3}{2}}_{2,\infty}$ regular. The proof relies on a difference quotient argument for the directions tangential to the crack. In order to obtain the regularity estimates also in the normal direction, an argument from [EFK02, KM07] is applied and slightly modified. This argument allows us to solve the equation for the missing derivatives in the normal direction. It is based on Lemma 2.4, by which it is possible to transfer estimates for finite differences that are available for tangential directions into estimates for the perpendicular direction. The idea originates from [EFK02, KM07], where the proof is given using Fourier techniques. We give here an alternative proof, which relies on direct estimates of the relevant quantities, and in this way also generalizes the argument to the case $p \neq 2$. The proof of the main regularity result is carried out in Section 2 for a class of systems of quasilinear elliptic partial differential equations that includes the system of linear elasticity as a special case.

The methods described here are then applied to contact problems with nonsmooth rigid foundations, to frictional contact problems with Tresca friction and to minimization problems with uniformly convex (nonsmooth) energies and nonsmooth constraints as they occur for instance in the modeling of shape memory alloys. These extensions are discussed in Section 3.

The results presented here extend the regularity investigations from [EF99, Sav98, EFK02, KM07, Kne06] for quasilinear elliptic systems on nonsmooth domains in the following aspects: More general systems of quasilinear elliptic partial differential equations are studied, meaning that we only assume that the nonlinearities are rank-one monotone, instead of the stronger monotonicity assumption in [EF99, Sav98, EFK02, KM07, Kne06]. Crack and contact problems are studied with nonsmooth crack fronts in three space dimensions and with nonsmooth rigid obstacles. Moreover, the geometric assumption in [KM07] on the interface between Dirichlet- and Neumann-boundary is weakened.

Based on Falk's approximation Theorem for variational inequalities, convergence rates for FE-discretizations of contact problems are derived relying on the proved regularity properties. While in two space dimensions a rich variety of geometric situations can be treated, in three dimensions the situation is more delicate. In this case it is an interesting task to construct an interpolation operator into the FE-space that preserves the contact conditions and that has optimal approximation properties. In particular, Lagrange interpolation is not applicable since in 3D the space $B^{\frac{3}{2}}_{2,\infty}$ is not embedded into the continuous functions. This issue will be discussed in detail in Section 4 for a special case.

In Section 5, the regularity results are confirmed by several numerical experiments. We study the convergence rate of the finite element approximation for representative numerical examples of contact problems with self-contact and unilateral contact constraints in 2D and 3D. Nonsmooth crack fronts and rigid foundations with nonsmooth boundaries are included. The convergence rate is estimated by using some extrapolated reference solutions on very fine finite element meshes. In all examples, convergence rates not less than 1/2 are determined which indicate the predicted $B^{3/2}_{2,\infty}$-regularity.

2 The main regularity result

In this section we investigate the regularity of the displacement fields in a neighborhood of the crack front of interior cracks. On the crack, self-contact conditions/Signorini conditions are imposed. In Section 3 we will then discuss further examples, where the crack intersects with the outer boundary of the physical body. For shortness it is assumed in this paper that the crack is contained in a hyperplane. Results for cracks, which are parts of smooth manifolds, then can be deduced with the usual transformation arguments. The abstract regularity results will be derived not only for functionals with quadratic energies describing linear elasticity, but for a more general class of non-quadratic, but uniformly convex functionals.

2.1 Notation

Let $\Omega \subset \mathbb{R}^d$ be a domain. For $k \in \mathbb{N}_0$ the spaces $H^k(\Omega)$ are the usual Sobolev spaces. For $m \in \mathbb{N}_0$ and $\sigma \in (0,1)$, the set $B^{m+\sigma}_{2,\infty}(\Omega)$ denotes the Besov space of differentiability order $m+\sigma$ and we refer to [Tri78] for a precise definition. Observe that the following identity holds true if Ω is a Lipschitz domain, [Nik75, Tri78]:

$$\begin{aligned} B^{m+\sigma}_{2,\infty}(\Omega) &= \{\, u \in L^2(\Omega)\,;\, \|u\|_{\mathcal{N}^{m+\sigma}(\Omega)} < \infty \,\}, \\ \|u\|^2_{\mathcal{N}^{m+\sigma}(\Omega)} &= \|u\|^2_{L^2(\Omega)} + \sum_{|\alpha|=m} \sup_{\substack{\eta>0 \\ h\in\mathbb{R}^d \\ 0<|h|<\eta}} \int_{\Omega_\eta} |h|^{-2\sigma} \left| D^\alpha u(x+h) - D^\alpha u(x)\right|^2 \,\mathrm{d}x \end{aligned}$$

and $\Omega_\eta = \{\, x \in \Omega\,;\, \mathrm{dist}(x,\partial\Omega) > \eta \,\}$. Furthermore, for every $\epsilon > 0$ and $s > 0$ not an integer, the following embeddings are continuous: $B^{s+\epsilon}_{2,\infty}(\Omega) \subset H^s(\Omega) \subset B^s_{2,\infty}(\Omega)$. Here, $H^s(\Omega)$ stands for the Sobolev-Slobodeckij space of order s.

In the sequel we assume that the crack is contained in the hyperplane $E_d := \{\, x \in \mathbb{R}^d\,;\, x_d = 0 \,\}$. For $x_0 \in E_d$ the set $B^{d-1}_R(x_0) = \{\, x \in E_d\,;\, |x - x_0| < R \,\}$ denotes the (with respect to E_d open) disc centered at x_0 of radius R. Moreover, $Q_R(x_0) = x_0 + (-R,R)^d$ is the cube of radius R centered at x_0, $Q^+_R(x_0) = x_0 + (-R,R)^{d-1} \times (0,R)$ and $Q^-_R(x_0)$ are its upper and lower part, and for $x_0 \in E_d$ we define $Q^{d-1}_R(x_0) = Q_R(x_0) \cap E_d$. Let $S \subset \partial B^{d-1}_1(0)$ be nonempty and open with respect to E_d. For $h_0 > 0$ the set $\mathcal{C}(h_0; S)$ denotes the flat finite open cone in E_d of height h_0 defined via

$$\mathcal{C}(h_0; S) = \{\, x \in B^{d-1}_{h_0}(0)\,;\, x/|x| \in S \,\}.$$

We will usually write $\mathcal{C}(h_0)$ instead of $\mathcal{C}(h_0; S)$.

For elements $A, B \in \mathbb{R}^{m\times d}$ the inner product is defined as $A : B = \mathrm{tr}\, B^\top A$.

2.2 Admissible geometries and energies

It is assumed that

(G1) $\widetilde{\Omega} \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain.

(G2) $\Gamma_C \subset E_d := \{\, x \in \mathbb{R}^d\,;\, x_d = 0 \,\}$ is a $d-1$ dimensional domain satisfying the uniform cone condition with respect to E_d. Moreover, $\overline{\Gamma_C} \subset \widetilde{\Omega}$ and $\Omega := \widetilde{\Omega} \backslash \overline{\Gamma_C}$.

The set Γ_C represents the crack, which is assumed to be in the interior of $\widetilde{\Omega}$, the set Ω describes the cracked domain. Observe that the uniform cone condition is equivalent to the property that the crack front $\partial\Gamma_C$ locally can be represented as the graph of a Lipschitz continuous function, [Gri85]. The purpose is to study the regularity of displacement fields in a neighborhood of the crack front $\partial\Gamma_C$. Condition (G2) implies that the sets

$$\Omega_+ := \{\, x \in \widetilde{\Omega}\,;\, x_d > 0\,\}, \quad \Omega_- := \{\, x \in \widetilde{\Omega}\,;\, x_d < 0\,\} \tag{2.1}$$

are not empty.

The energy density $W : \Omega \times \mathbb{R}^{d\times d} \to \mathbb{R}$ shall satisfy the following growth and convexity conditions:

(W1) $W : \Omega \times \mathbb{R}^{d\times d} \to [0,\infty)$ is a Carathéodory function and there exist constants $c_1 > 0$ and $\kappa \in \{0,1\}$ such that for all $A \in \mathbb{R}^{d\times d}$ and $x, y \in \Omega_+$ or $x, y \in \Omega_-$ it holds $W(x,A) \leq c_1(\kappa + |A|^2)$ and

$$|W(x,A) - W(y,A)| \leq c_1\, |x-y|\, (\kappa + |A|^2). \tag{2.2}$$

Moreover, there exist constants $c_2, \beta > 0$ such that for all $A, B \in \mathbb{R}^{d\times d}$ with $|B - \mathbb{I}| \leq \beta$ and for e.a. $x \in \Omega$ it holds

$$|W(x,AB) - W(x,A)| \leq c_2\, |B - \mathbb{I}|\, (\kappa + |A|^2). \tag{2.3}$$

(W2) The density W generates a uniformly convex functional on $H^1(\Omega;\mathbb{R}^d)$ in the following sense: For $v \in H^1(\Omega;\mathbb{R}^d)$ let $\mathcal{F}(v) = \int_\Omega W(\nabla v)\,\mathrm{d}x + \frac{1}{2}\,\|v\|^2_{L^2(\Omega)}$. There exists a constant $\alpha > 0$ such that for all $u, v \in H^1(\Omega;\mathbb{R}^d)$ and all $\lambda \in [0,1]$ it holds

$$\mathcal{F}(\lambda u + (1-\lambda)v) + \frac{\alpha}{2}\lambda(1-\lambda)\,\|u - v\|^2_{H^1(\Omega)} \leq \lambda\mathcal{F}(u) + (1-\lambda)\mathcal{F}(v). \tag{2.4}$$

By assumption (W1) energy densities of the type $W(x,A) := W_\pm(A)$ for $x \in \Omega_\pm$ are included. In other words, the subsequent analysis allows for transmission problems, where the crack is contained in the interface.

For $x \in \Omega$ and $A \in \mathbb{R}^{d\times d}$ the derivative of W with respect to A is denoted by $DW(x,A) \in \mathbb{R}^{d\times d}$ with $(DW(x,A))_{ij} = \frac{\partial}{\partial A_{ij}} W(x,A)$.

(W3) For every $x \in \Omega$ it holds $W(x,\cdot) \in C^1(\mathbb{R}^{d\times d})$. Moreover, there exists a constant $c > 0$ such that for all $x, y \in \Omega_+$ or $x, y \in \Omega_-$ and all $A, B \in \mathbb{R}^{d\times d}$ it holds

$$|DW(x,A) - DW(y,B)| \leq c\big(\,|A-B| + |x-y|\,(\kappa + |A| + |B|)\big). \tag{2.5}$$

(W4) The derivative DW is strongly rank-one monotone: There exists a constant $\beta > 0$ such that for every $x \in \Omega$, $A \in \mathbb{R}^{d\times d}$, $\xi, \eta \in \mathbb{R}^d$ it holds

$$\big(DW(x, A + \xi\otimes\eta) - DW(x,A)\big) : \xi\otimes\eta \geq \beta\,|\xi|^2\,|\eta|^2\,. \tag{2.6}$$

If W is twice differentiable in A and if all A-derivatives are continuous in x, then the uniform convexity assumption (2.4) implies that a Gårding inequality is satisfied, which in turn implies that DW is rank-one monotone, see for example [Val88, Thm. 6.1].

For $f \in L^2(\Omega, \mathbb{R}^d)$ and $v \in H^1(\Omega, \mathbb{R}^d)$ we define the following functional:

$$\mathcal{E}(v) := \int_\Omega W(x, \nabla v(x))\,\mathrm{d}x - \int_\Omega f(x) \cdot v(x)\,\mathrm{d}x.$$

The convex cone of functions satisfying non-penetration conditions/Signorini conditions on the crack Γ_C is given by

$$K = \{\, v \in H^1(\Omega, \mathbb{R}^d)\,;\, [v] \cdot \mathbf{n} \geq 0 \text{ on } \Gamma_C \,\}.$$

Here, the quantity $[v](x) = v\big|_{\Omega_+}(x) - v\big|_{\Omega_-}(x)$ with $x \in \Gamma_C$ denotes the jump of v across the crack Γ_C and $\mathbf{n} = e_d$ is the unit normal vector on Γ_C. The following regularity theorem is the main result of this paper:

Theorem 2.1. *Let (G1)–(G2) and (W1)–(W4) be satisfied and $f \in L^2(\Omega, \mathbb{R}^d)$. Assume further that $u \in K$ satisfies $\mathcal{E}(u) \leq \mathcal{E}(v)$ for all $v \in K$ such that* $\operatorname{supp}(u - v) \subset \widetilde{\Omega}$*. Then for every domain $\Omega_0 \Subset \widetilde{\Omega}$ with $\Gamma_C \Subset \Omega_0$ it holds $u\big|_{\Omega_0 \cap \Omega_\pm} \in B^{\frac{3}{2}}_{2,\infty}(\Omega_0 \cap \Omega_\pm)$ and there exists a constant $c_{\Omega_0} > 0$ such that with κ from (W1)*

$$\|u\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega_0 \cap \Omega_+)} + \|u\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega_0 \cap \Omega_-)} \leq c_{\Omega_0}(\kappa + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}).$$

If in addition (2.2) *is valid for all $x, y \in \Omega$, then $u \in B^{3/2}_{2,\infty}(\Omega \cap \Omega_0)$.*

Theorem 2.1 extends the results from [EFK02, KM07] in several respects: The class of energy functionals considered in Theorem 2.1 is more general (Legendre-Hadamard condition instead of the stronger convexity assumption in [EFK02]), non-penetration conditions are included and the crack front is allowed to be a (nonsmooth) Lipschitz-continuous “curve“.

Remark 2.2. In [KM07] the authors investigate the regularity of solutions close to regions, where the boundary conditions change from Dirichlet to Neumann. They assume that locally the boundary is smooth and that the interface separating the Dirichlet and the Neumann boundary is smooth. With the arguments that we apply in the proof of Theorem 2.1 one can also treat the case, when the interface between the Dirichlet and the Neumann boundary is Lipschitz, only. In order to carry over the result from [KM07] with a smooth separation line to Lipschitz-continuous separation lines, it suffices to repeat the arguments from the proof of Theorem 2.1, where now Γ_N plays the role of Γ_C and Γ_D plays the role of $(\partial\Omega_+ \cap E_d)\backslash\Gamma_C$. One obtains again that $u \in B^{\frac{3}{2}}_{2,\infty}$ in a neighborhood of $\overline{\Gamma}_D \cap \overline{\Gamma}_N$.

Remark 2.3. Under the conditions of Theorem 2.1 it holds $u\big|_{\Omega_\pm} \in H^2_{\mathrm{loc}}(\Omega_\pm \cup \Gamma_C)$, [BGK87, Neč83].

The proof of Theorem 2.1, which is carried out in Section 2.4, relies on the estimation of finite differences of the solutions taken tangential to the contact surface and on the application of Lemma 2.4 introduced in the next section. Thanks to Remark 2.3 we already know that $u\big|_{\Omega_\pm} \in H^2_{\text{loc}}(\Omega_\pm \cup \Gamma_C)$. Hence, we only have to investigate the behavior of the function u in a neighborhood of the crack front $\partial\Gamma_C$.

2.3 A technical lemma

For $r > 0$ let $Q_r^+ := (-r, r)^{d-1} \times (0, r)$ be the d-dimensional half cube with side length $2r$. We recall that e_i, $1 \le i \le d$, denote the canonical unit vectors in $\mathbb{R}^d$. Furthermore, the following notation for finite differences is used: $\triangle_h^i w(x) = w(x + he_i) - w(x)$ for $h > 0$. Finally, the Steklov regularization operators $\mathcal{M}_h^i$ are defined as

$$\mathcal{M}_h^i(w)(x) = \frac{1}{h}\int_0^h w(x + te_i)\,\mathrm{d}t.$$

Lemma 2.4. *Let $s \in (0, 1]$, $p \in (1, \infty)$ and assume that for $w \in L^p(Q_{2R}^+)$ there exist constants $c_1, c_2 > 0$ and $0 < h_* \le R/2$ such that*

$$\sup_{\substack{1\le i\le d-1\\ 0<h<h_*}} |h|^{-ps} \int_{Q^+_{3R/2}} \left|\triangle_h^i w(x)\right|^p \,\mathrm{d}x \le c_1, \tag{2.7}$$

$$\sup_{0<h<h_*} |h|^{-ps} \int_{Q^+_{R/2}} \left|\triangle_h^d\big(\mathcal{M}_h^{d-1}\cdots\mathcal{M}_h^1(w)\big)(x)\right|^p \,\mathrm{d}x \le c_2. \tag{2.8}$$

Then there exists a constant $c_3 > 0$ such that

$$\sup_{0<h<h_*} |h|^{-ps} \int_{Q^+_{R/2}} \left|\triangle_h^d w(x)\right|^p \,\mathrm{d}x \le c_3. \tag{2.9}$$

It holds $c_3 = c_{p,d}(c_2 + c_1(1+ps)^{-1})$, where $c_{d,p}$ depends only on the space dimension d and on p.

Lemma 2.4 is a generalization of [EFK02, Lemma 1] and of [KM07, Section 1.4] from the case $p = 2$ to the case $p \in (1, \infty)$. Our proof is done by estimating directly the integrals, whereas the proofs in [EFK02] and [KM07] use Fourier techniques. Lemma 2.4 implies that w belongs to the Nikolskii or Besov space $B^s_{p,\infty}(Q^+_{R/2})$ if $s \in (0, 1)$ and to $W^{1,p}(Q^+_{R/2})$ if $s = 1$.

Proof. Direct computations show that the operators $\mathcal{M}_h^i : L^p(Q_R^+) \to L^p(Q^+_{R/2})$ are uniformly bounded with respect to $h \in (0, R/2)$.

Let $h \in (0, R/2)$ and assume that $w \in L^p(Q^+_{2R})$ satisfies (2.7) and (2.8). Let furthermore $\tau_{he_d}(x) := x + he_d$. The triangle inequality implies that

$$\begin{aligned}\big\|\triangle_h^d w\big\|_{L^p(Q^+_{R/2})} &\le \big\|\big(w - (\mathcal{M}_h^{d-1}\cdots\mathcal{M}_h^1)w\big)\circ\tau_{he_d}\big\|_{L^p(Q^+_{R/2})} + \big\|w - (\mathcal{M}_h^{d-1}\cdots\mathcal{M}_h^1)w\big\|_{L^p(Q^+_{R/2})} \\ &\quad + \big\|\triangle_h^d\big(\mathcal{M}_h^{d-1}\cdots\mathcal{M}_h^1 w\big)\big\|_{L^p(Q^+_{R/2})} \\ &\le 2\big\|w - (\mathcal{M}_h^{d-1}\cdots\mathcal{M}_h^1)w\big\|_{L^p(Q^+_{R})} + \big\|\triangle_h^d\big(\mathcal{M}_h^{d-1}\cdots\mathcal{M}_h^1 w\big)\big\|_{L^p(Q^+_{R/2})}.\end{aligned}$$

By assumption (2.8) the second term on the right hand side is bounded:

$$\left\| \triangle_h^d \big(\mathcal{M}_h^{d-1} \cdots \mathcal{M}_h^1 w \big) \right\|_{L^p(Q^+_{R/2})} \leq h^s c_2^{\frac{1}{p}}.$$

The first term can be estimated as follows using a transformation of coordinates ($h\rho_i = t_i$) and applying Hölder's inequality

$$\begin{aligned}
&\left\| w - (\mathcal{M}_h^{d-1} \cdots \mathcal{M}_h^1) w \right\|^p_{L^p(Q^+_R)} \\
&= |h|^{-p(d-1)} \int_{Q^+_R} \left| \int_{(0,h)^{d-1}} w(x) - w(x + t_1 e_1 + \ldots + t_{d-1} e_{d-1}) \, \mathrm{d}t_{d-1} \ldots \mathrm{d}t_1 \right|^p \mathrm{d}x \\
&\leq \int_{(0,1)^{d-1}} \int_{Q^+_R} \big| w(x) - w(x + h(\rho_1 e_1 + \ldots \rho_{d-1} e_{d-1})) \big|^p \, \mathrm{d}x \, \mathrm{d}\rho_{d-1} \ldots \mathrm{d}\rho_1 \\
&=: S_h.
\end{aligned}$$

Adding and subtracting the term $\sum_{l=1}^{d-2} w(x + \sum_{j=1}^{l} h\rho_j e_j)$ under the integral we arrive at

$$S_h \leq c_{d,p} \sum_{l=0}^{d-2} \int_{(0,1)^{d-1}} \int_{Q^+_R} \Big| w\big(x + \sum_{j=1}^{l} h\rho_j e_j\big) - w\big(x + \sum_{j=1}^{l+1} h\rho_j e_j\big) \Big|^p \mathrm{d}x \, \mathrm{d}\rho_{d-1} \ldots \mathrm{d}\rho_1,$$

with a constant $c_{d,p}$ depending only on d and p. Let now $l \in \{0, \ldots, d-2\}$. With condition (2.7) and taking into account that $Q^+_R + h \sum_{j=1}^{l} \rho_j e_j \subset Q^+_{3R/2}$ we conclude that

$$\begin{aligned}
&\int_{(0,1)^{d-1}} \int_{Q^+_R} \Big| w\big(x + \sum_{j=1}^{l} h\rho_j e_j\big) - w\big(x + \sum_{j=1}^{l+1} h\rho_j e_j\big) \Big|^p \mathrm{d}x \, \mathrm{d}\rho_{d-1} \ldots \mathrm{d}\rho_1 \\
&\qquad = \int_{(0,1)^{d-1}} \int_{Q^+_{R/2} + h \sum_{j=1}^{l} \rho_j e_j} |w(y) - w(y + h\rho_{l+1} e_{l+1})|^p \, \mathrm{d}y \, \mathrm{d}\rho_{d-1} \ldots \mathrm{d}\rho_1 \\
&\qquad \leq c_1 \int_0^1 |h\rho_{l+1}|^{ps} \, \mathrm{d}\rho_{l+1} = c_1 h^{ps} (1 + ps)^{-1}.
\end{aligned}$$

Collecting all estimates finishes the proof of Lemma 2.4. □

2.4 Proof of Theorem 2.1

2.4.1 Regularity in tangential direction

In the first step we prove higher differentiability in directions parallel to the crack plane E_d. Chose $\Omega_0 \Subset \widetilde{\Omega}$ with $\Gamma_C \Subset \Omega_0$. Let $x_0 \in \partial\Gamma_C$. Since Γ_C satisfies the uniform cone condition, there exist $\widetilde{R} > 0$, $h_0 > 0$ and a ($d-1$ dimensional) cone $\mathcal{C}(h_0) \subset E_d$ with the property

$$x \in Q_{\widetilde{R}}(x_0) \backslash \Gamma_C, \; h \in \mathcal{C}(h_0) \quad \Longrightarrow \quad x + h \in \Omega. \tag{2.10}$$

Since by assumption $\Gamma_C \Subset \Omega_0$, we may further choose $R < \widetilde{R}/2$ such that the cube $Q_{2R}(x_0) \backslash \Gamma_C$ is contained in Ω_0 and that, possibly after a rotation of the whole domain $\widetilde{\Omega}$ with respect to the

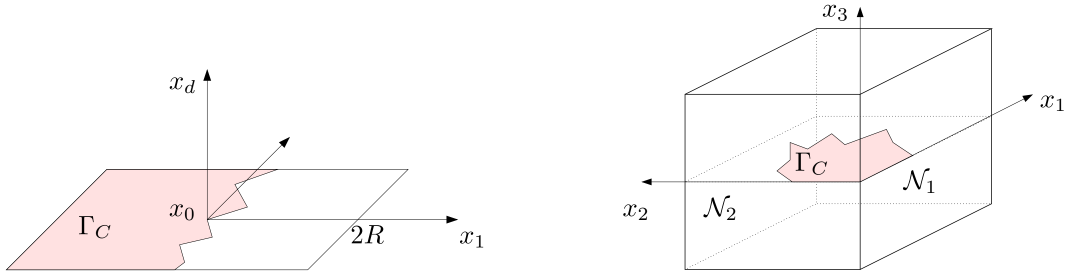


Figure 1: Left: Model problem in the proof of Theorem 2.1; Right: Crack geometry according to Section 3.1.

x_d-axis, it holds

$$\partial\Gamma_C \cap Q_{2R}(x_0) \;\subset\; x_0 + (-R,R)\times(-2R,2R)^{d-2}\times\{0\} =: U_R(x_0), \tag{2.11}$$

see Figure 1.

We next define special inner variations (transformations on $\mathbb{R}^d$) which then are used in the difference quotient argument. For this purpose we choose a cut-off function $\theta \in C_0^\infty(Q_{2R}(x_0))$, $0 \le \theta \le 1$, which satisfies $\theta(x) = 1$ for all $x \in Q_{15R/8}(x_0)$. For $h \in \mathcal{C}(h_0)$ we define

$$T_h : \mathbb{R}^d \to \mathbb{R}^d, \quad T_h(x) = x + \theta(x)h.$$

There exists a constant $0 < h_1 < R/8$ such that for all $h \in \mathcal{C}(h_1) := \mathcal{C}(h_0) \cap B_{h_1}^{d-1}(0)$ the mappings T_h are diffeomorphisms on $\mathbb{R}^d$ with $T_h(E_d) = E_d$ and $T_h(x) = x$ for $x \in \mathbb{R}^d \backslash Q_{2R}(x_0)$, see e.g. [GH96]. Observe that on $Q_{15R/8}(x_0)$ the diffeomorphisms act like local translations defined by the vector h. Furthermore, thanks to (2.10), for $x \in \Omega$ we have $T_h(x) \in \Omega$. Hence, for the cone of admissible displacements we obtain

$$v \in K,\; h \in \mathcal{C}(h_1) \quad \Rightarrow \quad v \circ T_h \in K$$

and there exists a constant $c > 0$ such that $\sup_{h\in\mathcal{C}(h_1)} \|v \circ T_h\|_{H^1(\Omega)} \le c\,\|v\|_{H^1(\Omega)}$, [GH96].

Lemma 2.5. *Assume that (G1)–(G2) and (W1)–(W2) are satisfied, $f \in L^2(\Omega)$ and let $u \in K$ satisfy $\mathcal{E}(u) \le \mathcal{E}(v)$ for all $v \in K$ with* $\operatorname{supp}(u - v) \subset \widetilde{\Omega}$. *Then there exist constants $c > 0$ and $h_* > 0$ such that with κ from (W1)*

$$\sup_{1\le i\le d-1}\;\sup_{0<h<h_*} h^{-\frac{1}{2}}\,\|u(\cdot + he_i) - u(\cdot)\|_{H^1(Q_{3R/2}(x_0)\backslash\Gamma_C)} \le c(\kappa + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}), \tag{2.12}$$

$$\sup_{0<h<h_*} |h|^{-\frac{1}{2}}\left\|\widetilde{\nabla} u(\cdot \pm he_d) - \widetilde{\nabla} u(\cdot)\right\|_{L^2(Q^\pm_{R/2})} \le c(\kappa + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}). \tag{2.13}$$

The constant c depends on $x_0 \in \partial\Gamma_C$, the geometry (i.e. the Lipschitz constant) of Γ_C and the chosen cut-off function θ. Furthermore, $\widetilde{\nabla} u$ is defined as $(\partial_1 u, \dots, \partial_{d-1} u, 0)$. Observe that for Lemma 2.5 also non-differentiable energies $\mathcal{E}$ are admissible.

Proof. Let $u \in K$ be given according to Lemma 2.5. From the uniform convexity assumption (W2) it follows for all $v \in K$ such that $\mathrm{supp}(u-v) \subset \widetilde{\Omega}$ and all $\lambda \in [0,1]$:

$$\mathcal{E}(u) \leq \mathcal{E}(v_\lambda) \leq \lambda \mathcal{E}(u) + (1-\lambda)\mathcal{E}(v) - \frac{\alpha}{2}\lambda(1-\lambda) \left\| u - v \right\|^2_{H^1(\Omega)},$$

where $v_\lambda := \lambda u + (1-\lambda)v \in K$. Subtracting $\mathcal{E}(u)$ from both sides, dividing by $1-\lambda$, one obtains for $\lambda \to 1$ that

$$\frac{\alpha}{2} \left\| u - v \right\|^2_{H^1(\Omega)} \leq \mathcal{E}(v) - \mathcal{E}(u). \tag{2.14}$$

Let now $(T_h)_{h \in \mathcal{C}(h_1)}$ be the above introduced family of diffeomorphisms. With $v = u \circ T_h$ one obtains

$$\frac{\alpha}{2} \left\| u \circ T_h - u \right\|^2_{H^1(\Omega)} \leq \int_\Omega W(x, \nabla(u \circ T_h)) - W(x, \nabla u) \,\mathrm{d}x - \int_\Omega f \cdot (u \circ T_h - u) \,\mathrm{d}x =: S_1 + S_2. \tag{2.15}$$

The next goal is to show that the right hand side is bounded by $c\,|h|\,(\kappa + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})^2$. Since $u \in H^1(\Omega)$, the last term can be estimated with

$$|S_2| \leq c_\theta \,|h|\, \|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)} \tag{2.16}$$

with a constant c_θ depending on $\|\theta\|_{C^1(\mathbb{R}^d)}$. Furthermore, let $q_h(y) := \left|\det \nabla T_h^{-1}(y)\right|$. Applying the transformation $y = T_h(x)$ to the first term in S_1 one finds

$$\begin{aligned} S_1 &= \int_{T_h(\Omega)} q_h W(T_h^{-1}(y), \nabla u \nabla T_h^{-1}) \,\mathrm{d}y - \int_\Omega W(x, \nabla u) \,\mathrm{d}x \\ &= \int_{T_h(\Omega)} q_h W(T_h^{-1}(y), \nabla u \nabla T_h^{-1}) - W(y, \nabla u \nabla T_h^{-1}) \,\mathrm{d}y \\ &\quad + \int_{T_h(\Omega)} W(y, \nabla u \nabla T_h^{-1}) - W(y, \nabla u) \,\mathrm{d}y - \int_{\Omega \setminus T_h(\Omega)} W(y, \nabla u) \,\mathrm{d}y. \end{aligned} \tag{2.17}$$

Due to (W1), the last term is not positive. As it is shown for instance in [GH96], there exists a constant $c_\theta > 0$ depending on $\|\theta\|_{C^1(\mathbb{R}^d)}$ such that $\|q_h - 1\|_{L^\infty(\mathbb{R}^d)} \leq c_\theta\,|h|$ and $\left\|\nabla T_h^{-1} - \mathbb{I}\right\|_{L^\infty(\mathbb{R}^d)} \leq c_\theta\,|h|$. Hence, taking into account assumption (W1) it follows that

$$S_1 \leq c_\theta \,|h|\, (\kappa + \|u\|^2_{H^1(\Omega)}).$$

Collecting all estimates we have shown that

$$\sup_{h \in \mathcal{C}(h_1)} |h|^{-\frac{1}{2}} \left\| u(\cdot + h) - u(\cdot) \right\|_{H^1(Q_{\frac{15}{8}R}(x_0) \setminus \Gamma_C)} \leq c \left(\kappa + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right). \tag{2.18}$$

From this we deduce that

$$\begin{aligned} \sup_{h \in \mathcal{C}(h_1)} |h|^{-\frac{1}{2}} \left\| u(\cdot - h) - u(\cdot) \right\|_{H^1(Q_{\frac{7}{4}R}(x_0) \setminus \Gamma_C)} &\leq \sup_{h \in \mathcal{C}(h_1)} |h|^{-\frac{1}{2}} \left\| u(\cdot + h) - u(\cdot) \right\|_{H^1(Q_{\frac{15}{8}R}(x_0) \setminus \Gamma_C)} \\ &\leq c \left(\kappa + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right). \end{aligned} \tag{2.19}$$

Next, we prove (2.12) for the standard basis $\{e_1,\dots,e_{d-1}\}$. Let $\{b_1,\dots,b_{d-1}\}$ be a basis of E_d with $|b_i|=1$ and $b_i/(2h_1)\in\mathcal{C}(h_1)$. Clearly, there exist constants $\alpha_{ij}\in\mathbb{R}$ such that the canonical basis $\{e_1,\dots,e_{d-1}\}$ of E_d can be represented as $e_i=\sum_{j=1}^{d-1}\alpha_{ij}b_j$. Choose $h_*=h_1\left(\sup_{1\le i\le d-1}\sum_{j=1}^{d-1}|\alpha_{ij}|\right)^{-1}$. Observe that for $|h|<h_*$ and $1\le l\le d-1$ it holds that $Q_{\frac{3R}{2}}(x_0)+\sum_{j=1}^{l-1}h\alpha_{ij}b_j\subset Q_{7R/4}(x_0)$ and that $h\alpha_{il}b_l\in\mathcal{C}(h_*)\cup-\mathcal{C}(h_*)$. Hence, by (2.18) and (2.19) we find

$$\begin{aligned}\left\|\triangle_h^i u\right\|_{H^1(Q_{\frac{3R}{2}}(x_0)\setminus\Gamma_C)} &\le \sum_{l=1}^{d-1}\left\|u(\cdot+h\sum_{j=1}^{l}\alpha_{ij}b_j)-u(\cdot+h\sum_{j=1}^{l-1}\alpha_{ij}b_j)\right\|_{H^1(Q_{\frac{3R}{2}}(x_0)\setminus\Gamma_C)}\\ &\le \sum_{l=1}^{d-1} c\left(\kappa+\|f\|_{L^2(\Omega)}+\|u\|_{H^1(\Omega)}\right)|h|^{-\frac12}\,|\alpha_{il}|^{\frac12},\end{aligned}$$

where the finite difference $\triangle_h^i$ is taken with respect to the basis vector e_i. This implies (2.12).

Estimate (2.13) is a straightforward application of Lemma 2.4 based on estimate (2.12) and the identity $\triangle_h^d\mathcal{M}_h^{d-1}\cdots\mathcal{M}_h^1\partial_1 u=\triangle_h^1\mathcal{M}_h^d\cdots\mathcal{M}_h^2\partial_d u$ (and similar for the other indices $i\in\{1,\dots,d-1\}$). □

2.4.2 Regularity in normal direction

We now prove the higher differentiability in normal direction. For $x_0\in\partial\Gamma_C$ let Q_{2R} be the cube introduced in Section 2.4.1.

Lemma 2.6. *Assume that (G1)–(G2) and (W1)–(W4) are satisfied, $f\in L^2(\Omega)$ and let $u\in K$ satisfy $\mathcal{E}(u)\le\mathcal{E}(v)$ for all $v\in K$ with $\operatorname{supp}(u-v)\subset\widetilde{\Omega}$. Then there exist constants $c>0$ and $h_*>0$ such that*

$$\sup_{0<h<h_*} h^{\frac12}\left\|u(\cdot+he_d)-u\right\|_{H^1(Q_{R/2}(x_0)\cap\Omega_+)}\le c(\kappa+\|f\|_{L^2(\Omega)}+\|u\|_{H^1(\Omega)})$$

and similar on $Q_{R/2}(x_0)\cap\Omega_-$.

Again, the constants c and h_* depend on $x_0\in\partial\Gamma_C$, the geometry of $\partial\Gamma_C$ and the chosen cut-off function θ. The proof of Lemma 2.6 relies on the technical Lemma 2.3.

Proof. Thanks to (W3) the functional $\mathcal{E}$ is Gâteaux-differentiable and u satisfies the variational inequality

$$0\le D_u\mathcal{E}(u)[v-u]=\int_\Omega DW(x,\nabla u):\nabla(v-u)\,\mathrm{d}x-\int_\Omega f\cdot(v-u)\,\mathrm{d}x$$

for all $v\in K$ with $\operatorname{supp}(u-v)\subset\widetilde{\Omega}$. By choosing $v=u\pm w$ with $w\in H_0^1(\Omega_0\setminus\overline{\Gamma}_C)$ (in particular, $w\big|_{\Gamma_C}=0$), where Ω_0 is a set according to Theorem 2.1, we find that

$$\int_{\Omega_0\setminus\Gamma_C} DW(x,\nabla u):\nabla w\,\mathrm{d}x=\int_{\Omega_0\setminus\Gamma_C} f\cdot w\,\mathrm{d}x \tag{2.20}$$

for all $w \in H^1_0(\Omega_0\backslash\overline{\Gamma}_C)$. Hence, standard regularity theory for strongly monotone elliptic differential operators, see e.g. [Neč83], shows that $u \in H^2_{\mathrm{loc}}(\Omega_0)$. From equation (2.20) it follows that

$$\partial_d DW_d(x, \nabla u) = -f - \sum_{i=1}^{d-1} \partial_i DW_i(x, \nabla u). \tag{2.21}$$

Here, $DW_i(x, \nabla u) \in \mathbb{R}^d$ denotes the i-th column of $DW(x, \nabla u)$.

The next estimates are carried out on the domain $\Omega_+ \cap Q_{R/2} =: Q^+_{R/2}$. We will show by applying again Lemma 2.4 that for $0 < h < h_*$ it holds

$$\left\| \triangle^d_h DW_d(\cdot, \nabla u) \right\|_{L^2(Q^+_{R/2})} \leq c h^{\frac{1}{2}} (\kappa + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}). \tag{2.22}$$

Indeed, due to the Lipschitz continuity of DW, cf. (W3), together with estimate (2.12) the function $w := DW_d(\cdot, \nabla u)$ satisfies condition (2.7) with $c_1 \leq c(\kappa + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})^2$. In order to verify condition (2.8) observe that using the identity $\triangle^i_h \xi = h\mathcal{M}^i_h \partial_i \xi$ and (2.21) it holds

$$\begin{aligned} \triangle^d_h \mathcal{M}^{d-1} \dots \mathcal{M}^1_h DW_d(\cdot, \nabla u) &= h\mathcal{M}^d_h \dots \mathcal{M}^1_h \partial_d DW_d(\cdot, \nabla u) \\ &= -h\mathcal{M}^d_h \dots \mathcal{M}^1_h \big(f + \sum_{i=1}^{d-1} \partial_i DW_i(\cdot, \nabla u) \big). \end{aligned}$$

Since the operators $\mathcal{M}^i_h$ are uniformly bounded it follows for $0 < h < h_*$ that

$$\left\| h\mathcal{M}^d_h \dots \mathcal{M}^1_h f \right\|_{L^2(Q^+_{R/2})} \leq ch \|f\|_{L^2(\Omega)}.$$

Moreover, using again the identity $h\mathcal{M}^i_h \partial_i \xi = \triangle^i_h \xi$, condition (W3) and estimate (2.12), it follows that for $1 \leq i \leq d-1$ we have

$$\left\| h\mathcal{M}^d_h \dots \mathcal{M}^1_h \partial_i DW_i(\cdot, \nabla u) \right\|_{L^2(Q^+_{R/2})} \leq c h^{\frac{1}{2}} (\kappa + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}).$$

Collecting the estimates shows that $w = DW_d(\cdot, \nabla u)$ satisfies condition (2.8). Hence, Lemma 2.4 implies (2.22). The strong rank-one monotonicity of DW, see assumption (W4), now yields

$$\begin{aligned} \beta &\left\| \triangle^d_h \partial_d u \right\|^2_{L^2(Q^+_{R/2})} \\ &\leq \int_{Q^+_{R/2}} \big(DW(x, \widetilde{\nabla} u + \partial_d u(x + he_d) \otimes e_d) - DW(x, \nabla u) \big) : \triangle^d_h \partial_d u \otimes e_d \, \mathrm{d}x \\ &= \int_{Q^+_{R/2}} \triangle^d_h DW_d(\cdot, \nabla u) \cdot \triangle^d_h \partial_d u \, \mathrm{d}x \\ &\quad + \int_{Q^+_{R/2}} \big(DW_d(x, \widetilde{\nabla} u + \partial_d u(x + he_d) \otimes e_d) - DW_d(x + he_d, \nabla u(x + he_d)) \big) \cdot \triangle^d_h \partial_d u \, \mathrm{d}x. \end{aligned} \tag{2.23}$$

Taking into account (2.22), the Lipschitz continuity of DW and estimates (2.12)–(2.13) one finally obtains that

$$\beta \left\| \triangle_h^d \partial_d u \right\|^2_{L^2(Q^+_{R/2})} \leq ch^{\frac{1}{2}} (\kappa + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}) \left\| \triangle_h^d \partial_d u \right\|_{L^2(Q^+_{R/2})},$$

which finishes the proof of Lemma 2.6. $\square$

Proof of Theorem 2.1. Covering the crack front $\partial\Gamma_C$ by finitely many of the cubes discussed in Lemma 2.5 and Lemma 2.6 the claim of Theorem 2.1 follows. If in addition (2.2) is valid for all $x, y \in \Omega$, then the considerations of Lemma 2.6 can also be carried out on the set $\{ x = (x', x_d) \in Q_{R/2} \,;\, |x_d| < \frac{R}{2},\, (x', 0) = y + h, \text{ where } y \in \partial\Gamma_C \cap Q_{R/2},\, h \in \mathcal{C}(h_0) \}$, and $\mathcal{C}(h_0)$ is the flat cone from (2.10). $\square$

3 Examples and extensions

3.1 Linear elastic body with crack and self-contact

In this example we extend the result from the previous section to the situation, where a plane crack intersects with the exterior boundary $\partial\widetilde{\Omega}$. Thereby we restrict ourself to a simple model problem. For $L > 0$ let $\widetilde{\Omega} = (0, 2L)^2 \times (-L, L) \subset \mathbb{R}^3$ be a cube. Let furthermore $\ell_1, \ell_2 \in (0, 2L)$ and let $\widetilde{\Gamma}_C \subset E_3$ satisfy the uniform cone condition with respect to E_3. Assume finally that $\partial\widetilde{\Gamma}_C \cap \partial\widetilde{\Omega}$ coincides with the segments between $(\ell_1, 0, 0)^\top$ and 0 and between 0 and $(0, \ell_2, 0)^\top$. We define $\Omega := \widetilde{\Omega}\backslash\overline{\widetilde{\Gamma}_C}$ to be the cracked domain with crack $\Gamma_C = \widetilde{\Omega}\backslash\Omega$, see Fig. 1. Assume that Neumann conditions are prescribed on the faces $\mathcal{N}_1 := (0, 2L) \times \{0\} \times (-L, L)$ and $\mathcal{N}_2 := \{0\} \times (0, 2L) \times (-L, L)$ and that the Dirichlet boundary $\Gamma_D \subset \partial\Omega$ has positive measure. We define

$$\mathcal{K} = \{ v \in H^1(\Omega, \mathbb{R}^3) \,;\, [v]e_3 \geq 0 \text{ on } \Gamma_C,\, v\big|_{\Gamma_D} = 0 \}. \tag{3.1}$$

Let $\mathbf{C} \in C^{\mathrm{Lip}}(\Omega; \mathrm{Lin}(\mathbb{R}^{3\times3}, \mathbb{R}^{3\times3}))$ denote the elasticity tensor with the usual symmetry and positivity properties, i.e. for all $\xi, \eta \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ and $x \in \Omega$ it holds $\mathbf{C}(x)\xi : \eta = \mathbf{C}(x)\eta : \xi$ and $\mathbf{C}(x)\xi : \xi \geq \alpha |\xi|^2$ with some positive constant α. For given $f \in L^2(\Omega, \mathbb{R}^3)$ we study the regularity properties of the minimizer of the problem

$$u = \operatorname{argmin}\{ \mathcal{E}(v) \,;\, v \in \mathcal{K} \}, \tag{3.2}$$

$$\mathcal{E}(v) = \int_\Omega \frac{1}{2} \mathbf{C}\varepsilon(v) : \varepsilon(v) \,\mathrm{d}x - \int_\Omega f \cdot v \,\mathrm{d}x. \tag{3.3}$$

Here, $\varepsilon(v) = \frac{1}{2}(\nabla v + (\nabla v)^\top)$ denotes the symmetrized strain tensor. For $\rho > 0$ let $\Omega_\rho := \{ x \in \Omega \,;\, \mathrm{dist}(x, \Gamma_C) < \rho) \}$.

Proposition 3.1. *For every $\rho > 0$ such that $\overline{\Omega}_\rho \cap (\partial\widetilde{\Omega} \backslash \overline{\mathcal{N}_1 \cup \mathcal{N}_2}) = \emptyset$ there exists a constant $c_\rho > 0$ such that for all $f \in L^2(\Omega, \mathbb{R}^3)$ the corresponding minimizer u of $\mathcal{E}$ satisfies*

$$u \in B^{\frac{3}{2}}_{2,\infty}(\Omega_\rho), \quad \|u\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega_\rho)} \le c_\rho \|f\|_{L^2(\Omega)} .$$

Proof. Observe first that $u \in H^2_{\mathrm{loc}}(\Omega)$. Moreover, let $x_0 \in \mathrm{int}(\Gamma_C)$ (relative to E_3). Then there exists $R > 0$ such that $u\big|_{Q^\pm_R(x_0)} \in H^2(Q^\pm_R(x_0))$, cf. [BGK87]. Hence, it suffices to investigate the regularity of u in a neighborhood of $\partial\Gamma_C$. Here, we distinguish two cases.

If $x_0 \in \partial\Gamma_C \backslash \overline{\mathcal{N}_1 \cup \mathcal{N}_2}$, then there exists $R > 0$ such that $Q_R(x_0) \Subset \widetilde{\Omega}$. Hence, Theorem 2.1 can be applied showing that $u\big|_{\Omega \cap Q_R(x_0)} \in B^{\frac{3}{2}}_{2,\infty}(\Omega \cap Q_R(x_0))$.

If $x_0 \in \overline{\mathcal{N}_1 \cup \mathcal{N}_2} \cap \partial\Gamma_C$, then due to the assumed uniform cone property of $\widetilde{\Gamma}_C$ and the exterior geometry of $\widetilde{\Omega}$, there exist $R > 0$, $h_0 > 0$ and a flat cone $\mathcal{C}(h_0) \subset E_3$ such that it holds

$$x \in Q_{2R}(x_0) \cap \Omega,\ h \in \mathcal{C}(h_0) \ \Rightarrow\ x + h \in \Omega.$$

As in the proof of Lemma 2.5 we now construct the mappings $T_h(x) = x + \theta(x)h$ for $h \in \mathcal{C}(h_0)$ and a suitable cut-off function $\theta \in C^\infty_0(Q_{2R}(x_0))$, which are diffeomorphisms for $h \in \mathcal{C}(h_1)$, if $h_1 \le h_0$ is small enough. Observe that it holds $v \in \mathcal{K} \Rightarrow v \circ T_h \in \mathcal{K}$ for $h \in \mathcal{C}(h_1)$. Hence, as in the proof of Lemma 2.5 we may use $u \circ T_h$ as test function for the minimization problem (3.2) and find, in analogy to (2.14)–(2.17), that

$$\frac{\alpha}{2} \|u \circ T_h - u\|^2_{H^1(\Omega)} \le c_\theta\, |h|\, (\|f\|_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)}).$$

Arguing now analogously to Lemma 2.6 we finally deduce that $u\big|_{\Omega \cap Q_{R/2}(x_0)} \in B^{3/2}_{2,\infty}(\Omega \cap Q_{R/2}(x_0))$. Combining the above considerations for the different positions of x_0 finishes the proof of Proposition 3.1. □

3.2 Contact with a rigid foundation

In this section we apply the regularity techniques from Section 2 to derive regularity results for contact problems with a rigid foundation. Again, we restrict ourself to a model problem. As before, for $L > 0$ let $\Omega = (-L, L)^{d-1} \times (0, L) \subset \mathbb{R}^d$ be a cuboid. It is assumed that at one part Γ_C of $\Gamma = (-L, L)^{d-1} \times \{0\}$, the body can be in contact with a rigid foundation, whereas on the remaining part of Γ Neumann boundary conditions shall be imposed: $\overline{\Gamma} = \overline{\Gamma_C} \cup \overline{\Gamma_N}$. (The case $\Gamma = \Gamma_C$ can be treated in a similar way). It is assumed that $\Gamma_C \Subset \Gamma$ and satisfies the uniform cone condition with respect to E_d. The obstacle is described by the graph of a function $g : \Gamma_C \to \mathbb{R}$. Finally it is assumed that the Dirichlet boundary $\Gamma_D \subset \partial\Omega$ has positive measure.

3.2.1 Frictionless contact

For modeling frictionless contact with a rigid foundation the convex cone of admissible displacement fields is given by

$$\mathcal{K} = \{\, v \in H^1(\Omega; \mathbb{R}^d)\,;\, v(x) \cdot e_d \ge g(x) \text{ a.e. on } \Gamma_C,\ v\big|_{\Gamma_D} = 0 \,\}. \tag{3.4}$$

The inequality is to be understood in the sense of traces. Given $f \in L^2(\Omega, \mathbb{R}^d)$ we investigate the regularity of minimizers of the problem

$$u = \operatorname{argmin}\{\, \mathcal{E}(v)\, ;\, v \in \mathcal{K}\,\}, \tag{3.5}$$

$$\mathcal{E}(v) = \int_\Omega W(\nabla v)\,\mathrm{d}x - \int_\Omega f \cdot v \,\mathrm{d}x, \tag{3.6}$$

with an energy density $W : \mathbb{R}^{3\times3} \to \mathbb{R}$ satisfying (W1)–(W4). The following regularity property is assumed on the function g: Let $g_\infty : \Gamma \to \mathbb{R} \cup \{-\infty\}$ be defined by $g_\infty(x) = g(x)$ if $x \in \Gamma_C$ and $g_\infty(x) = -\infty$ if $x \in \Gamma\backslash\Gamma_C$. Then

(R1) For every $x_0 \in \Gamma$ exist constants $R > 0$, $h_0 > 0$ and a flat cone $\mathcal{C}(h_0) \subset E_d$ such that $Q^{d-1}_{2R}(x_0) \Subset \Gamma$ and that the following implication is valid:

$$x \in \Gamma \cap Q^{d-1}_{2R}(x_0),\ h \in \mathcal{C}(h_0) \ \Rightarrow\ x + h \in \Gamma \text{ and } g_\infty(x+h) \geq g_\infty(x).$$

Observe that we do not require that g is continuous.

As in the previous section, for $\rho > 0$ we define the set $\Omega_\rho = \{\, x \in \Omega\, ;\, \operatorname{dist}(x, \Gamma_C) < \rho\,\}$.

Proposition 3.2. *Assume that (W1)–(W4) and (R1) are valid and that $\mathcal{K} \neq \emptyset$. Then, for every $\rho > 0$ such that $\overline{\Omega_\rho} \cap \partial\Omega\backslash\Gamma = \emptyset$ there exists a constant $c_\rho > 0$ such that for all $f \in L^2(\Omega)$ the corresponding minimizer $u \in \mathcal{K}$ satisfies $u \in B^{3/2}_{2,\infty}(\Omega_\rho)$ and $\|u\|_{B^{3/2}_{2,\infty}(\Omega_\rho)} \leq c_\rho(\kappa + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})$.*

Proof. Let $x_0 \in \Gamma$ and $R, h_0 > 0$ be given according to condition (R1). Let $\theta \in C_0^\infty(Q_{2R})$ be a cut-off function with $\theta\big|_{Q_R(x_0)} = 1$. As in the proof of Lemma 2.5 for small enough $h_1 \leq h_0$ and $h \in \mathcal{C}(h_1)$ the mappings $T_h : \mathbb{R}^d \to \mathbb{R}^d$, $T_h(x) = x + \theta(x)h$ are diffeomorphisms. Due to (R1) it holds $v \in \mathcal{K} \Rightarrow v \circ T_h \in \mathcal{K}$. Hence, the same arguments as in the proof of Lemma 2.5 and Lemma 2.6 yield Proposition 3.2. □

Example 3.3. Let $\Gamma_C \Subset \Gamma$ satisfy the uniform cone condition and choose $g(x) = \mathit{const}$ for $x \in \Gamma_C$. Then condition (R1) is satisfied and Proposition 3.2 is applicable.

Alternatively, let $\Gamma_C = B^d_{L/2}(0) \cap \Gamma$ and define $g(x) := \widetilde{g}(|x|)$, where $\widetilde{g} : [0, L/2) \to \mathbb{R}$ is a non-increasing function with $\widetilde{g}(r) = \mathit{const}$ for $0 \leq r \leq \delta < R/2$ with some $\delta > 0$. In this case, (R1) is satisfied as well, and Proposition 3.2 is applicable. Observe that we do not require that $\widetilde{g}$ is continuous.

If the obstacle is described by the graph of an $H^{1/2+\mu}$-smooth function, then the following holds:

Proposition 3.4. *Assume (W1)–(W4). Let furthermore $g \in H^{1/2+\mu}(\Gamma)$ for some $\mu \in (0, 1]$ and assume that $\Gamma_C \Subset \Gamma$ satisfies the uniform cone condition. Then, for every $\rho > 0$ such that $\overline{\Omega_\rho} \cap \partial\Omega\backslash\Gamma = \emptyset$ there exists a constant $c_\rho > 0$ such that for all $f \in L^2(\Omega)$ the corresponding minimizer $u \in \mathcal{K}$ satisfies $u \in B^{1+\mu/2}_{2,\infty}(\Omega_\rho)$ and $\|u\|_{B^{1+\mu/2}_{2,\infty}(\Omega_\rho)} \leq c_\rho(\kappa + \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} + \|g\|_{H^{1/2+\mu}(\Gamma)})$.*

Remark 3.5. For $\rho,\delta>0$ let $\Omega_{\rho,\delta}=\{\,x\in\Omega_\rho\,;\,\mathrm{dist}(x,\partial\Gamma_C)>\delta\,\}$. Combining regularity results for solutions of elliptic systems with respect to smooth Neumann boundaries with regularity results for contact problems with smooth obstacles, [Kin81], the following regularity is valid: Let $g\in C^{2,1}(\overline{\Gamma})$. Then, for every $\rho,\delta>0$ it holds $u\big|_{\Omega_{\rho,\delta}}\in H^2(\Omega_{\rho,\delta})$. Proposition 3.4 here above in particular treats the regularity of u in a neighborhood of the boundary between the set Γ_C and the surrounding Neumann part and thus completes the results from [Kin81].

Proof of Proposition 3.4. We investigate the regularity of the minimizer in a neighborhood of a point $x_0\in\partial\Gamma_C$. The other cases are covered by the previous remark.

Let $x_0\in\partial\Gamma_C$. The proof consists in constructing a suitable test function v_h so that arguments similar to the proof of Lemma 2.5 can be applied. Due to the uniform cone condition there exist $R>0$, $h_0>0$ and a flat cone $\mathcal{C}(h_0)$ such that for all $x\in Q_{2R}(x_0)\cap\Gamma_C$ and all $h\in\mathcal{C}(h_0)$ it holds $x+h\in\Gamma_C$. Let $\theta\in C_0^\infty(Q_{2R}(x_0))$ be a cut-off function with $\theta(x)=1$ on $Q_R(x_0)$. There exists $h_1\le h_0$ such that for all $h\in\mathcal{C}(h_1)\subset\mathcal{C}(h_0)$ the mappings $T_h:\mathbb{R}^d\to\mathbb{R}^d$ are diffeomorphisms with $x\in\Gamma_C\Rightarrow T_h(x)\in\Gamma_C$. Let $\widetilde{g}\in H^{1+\mu}(\Omega)$ such that $\widetilde{g}\big|_\Gamma=g$ and $\widetilde{g}\big|_{\Gamma_D}=0$. For $h\in\mathcal{C}(h_1)$ we define $v_h=u\circ T_h+\widetilde{g}-\widetilde{g}\circ T_h$. Obviously, $v_h\big|_{\Gamma_D}=0$. Moreover, for $x\in\Gamma_C$ it holds $v_h(x)\cdot e_d\ge g(x)$, and hence, $v_h\in\mathcal{K}$. Choosing $v=v_h$ in (2.14) yields after rearranging the terms (as in (2.15)–(2.17) and using that $T_h(\Omega)=\Omega$))

$$
\begin{aligned}
&\frac{\alpha}{2}\left\|u-u\circ T_h\right\|^2_{H^1(\Omega)}\le c\left\|\widetilde{g}-\widetilde{g}\circ T_h\right\|^2_{H^1(\Omega)}+\mathcal{E}(v_h)-\mathcal{E}(u)\\
&\quad=\int_{T_h(\Omega)}q_hW(T_h^{-1}(y),\nabla u\nabla T_h^{-1}+(\nabla\widetilde{g})\circ T_h^{-1}-\nabla\widetilde{g}\nabla T_h^{-1})-W(y,\nabla u)\,\mathrm{d}y\\
&\quad\quad-\int_\Omega f\cdot(u\circ T_h-u+\widetilde{g}-\widetilde{g}\circ T_h)\,\mathrm{d}x\\
&\quad\quad+c\left\|\widetilde{g}-\widetilde{g}\circ T_h\right\|^2_{H^1(\Omega)}\\
&\quad=S_1+S_2+S_3.
\end{aligned}
$$

As in (2.16) it holds $|S_2|\le c\,|h|\,\|f\|_{L^2(\Omega)}\,(\|u\|_{H^1(\Omega)}+\|\widetilde{g}\|_{H^1(\Omega)})$. From the regularity assumption on g we conclude that $|S_3|\le c\,|h|^{2\mu}\,\|\widetilde{g}\|^2_{H^{1+\mu}(\Omega)}$. The term S_1 can be treated in the same way as in (2.17) using the regularity of $\widetilde{g}$ and taking into account that (W3) implies the estimate $|W(x,A)-W(x,B)|\le c(\kappa+|A|+|B|)\,|A-B|$:

$$
|S_1|\le c(|h|+|h|^\mu)(\kappa+\|u\|^2_{H^1(\Omega)}+\|\widetilde{g}\|^2_{H^{1+\mu}(\Omega)}).
$$

Hence, altogether for $h\in\mathcal{C}(h_1)$ and h_1 small enough we find

$$
\|u\circ T_h-u\|_{H^1(\Omega)}\le c\,|h|^{\mu/2}\,(\kappa+\|f\|_{L^2(\Omega)}+\|u\|_{H^1(\Omega)}+\|\widetilde{g}\|_{H^{1+\mu}(\Omega)}).
$$

We may now proceed using the arguments in the proof of Lemma 2.6 in order to finally obtain Proposition 3.4. □

3.2.2 Tresca friction

The above described techniques are also applicable to derive higher regularity results for models with Tresca friction. Again, we study a model problem for a cuboid $\Omega \subset \mathbb{R}^d$ with $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C$ as described at the beginning of Section 3.2. On the boundary part Γ_C the Tresca friction law will be imposed. As in [ACF02, HMW05], a nonseparation condition on Γ_C is assumed leading to the following space of admissible displacement fields

$$V := \{ u \in H^1(\Omega;\mathbb{R}^d) \,;\, u\big|_{\Gamma_D} = 0,\ u \cdot e_d\big|_{\Gamma_C} = 0 \}. \tag{3.7}$$

In [ACF02] an evolutionary Tresca friction model is set-up in the framework of rate independent processes. It is based on the following energy functional $\mathcal{E}$ and dissipation potential $\mathcal{R}$:

$$\mathcal{E}(t,v) := \int_\Omega W(\nabla v) - f(t)\cdot v \,\mathrm{d}x, \quad W(\nabla v) := \frac{1}{2}\mathbf{C}\varepsilon(v) : \varepsilon(v),$$

where $f \in H^1(0,T;L^2(\Omega,\mathbb{R}^d))$ is a given time dependent volume force density, and

$$\mathcal{R}(v) := \int_{\Gamma_C} \kappa(x)\,|v_\tau|\,\mathrm{d}s_x.$$

Here, $\kappa \in L^\infty(\Omega)$ with $\kappa \geq \kappa_0 > 0$ is the friction coefficient and $v_\tau = v - (v\cdot e_d)e_d$ denotes the tangential part of v. For fixed $w \in V$ and $t \in [0,T]$ let $\mathcal{F}_w(t,v) := \mathcal{E}(t,v) + \mathcal{R}(w - v)$. The so-called set of stable states, cf. [Mie05], is defined as $\mathcal{S}(t) := \{ w \in V \,;\, \mathcal{F}_w(t,w) \leq \mathcal{F}_w(t,v) \text{ for all } v \in V \}$. On the basis of [Mie05, Ch. 2], the Tresca-type evolution law from [ACF02] can equivalently be reformulated as follows: Given $f \in H^1(0,T;L^2(\Omega,\mathbb{R}^d))$ and $u_0 \in V \cap \mathcal{S}(0)$ find $u \in H^1(0,T;V)$ with $u(0) = u_0$ such that $u(t) \in \mathcal{S}(t)$ for all t and

$$\mathcal{E}(t_1,u(t_1)) + \int_{t_0}^{t_1} \mathcal{R}(\partial_t u(\tau))\mathrm{d}\tau = \mathcal{E}(t_0,u(t_0)) + \int_{t_0}^{t_1}\int_\Omega -\partial_t f(\tau)\cdot u(\tau)\,\mathrm{d}x\mathrm{d}\tau$$

for all $t_0, t_1 \in [0,T]$. According to [ACF02] for every $f \in H^1(0,T;L^2(\Omega,\mathbb{R}^d))$ and $u_0 \in V \cap \mathcal{S}(0)$ there exists a unique function $u \in H^1(0,T;V)$ satisfying the above conditions.

Proposition 3.6. *Let Ω be the model domain described above and assume that $\Gamma_C \Subset \Gamma$ satisfies the uniform cone condition with respect to E_d. Let furthermore $f : [0,T] \to L^2(\Omega;\mathbb{R}^d)$ be a given function. Then for every $\rho > 0$ such that $\overline{\Omega_\rho} \cap \partial\Omega\backslash\Gamma = \emptyset$ and for all $t \in [0,T]$ it holds: if $u \in \mathcal{S}(t)$, then $u\big|_{\Omega_\rho} \in H^{\frac{3}{2}-\delta}(\Omega_\rho)$ for all $\delta > 0$. Moreover, there exists a constant $c_{\rho,\delta} > 0$ such that for all $t \in [0,T]$ and $u \in \mathcal{S}(t)$ it holds $\|u\|_{H^{\frac{3}{2}-\delta}(\Omega_\rho)} \leq c_{\rho,\delta}(\|f(t)\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})$.*

This implies that solutions u of the Tresca friction model satisfy $u \in L^\infty(0,T;H^{\frac{3}{2}-\delta}(\Omega_\rho))$.

Proof. We follow again the ideas presented in the proofs of Lemma 2.5 and 2.6. Observe first that the set Γ_C satisfies condition (R1) if one chooses $g_\infty(x) = 0$ for $x \in \Gamma_C$ and $g_\infty(x) = -\infty$ for $x \in \Gamma\backslash\Gamma_C$. Hence, the test-functions $u \circ T_h$ constructed in the proof of Proposition 3.2 are admissible competitors for the minimization problems characterizing the set $\mathcal{S}(t)$. Similar to the arguments in (2.14)–(2.15), for elements $u \in \mathcal{S}(t)$ we find the estimate (2.15) with an additional term on the right hand side due to the functional $\mathcal{R}$:

$$\frac{\alpha}{2}\left\|u \circ T_h - u\right\|_{H^1(\Omega)}^2 \leq S_1 + S_2 + \int_{\Gamma_3} \kappa\big(\left|u_\tau - (u \circ T_h)_\tau\right| - \left|u_\tau - u_\tau\right|\big)\,\mathrm{d}s_x, \tag{3.8}$$

S_1, S_2 having the same meaning as in (2.15). As in the proof of Lemma 2.5 it holds $S_1 + S_2 \leq c\,|h|\,(\|f(t)\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})$. It remains to estimate the boundary term. Since $H^{\frac{1}{2}}(\Gamma) \subset B^{\frac{1}{2}}_{2,\infty}(\Gamma)$, by the trace theorem we find

$$\int_\Gamma \kappa\left|u_\tau - (u \circ T_h)_\tau\right|\,\mathrm{d}s_x \leq c_\kappa\,|h|^{\frac{1}{2}}\,\|u\|_{B^{\frac{1}{2}}_{2,\infty}(\Gamma)} \leq c_\kappa c_\gamma\,|h|^{\frac{1}{2}}\,\|u\|_{H^1(\Omega)}\,,$$

where c_γ comprises the embedding and trace constants. Following the arguments in the proof of Lemma 2.6 we therefore obtain $u \in B^{1+\frac{1}{4}}_{2,\infty}(\Omega_\rho)$. Now, a recursive argument can be applied starting with $s_0 = \frac{1}{4}$: Assume that $u \in B^{1+s_{k-1}}_{2,\infty}(\Omega_\rho)$. Then

$$\|(u - u \circ T_h)_\tau\|_{L^1(\Gamma)} \leq c_\rho h^{\frac{1}{2}+s_{k-1}}\,\|u\|_{B^{\frac{1}{2}+s_{k-1}}_{2,\infty}(\Gamma\cap\partial\Omega_\rho)} \leq c_\rho c_{k-1} h^{\frac{1}{2}+s_{k-1}}\,\|u\|_{B^{1+s_{k-1}}_{2,\infty}(\Omega_\rho)}\,,$$

where c_{k-1} is the constant of the corresponding trace theorem. Taking into account (3.8) and using arguments as in the proof of Lemma 2.6 it follows that $u \in B^{1+s_k}(\Omega_\rho)$ with $s_k = \frac{1}{2}(\frac{1}{2} + s_{k-1})$. Since $\lim_{k\to\infty} s_k = \frac{1}{2}$, it follows that for all $\delta > 0$ the function u belongs to $H^{\frac{3}{2}-\delta}(\Omega_\rho)$ and satisfies the estimate (after iteration) $\|u\|_{H^{\frac{3}{2}-\delta}(\Omega_\rho)} \leq c_{\rho,\delta}(\|f(t)\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})$. □

Remark 3.7. The result and proof of Proposition 3.6 remain unchanged if one formulates the problem with respect to the set $\mathcal{K}$ defined in (3.4) instead of the space V from (3.7).

3.3 Further nonsmooth energies

With similar arguments, convex, but nonsmooth energies can be treated as well. To the energy density W we add a possibly nonsmooth lower order term of the following type

(H1) $w : \Omega \times \mathbb{R}^m \to [0, \infty]$ is a Carathéodory function satisfying

- (a) For a.e. $x \in \Omega$ the function $z \mapsto w(x, z)$ is convex.
- (b) If $x \in \Omega$ and $z \in \mathbb{R}^m$ with $w(x, z) < \infty$, then for a.e. $y \in \Omega$ it holds $w(y, z) < \infty$.
- (c) There exists a constant $c_w > 0$ such that for all $x, y \in \Omega$ and $z \in \mathbb{R}^m$ it holds $w(x, z) < \infty \Rightarrow |w(x, z) - w(y, z)| \leq c_w\,|x - y|\,(1 + |z|^p)$. Here, $p \in [1, \infty)$ is such that $H^1(\Omega)$ is continuously embedded in $L^p(\Omega)$.

Given $f \in L^2(\Omega, \mathbb{R}^m)$ and $\xi \in L^1(\Omega, \mathbb{R}^m)$ we consider the energy defined for $z \in H^1(\Omega, \mathbb{R}^m)$

$$\mathcal{E}(z) = \int_\Omega W(x, \nabla z(x)) + w(x, z(x)) - f(x) \cdot z(x)\,\mathrm{d}x + \int_\Omega \rho(x)\,|z(x) - \xi(x)|\,\mathrm{d}x, \tag{3.9}$$

where $W : \Omega \times \mathbb{R}^{m\times d} \to [0, \infty)$ is given according to (W1) and (W2) and $\rho \in L^\infty(\Omega)$ with $\rho(x) \geq \rho_0 > 0$ a.e. Clearly, the energy $\mathcal{E}$ satisfies the convexity condition (W2). We study the spatial regularity of minimizers of the following problem

$$z = \operatorname{argmin}\{\, \mathcal{E}(v)\,;\, v \in H^1(\Omega, \mathbb{R}^m)\,\}. \tag{3.10}$$

Theorem 3.8. *Assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain which satisfies the uniform cone condition, that W satisfies (W1) and (W2) and that w is given according to (H1). Let $z \in H^1(\Omega)$ be a minimizer from (3.10). If $f \in L^2(\Omega, \mathbb{R}^m)$ and $\xi \in L^1(\Omega, \mathbb{R}^m)$, then $z \in B^{3/2}_{2,\infty}(\Omega, \mathbb{R}^m)$ and the following estimate is valid*

$$\|z\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega)} \leq c \left(1 + \|f\|_{L^2(\Omega)} + \|z\|_{H^1(\Omega)}^{\max\{1,\frac{p}{2}\}} + \left(\int_\Omega w(x, z(x))\,\mathrm{d}x \right)^{\frac{1}{2}} \right).$$

The constant c is independent of f, ξ and z.

Proof. It is sufficient to investigate the regularity of minimizers in a neighborhood of points on $\partial\Omega$. Let $x_0 \in \partial\Omega$. Due to the uniform cone condition there exists a (d-dimensional) cone $\mathcal{C}(h_0) \subset \mathbb{R}^d$ and a radius $R > 0$ such that for all $x \in Q_R(x_0) \cap \overline{\Omega}$ and $h \in \mathcal{C}(h_0)$ we have $x + h \in \Omega$. As in the previous proofs we introduce the family of mappings $T_h : \Omega \to \mathbb{R}^d$, $T_h(x) = x + \theta(x)h$ for $h \in \mathcal{C}(h_0)$. Let $h_1 \leq h_0$ such that for $h \in \mathcal{C}(h_1)$ the mappings T_h are diffeomorphisms on $\mathbb{R}^d$. Observe that $T_h(\Omega) \subset \Omega$. Let $z \in H^1(\Omega)$ be a minimizer from (3.10). For the function $z \circ T_h \in H^1(\Omega)$ it holds $\mathcal{E}(z \circ T_h) < \infty$. In the same way as in (2.14), the following estimate is valid

$$\begin{aligned}\frac{\alpha}{2}\,\|z \circ T_h - z\|^2_{H^1(\Omega)} \leq{}& \int_\Omega W(x, \nabla(z \circ T_h)) - W(x, \nabla z)\,\mathrm{d}x + \int_\Omega w(x, z \circ T_h) - w(x, z)\,\mathrm{d}x \\ &- \int_\Omega f \cdot (z \circ T_h - z)\,\mathrm{d}x + \int_\Omega \rho(x)\big(\,|z \circ T_h - \xi| - |z - \xi|\,\big)\,\mathrm{d}x \\ ={}& S_1 + \ldots + S_4.\end{aligned}$$

Observe that $S_4 \leq \|\rho\|_{L^\infty(\Omega)} \int_\Omega |z \circ T_h - z|\,\mathrm{d}x$. Now, arguments similar to (2.16)–(2.17) show that

$$\frac{\alpha}{2}\,\|z \circ T_h - z\|^2_{H^1(\Omega)} \leq c\,|h| \left(1 + \|f\|^2_{L^2(\Omega)} + \|z\|_{H^1(\Omega)}^{\max\{2,p\}} + \int_\Omega w(x, z(x))\,\mathrm{d}x \right).$$

and the proof of Theorem 3.8 is finished. □

Example 3.9. Energies of the type (3.9) occur for instance in the Souza-Auricchio model describing shape memory alloys, [AMS08, MPPS10]. Let $u : \Omega \to \mathbb{R}^d$ denote the displacement field and $z : \Omega \to \mathbb{R}^{d\times d}_{\text{sym, dev}}$ the transformation strain. The time-dependent energy is defined by

$$\mathcal{W}(t, u, z) := \int_\Omega \frac{1}{2}\mathbf{C}(\varepsilon(u) - z) : (\varepsilon(u) - z)\,\mathrm{d}x - \int_\Omega \ell(t) \cdot u\,\mathrm{d}x + \frac{\sigma}{2}\,\|\nabla z\|^2_{L^2(\Omega)} + \int_\Omega w(x, z)\,\mathrm{d}x,$$

where $t \mapsto \ell(t)$ are given time-dependent loadings. A typical choice for w is given by $w(x,z) = c_1(x)\,|z| + c_2(x)\,|z|^2 + \chi_M(z)$. Here, $M \subset \mathbb{R}^{d\times d}_{\text{sym, dev}}$ is closed and convex and $\chi_M(z) = 0$ if $z \in M$ and $\chi_M(z) = \infty$ otherwise. As in the example with Tresca friction, a dissipation pseudo-potential is needed to set up an evolution model for shape memory alloys. In the rate independent framework, for shape-memory alloys the dissipation potential is typically given by $\mathcal{R}(z) = \int_\Omega \rho(x)\,|z|\,\mathrm{d}x$ for some L^∞-coefficient $\rho \geq \rho_0 > 0$. The evolution law is formulated in terms of a global stability criterion based on stable sets $\mathcal{S}(t)$ and an energy balance, [Mie05, MPPS10]. Here, for fixed time t the stable set $\mathcal{S}(t)$ is defined as $\mathcal{S}(t) = \{\,(u,z) \in H^1_{\Gamma_D}(\Omega) \times H^1(\Omega)\,;\, \mathcal{W}(u,z) \leq \mathcal{W}(v,\zeta) + \mathcal{R}(z-\zeta),\ (v,\zeta) \in H^1_{\Gamma_D}(\Omega) \times H^1(\Omega)\,\}$. Theorem 3.8 guarantees that for every t and every pair $(u,z) \in \mathcal{S}(t)$ we have $z \in B^{3/2}_{2,\infty}(\Omega)$ and there exists a constant c, which is independent of t, such that $\|z\|_{B^{3/2}_{2,\infty}(\Omega)} \leq c\big(1 + \|\ell(t)\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} + \|z\|_{H^1(\Omega)}\big)$. The regularity of the displacement field u now can be investigated applying for example the techniques from [EF99, Kne06] or [Dau88, Gri87, Kon67].

4 FE-convergence rates based on Falk's approximation theorem

We will now use the above proved regularity results to derive convergence rates for finite element discretizations of contact problems. We restrict ourself to linear elasticity and polygonal or polyhedral domains. The main goal is to characterize the function spaces occurring in the Falk approximation theorem for variational inequalities and to discuss suitable interpolation operators. We assume that a conforming discretization $\mathcal{K}_h$ of the convex set $\mathcal{K}$ is used, i.e. that $\mathcal{K}_h \subset \mathcal{K}$.

First, for a quite general geometric setting we derive the spaces W and W^* occurring in the Falk approximation theorem (Section 4.1). Subsequently, in Section 4.2, we discuss for two- and three-dimensional model problems, how the relevant convergence rates can be obtained on the basis of the regularity results proved in the previous section. The theoretical results will be confirmed by numerical simulations in Section 5.

4.1 Falk approximation theorem, the spaces W and W^*

Let $\widetilde{\Omega}$, Γ_C and Ω be given according to (G1) and (G2) and assume that the Dirichlet boundary $\Gamma_D \subset \partial\Omega$ has positive measure. Observe that $\widetilde{\Omega} = \operatorname{int}\overline{\Omega}$. For $\mathcal{K}$ and $\mathcal{E}$ as in (3.1) and (3.2), we consider the following linear elastic minimization problem: For given $f \in L^2(\Omega, \mathbb{R}^d)$ find

$$u = \operatorname{argmin}\{\, \mathcal{E}(v)\,;\, v \in \mathcal{K}\,\}, \tag{4.1}$$

$$\mathcal{E}(v) = \int_\Omega \frac{1}{2}\mathbf{C}\varepsilon(v) : \varepsilon(v)\,\mathrm{d}x - \int_\Omega f \cdot v\,\mathrm{d}x. \tag{4.2}$$

Throughout the whole section we assume that the coefficient tensor $\mathbf{C}$ is constant and satisfies the symmetry and positivity conditions from Section 3.1.

Let $V := H^1_{\Gamma_D}(\Omega;\mathbb{R}^d) = \{ v \in H^1(\Omega;\mathbb{R}^d)\,;\, v\big|_{\Gamma_D} = 0 \}$ with dual space V^*. For $h > 0$ let $V_h \subset V$ be a closed subspace and $\mathcal{K}_h := \mathcal{K} \cap V_h$. It is assumed that $\mathcal{K}_h \neq \emptyset$. Let $u_h \in \mathcal{K}_h$ be defined as

$$u_h = \operatorname{argmin}\{ \mathcal{E}(v_h)\,;\, v_h \in \mathcal{K}_h \}. \tag{4.3}$$

We introduce the bilinear form $a(u,v) = \int_\Omega \mathbf{C}\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x$ and define the corresponding linear operator in the usual way by

$$A : V \to V^*, \qquad \langle Au, v\rangle = a(u,v) \quad \text{for all } u, v \in V. \tag{4.4}$$

The minimization problem (4.3) is equivalent to solving the following variational inequality: Find $u_h \in \mathcal{K}_h$ such that

$$a(u_h, v_h - u_h) \geq \int_\Omega f \cdot (v_h - u_h)\,\mathrm{d}x \quad \text{for all } v_h \in \mathcal{K}_h. \tag{4.5}$$

Since $\mathcal{K}_h \subset \mathcal{K}$, by Falk's approximation theorem [Fal74, Theorerem 1] there exists a constant $c > 0$, which is independent of the choice of V_h, such that for all $v_h \in \mathcal{K}_h$ it holds

$$\|u - u_h\|_V \leq c \left(\|u - v_h\|_V + \|f - Au\|_W^{\frac{1}{2}} \|u - v_h\|_{W^*}^{\frac{1}{2}} \right). \tag{4.6}$$

Thereby, W is a Hilbert space that is dense in V^* and has to be chosen suitably. The next goal is to suggest an admissible choice of W on the basis of the above proven regularity results.

For $\sigma \in (0,1)$ and $k \in \mathbb{N}_0$ let the spaces $H^{k+\sigma}(\Omega)$ be defined as complex interpolation spaces, [LM72]:

$$H^{k+\sigma}(\Omega) := [H^{k+1}(\Omega), H^k(\Omega)]_{(1-\sigma)}, \quad H^\sigma_{\Gamma_D}(\Omega) := [V, L^2(\Omega)]_{(1-\sigma)},$$

while for $s \geq 1$ we set $H^s_{\Gamma_D}(\Omega) := H^s(\Omega) \cap V$.

For the next argument we use a regularity assumption for solutions of Dirichlet-boundary value problems defined on Ω_+ and Ω_- separately. Let the operator $B_\pm : H^1_0(\Omega_\pm) \to (H^1_0(\Omega_\pm))^*$ be defined as follows: For all $u, v \in H^1_0(\Omega_\pm)$

$$\langle B_\pm(u), v\rangle = \int_{\Omega_\pm} \mathbf{C}\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x. \tag{4.7}$$

The regularity assumption reads

(D1) The sets Ω_+ and Ω_- are bounded with Lipschitz-boundary (local bi-Lipschitz mappings). Moreover, there exists a constant $s_0 \in (1,2)$ such that the differential operators $B_\pm$ defined in (4.7) are isomorphisms from $H^{s_0}(\Omega_\pm) \cap H^1_0(\Omega_\pm)$ onto $\big([H^1_0(\Omega_\pm), L^2(\Omega_\pm)]_{s_0-1}\big)^*$.

Remark 4.1. In the main result of this section, Corollary 4.4, we need $s_0 > \frac{3}{2}$. This can be obtained if for example Ω_+ and Ω_- are polyhedral Lipschitz-domains (in the sense of local Lipschitz-graphs), see [Dau88, Nic92, KM88].

For $s \in (1, 2)$ we define the space

$$\mathcal{M}^s_{\Gamma_D}(\Omega) = \{\, v \in H^s_{\Gamma_D}(\Omega)\,;\, \operatorname{div} \mathbf{C}\varepsilon(v) \in L^2(\Omega)\,\},$$

which is endowed with the graph norm $|||w|||_{s,\Omega} = \|w\|_{H^s(\Omega)} + \|\operatorname{div} \mathbf{C}\varepsilon(u)\|_{L^2(\Omega)}$. The spaces $\mathcal{M}^s_{\Gamma_D}(\Omega_\pm)$ are defined in a similar way.

Lemma 4.2. *Let (G1), (G2) and (D1) be satisfied. For all $s \in (1, s_0)$ the following identity holds true for Ω_+ and Ω_- with $\theta = \frac{s_0 - s}{s_0 - 1}$:*

$$\mathcal{M}^s_{\Gamma_D}(\Omega_\pm) = [\mathcal{M}^{s_0}_{\Gamma_D}(\Omega_\pm), \mathcal{M}^1_{\Gamma_D}(\Omega_\pm)]_\theta.$$

The proof is given in the Appendix and relies on Theorem 14.3 in [LM72, Chapter 1].

We next investigate the mapping properties of the operators $A_\pm : H^1_{\Gamma_D}(\Omega_\pm) \to \big(H^1_{\Gamma_D}(\Omega_\pm)\big)^*$ defined by

$$\forall\, u, v \in H^1_{\Gamma_D}(\Omega_\pm) : \quad \langle A_\pm u, v\rangle = \int_{\Omega_\pm} \mathbf{C}\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x. \tag{4.8}$$

Here, $H^1_{\Gamma_D}(\Omega_\pm) = \{\, v \in H^1(\Omega_\pm)\,;\, \exists\, \widetilde{v} \in H^1_{\Gamma_D}(\Omega) \text{ with } v = \widetilde{v}\big|_{\Omega_\pm}\,\}$.

Lemma 4.3. *Let (G1), (G2) and (D1) be satisfied with $s_0 > \frac{3}{2}$. Then there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ there exists $\widetilde{\delta} > 0$ for which the differential operators $A_\pm$ defined in (4.8) are linear and continuous from $\mathcal{M}^{\frac{3}{2}-\widetilde{\delta}}_{\Gamma_D}(\Omega_\pm) \to \big(H^{\frac{1}{2}+\delta}_{\Gamma_D}(\Omega_\pm)\big)^*$, and $\widetilde{\delta} \to 0$ if $\delta \to 0$.*

Proof. Let $s_0 > \frac{3}{2}$. Then for every $\epsilon \in (0, \frac{1}{2})$ and $s \in (\frac{3}{2}, s_0]$ the operator $A_\pm$ is continuous from $\mathcal{M}^s_{\Gamma_D}(\Omega_\pm) \to (H^{\frac{1}{2}+\epsilon}_{\Gamma_D}(\Omega_\pm))^*$. This is a direct consequence of the Gauss-Theorem and the fact that $H^1_{\Gamma_D}(\Omega_\pm)$ is dense in $H^{\frac{1}{2}+\epsilon}_{\Gamma_D}(\Omega_\pm)$. Indeed, for all $v \in \mathcal{M}^s_{\Gamma_D}(\Omega_\pm)$ and all $w \in H^1_{\Gamma_D}(\Omega_\pm)$ it holds

$$\begin{aligned}\langle A_\pm(v), w\rangle &= \int_{\Omega_\pm} -\operatorname{div} \mathbf{C}\varepsilon(v) \cdot w\,\mathrm{d}x + \int_{\partial\Omega_\pm} \mathbf{C}\varepsilon(v)n \cdot w\,\mathrm{d}s \\ &\le c\,|||v|||_{s,\Omega_\pm} \big(\|w\|_{L^2(\Omega_\pm)} + \|w\|_{L^2(\partial\Omega_\pm)}\big) \le c_\epsilon\,|||v|||_{s,\Omega_\pm} \|w\|_{H^{\frac{1}{2}+\epsilon}(\Omega_\pm)}.\end{aligned}$$

By complex interpolation for all $\theta \in (0, 1)$ it follows that

$$A_\pm : [\mathcal{M}^s_{\Gamma_D}(\Omega_\pm), \mathcal{M}^1_{\Gamma_D}(\Omega_\pm)]_\theta \to [\big(H^{\frac{1}{2}+\epsilon}_{\Gamma_D}(\Omega_\pm)\big)^*, \big(H^1_{\Gamma_D}(\Omega_\pm)\big)^*]_\theta = [H^1_{\Gamma_D}(\Omega_\pm), H^{\frac{1}{2}+\epsilon}_{\Gamma_D}(\Omega_\pm)]^*_{1-\theta} \tag{4.9}$$

is continuous. For $\delta \in (0, \frac{1}{2})$ we set $\epsilon = \delta/2$, $\theta = \delta(1-\delta)^{-1}$, $s = \frac{3}{2} + \delta(4 - 8\delta)^{-1}$ and $\widetilde{\delta} = \delta(4 - 4\delta)^{-1}$. Observe that there exists $\delta_0 \in (0, \frac{1}{2})$ such that for all $\delta \in (0, \delta_0)$ it holds $s \in (\frac{3}{2}, s_0)$. With this choice of s and θ it follows with (4.9) that $A_\pm : \mathcal{M}^{\frac{3}{2}-\widetilde{\delta}}_{\Gamma_D}(\Omega_\pm) \to (H^{\frac{1}{2}+\delta}(\Omega_\pm))^*$ is linear and continuous. □

According to Proposition 3.1 the minimizer u of problem (4.1) is $B^{\frac{3}{2}}_{2,\infty}$-regular in a neighborhood of the crack Γ_C. We assume now that the exterior boundary of Ω and the interface separating Γ_D and Γ_N are regular enough such that $u \in B^{\frac{3}{2}}_{2,\infty}(\Omega)$, globally. Sufficient conditions that guarantee this global regularity are described for example in [Ebm99, EF99, Kne06] and in Remark 4.1.

Corollary 4.4. *Let (G1), (G2) and (D1) be satisfied with* $s_0 > \frac{3}{2}$. *Let furthermore* $u \in \mathcal{K}$ *with* $u \in B^{\frac{3}{2}}_{2,\infty}(\Omega)$ *and* $u_h \in \mathcal{K}_h$ *be minimizers of* (4.1) *and* (4.3) *with* $f \in L^2(\Omega)$. *Then there exists a constant* $\delta_0 > 0$ *such that for every* $\delta \in (0, \delta_0)$ *the choice* $W^* = H^{\frac{1}{2}+\delta}_{\Gamma_D}(\Omega)$ *is admissible for the estimate* (4.6).

Proof. Let $\delta_0 > 0$ be the constant from Lemma 4.3 and let $\delta \in (0, \delta_0)$ be arbitrary. By Lemma 4.3 there exists $\widetilde{\delta} > 0$ such that the operators $A_\pm : \mathcal{M}^{\frac{3}{2}-\widetilde{\delta}}_{\Gamma_D}(\Omega_\pm) \to \big(H^{\frac{1}{2}+\delta}_{\Gamma_D}(\Omega_\pm)\big)^*$ are well defined and continuous. Since $u \in B^{\frac{3}{2}}_{2,\infty}(\Omega)$ with $\operatorname{div} \mathbf{C}\varepsilon(u) = -f \in L^2(\Omega)$ it follows that for every $\widetilde{\delta} > 0$ we have $u \in \mathcal{M}^{\frac{3}{2}-\widetilde{\delta}}_{\Gamma_D}(\Omega)$ and in particular, $u\big|_{\Omega_\pm} \in \mathcal{M}^{\frac{3}{2}-\widetilde{\delta}}_{\Gamma_D}(\Omega_\pm)$. Hence, using Lemma 4.3. the following estimate is valid for all $v \in H^1_{\Gamma_D}(\Omega)$:

$$
\begin{aligned}
|\langle Au, v\rangle| &\leq \sum_{i\in\{+,-\}} \big|\langle A_i(u|_{\Omega_i}), v|_{\Omega_i}\rangle\big| \leq c_\delta \sum_{i\in\{+,-\}} \big\|u|_{\Omega_i}\big\|_{\mathcal{M}^{\frac{3}{2}-\widetilde{\delta}}_{\Gamma_D}(\Omega_i)} \big\|v|_{\Omega_i}\big\|_{H^{\frac{1}{2}+\delta}(\Omega_i)} \\
&\leq c(\delta)\, \|u\|_{\mathcal{M}^{\frac{3}{2}-\widetilde{\delta}}_{\Gamma_D}(\Omega)} \|v\|_{H^{\frac{1}{2}+\delta}(\Omega)} .
\end{aligned}
$$

By density of $H^1_{\Gamma_D}(\Omega)$ in $H^{\frac{1}{2}+\delta}_{\Gamma_D}(\Omega)$ this estimate can be extended to all $v \in H^{\frac{1}{2}+\delta}_{\Gamma_D}(\Omega)$. This shows that $W := \big(H^{\frac{1}{2}+\delta}_{\Gamma_D}(\Omega)\big)^*$ is an admissible choice in the estimate (4.6). □

Remark 4.5. It is straightforward to extend these arguments to the problems described in Sections 3.1–3.2.

4.2 Convergence rates for discretizations with standard finite elements

We next discuss possible choices of interpolation operators that allow us to deduce the rate of convergence of FE-discretizations on the basis of the Falk estimate (4.6) and Corollary 4.4. The interpolation operator should preserve the contact conditions and it should lead to estimates of the type

$$
\|u - \mathcal{I}_h u\|_{H^1(\Omega)} \leq ch^\alpha \|u\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega)}, \qquad \|u - \mathcal{I}_h u\|_{H^{\frac{1}{2}+\delta}(\Omega)} \leq ch^{\beta_\delta} \|u\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega)},
$$

where h is associated with the mesh size. As can be seen from (4.6), the choice is optimal if $2\alpha \approx \beta_\delta$. Due to the regularity results and the embedding theorems, in two dimensions one may use Lagrange interpolation. However, in three dimensions the Lagrange interpolation is not meaningful for elements from $B^{\frac{3}{2}}_{2,\infty}(\Omega)$ since this space is not contained in the continuous functions. Hence, interpolation operators based on local averaging should be used.

4.2.1 The two-dimensional case

Assume that $\widetilde{\Omega} \subset \mathbb{R}^2$ is a polygonal domain with Lipschitz boundary. Let Ω_+, Ω_- be defined as in (2.1) and assume that the crack $\Gamma_C = \{\, (x_1, 0)\,;\, -\ell \leq x_1 \leq \ell\,\}$ is either completely contained in $\widetilde{\Omega}$ or that it intersects with $\partial\widetilde{\Omega}$ in exactly one point. As before, $\Omega = \widetilde{\Omega} \backslash \Gamma_C$. It is assumed that $\Gamma_D \subset \partial\Omega$ is closed and not empty, that $\Gamma_D \cap \Gamma_C = \emptyset$ and that the geometry is chosen in such a way that condition (D1) is satisfied. For shortness we assume that the Neumann data as well as the Dirichlet data vanish and that the volume term f is from $L^2(\Omega)$. Moreover, we assume that the minimizer u in (4.1) belongs to the space $B^{\frac{3}{2}}_{2,\infty}(\Omega)$. This is exactly the regularity that is available in a neighborhood of the crack tip, see the discussion in Chapters 2 and 3.

Theorem 4.6. *Assume that in the above described setting the minimization problem* (4.1) *is discretized with continuous, piecewise linear ansatz functions on quasiuniform triangular finite element meshes or with continuous, piecewise bilinear ansatz functions on quasiuniform quadrilateral meshes with mesh size h. It is further assumed that on the crack Γ_C the nodes belonging to the upper half (i.e. Ω_+) and the nodes belonging to the lower half (i.e. Ω_-) coincide.*

For the finite element approximation u_h satisfying (4.5) *and the minimizer u of* (4.1) *the following error estimate is valid: For every $\delta > 0$ there exists $c_\delta > 0$ such that*

$$\|u - u_h\|_{H^1(\Omega)} \leq c_\delta h^{\frac{1}{2}-\delta} \left(\|u\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega)} + \|f\|_{L^2(\Omega)} \right). \tag{4.10}$$

Proof. Let $\mathcal{I}_h$ denote the Lagrange interpolation operator on linear (in case of triangles) or bilinear elements (in case of quadrilaterals). According to [SA84] for $\delta > 0$ (small) and $k \in \{0, 1\}$ the estimate $\|v - \mathcal{I}_h v\|_{H^k(\Omega)} \leq c_\delta h^{\frac{3}{2}-k-\delta} \|v\|_{H^{\frac{3}{2}-\delta}(\Omega)}$ is valid for all $v \in H^{\frac{3}{2}-\delta}(\Omega)$. Hence, by interpolation, we obtain $\|v - \mathcal{I}_h v\|_{H^{\frac{1}{2}+\delta}(\Omega)} \leq c_\delta h^{1-2\delta} \|v\|_{H^{\frac{3}{2}-\delta}(\Omega)}$. Combining this estimate with estimate (4.6) and Corollary 4.4 yields (4.10). □

Remark 4.7. Without any serious changes, the above arguments can be extended to two-dimensional contact problems as described in Section 3.2 and lead to the same convergence rates as in Theorem 4.6.

4.2.2 The three-dimensional case

In three space dimensions we restrict the discussion to the case with an interior crack. Let $\widetilde{\Omega} \subset \mathbb{R}^3$ be a polyhedral domain and $\Gamma_C \subset E_3$ a polygonal set such that (G1), (G2) and (D1) are satisfied. As in the two-dimensional case it is assumed that $\Gamma_D \subset \partial\Omega$ is closed and not empty and that $\Gamma_D \cap \Gamma_C = \emptyset$. We assume that the Neumann data as well as the Dirichlet data vanish, that the volume term f is from $L^2(\Omega)$ and that the minimizer u in (4.1) belongs to the space $B^{\frac{3}{2}}_{2,\infty}(\Omega)$. For $h > 0$ let $\mathcal{T}_h$ denote a family of quasiuniform regular tetrahedral meshes on Ω. It is assumed that on the crack Γ_C the nodes belonging to the upper half (i.e. Ω_+) and the nodes belonging to the lower half (i.e. Ω_-) coincide.

Theorem 4.8. *Assume that in the above described setting the minimization problem* (4.1) *is discretized with continuous, piecewise linear ansatz functions on the above introduced tetrahedral finite element meshes with mesh size h.*

For the finite element approximation u_h satisfying (4.5) *and the minimizer u of* (4.1) *the following error estimate is valid: For every $\delta > 0$ (small) there exists $c_\delta > 0$ such that*

$$\|u - u_h\|_{H^1(\Omega)} \leq c_\delta h^{\frac{1}{2}-\delta} \left(\|u\|_{B^{\frac{3}{2}}_{2,\infty}(\Omega)} + \|f\|_{L^2(\Omega)} \right). \tag{4.11}$$

Proof. Inspired by [HN07], where a two-dimensional situation is investigated, in order to prove Theorem 4.8 we choose an interpolation operator that coincides with the Scott-Zhang operator, cf. [SZ90], on nodes from $\overline{\Omega} \setminus \operatorname{int} \Gamma_C$. For the nodes on $\operatorname{int} \Gamma_C$ a Chen-Nochetto type ansatz is used, cf. [CN00]. To be more precise, let $\mathcal{N}_h = \{\, a_i \,;\, 1 \leq i \leq N_h \,\}$ denote the nodes of the finite element mesh and $\mathcal{N}_h^C := \{a_i \in \operatorname{int} \Gamma_C\}$ the nodes inside Γ_C. Let ϕ_a denote the nodal basis function associated with the node a. If a belongs to $\operatorname{int} \Gamma_C$ we distinguish between ϕ_a^+, which has its support in $\overline{\Omega_+}$, and ϕ_a^- with support in $\overline{\Omega_-}$.

The definition of the Scott-Zhang operator as well as the Chen-Nochetto operators are based on the averaging of Sobolev functions on suitable sets σ_a containing the node a. The sets σ_a are chosen in the following way, where (i)–(iii) correspond to an ansatz of Scott-Zhang type, while (iv) is of Chen-Nochetto type:

(i) If $a \in \overline{\Gamma}_D$, we choose a tetrahedron that intersects with the Dirichlet boundary at least with one whole face containing a. Then σ_a is chosen as one of these Dirichlet-faces.

(ii) If $a \in \partial \Gamma_C$ (i.e. a is situated on the crack front), we choose a tetrahedron that contains a and define σ_a as one of the faces containing a but not lying on the crack.

(iii) For $a \in \Omega \cup \Gamma_N$ we choose σ_a as one of the faces containing a of one of the tetrahedra around a.

(iv) For $a \in \operatorname{int} \Gamma_C$ let $M_a := \cup \{\, \overline{\tau}\big|_{\Gamma_C} \,;\, \tau \in \mathcal{T}_h \text{ with } a \in \overline{\tau} \,\}$. Let $\triangle_a$ be the maximum two-dimensional disk centered in a and contained in M_a. Then $\sigma_a = \triangle_a$.

Next we assign to $u \in H^1(\Omega)$ its nodal value $\pi_a(u)$ for $a \in \mathcal{N}_h$ in the following way: For $a \in \mathcal{N}_h^C$ we define $\pi_a^\pm(u) := |\sigma_a|^{-1} \int_{\sigma_a} u\big|_{\Omega_\pm} \,\mathrm{d}s_x$. For $a \in \mathcal{N}_h \setminus \mathcal{N}_h^C$ we proceed as in [SZ90]: Let $a_1 := a$ and let $a_2, a_3 \in \mathcal{N}_h$ be the two remaining vertices of σ_a. The dual nodal basis functions ψ_a^j, $1 \leq j \leq 3$, are defined through the relation $\int_{\sigma_a} \psi_a^j \phi_{a_k} \,\mathrm{d}s_x = \delta_{jk}$ for $1 \leq k \leq 3$. Then, $\pi_a(u) := |\sigma_a|^{-1} \int_{\sigma_a} u \psi_a^1 \,\mathrm{d}s_x$. Finally, we define the interpolation operator as follows:

$$\mathcal{I}_h(u)(x) := \sum_{a \in \mathcal{N}_h \setminus \mathcal{N}_h^C} \pi_a(u) \phi_a(x) + \sum_{a \in \mathcal{N}_h^C} \left(\pi_a^+(u) \phi^+(x) + \pi_a^-(u) \phi^-(x) \right).$$

Claim 4.9. The above defined interpolation operator $\mathcal{I}_h$ is well defined as a mapping from $\mathcal{K}$ to $\mathcal{K}_h$ (i.e. it preserves the contact condition and the Dirichlet-condition). Moreover, there exists a constant $c > 0$ such that for all $m \in \{1, 2\}$ and $k \in \{0, 1\}$ and all $u \in H^m(\Omega)$ it holds

$$\|u - \mathcal{I}_h(u)\|_{H^k(\Omega)} \leq ch^{m-k} \|u\|_{H^m(\Omega)} . \tag{4.12}$$

Proof of Claim 4.9. The fact that $\mathcal{I}_h$ preserves the Dirichlet-conditions as well as the contact conditions on Γ_C follows immediately from the definition of $\mathcal{I}_h$.

For $\tau \in \mathcal{T}_h$ let $\widetilde{\tau} = \cup\{\, \tau_* \in \mathcal{T}_h\, ;\, \overline{\tau_*} \cap \overline{\tau} \neq \emptyset \,\}$. Following the arguments in [SZ90, Theorem 3.1 and (4.1)–(4.3)], for all those $\tau \in \mathcal{T}_h$, where none of the vertices of τ belongs to $\operatorname{int}\Gamma_C$, one proves that

$$\|v - \mathcal{I}_h(v)\|_{H^k(\tau)} \leq ch^{m-k} \|v\|_{H^m(\widetilde{\tau})} \tag{4.13}$$

for $k \in \{0, 1\}$, $m \in \{1, 2\}$ and all $v \in H^m(\widetilde{\tau})$.

Let now $\tau \in \mathcal{T}_h$ and assume that at least one of the vertices belongs to $\mathcal{N}_h^C$. Then following the arguments in the proof of Theorem 3.1 in [SZ90] it is possible to show that $|\pi_a(v)| \leq c\sum_{i=0}^m h^{-\frac{3}{2}+i} |v|_{W^{i,2}(\widetilde{\tau})}$ for all vertices a of τ. This in turn implies, again as in the proof of Theorem 3.1 in [SZ90], that $\|\mathcal{I}_h v\|_{H^k(\tau)} \leq c\sum_{i=0}^{\ell} h^{i-k} |v|_{H^i(\widetilde{\tau})}$ for $\ell \in \mathbb{N}\backslash\{0\}$. Observe next that for all $\eta \in \mathcal{P}^1(\widetilde{\tau})$ (i.e. η is affine on the set $\widetilde{\tau}$) and all vertices a_j of τ, $1 \leq j \leq 4$, we have $\mathcal{I}_h(\eta)(a_j) = \eta(a_j)$, and hence $\mathcal{I}_h(\eta)(x) = \eta(x)$ for all $x \in \tau$. But this is exactly property (3.3) in [CN00]. Hence we may now use the arguments from the proof of [CN00, Lemma 3.2] in order to finally arrive at estimate (4.13) also for this case.

Summing up the estimate (4.13) over all $\tau \in \mathcal{T}_h$ and taking into account that due to the assumptions on the mesh the number of elements belonging to $\widetilde{\tau}$ is bounded independently of h, we finally arrive at (4.12). □

We return to the proof of Theorem 4.8. By complex interpolation it follows from (4.12) that for $\delta > 0$ (small) and $k \in \{0, 1\}$ the estimate $\|v - \mathcal{I}_h v\|_{H^k(\Omega)} \leq c_\delta h^{\frac{3}{2}-k-\delta} \|v\|_{H^{\frac{3}{2}-\delta}(\Omega)}$ is valid for all $v \in H^{\frac{3}{2}-\delta}(\Omega)$. Hence, again by interpolation, we obtain $\|v - \mathcal{I}_h v\|_{H^{\frac{1}{2}+\delta}(\Omega)} \leq c_\delta h^{1-2\delta} \|v\|_{H^{\frac{3}{2}-\delta}(\Omega)}$. Combining this estimate with estimate (4.6) and Corollary 4.4 yields (4.10). □

Remark 4.10. In the literature only very few references are available, where an extension of the Scott-Zhang operator or the Chen-Nochetto operator to hexahedral finite elements is discussed. We only are aware of [HS07], where a Scott-Zhang-type operator is defined for hexahedral elements. But there the operator is defined in such a way that an estimate of the type $\|u - \mathcal{I}_h u\|_{H^1(\Omega)} \leq ch \|u\|_{H^2(\Omega)}$ is not valid. For this reason, we decided to formulate Theorem 4.8 for tetrahedral elements, only, although the three-dimensional simulations in Section 5 are carried out on meshes with hexahedral elements. The three-dimensional simulations indicate that Theorem 4.8 is also valid for discretizations based on hexahedral elements.

Remark 4.11. If the crack is allowed to intersect with the exterior boundary it is not clear how to define an interpolation operator with the properties described in Claim 4.9. Due to a non-existence result by Nochetto and Wahlbin [NW02] in this case it is not possible to construct an interpolation operator that is based on averaging, that respects the contact condition also on $\Gamma_C \cap \partial\widetilde{\Omega}$ and that has optimal approximation properties as described in Claim 4.9. The difficulties arise in extremal points of $\partial\Omega$. For the contact problems studied in Section 3.2 the same difficulty occurs along the line $\overline{\Gamma_N} \cap \overline{\Gamma_C}$.

5 Numerical verification

Theorem 2.1 predicts that the minimizer of an energy functional with self-contact constraints is at least in $B^{3/2}_{2,\infty}(\Omega)$. Using Falk's theorem in conjunction with some interpolation operators we show that this regularity leads to the convergence rate $\mathcal{O}(h^{1/2-\delta})$ for finite element approximations, see Theorems 4.6 and 4.8. The derivation of similar results is also possible for unilateral contact conditions, at least in 2D, see Remark 4.7. In this Section, we second these regularity results with some numerical experiments. We study several representative numerical examples of contact problems with self-contact and unilateral contact constraints in 2D and 3D.

We start our investigations with the observation that

$$\begin{aligned} C\|u-u_h\|^2_{H^1(\Omega)} &\le \int_\Omega \mathbf{C}\varepsilon(u-u_h) : \varepsilon(u-u_h)\,dx \\ &= 2(\mathcal{E}(u_h)-\mathcal{E}(u)) - 2\left(\int_\Omega \mathbf{C}\varepsilon(u) : \varepsilon(u_h-u)\,dx - 2\int_\Omega f\cdot(u_h-u)\,dx\right) \\ &\le 2(\mathcal{E}(u_h)-\mathcal{E}(u)) \end{aligned}$$

for some constant $C > 0$. Hence, we expect that $(\mathcal{E}(u_h)-\mathcal{E}(u))^{1/2}$ behaves like $\mathcal{O}(h^\varrho)$ with $\varrho \ge 1/2$. Assuming $(\mathcal{E}(u_h)-\mathcal{E}(u))^{1/2} \approx Ch^\varrho$, we obtain

$$\varrho \approx \varrho_k := \frac{\log(Q_k/Q_{k+1})}{\log(2)}$$

with $h_k := \bar{h}2^{-k}$, $\bar{h} > 0$, $Q_k := (\mathcal{E}_k - \mathcal{E}_{\mathrm{ref}})^{1/2}$, $\mathcal{E}_k := \mathcal{E}(u_{h_k})$ and $\mathcal{E}_{\mathrm{ref}} \approx \mathcal{E}(u)$. Note that the mesh size h_k can be realized if, for instance, uniform refinements are applied. With the maximum number of refinements $k_{\max}$ for which a finite element approximation is available, we determine an appropriate reference value $\mathcal{E}_{\mathrm{ref}}$ by simply setting $\mathcal{E}_{\mathrm{ref}} := \mathcal{E}_{k_{\max}}$ or by computing $\mathcal{E}_{\mathrm{ref}}$ as an extrapolation of the values $\mathcal{E}_r, \dots, \mathcal{E}_{k_{\max}}$ with $r \ge 0$. This extrapolation is given by $\tilde{\mathcal{E}}_r := a_{k_{\max}-r,k_{\max}-r}$, where $a_{i,0} := \mathcal{E}_{i+r}$, $i = 0, \dots, k_{\max}-r$, and

$$a_{ij} := a_{i,j-1} + \frac{a_{i,j-1} - a_{i-1,j-1}}{2^j - 1}, \quad i = 1, \dots, k_{\max}-r, \quad j = 1, \dots, i.$$

For instance, we obtain the linear extrapolation $\tilde{\mathcal{E}}_{k_{\max}-1} = 2\mathcal{E}_{k_{\max}} - \mathcal{E}_{k_{\max}-1}$ with $r = k_{\max}-1$. To demonstrate that ϱ_k can be used to estimate the asymptotic convergence rate ϱ, we consider a simple example without contact constraints: It is well-known that $u(r,\phi) = r^{2/3}\sin((2\phi-\pi)/3))$ solves Poisson's equation $-\Delta u = 0$ on the L-shaped domain $\Omega := (-0.5, 0.5)^2\setminus[0, 0.5)^2$ where $u = 0$ on $\Gamma_D := [0, 0.5] \times \{0\} \cup \{0\} \times [0, 0.5]$. It is also well-known that $u \in B^{5/3}_{2,\infty}(\Omega)$ and that the convergence rate of the finite element approximation is $\mathcal{O}(h^{2/3})$. Thus, we expect $\varrho_k \approx \varrho = 2/3$ which is, indeed, confirmed by the numerical experiments: In Table 1, the estimated convergence rate ϱ_k is tabulated for several reference values $\mathcal{E}_{\text{ref}}$. As we can see, we always obtain an approximation of the expected value 2/3. Apart from the second column where the exact reference value $\mathcal{E}(u)$ is used, the extrapolation with $r = 0$ in the last column yields the closest values to the exact asymptotic convergence rate. For coarser mesh sizes, the assumption $(\mathcal{E}(u_h) - \mathcal{E}(u))^{1/2} \approx Ch^{\varrho}$ may not be justified so that the convergence rate ϱ is not accurately predicted in the first rows. Moreover, if k is close to $k_{\max}$, we may also get inappropriate values for ϱ, since the reference value $\mathcal{E}_{\text{ref}}$ may possibly be too close to $\mathcal{E}_{k_{\max}}$. Hence, the second or third to last entry may give the most reliable value to estimate the exact convergence rate and, therewith, the regularity of the solution.

h_k	$\mathcal{E}_{\text{ref}} := \mathcal{E}(u)$	$\mathcal{E}_{\text{ref}} := \mathcal{E}_{k_{\max}}$	$\mathcal{E}_{\text{ref}} = \tilde{\mathcal{E}}_8$	$\mathcal{E}_{\text{ref}} = \tilde{\mathcal{E}}_0$
0.5000	0.5387	0.5390	0.5386	0.5387
0.2500	0.6171	0.6178	0.6168	0.6170
0.1250	0.6393	0.6411	0.6384	0.6390
0.0625	0.6507	0.6551	0.6484	0.6500
0.0312	0.6570	0.6681	0.6515	0.6552
0.0156	0.6608	0.6894	0.6470	0.6563
0.0078	0.6630	0.7403	0.6296	0.6519
0.0039	0.6644	0.9057	0.5866	0.6372
0.0020	0.6653	-	0.5000	0.6011

Table 1: Regularity index ϱ_k for Poisson's equation and several reference values $\mathcal{E}_{\text{ref}}$.

-	α	β	f_0	f_1	f_2	f_3	scale
2(a)	0	0	0	0	-1	1	50
2(b)	$\pi/4$	0	-1	2	0	0	50
2(c)	$\pi/4$	0	-1	-1	0	0	20
2(d)	$\pi/4$	0	1	-5	0	0	10
2(e)	$\pi/2$	0	0.5	0.5	0	0	10
2(f)	$\pi/2$	0	-1	-1	0	0	50

Table 2: Configuration for self-contact in 2D.

5.1 Self-contact

As a first example, we consider contact problems in linear elasticity with self-contact constraints in 2D. We use Hooke's law and plane stress with Young's modulus $E = 210,000$ and Poisson number $\nu = 0.28$. The domain is set to $\Omega := (-2,1) \times (-1,1) \backslash \Gamma_C$ with the Dirichlet boundary $\Gamma_D := \{1\} \times [-1,1]$, where homogeneous boundary conditions are presumed. The crack is given by $\Gamma_C := (-2,1) \times \{0\}$. Furthermore, we define Neumann conditions on $\Gamma_{1,\pm} := (-2,1) \times \{\pm 1\}$ and $\Gamma_2 := \{-2\} \times (0,1)$ as well as $\Gamma_3 := \{-2\} \times (-1,0)$. The Neumann data for $(x_0, x_1) \in \Gamma_{1,\pm}$ is given by $\pm f(x_0, x_1)(\cos(\alpha), \sin(\alpha))$ with $f(x_0, x_1) = \frac{1}{3}(x_0 + 2)f_1 - \frac{1}{3}(x_0 - 1)f_0$, $f_0, f_1 \in \mathbb{R}$ and on Γ_i by $f_i \cdot (\cos(\beta), \sin(\beta))$ with $f_2, f_3 \in \mathbb{R}$. We study six different configurations with the Neumann data as tabulated in Table 2 and use conforming finite elements with piecewise bilinear ansatz functions on quadrilateral meshes. As assumed in Theorem 4.6 the mesh nodes on both sides of Γ_C coincide. In this way, the contact constraints are ensured by simply imposing them in the mesh nodes of Γ_C. For instance, this assumption on the mesh nodes is fulfilled, if the crack is generated via doubling of edges, cf. [KS11]. The maximum number of refinements is $k_{\max} := 8$ which corresponds to $3,150,848$ degrees of freedom.

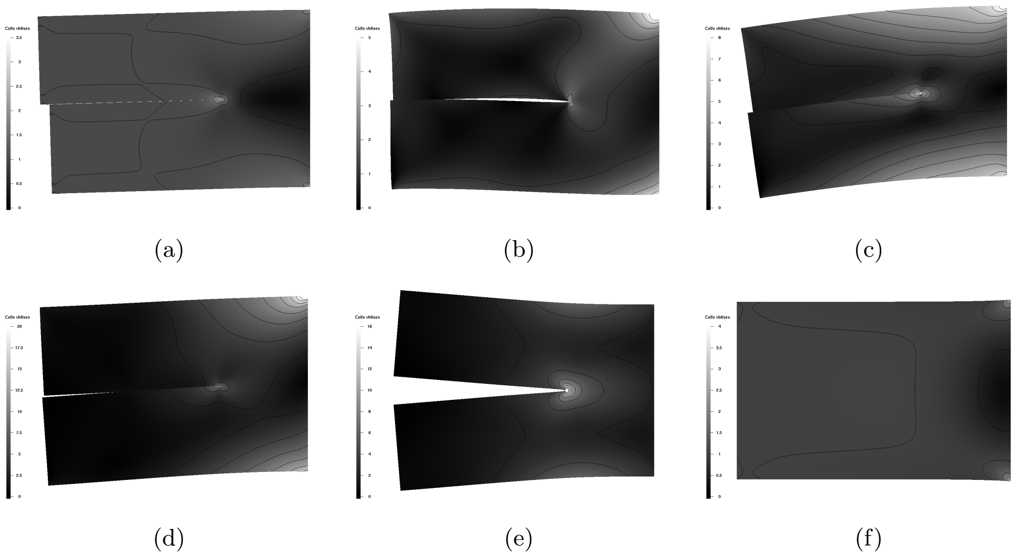


Figure 2: Solutions of self contact in 2D.

In the Figures 2(a)-(f), the resulting deformations and Von Mises stresses are depicted. In these figures, the deformations are scaled by the scaling factor as given in the last column of Table 2. The regularity of the solutions in the Figures 2(e) and (f) are known since they do not actually solve contact problems. The Neumann data in the configuration 2(e) prevents the contact along the crack Γ_C. The regularity of the solution is dominated by the singularity at the crack tip. It is well-known that the singular exponent is $1/2$ which means that the solution is in $B^{3/2}_{2,\infty}(\Omega)$. The convergence rate of the finite element approximation is, therefore, $\mathcal{O}(h^{1/2})$

which is sharp. Indeed, this fact is reproduced by Q_k and, in particular, by ϱ_k in the numerical experiments as shown in Figure 3 and Table 3. The Neumann data in the configuration 2(f) leads to complete self-contact along Γ_C. Due to the symmetry of the Neumann data, we obtain the same solution as in the case without the crack Γ_C in the interior of the domain. In this case, the regularity is dominated by the singularities resulting from the change from Neumann to Dirichlet boundary conditions with the interior angle $\pi/4$. The singular exponent can be estimated with $\alpha := 0.767075\ldots$, cf. [Nic92], leading to the convergence rate $\mathcal{O}(h^\alpha)$. We observe in Table 3 that ϱ_k approximatively predict this convergence rate.

In the Figures 2(c) and (d), the upper and lower parts of the domain Ω are in contact at the crack tip. We see that in both cases the stresses at the crack tip are dominated by the singularities at the corners. Nevertheless, a singularity resulting from a minimal shearing could be present with singular exponent 1/2. Such a singularity does not seem to be resolved by the finite element approximation so that ϱ_k indicates a higher regularity. In contrast, the configuration 2(a) leads to a considerable shearing and to a singularity at the crack tip. In this case, ϱ_k clearly indicates the convergence rate 1/2 and, therefore, $B^{3/2}_{2,\infty}$-regularity as expected. In Figure 2(b), the Neumann data results in tearing at the crack tip and in self-contact on the opposite side. We expect a singularity with singular exponent 1/2 at the crack tip and, therefore, merely $B^{3/2}_{2,\infty}$-regularity of the solution. However, this expectation is not confirmed by ϱ_k in Table 3. Again, this effect may be explained by the insufficient resolution of the finite element approximation. Anyway, we obtain $\varrho_k \geq 0.5$ for all configurations as desired.

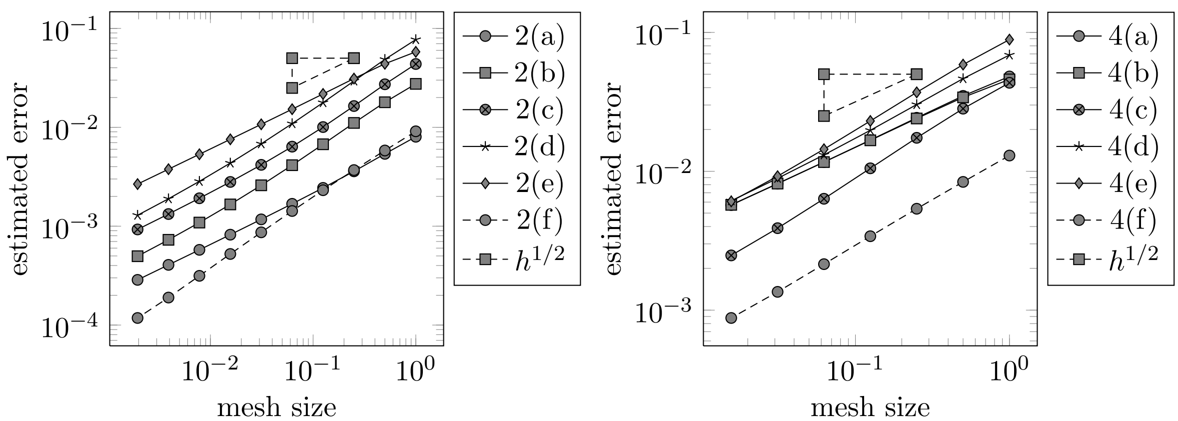


Figure 3: Q_k for several contact problems with self-contact constraints in 2D (left) and in 3D (right).

We also consider contact problems with self-contact constraints in 3D. Again, we assume linear elasticity by Hooke's law and the same material parameters as in the 2D case. The domain is given by $\Omega := [-2,1]\times[-1,1]^2$ and the Dirichlet boundary by $\Gamma_D := \{1\}\times(-1,1)^2$, where homogeneous boundary conditions are prescribed. The crack is given by $\Gamma_C := (-2,0)\times(-1,0)\times\{0\}\cup(-2,-1)\times(0,1)\times\{0\}$ which has a Lipschitz continuous, but not differentiable boundary. The

h	2(a)	2(b)	2(c)	2(d)	2(e)	2(f)
0.5000	0.5746	0.6170	0.6810	0.6659	0.4018	0.6445
0.2500	0.5865	0.6952	0.7310	0.7251	0.4967	0.6578
0.1250	0.5574	0.7167	0.7062	0.7263	0.5161	0.6772
0.0625	0.5348	0.7030	0.6582	0.7062	0.5128	0.6984
0.0312	0.5219	0.6756	0.6113	0.6776	0.5077	0.7161
0.0156	0.5145	0.6427	0.5744	0.6465	0.5042	0.7280
0.0078	0.5097	0.6098	0.5482	0.6152	0.5023	0.7314
0.0039	0.5064	0.5788	0.5302	0.5849	0.5013	0.7194
0.0020	0.5038	0.5497	0.5174	0.5544	0.5007	0.6744

Table 3: Convergence rate ϱ_k for several contact problems with self-contact constraints in 2D.

Neumann data is defined on $\Gamma_{1,\pm} := (-2,-1)\times(-1,1)\times\{\pm 1\}$, $\Gamma_2 := \{-2\}\times(-1,1)\times(0,1)$ and $\Gamma_3 := \{-2\}\times(-1,1)\times(-1,0)$ and is given by $\pm f(x_0,x_1,x_2)(\sin(\alpha_0)\cos(\alpha_1),\sin(\alpha_0)\sin(\alpha_1),\cos(\alpha_0))$ for $(x_0,x_1,x_2)\in\Gamma_{1,\pm}$ with $f(x_0,x_1,x_2)=\frac{1}{3}(x_0+2)f_1-\frac{1}{3}(x_0-1)f_0$, $f_0,f_1\in\mathbb{R}$, and $f_i\cdot(\sin(\beta_0)\cos(\beta_1),\sin(\beta_0)\sin(\beta_1),\cos(\beta_0))$ on Γ_i with $f_i\in\mathbb{R}$, $i=2,3$. Again, we consider six different configurations which are given by the Neumann data as tabulated in Table 4. We use conforming finite element approximations with piecewise trilinear ansatz functions on hexahedrons. The maximum number of refinements is $k_{\max}=6$ which yields $9,622,272$ degrees of freedom.

-	α_0	α_1	β_0	β_1	f_0	f_1	f_3	f_4	scale
4(a)	0	0	$\pi/2$	$\pi/4$	0	0	-1	1	40
4(b)	$\pi/4$	$\pi/4$	0	0	0.5	0.5	0	0	40
4(c)	$\pi/4$	$\pi/4$	0	0	0.5	-2	0	0	40
4(d)	$\pi/4$	$\pi/4$	0	0	-1	-1	0	0	40
4(e)	$\pi/4$	$\pi/4$	0	0	-1	4	0	0	40
4(f)	0	0	$\pi/2$	$\pi/4$	-1	-1	0	0	40

Table 4: Configurations with self-contact in 3D.

In Figure 4, the deformations and Von Mises stresses of the solutions are depicted. In Figure 5, the lower side of the crack layer $[-2,1]\times[-1,1]\times\{0\}$ is shown. The estimated convergence rate ϱ_k is tabulated in Table 5. Because of the tearing along Γ_C, we expect that the solution in Figure 4(b) is in $B^{3/2}_{2,\infty}(\Omega)$ only and, hence, the convergence rate ϱ is 1/2. In fact, this is indicated by ϱ_k in Table 5. As a consequence of the symmetric Neumann data leading to complete self-contact along Γ_C, the regularity of the solution in Figure 4(f) should be strictly greater than 1/2. This is clearly confirmed by the estimated convergence rate ϱ_k in Table 5. The configuration 4(a) leads to considerable shearing so that the solution should be not more than $B^{3/2}_{2,\infty}$-regular. Thus,

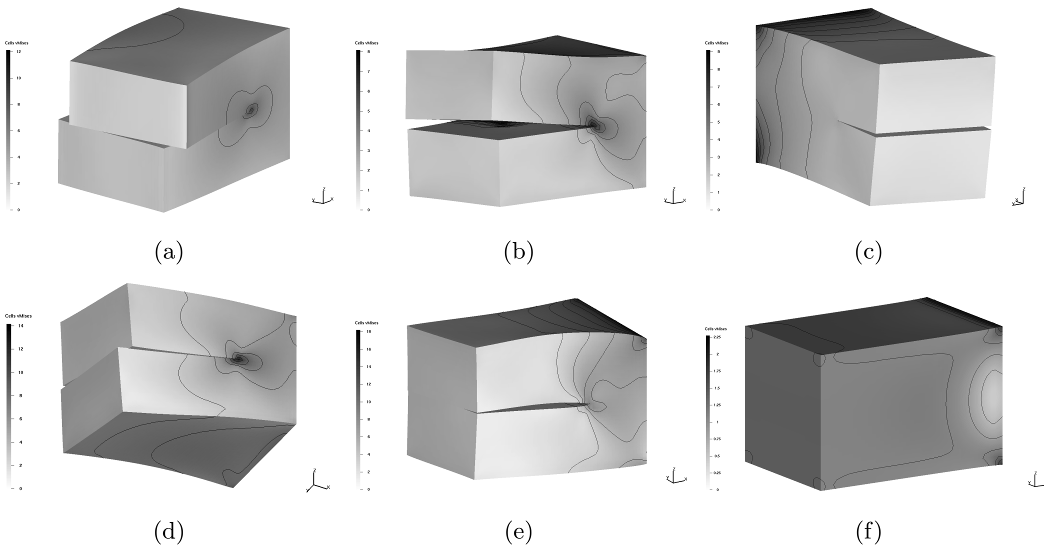


Figure 4: Solutions of self contact in 3D

the convergence rate should be 1/2. We observe in Table 5 that the estimated convergence rate ϱ_k nearly predict this expectation. For the other configurations 4(c), 4(d) and 4(e), we expect that tearing or shearing also leads to $B^{3/2}_{2,\infty}$-regularity of the solution and, hence, to the convergence rate $\varrho = 1/2$. Unfortunately, this expectation is not reflected in Table 5. For these configurations, the estimated convergence rate ϱ_k may overestimate the asymptotic convergence rate ϱ. As in the 2D case 2(b), this effect may be explained by the insufficient resolution of the finite element approximation. However, the expected behavior $\varrho_k \geq 1/2$ can be observed in all cases.

h	4(a)	4(b)	4(c)	4(d)	4(e)	4(f)	6(a)	6(c)
0.5000	0.4645	0.4299	0.6185	0.5653	0.5982	0.6307	0.5914	0.6427
0.2500	0.5243	0.5046	0.6974	0.6197	0.6634	0.6422	0.6142	0.6528
0.1250	0.5324	0.5223	0.7293	0.6158	0.6849	0.6573	0.6036	0.6576
0.0625	0.5261	0.5206	0.7287	0.5892	0.6754	0.6681	0.5795	0.6492
0.0312	0.5176	0.5145	0.7032	0.5605	0.6460	0.6630	0.5511	0.6117
0.0156	0.5104	0.5087	0.6519	0.5363	0.6017	0.6256	-	-

Table 5: Convergence rate ϱ_k for several contact problems with self-contact constraints in 3D.

In the next experiment, we study a problem with self-contact constraints where the crack does not intersect with the Neumann or Dirichlet boundary. The domain is given by $\Omega := [0,5]^2 \times [0,2] \backslash \Gamma_C$ and Dirichlet boundary by $\Gamma_D := [0,5]^2 \times \{0\}$, where homogeneous Dirichlet conditions are prescribed. We choose the same material parameters for linear elasticity based on Hooke's law as before. The crack is described by $\Gamma_C := (1,3)^2 \times \{1\} \cup (2,4)^2 \times \{1\}$. The Neumann data is

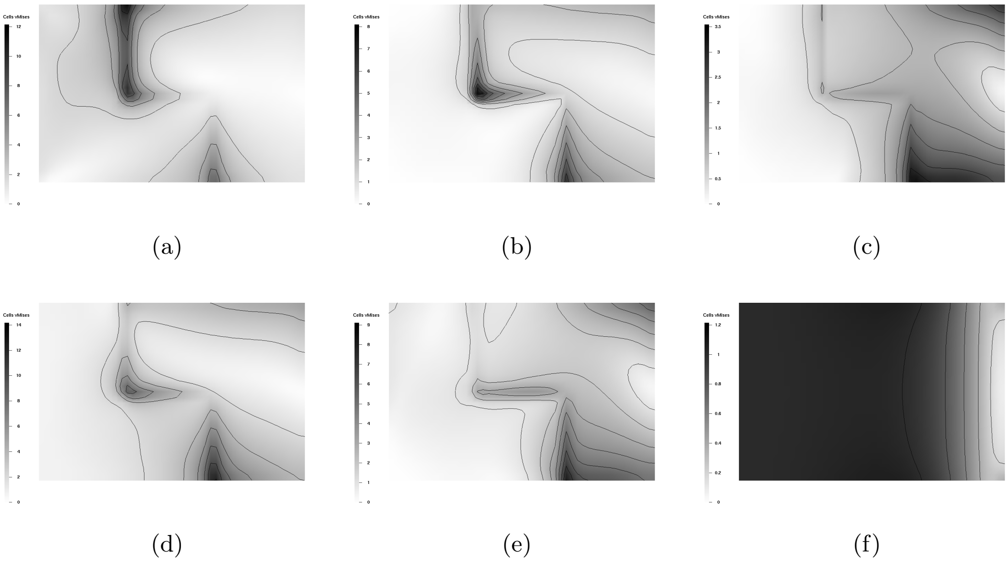


Figure 5: Lower side of the crack layer.

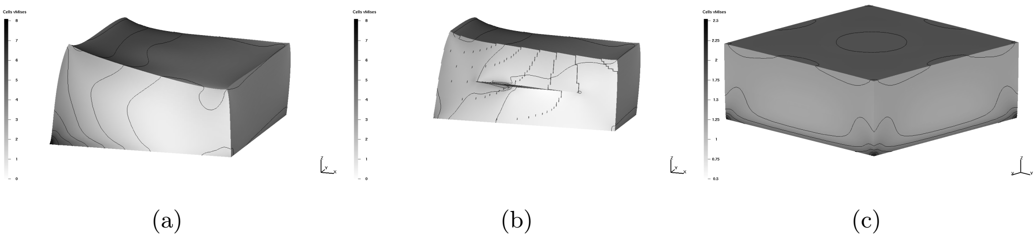
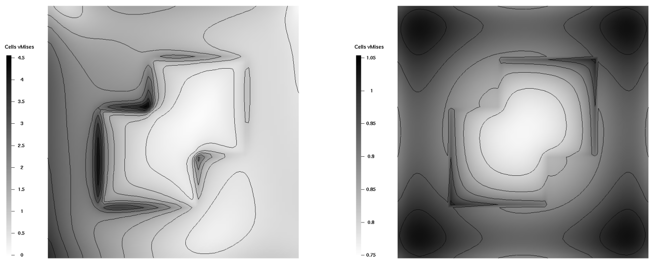


Figure 6: Solutions of a self-contact problem in 3D with interior crack.

given by $f(x)(\sin(\alpha_0)\cos(\alpha_1), \sin(\alpha_0)\sin(\alpha_1), \cos(\alpha_0))$ with $f(x_0, x_1, x_2) = \frac{1}{5}x_0 f_1 - \frac{1}{5}(x_0 - 5)f_0$ for $(x_0, x_1, x_2) \in \Gamma_1 := (0,5) \times (0,5) \times \{2\}$. We consider two configurations: $\alpha_0 = \alpha_1 = \pi/4$, $f_0 = 2$, $f_1 = -1$ as well as $\alpha_0 = 0$, $\alpha_1 = \pi/2$, $f_0 = f_1 = -1$. The first configuration leads to self-contact as well as tearing, see Figures 6(a),(b) and 7(a), whereas the second configuration implies a complete self-contact along Γ_C, see Figures 6(c) and 7(b). The finite element solutions of both configurations are scaled by a factor of 150 in these figures. The maximum number of refinements is set to $k_{\max} := 5$ with $4,997,763$ degrees of freedom. Due to the symmetric Neumann data of the second configuration, we expect that the solution is of higher regularity. Indeed, this is confirmed by the estimated convergence rate ϱ_k in Table 5. The solution of the first configuration is presumably not of higher regularity. Unfortunately, this is not reflected by the estimated convergence rate ϱ_k in Table 5 which seems to be too large. Again, a higher resolution of the finite element approximation may lead to clearer results.



(a) (b)

Figure 7: Lower sides of the crack layer of the self-contact problem with interior crack.

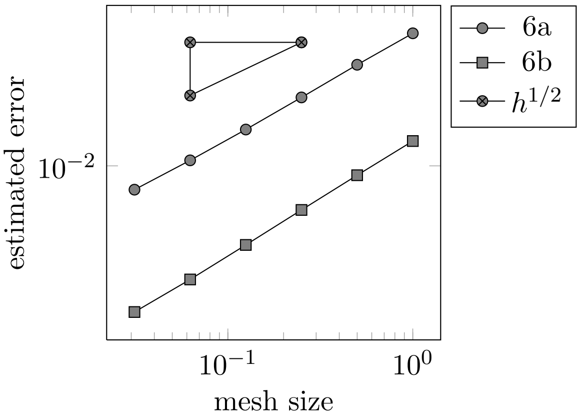


Figure 8: Q_k for the contact problem with interior crack.

5.2 Unilateral contact

We also consider contact problems with unilateral contact constraints. First, we study a 2D problem in linear elasticity with plane stress and the same material parameters as before. The domain is $\Omega := (0, 4) \times (0, 1)$ and the Dirichlet boundary is $\Gamma_D := (0, 4) \times \{0\}$ with prescribed homogeneous conditions. Again, homogeneous Dirichlet boundary conditions are prescribed. The rigid obstacle is given by the piecewise constant function $\psi(x) = 1 - d_i$ for $x \in (1 + i, 2)$, $i = 1, 2$ so that the contact boundary is $\Gamma_C := (1, 3) \times \{1\}$. Note that the constraints are exactly fulfilled by the finite element approximation and, hence, no additional approximation errors resulting, e.g., from a curved obstacle or non-matching grids has to be taken into account. The infeed of the obstacle $d_i \in \mathbb{R}$ is given in Table 6. The resulting deformations are shown in Figure 9. The maximum number of refinements is $k_{\max} := 9$ which corresponds to $8, 390, 656$ degrees of freedom.

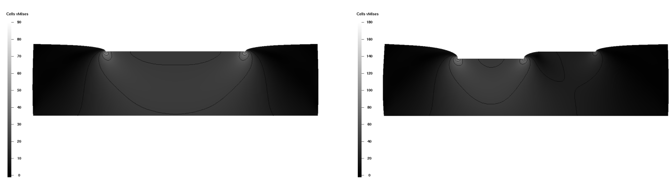
The second contact problem with unilateral contact constraints is a 3D problem. The domain is set to $\Omega := (0, 5)^2 \times (0, 2)$ and the Dirichlet boundary is $\Gamma_D := [0, 5]^2 \times \{0\}$, where homogeneous boundary conditions are assumed. Here, the obstacle is described by the piecewise constant function $\psi(x) = 1 - d_i$ for $x \in \Gamma_{C,i}$ with $\Gamma_{C_0} := (1,3)^2 \times \{2\}$ and $\Gamma_{C,1} := (2,4)^2 \times \{2\} \backslash \Gamma_{C,0}$. Thus, the contact boundary is $\Gamma_C := \Gamma_{C,0} \cup \Gamma_{C,1}$. The infeed d_i is given in Table 6 and the resulting deformations are shown in Figure 10. The maximum number of refinements is $k_{\max} := 4$ with $4,976,832$ degrees of freedom.

The estimated convergence rate ϱ_k is given in Table 7. We observe that the convergence rate ϱ is estimated by 1/2. Observe that in Examples 10(a) and 10(b) the geometric condition (R1) is satisfied, while in 10(c) this condition is violated in a neighborhood of those triple points, where the Neumann boundary meets the discontinuity line of the obstacle. Hence, in Example 10(c) the solution might be less regular in a neighborhood of these points. This is also reflected in the lower convergence rate, cf. Table 7.

In contrast to the contact problems with self-contact, these numerical results comply with the expectation that the solutions are solely in $B^{3/2}_{2,\infty}(\Omega)$ and not of higher regularity. The reason for this accordance possibly lies in the fact that the configurations used in the experiments for unilateral contact are considerably simpler than the configurations for self-contact. The deformations in the unilateral contact only result from the infeed of the obstacle, whereas complicated Neumann boundary data is prescribed in the configurations for self-contact. We may conclude that the finite element approximations seems to better resolve the significant singularities arising from unilateral contact.

-	d_0	d_1	scale
9(a)	-0.01	0.01	10
9(b)	-0.02	0.01	10
10(a)	-0.01	-0.01	10
10(b)	-0.02	-0.01	10
10(c)	-0.01	-0.02	10

Table 6: Infeed of the obstacle for unilateral contact in 2D.



(a) (b)

Figure 9: Solutions of unilateral contact in 2D.

In summary, we observe that the convergence rates are estimated by $\varrho_k \geq 1/2$ for all configurations which indicates that their solutions are at least in $B^{3/2}_{2,\infty}(\Omega)$. This regularity is predicted

(a) (b) (c)

Figure 10: Solutions of unilateral contact in 3D

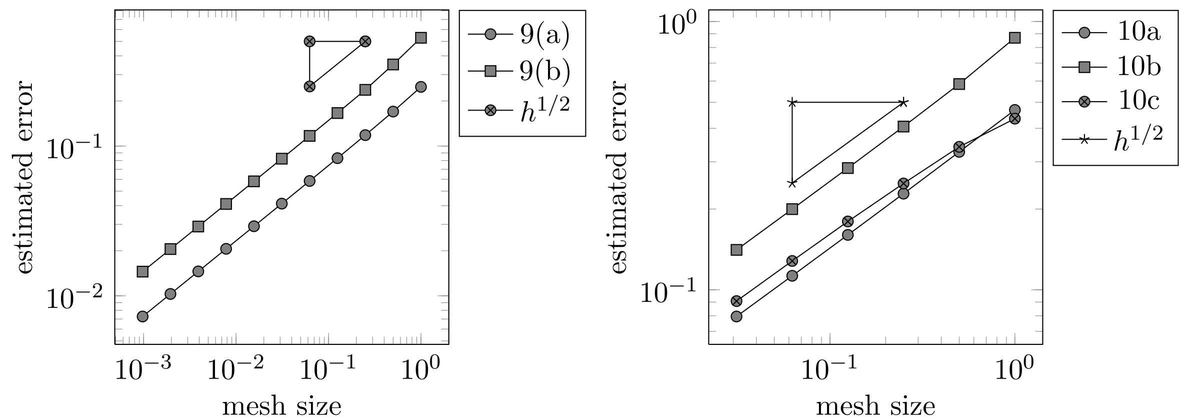


Figure 11: Q_k for contact problems with unilateral contact constraints in 2D (left) and in 3D (right).

by Theorem 2.1. In several configurations, a higher regularity is indicated by the estimated convergence rate ϱ_k where singularities resulting from shearing or tearing should actually prevent this. The overestimation of the asymptotic convergence rate ϱ by ϱ_k may result from possibly insufficient finite element approximations. The contribution of the solution with singular exponent 1/2 may be too small to be adequately resolved by the finite elements. The overestimation effects do not occur in the experiments where unilateral contact conditions are prescribed. Here, the expected convergence rate is 1/2 and, hence, the $B^{3/2}_{2,\infty}$-regularity is predicted as desired.

A An interpolation result

We prove Lemma 4.2 on the basis of Theorem 14.3 from [LM72, Chapter 1]. It is assumed that

(A1) $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain (local bi-Lipschitz mappings) and that $\Gamma_D \subset \partial\Omega$ is a closed, possibly empty subset.

As in Section 4.1, for $s \in (1,2)$ we define the space $\mathcal{M}^s_{\Gamma_D}(\Omega) = \{\, v \in H^s_{\Gamma_D}(\Omega)\,;\, \operatorname{div} \mathbf{C}\varepsilon(v) \in L^2(\Omega)\,\}$, which is endowed with the graph norm $|||w|||_s = \|w\|_{H^s(\Omega)} + \|\operatorname{div} \mathbf{C}\varepsilon(u)\|_{L^2(\Omega)}$. Let the

Table 7: Convergence rates, unilateral

h	9a	9b	10a	10b	10c
0.5000	0.5476	0.5924	0.5192	0.5728	0.3490
0.2500	0.5228	0.5623	0.5173	0.5266	0.4537
0.1250	0.5107	0.5144	0.5107	0.5139	0.4701
0.0625	0.5057	0.5073	0.5065	0.5083	0.4894
0.0312	0.5030	0.5053	0.5037	0.5047	0.4969
0.0156	0.5016	0.5026	-	-	-
0.0078	0.5008	0.5014	-	-	-
0.0039	0.5005	0.5007	-	-	-
0.0020	0.5002	0.5004	-	-	-
0.0010	0.5001	0.5002	-	-	-

operator $B : H_0^1(\Omega) \to (H_0^1(\Omega))^*$ be defined as follows: For all $u, v \in H_0^1(\Omega)$

$$\langle B(u), v\rangle = \int_\Omega \mathbf{C}\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x =: b(u,v) \tag{A.1}$$

with $\mathbf{C}$ as in Section 3.1. The regularity assumption reads

(AR1) There exists a constant $s_0 \in (1,2)$ such that the differential operator B defined in (A.1) is a topological isomorphism from $H_0^{s_0}(\Omega)$ onto $\big([H_0^1(\Omega), L^2(\Omega)]_{s_0-1}\big)^*$.

Lemma A.1. *Let (A1) and (AR1) be satisfied. Then for all* $s \in (1, s_0)$ *the following identity holds true with* $\theta = \frac{s_0-s}{s_0-1}$*:*

$$\mathcal{M}^s_{\Gamma_D}(\Omega) = [\mathcal{M}^{s_0}_{\Gamma_D}(\Omega), \mathcal{M}^1_{\Gamma_D}(\Omega)]_\theta. \tag{A.2}$$

Proof. In the notation of Theorem 14.3 of [LM72, Chapter 1] we set

$$\Phi = Y = H^1_{\Gamma_D}(\Omega), \quad \mathcal{Y} = \mathcal{X} = L^2(\Omega), \quad \Psi = \widetilde{\mathcal{Y}} = (H_0^1(\Omega))^*,$$
$$X = H^{s_0}_{\Gamma_D}(\Omega), \quad \widetilde{\mathcal{X}} = \big([H_0^1(\Omega), L^2(\Omega)]_{s_0-1}\big)^*.$$

Further, we define $\partial : Y \to \widetilde{\mathcal{Y}}$ via the following relation: For all $u \in Y, v \in H_0^1(\Omega)$ we set $\langle \partial u, v\rangle = \int_\Omega \mathbf{C}\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x$, i.e. $\partial u = -\operatorname{div}\mathbf{C}\varepsilon(u)$ in the distributional sense. Still in the notation of [LM72] we have

$$Y_{\partial,\mathcal{Y}} = \{\, v \in Y \,;\, \partial v \in \mathcal{Y}\,\} = \mathcal{M}^1_{\Gamma_D}(\Omega), \quad X_{\partial,\mathcal{X}} = \{\, v \in X \,;\, \partial v \in \mathcal{X}\,\} = \mathcal{M}^{s_0}_{\Gamma_D}(\Omega).$$

Finally, we define $\mathcal{G} : \widetilde{\mathcal{Y}} \to Y$, $f \mapsto \omega_f \in H_0^1(\Omega)$ as a solution operator via the relation $b(\omega_f, v) = \langle f, v\rangle$ for all $v \in H_0^1(\Omega)$. By assumption (AR1), $\mathcal{G}$ is also well defined, linear and continuous as a mapping from $\big([H_0^1(\Omega), L^2(\Omega)]_{s_0-1}\big)^*$ to $H_0^1(\Omega) \cap H^{s_0}(\Omega) \subset H^{s_0}_{\Gamma_D}(\Omega)$. Moreover, for all $f \in \widetilde{\mathcal{X}} + \widetilde{\mathcal{Y}}$ it holds that $\partial\mathcal{G}f = f$. Indeed, for all $v \in H_0^1(\Omega)$ we have $\langle \partial\mathcal{G}f, v\rangle = \int_\Omega \mathbf{C}\varepsilon(\omega_f) : \varepsilon(v)\,\mathrm{d}x =$

$\langle f, v\rangle$, and by density of $H_0^1(\Omega)$ in $[H_0^1(\Omega), L^2(\Omega)]_{s_0-1}$ this is true for all $\psi \in [H_0^1(\Omega), L^2(\Omega)]_{s_0-1}$ provided that $f \in \widetilde{\mathcal{X}}$. Hence, Theorem 14.3 in [LM72, Chapter 1] is applicable and implies that for all $\theta \in (0,1)$ the identity $[X_{\partial,\mathcal{X}}, Y_{\partial,\mathcal{Y}}]_\theta = \big([X,Y]_\theta\big)_{\partial,[\mathcal{X},\mathcal{Y}]_\theta}$ is valid, which is (A.2). □

References

[ACF02] A. Amassad, D. Chenais, and C. Fabre. Optimal control of an elastic contact problem involving Tresca friction law. *Nonlinear Anal., Theory Methods Appl.*, 48:1107–1135, 2002.

[AMS08] F. Auricchio, A. Mielke, and U. Stefanelli. A rate-independent model for the isothermal quasi-static evolution of shape-memory materials. *Math. Models Methods Appl. Sci.*, 18(1):125–164, 2008.

[BGK87] P. Boieri, F. Gastaldi, and D. Kinderlehrer. Existence, uniqueness, and regularity results for the two-body contact problem. *Appl. Math. Optimization*, 15:251–277, 1987.

[CD93] Martin Costabel and Monique Dauge. General edge asymptotics of solutions of second-order elliptic boundary value problems. I,II. *Proc. R. Soc. Edinb., Sect. A*, 123(1):109–184, 1993.

[CDY04] Martin Costabel, Monique Dauge, and Zohar Yosibash. A quasi-dual function method for extracting edge stress intensity functions. *SIAM J. Math. Anal.*, 35(5):1177–1202, 2004.

[CN00] Zhiming Chen and Ricardo H. Nochetto. Residual type a posteriori error estimates for elliptic obstacle problems. *Numer. Math.*, 84(4):527–548, 2000.

[Dau88] M. Dauge. *Elliptic Boundary Value Problems on Corner Domains. Smoothness and Asymptotic Expansion*, volume 1341 of *Lecture Notes in Mathematics.* Springer-Verlag, 1988.

[Ebm99] C. Ebmeyer. Mixed boundary value problems for nonlinear elliptic systems in n-dimensional Lipschitzian domains. *Z. Anal. Anwend.*, 18(3):539–555, 1999.

[EF99] C. Ebmeyer and J. Frehse. Mixed boundary value problems for nonlinear elliptic equations in multidimensional non-smooth domains. *Math. Nachrichten*, 203:47–74, 1999.

[EFK02] C. Ebmeyer, J. Frehse, and M. Kassmann. Boundary regularity for nonlinear elliptic systems: Applications to the transmission problem. In S.Hildebrandt and H.Karcher, editors, *Geometric Analysis and Nonlinear Partial Differential Equations*, pages 505–517. Springer Verlag, 2002.

[Fal74] Richard S. Falk. Error estimates for the approximation of a class of variational inequalities. *Math. Comput.*, 28:963–971, 1974.

[GH96] M. Giaquinta and S. Hildebrandt. *Calculus of Variations I.* Springer-Verlag, Berlin, Heidelberg, 1996.

[Gri85] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Pitman Publishing Inc, Boston, 1985.

[Gri87] P. Grisvard. Le problème de Dirichlet pour les équations de Lamé. *C. R. Acad. Sci. Paris, Série 1*, 304:71–73, 1987.

[HMW05] S. Hüeber, A. Matei, and B. I. Wohlmuth. A mixed variational formulation and an optimal a priori error estimate for a frictional contact problem in elasto-piezoelectricity. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, 48(96)(2):209–232, 2005.

[HN07] Patrick Hild and Serge Nicaise. Residual a posteriori error estimators for contact problems in elasticity. *ESAIM, Math. Model. Numer. Anal.*, 41(5):897–923, 2007.

[HS07] V. Heuveline and F. Schieweck. H^1-interpolation on quadrilateral and hexahedral meshes with hanging nodes. *Computing*, 80(3):203–220, 2007.

[Kin81] David Kinderlehrer. Remarks about Signorini's problem in linear elasticity. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, 8:605–645, 1981.

[KM88] V. A. Kozlov and V. G. Maz'ya. Spectral properties of the operator bundles generated by elliptic boundary-value problems in a cone. *Func. Anal. Appl.*, 22:114–121, 1988.

[KM07] Moritz Kassmann and W.R. Madych. Difference quotients and elliptic mixed boundary problems of second order. *Indiana Univ. Math. J.*, 56(3):1047–1082, 2007.

[Kne06] D. Knees. Global regularity of the elastic fields of a power-law model on Lipschitz domains. *Math. Methods Appl. Sci.*, 29:1363–1391, 2006.

[Kon67] V. A. Kondrat'ev. Boundary value problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.*, 10:227–313, 1967.

[KS11] Dorothee Knees and Andreas Schröder. Computational aspects of quasi-static crack propagation. WIAS preprint 1611, 2011. accepted in DCDS-S.

[LM72] J.L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I. Translated from the French by P. Kenneth.* Grundlehren der Mathematischen Wissenschaften, Band 181, Springer-Verlag, 1972.

[Mie05] A. Mielke. Evolution of rate-independent systems (ch. 6). In C.M. Dafermos and E. Feireisl, editors, *Handbook of Differential Equations, Evolutionary Equations II*, pages 461–559. Elsevier B.V., 2005.

[MPPS10] A. Mielke, L. Paoli, A. Petrov, and U. Stefanelli. Error estimates for space-time discretizations of a rate-independent variational inequality. *SIAM J. Numer. Anal.*, 48(5):1625–1646, 2010.

[Neč83] J. Nečas. *Introduction to the Theory of Nonlinear Elliptic Equations.* Teubner Verlagsgesellschaft, Leipzig, 1983.

[Nic92] S. Nicaise. About the Lamé system in a polygonal or a polyhedral domain and a coupled problem between the Lamé system and the plate equation, I: Regularity of the solution. *Ann. della Scuola Normale Sup. di Pisa*, 19:327–361, 1992.

[Nik75] S. M. Nikol'skii. *Approximation of functions of several variables and imbedding theorems.* Springer-Verlag, 1975.

[NW02] Ricardo H. Nochetto and Lars B. Wahlbin. Positivity preserving finite element approximation. *Math. Comput.*, 71(240):1405–1419, 2002.

[SA84] A.M. Sanchez and R. Arcangeli. Estimations des erreurs de meilleure approximation polynomiale et d'interpolation de Lagrange dans les espaces de Sobolev d'ordre non entier. *Numer. Math.*, 45:301–321, 1984.

[Sav98] G. Savaré. Regularity results for elliptic equations in Lipschitz domains. *J. Funct. Anal.*, 152:176–201, 1998.

[Sch89] Rainer Schumann. A remark on a boundary contact problem in linear elasticity. *Manuscr. Math.*, 63(4):455–468, 1989.

[SZ90] L.Ridgway Scott and Shangyou Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comput.*, 54(190):483–493, 1990.

[Tri78] H. Triebel. *Spaces of Besov-Hardy-Sobolev type.* Teubner-Texte zur Mathematik. Teubner, 1978.

[Val88] T. Valent. *Boundary Value Problems of Finite Elasticity.* Springer Verlag Inc., New York, 1988.