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**Quasiconvexity equals rank-one convexity for isotropic sets  
of 2x2 matrices**

Sebastian Heinz

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Weierstraß-Institut  
Mohrenstr. 39  
10117 Berlin  
Germany  
E-Mail: [sebastian.heinz@wias-berlin.de](mailto:sebastian.heinz@wias-berlin.de)

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)

Leibniz-Institut im Forschungsverbund Berlin e. V.

Mohrenstraße 39

10117 Berlin

Germany

Fax: +49 30 2044975

E-Mail: [preprint@wias-berlin.de](mailto:preprint@wias-berlin.de)

World Wide Web: <http://www.wias-berlin.de/>

## Abstract

Let  $K$  be a given compact set of real  $2 \times 2$  matrices that is isotropic, meaning invariant under the left and right action of the special orthogonal group. Then we show that the quasicconvex hull of  $K$  coincides with the rank-one convex hull (and even with the lamination convex hull of order 2). In particular, there is no difference between quasicconvexity and rank-one convexity for  $K$ . This is a generalization of a known result for connected sets.

## 1 Introduction

We study quasicconvexity in the calculus of variations. Morrey [Mor52] introduced it as the essential property for functions in the context of sequentially weakly lower semicontinuity for multiple integrals. He also conjectured that quasicconvexity is a “non-local” property, which was later shown to be true by Kristensen [Kri99]. At the heart of Kristensen’s proof lies Šverák’s counterexample of a rank-one convex function that fails to be quasicconvex [Šve92]. However, this counterexample works only in the case of an underlying space  $\mathbb{M}^{m \times n}$  with  $m \geq 3$ ,  $n \geq 2$ . Müller [Mül99a] showed that rank-one convexity implies quasicconvexity on diagonal  $2 \times 2$  matrices. The general situation in  $\mathbb{M}^{2 \times 2}$  remains unknown.

Closely related to the quasicconvexity for functions is the corresponding concept for sets. Basically, quasicconvex sets are lower-level sets of quasicconvex continuous functions. We focus on quasicconvexity for isotropic sets in  $\mathbb{M}^{2 \times 2}$  and prove the following result (see Theorem 7.2).

**Theorem (Equivalence).** *Let  $K \subseteq \mathbb{M}^{2 \times 2}$  be a given compact and isotropic set. Then  $K$  is lamination convex if and only if  $K$  is quasicconvex.*

As long as the set  $K$  is connected, there is even equivalence between lamination convexity and polyconvexity. This was shown by Conti et al. [CDLMR03] and, before, by Cardaliaguet and Tahraoui [CT00, CT02a, CT02b] in the case when  $K$  contains only matrices with non-negative determinant. Conti et al. [CDLMR03] give also an example of a disconnected  $K$  that is lamination convex but not polyconvex. In addition, we will characterize the structure of the quasicconvex hull of  $K$ . Our main result reads (see Theorem 7.3)

**Theorem (Characterization of  $K^{\text{qc}}$ ).** *Let  $K \subseteq \mathbb{M}^{2 \times 2}$  be compact and isotropic. Then its quasicconvex hull coincides with its lamination convex hull of order 2.*

The paper is organized as follows:

In Section 2 we will fix the notations and recall definitions of the convexity notions that are used later on. Preliminaries can be found in Section 3 and 4. Then we refine a result by Conti et

al. [CDLMR03] for connected  $K$  in Section 5. Section 6 is dedicated to the closed lamination convex hull  $K^{\text{clc}}$  and its structure. The key observation is that the principle structure of  $K^{\text{clc}}$  is already determined by the lamination convex hull of order one. In Section 7 we deal with the equivalence of lamination convexity and quasiconvexity. The main step is to show that what is disconnected in  $K^{\text{clc}}$  remains so in  $K^{\text{pc}}$ . Then we apply a deep result by Faraco and Székelyhidi [FS08] saying that the quasiconvex hull for the support of a homogeneous gradient Young measure is connected.

## 2 Functions, measures, and hulls

We are going to recall some convexity notions that play an important role in this paper. Our focus lies on dimension 2. A detailed discussion, also for higher dimensions, can be found in Dacorogna [Dac89, 4.1], Ball [Bal77] and Müller [Mül99b].

We denote by  $\mathbb{M}^{2 \times 2}$  the vector space of all real  $2 \times 2$  matrices equipped with the Euclidean structure of  $\mathbb{R}^4$ . The corresponding matrix norm is denoted by  $|\cdot|$ , the identity matrix by  $I$ . Let  $f: \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$  be a given continuous function. Then  $f$  is *convex* if for every  $A, B \in \mathbb{M}^{2 \times 2}$  we have

$$\forall \lambda \in [0, 1] \quad f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B). \quad (1)$$

The function  $f$  is *polyconvex* if there exists a convex function  $g: \mathbb{R}^5 \rightarrow \mathbb{R}$  such that for every  $A \in \mathbb{M}^{2 \times 2}$  we have  $f(A) = g(A, \det(A))$ , where  $\det(A)$  denotes the determinant of  $A$ . We will often use that for every real number  $\alpha \in \mathbb{R}$  the function  $\alpha \det$  is polyconvex. The function  $f$  is *quasiconvex* (in the sense of Morrey [Mor52]), if for every  $A \in \mathbb{M}^{2 \times 2}$  and every smooth function  $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with compact support we have

$$0 \leq \int_{\mathbb{R}^2} (f(A + D\phi(x)) - f(A)) dx.$$

The function  $f$  is *rank-one convex* if (1) holds for every  $A, B \in \mathbb{M}^{2 \times 2}$  that are *rank-one connected*, meaning  $A - B$  equals the tensor product  $a \otimes b$  for some vectors  $a, b \in \mathbb{R}^2$ . Polyconvexity and rank-one convexity were introduced by Ball [Bal77].

With the help of the convexity notions for functions, we now define the convexity notions for sets. Let  $K \subseteq \mathbb{M}^{2 \times 2}$  be a given set and  $A \in \mathbb{M}^{2 \times 2}$  a matrix. Then  $A$  lies in the *polyconvex hull* of  $K$  and we write  $A \in K^{\text{pc}}$  whenever  $f(A) \leq \sup\{f(B) \mid B \in K\}$  holds for every polyconvex function  $f: \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$ . The set  $K$  is called *polyconvex* whenever  $K = K^{\text{pc}}$  holds. The *quasiconvex hull* and the *rank-one convex hull* as well as *quasiconvexity* and *rank-one convexity* for sets are defined correspondingly.

We will give an alternative characterization in the case of compact sets. Therefore, denote by  $\mathcal{P}_0(\mathbb{M}^{2 \times 2})$  the set of all compactly supported probability measures that are defined over the Borel sets of  $\mathbb{M}^{2 \times 2}$ . Let  $\nu \in \mathcal{P}_0(\mathbb{M}^{2 \times 2})$  be a given element. We write  $\bar{\nu}$  for its mean value and  $\text{supp}(\nu)$  for its support, meaning the compliment of the set  $\cup\{U \subseteq \mathbb{M}^{2 \times 2} \mid \nu(U)=0 \wedge U \text{ open}\}$ . In addition, let  $f: \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$  be a continuous function. Then the following pairing is finite and well-defined

$$\langle \nu, f \rangle = \int_{\mathbb{M}^{2 \times 2}} f(A) d\nu(A).$$

We define the sets  $\mathcal{P}^{\text{pc}}$ ,  $\mathcal{P}^{\text{qc}}$  and  $\mathcal{P}^{\text{rc}}$ . A probability measure  $\nu \in \mathcal{P}_0(\mathbb{M}^{2 \times 2})$  lies in  $\mathcal{P}^{\text{pc}}$  ( $\mathcal{P}^{\text{qc}}$  or  $\mathcal{P}^{\text{rc}}$ ) if and only if Jensen's inequality  $f(\bar{\nu}) \leq \langle \nu, f \rangle$  is fulfilled for every polyconvex (quasiconvex or rank-one convex) continuous function  $f: \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$ . Kinderlehrer and Pedregal [KP91] show that every  $\nu \in \mathcal{P}^{\text{qc}}$  is a homogenous gradient Young measure. Whereas every  $\nu \in \mathcal{P}^{\text{rc}}$  is a laminate, see Pedregal [Ped93].

**Remark 2.1.** Let  $K \in \mathbb{M}^{2 \times 2}$  be a given compact set. Then the set  $K^{\text{pc}}$  coincides with  $\{\bar{\nu} \mid \nu \in \mathcal{P}^{\text{pc}} \wedge \text{supp}(\nu) \subseteq K\}$  and  $K^{\text{qc}}$  as well as  $K^{\text{rc}}$  can be characterized in a corresponding way.

As in Müller and Šverák [MŠ96],  $K$  is called *lamination convex* if for every rank-one connected  $A, B \in K$  and every real number  $\lambda \in [0, 1]$  we have that  $\lambda A + (1-\lambda)B$  lies in  $K$ . The *closed lamination convex hull*  $K^{\text{clc}}$  is the intersection of all closed lamination convex subsets in  $\mathbb{M}^{2 \times 2}$  that contain  $K$ . Note that  $\{A, B\}^{\text{clc}}$  equals  $\{\lambda A + (1-\lambda)B \mid \lambda \in [0, 1]\}$  and, hence, is a connected set if  $A, B \in \mathbb{M}^{2 \times 2}$  are rank-one connected. Otherwise  $\{A, B\}^{\text{clc}} = \{A, B\}$  is disconnected. Here we call a given set  $S \subseteq \mathbb{M}^{2 \times 2}$  *connected* if there is no way to write  $S$  as the union of two disjoint nonempty relatively-open subsets of  $S$ . Moreover, we set  $K^{\text{lc},1} = \cup\{\{A, B\}^{\text{clc}} \mid A, B \in K\}$  as well as  $K^{\text{lc},2} = (K^{\text{lc},1})^{\text{lc},1}$ , which are called the *lamination convex hulls of order one and two*, respectively. We would like to remark that, in general, the set  $K^{\text{clc}}$  and the lamination convex hull of  $K$  (which is not defined here) are different as has been shown by Kolář [Kol03]. The previous definitions together with the hierarchy of convexity notions on the level of functions imply that

$$K \subseteq K^{\text{lc},1} \subseteq K^{\text{lc},2} \subseteq K^{\text{clc}} \subseteq K^{\text{rc}} \subseteq K^{\text{qc}} \subseteq K^{\text{pc}}.$$

Finally, we denote by  $\text{cc}(K)$  the set of all connected components (meaning maximal connected subsets) of  $K$ .

### 3 Compatible isotropic sets

We give a characterization of compatible isotropic sets. The general result for  $\mathbb{M}^{n \times n}$ ,  $n \geq 1$ , is due to Šilhavý [Šil01, Pro. 3.1]. In our case  $\mathbb{M}^{2 \times 2}$ , this was already done by Aubert and Tahraoui [AT87, Thé. 2.8], if only for matrices with non-negative determinant. The proofs of Lemma 3.1 and Lemma 3.2 are given for the convenience of the reader.

We call a set  $M \subseteq \mathbb{M}^{2 \times 2}$  *isotropic* whenever it is invariant under the left and right action of the special orthogonal group  $\text{SO}(2)$ , meaning  $M = M^{\text{iso}}$  where

$$M^{\text{iso}} = \{QAR \mid Q, R \in \text{SO}(2) \wedge A \in M\}.$$

Here we consider  $\text{SO}(2)$  as a subset of  $\mathbb{M}^{2 \times 2}$  so that the group action becomes just matrix multiplication. The following notation works well in the context of isotropic sets and has been used before by many authors. Let  $A \in \mathbb{M}^{2 \times 2}$  be a given matrix, then we define  $\lambda(A) = (\lambda_1(A), \lambda_2(A)) \in \mathbb{R}^2$  as the only pair of real numbers such that  $\{|\lambda_1(A)|, \lambda_2(A)\}$  is the set

of singular values of  $A$  and, in addition,  $|\lambda_1(A)| \leq \lambda_2(A)$  as well as  $\det(A) = \lambda_1(A)\lambda_2(A)$  holds. In fact, we have that

$$\{A\}^{\text{iso}} = \{B\}^{\text{iso}} \Leftrightarrow \lambda(A) = \lambda(B) \Leftrightarrow (|A| = |B| \wedge \det(A) = \det(B)).$$

We say that two subsets  $M_1, M_2 \subseteq \mathbb{M}^{2 \times 2}$  are *compatible* whenever there exist rank-one connected matrices  $A_1 \in M_1$  and  $A_2 \in M_2$ . Otherwise  $M_1$  and  $M_2$  are called *incompatible*.

**Lemma 3.1.** *Let  $A \in \mathbb{M}^{2 \times 2}$  be a given matrix. Then  $\{A\}^{\text{iso}}$  and  $\text{SO}(2)$  are compatible if and only if  $|\lambda_1(A)| \leq 1 \leq \lambda_2(A)$  holds.*

**Proof.** Assume that  $|\lambda_1(A)| \leq 1 \leq \lambda_2(A)$ . Then the following matrices are rank-one connected:  $I \in \text{SO}(2)$  and

$$I + \begin{pmatrix} \lambda_1(A)\lambda_2(A) - 1 & \sqrt{(1 - \lambda_1(A)^2)(\lambda_2(A)^2 - 1)} \\ 0 & 0 \end{pmatrix} \in \{A\}^{\text{iso}}.$$

Now assume that  $\{A\}^{\text{iso}}$  and  $\text{SO}(2)$  are compatible. Then there exist vectors  $a, b \in \mathbb{R}^2$  and a matrix  $C \in \{A\}^{\text{iso}}$  such that  $C = I + a \otimes b$ . We know that  $\det(C) = 1 + \langle a, b \rangle$  and  $|C|^2 = 2 + 2\langle a, b \rangle + |a|^2|b|^2$ . Together with the Cauchy-Schwarz inequality, we obtain the estimate  $|C|^2 - \det(C)^2 - 1 \geq 0$ . This implies that

$$\lambda_1(A)^2 + \lambda_2(A)^2 - \lambda_1(A)^2\lambda_2(A)^2 - 1 = (1 - \lambda_1(A)^2)(\lambda_2(A)^2 - 1) \geq 0.$$

Hence, we must have  $|\lambda_1(A)| \leq 1 \leq \lambda_2(A)$ .  $\square$

**Lemma 3.2.** *Let  $A, B \in \mathbb{R}^{2 \times 2}$  be given matrices. Then  $\{A\}^{\text{iso}}$  and  $\{B\}^{\text{iso}}$  are compatible if and only if  $|\lambda_1(A)| \leq \lambda_2(B)$  and, at the same time,  $|\lambda_1(B)| \leq \lambda_2(A)$ .*

**Proof.** Clearly, the lemma is true for  $\det(A) = \det(B) = 0$ . By symmetry, we can and we will assume that  $\det(B) > 0$  for the rest of the proof. If necessary, we replace  $A$  and  $B$  by  $-A$  and  $-B$ , respectively. In particular, we then have  $0 < \lambda_1(B)$ .

First, we start with  $|\lambda_1(A)| \leq \lambda_2(B)$  and  $|\lambda_1(B)| \leq \lambda_2(A)$ . Then we obtain the inequality  $|\lambda_1(A)/\lambda_2(B)| \leq 1 \leq \lambda_2(A)/\lambda_1(B)$ . By Lemma 3.1, we conclude that the sets  $\{C\}^{\text{iso}}$  and  $\text{SO}(2)$  are compatible where  $C = \text{diag}(\lambda_1(A)/\lambda_2(B), \lambda_2(A)/\lambda_1(B))$ . Hence, there exist a rotation  $R \in \text{SO}(2)$  and vectors  $a, b \in \mathbb{R}^2$  such that  $R + a \otimes b = C$ . If we multiply both sides from the right by  $\text{diag}(\lambda_2(B), \lambda_1(B))$ , we get

$$R \text{diag}(\lambda_2(B), \lambda_1(B)) + a \otimes b = \text{diag}(\lambda_1(A), \lambda_2(A)).$$

This shows that  $\{A\}^{\text{iso}}$  and  $\{B\}^{\text{iso}}$  are compatible.

Second, we start with  $\{A\}^{\text{iso}}$  and  $\{B\}^{\text{iso}}$  being compatible. Then we can write

$$R \text{diag}(\lambda_2(B), \lambda_1(B)) + a \otimes b = \text{diag}(\lambda_1(A), \lambda_2(A))Q$$

for some rotations  $R, Q \in \text{SO}(2)$  and vectors  $a, b \in \mathbb{R}^2$ . Multiplying both sides from the right by  $\text{diag}(1/\lambda_2(B), 1/\lambda_1(B))$ , we see that  $\text{SO}(2)$  and the set  $\{D\}^{\text{iso}}$  are compatible where

$$D = \text{diag}(\lambda_1(A), \lambda_2(A))Q \text{diag}(1/\lambda_2(B), 1/\lambda_1(B)). \quad (2)$$

Hence, by Lemma 3.1, we must have  $|\lambda_1(D)| \leq 1 \leq \lambda_2(D)$ . This implies, in particular, that we can fix a vector  $x_0 \in \mathbb{R}^2$  with  $|x_0| = 1$  such that  $|Dx_0| = 1$ .

The rest of the proof is by contradiction. Suppose that  $|\lambda_1(A)| > \lambda_2(B)$ . In view of (2), we obtain the inequality  $|Dx_0| \geq |\lambda_1(A)|/\lambda_2(B) > 1$ . Now suppose that  $|\lambda_1(B)| > \lambda_2(A)$ . Then we have  $|Dx_0| \leq \lambda_2(A)/|\lambda_1(B)| < 1$ . In both cases, we get a contradiction to the choice of  $x_0$ .  $\square$

In Figure 1(a), you see a given set  $\{A\}^{\text{iso}}$  and the region of all  $\{B\}^{\text{iso}}$  such that  $\{A\}^{\text{iso}}$  and  $\{B\}^{\text{iso}}$  are compatible.

The lemma and remark are taken from Conti et al. [CDLMR03, Lem. 2.2, Rem. 2].

**Lemma 3.3.** *Let  $c \in \mathbb{R} \setminus \{0\}$  be a real number. Then the functions  $\varphi_c^\pm: \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$  given by*

$$\varphi_c^\pm(A) = \lambda_2(A) \pm \lambda_1(A) - \det(A)/c$$

*are polyconvex. The same holds for the functions  $\varphi_0^\pm = -\det$ .*

**Proof.** The lemma follows from the convexity of the functions  $\lambda_2 \pm \lambda_1$ , which in turn is proved by the explicit computation

$$\lambda_2(A) \pm \lambda_1(A) = \sqrt{|A|^2 \pm 2 \det(A)} = \sqrt{(A_{11} \pm A_{22})^2 + (A_{21} \mp A_{12})^2}.$$

The functions  $-\det/c$  as well as  $-\det$  are polyconvex by definition.  $\square$

**Remark 3.4.** *Let  $A \in \mathbb{M}^{2 \times 2}$  be given. Consider the matrices  $A_+, A_- \in \mathbb{M}^{2 \times 2}$  defined via*

$$A_\pm = \begin{pmatrix} |\det(A)|^{1/2} & \pm \sqrt{|A|^2 - 2|\det(A)|} \\ 0 & |\det(A)|^{-1/2} \det(A) \end{pmatrix}.$$

*The matrices  $A_+$  and  $A_-$  are rank-one connected and  $A_+, A_- \in \{A\}^{\text{iso}}$  holds. Thus, the matrix  $(A_+ + A_-)/2 = \text{diag}(|\det(A)|^{1/2}, |\det(A)|^{-1/2} \det(A))$  as well as every other matrix  $B \in \mathbb{M}^{2 \times 2}$  with  $\det(A) = \det(B)$  and  $\lambda_2(B) \leq \lambda_2(A)$  lies in  $(\{A\}^{\text{iso}})^{\text{lc},1}$ .*

## 4 Lamination convex sets

We will introduce the sets  $L_\alpha^\pm, L_\beta^0, \Delta_\pm(\alpha, \beta)$  and  $\Delta_0(\beta)$ . With the help of these sets, the proof of our results is becoming much simpler.

The following lemma can be used to construct compact lamination convex sets.

**Lemma 4.1.** *Let  $\alpha, \beta \geq 0$  be given real numbers. Then the following three sets are closed, isotropic and lamination convex*

$$L_\alpha^\pm = \{A \in \mathbb{M}^{2 \times 2} \mid \alpha \leq \pm \lambda_1(A)\}, \quad L_\beta^0 = \{A \in \mathbb{M}^{2 \times 2} \mid \lambda_2(A) \leq \beta\}.$$

**Proof.** By definition, the sets  $L_\alpha^+$ ,  $L_\alpha^-$  and  $L_\beta^0$  are closed as well as isotropic. The set  $L_\beta^0$  is even convex, in fact, we have that  $L_\beta^0 = \{A \in \mathbb{M}^{2 \times 2} \mid \|A\|_s \leq \beta\}$  where  $\|\cdot\|_s$  denotes the spectral norm. Since for  $L_\alpha^-$  we can exploit the fact  $L_\alpha^- = -L_\alpha^+$ , it remains to show that  $L_\alpha^+$  is lamination convex. Suppose that this is not the case. Then there exist rank-one connected matrices  $A_1, A_2 \in \mathbb{M}^{2 \times 2}$  and a real number  $\mu \in [0, 1]$  such that  $\lambda_1(A_1), \lambda_1(A_2) \geq \alpha$  and  $\alpha_0 = \lambda_1(\mu A_1 + (1-\mu)A_2) < \alpha$ . On the one hand, since  $\alpha_0$  is a singular value of the matrix  $\mu A_1 + (1-\mu)A_2$ , there exist a normalized vector  $x_0 \in \mathbb{R}^2$  with  $|x_0| = 1$  and a rotation  $R \in \text{SO}(2)$  such that

$$x_0^t R(\mu A_1 + (1-\mu)A_2)x_0 = \alpha_0.$$

On the other hand, we know that  $|x_0^t R A_i x_0| \geq \lambda_1(A_i) \geq \alpha$  for  $i = 1, 2$ . We conclude that  $x_0^t R A_1 x_0$  and  $x_0^t R A_2 x_0$  have different signs. Hence, we can fix a real number  $\mu_0 \in [0, 1]$  such that  $x_0^t R(\mu_0 A_1 + (1-\mu_0)A_2)x_0 = 0$  and  $\det(\mu_0 A_1 + (1-\mu_0)A_2) = 0$ . This forms a contradiction, since the function  $-\det$  is rank-one convex (even polyconvex) and  $-\det(A_i) \leq -\alpha^2 < 0$  holds for  $i = 1, 2$ .  $\square$

For given non-negative real numbers  $\alpha, \beta \geq 0$  we consider the following isotropic and compact (possibly empty) sets

$$\begin{aligned} \Delta_\pm(\alpha, \beta) &= \{A \in \mathbb{M}^{2 \times 2} \mid \alpha \leq \pm \lambda_1(A) \wedge \lambda_2(A) \leq \beta\}, \\ \Delta_0(\beta) &= \{A \in \mathbb{M}^{2 \times 2} \mid \lambda_2(A) \leq \beta\}. \end{aligned}$$

We collect some properties of these sets.

**Lemma 4.2.** *The sets  $\Delta_+(\alpha, \beta)$ ,  $\Delta_-(\alpha, \beta)$  as well as  $\Delta_0(\beta)$  are compact, isotropic and lamination convex. Consider the matrices  $A_1^\pm = \text{diag}(\pm\alpha, \alpha)$ ,  $A_2^\pm = \text{diag}(\pm\alpha, \beta)$  and  $A_3^\pm = \text{diag}(\pm\beta, \beta)$ . Then we have  $\Delta_\pm(\alpha, \beta) = (\{A_1^\pm\}^{\text{iso}} \cup \{A_2^\pm\}^{\text{iso}} \cup \{A_3^\pm\}^{\text{iso}})^{\text{clc}}$  as well as  $\Delta_0(\beta) = (\{A_3^-\}^{\text{iso}} \cup \{A_3^+\}^{\text{iso}})^{\text{clc}}$ .*

**Proof.** The sets  $\Delta_+(\alpha, \beta)$ ,  $\Delta_-(\alpha, \beta)$  as well as  $\Delta_0(\beta)$  can be written as the intersection of  $L_\alpha^+$ ,  $L_\alpha^-$  and  $L_\beta^0$  from Lemma 4.1, which implies the first part. The second part exploits that  $\{A_1^\pm\}^{\text{iso}}$  and  $\{A_2^\pm\}^{\text{iso}}$ ,  $\{A_2^\pm\}^{\text{iso}}$  and  $\{A_3^\pm\}^{\text{iso}}$  as well as  $\{A_3^-\}^{\text{iso}}$  and  $\{A_3^+\}^{\text{iso}}$  are compatible, see Lemma 3.2.  $\square$

Let  $Z \subseteq \mathbb{M}^{2 \times 2}$  be a given compact and isotropic set. Using the pair  $\sigma(Z) = (\sigma_1(Z), \sigma_2(Z))$  given by  $\sigma_1(Z) = \min\{|\lambda_1(A)| \mid A \in Z\}$  and  $\sigma_2(Z) = \max\{\lambda_2(A) \mid A \in Z\}$ , we define the set  $Z^\Delta \subseteq \mathbb{M}^{2 \times 2}$  (see Figure 1(b)) via

$$Z^\Delta = \begin{cases} \Delta_\pm(\sigma(Z)) & \text{if } \forall A \in Z \pm \lambda_1(A) > 0 \\ \Delta_0(\sigma_2(Z)) & \text{otherwise.} \end{cases} \quad (3)$$

In view of Lemma 4.2, we obtain  $Z \subseteq Z^{\text{clc}} \subseteq Z^\Delta$ .



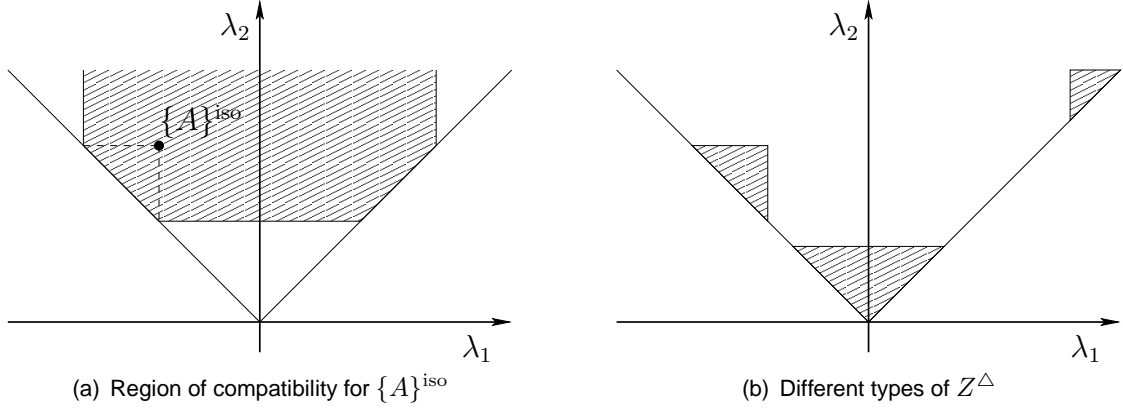


Figure 1: Subsets in the cone  $\{|\lambda_1| \leq \lambda_2\}$

## 5 A refinement for the connected case

Conti et al. [CDLMR03] show that polyconvexity and lamination convexity are the same for isotropic compact subsets of  $\mathbb{M}^{2 \times 2}$  that are connected. Their idea can be used to prove a bit more. In order to see that, we will sketch their proof and give the details where minor changes are necessary.

**Theorem 5.1.** *Let  $K \subseteq \mathbb{M}^{2 \times 2}$  be a given isotropic and compact set and  $Z \in \text{cc}(K^{\text{lc},1})$  a connected component. Then  $Z^{\text{lc},1}$  is polyconvex.*

**Proof.** Let  $Z \in \text{cc}(K^{\text{lc},1})$  be an arbitrary but fixed connected component. Then we have

$$Z^{\text{lc},1} \supseteq \{B \in \mathbb{M}^{2 \times 2} \mid \exists C \in Z^{\text{lc},1} \det(B) = \det(C) \wedge |\lambda_1(B)| = \lambda_2(B)\}. \quad (4)$$

In fact, set  $d_1 = \min\{\det(B) \mid B \in Z\}$  and  $d_2 = \max\{\det(B) \mid B \in Z\}$ . By definition, the set  $\{B \in \mathbb{M}^{2 \times 2} \mid d_1 \leq \det(B) \leq d_2\}$  is polyconvex and, hence,  $Z^{\text{lc},1}$  is a subset of it. The connectedness of  $Z$  together with Remark 3.4 implies that every matrix  $B \in \mathbb{M}^{2 \times 2}$  with  $d_1 \leq \det(B) \leq d_2$  and  $|\lambda_1(B)| = \lambda_2(B)$  lies in  $Z^{\text{lc},1}$ .

We show that for every matrix  $A \in \mathbb{M}^{2 \times 2} \setminus Z^{\text{lc},1}$  there is a polyconvex function  $\varphi: \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$  that separates  $A$  from  $Z^{\text{lc},1}$ , meaning  $\varphi(A) > \max\{\varphi(B) \mid B \in Z^{\text{lc},1}\}$ . In order to do that, we follow Conti et al. [CDLMR03]. They show that it is sufficient to check every  $A \in \mathbb{M}^{2 \times 2} \setminus Z^{\text{lc},1}$  such that  $A = \text{diag}(\sigma_1, \sigma_2)$  holds for some real numbers  $0 \leq \sigma_1 \leq \sigma_2$ . Fix such a matrix  $A$ . If  $\sigma_1 = \sigma_2$  holds, the set  $\{B \in \mathbb{M}^{2 \times 2} \mid \det(B) = \det(A)\}$  does not intersect  $Z^{\text{lc},1}$ . Otherwise (4) yields that  $A$  must lie in  $Z^{\text{lc},1}$ , a contradiction. Thus, the connectedness of  $Z^{\text{lc},1}$  implies that we can either put  $\varphi = \det$  or  $\varphi = -\det$  and are done.

Assume that  $\sigma_2 > \sigma_1$ . Given a real number  $c \in [-\sigma_2, \sigma_2]$ , they consider the level set

$$L_c = \begin{cases} \{B \in \mathbb{M}^{2 \times 2} \mid \varphi_c^-(B) = \varphi_c^-(A)\} & \text{for } c \in [-\sigma_2, \sigma_1[ \\ \{B \in \mathbb{M}^{2 \times 2} \mid \varphi_c^+(B) = \varphi_c^+(A)\} & \text{for } c \in [\sigma_1, \sigma_2], \end{cases} \quad (5)$$

see Lemma 3.3 for the definition of  $\varphi_c^\pm$ . They show that there exists a polyconvex  $\varphi$  that separates  $A$  from  $Z^{\text{lc},1}$  whenever at least one of the  $L_c$  does not intersect  $Z^{\text{lc},1}$ . In fact, by a nice

argument, they can reduce this further. Let  $\tilde{Z} \subseteq \mathbb{M}^{2 \times 2}$  be any compact, connected and isotropic set. They prove that there exists one  $L_{\tilde{c}}$  that does not intersect  $\tilde{Z}$  if for every  $c \in [-\sigma_2, \sigma_2]$  at least one of the sets  $L_c^> \cap \tilde{Z}$  and  $L_c^< \cap \tilde{Z}$  is empty, where  $L_c^>$  and  $L_c^<$  are the connected components of  $L_c \setminus \{A\}^{\text{iso}}$ . This can be used for  $\tilde{Z} = Z^{\text{lc},1}$ . Fix  $c \in [-\sigma_2, \sigma_2]$  and suppose that both sets  $L_c^> \cap Z^{\text{lc},1}$  and  $L_c^< \cap Z^{\text{lc},1}$  are non-empty. Then there show that  $A$  must lie in  $\{B, C\}^{\text{lc},1}$  for some rank-one connected matrices  $B \in L_c^> \cap Z^{\text{lc},1}$  and  $C \in L_c^< \cap Z^{\text{lc},1}$ . This forms a contradiction as long as  $Z^{\text{lc},1}$  is lamination convex and, hence, completes their proof.

In our case, we use the following argument. We still have  $A \in \{B, C\}^{\text{lc},1} \subseteq Z^{\text{lc},2}$  and conclude that  $d_1 \leq \det(A) \leq d_2$ . Connectedness of  $Z$  implies that there exists a matrix  $A' \in Z$  with  $\det(A') = \det(A)$ . We know that  $A \notin Z^{\text{lc},1}$  holds and, hence, we conclude that  $\lambda_2(A') < \lambda_2(A)$  by Remark 3.4. A simple computation shows that

$$\forall c \in [-\sigma_2, \sigma_2] \quad \varphi_c^\pm(A) \geq \varphi_c^\pm(A'). \quad (6)$$

If for every  $c \in [-\sigma_2, \sigma_2]$  at least one of the sets  $L_c^> \cap Z$  and  $L_c^< \cap Z$  is empty, then we use the above argument for  $\tilde{Z} = Z$ . Hence, we can fix a real number  $\tilde{c} \in [-\sigma_2, \sigma_2]$  such that  $L_{\tilde{c}}$  does not intersect  $Z$ . Let  $\varphi \in \{\varphi_{\tilde{c}}^+, \varphi_{\tilde{c}}^-\}$  be the function that defines  $L_{\tilde{c}}$  in (5). Then connectedness of  $Z$  implies that either  $\varphi(A) < \min\{\varphi(B) \mid B \in Z\}$  or  $\varphi(A) > \max\{\varphi(B) \mid B \in Z\}$ . In view of (6), the second alternative must hold, meaning  $\varphi$  separates  $A$  from  $Z$ . Polyconvexity of  $\varphi$  implies that  $\varphi$  also separates  $A$  from  $Z^{\text{lc},1}$  and we are done. Now if there exists a real number  $c \in [-\sigma_2, \sigma_2]$  such that both sets  $L_c^> \cap Z$  and  $L_c^< \cap Z$  are non-empty, then, as before, there exist  $B \in L_c^> \cap Z$  and  $C \in L_c^< \cap Z$  such that  $A$  lies in  $\{B, C\}^{\text{lc},1}$ . But  $\{B, C\}^{\text{lc},1}$  is contained in  $Z^{\text{lc},1}$  and, hence, we must have  $A \in Z^{\text{lc},1}$ , a contradiction.  $\square$

## 6 Closed lamination convex hull

We are going to characterize the closed lamination convex hull of an isotropic and compact set of  $2 \times 2$  matrices. The key ingredients are the following two lemmas. The first shows that the laminates of order one fully describe the topology of the closed lamination convex hull.

**Lemma 6.1.** *Let  $K \subseteq \mathbb{M}^{2 \times 2}$  be compact and isotropic. Let  $Z_1, Z_2 \in \text{cc}(K^{\text{lc},1})$  be arbitrary but fixed connected components with  $Z_1 \neq Z_2$ . Then  $(Z_1)^\Delta$  and  $(Z_2)^\Delta$  are incompatible and so are  $(Z_1)^{\text{clc}}$  and  $(Z_2)^{\text{clc}}$  as well as  $Z_1$  and  $Z_2$ .*

**Proof.** Since  $K$  is compact, so are the sets  $K^{\text{lc},1}$ ,  $Z_1$  and  $Z_2$ . The compact and isotropic sets given via  $K_i = Z_i \cap K$  fulfill  $(K_i)^{\text{clc}} = (Z_i)^{\text{clc}}$  for  $i = 1, 2$ . In addition, the sets  $K_1$  and  $K_2$  are incompatible. Otherwise there exist rank-one connected matrices  $B_1 \in K_1$  and  $B_2 \in K_2$  such that  $\{B_1, B_2\}^{\text{clc}}$  connects  $Z_1$  and  $Z_2$ , which forms a contradiction.

We know that  $Z_i \subseteq (Z_i)^{\text{clc}} \subseteq (Z_i)^\Delta$  as well as  $(K_i)^{\text{clc}} \subseteq (K_i)^\Delta$  and, hence,  $(K_i)^\Delta = (Z_i)^\Delta$  holds for  $i = 1, 2$ . It suffices to prove that  $(K_1)^\Delta$  and  $(K_2)^\Delta$  are incompatible. Without loss of generality, we set  $\sigma_2(K_1) \leq \sigma_2(K_2)$ . We distinguish two cases. First, suppose that  $\sigma_1(K_2) > \sigma_2(K_1)$ . Then, by Lemma 3.2, the sets  $(K_1)^\Delta$  and  $(K_2)^\Delta$  are incompatible. Second, suppose that  $\sigma_1(K_2) \leq \sigma_2(K_1)$ . Fix matrices  $B_1 \in K_1$  and  $B_2, B_2' \in K_2$  such that  $\lambda_2(B_1) =$

$\sigma_2(K_1), |\lambda_1(B_2)| = \sigma_1(K_2)$  and  $\lambda_2(B'_2) = \sigma_2(K_2)$ . Since  $K_1$  and  $K_2$  are incompatible, so are  $\{B_1\}$  and  $\{B_2\}$  as well as  $\{B_1\}$  and  $\{B'_2\}$ . We conclude that

$$|\lambda_1(B_2)| \leq \lambda_2(B_2) < |\lambda_1(B_1)| \leq \lambda_2(B_1) < |\lambda_1(B'_2)| \leq \lambda_2(B'_2).$$

But then the set  $K_2$  decomposes into at least two incompatible subsets and, hence,  $Z_2$  is not connected. This is a contradiction.  $\square$

The next lemma gives a candidate for the closed lamination convex hull.

**Lemma 6.2.** *Let  $K \subseteq \mathbb{M}^{2 \times 2}$  be a given compact and isotropic set. Then the set  $T = \cup\{Z^{\text{clc}} \mid Z \in \text{cc}(K^{\text{lc},1})\}$  is compact, lamination convex and contains  $K$ .*

**Proof.** By definition,  $T$  contains  $K$ . We show that  $T$  is compact. Let  $A_1, A_2, \dots$  be a given sequence in  $T$ . Since  $T$  is a bounded set, we can and we will assume that  $A_k \rightarrow A$  in  $\mathbb{M}^{2 \times 2}$  holds for some matrix  $A \in \mathbb{M}^{2 \times 2}$ . If necessary, we replace  $A_1, A_2, \dots$  by a subsequence. Let  $Z_1, Z_2, \dots \in \text{cc}(K^{\text{lc},1})$  be the sequence of connected components such that  $A_k \in (Z_k)^{\text{clc}} \subseteq (Z_k)^\Delta$  holds for every  $k = 1, 2, \dots$ . First, suppose that there exists a real number  $\epsilon > 0$  such that for every  $k = 1, 2, \dots$  we have  $|(Z_k)^\Delta| \geq \epsilon$  where  $|\cdot|$  denotes the Lebesgue measure of a set. Boundedness of  $T$  implies that there exists a connected component  $Z_0 \in \text{cc}(K^{\text{lc},1})$  and a subsequence (not relabeled) such that  $Z_k = Z_0$  for every  $k$ . Since the set  $(Z_0)^{\text{clc}}$  is compact,  $A$  lies in  $(Z_0)^{\text{clc}} \subseteq T$ . Second, suppose that there is no such  $\epsilon > 0$  as before. Then there exists a subsequence (not relabeled) such that  $|(Z_k)^\Delta| \rightarrow 0$  holds. In view of (3), this means that

$$\sup\{|B_1 - B_2| \mid B_1 \in (Z_k)^\Delta \wedge B_2 \in Z_k\} \rightarrow 0.$$

We take any sequence  $A'_1, A'_2, \dots$  in  $K^{\text{lc},1}$  such that  $A'_k \in Z_k$  for every  $k = 1, 2, \dots$ . Then we must have  $A'_k \rightarrow A$  and, hence, compactness of  $K^{\text{lc},1}$  implies that  $A \in K^{\text{lc},1} \subseteq T$ .

Now we show that  $T$  is lamination convex. Let  $A_1, A_2 \in T$  be given matrices. First, suppose that  $A_1, A_2 \in Z^{\text{clc}}$  for some  $Z \in \text{cc}(K^{\text{lc},1})$ . Then we have  $\{A_1, A_2\}^{\text{clc}} \subseteq Z^{\text{clc}} \subseteq T$ . Second, suppose that  $A_i \in Z_i^{\text{clc}}$  for  $i = 1, 2$  such that  $Z_1, Z_2 \in \text{cc}(K^{\text{lc},1})$  and  $Z_1 \neq Z_2$ . We know from Lemma 6.1 that  $(Z_1)^{\text{clc}}$  and  $(Z_2)^{\text{clc}}$  are incompatible and so are  $\{A_1\}$  and  $\{A_2\}$ . We conclude that  $\{A_1, A_2\}^{\text{clc}} = \{A_1, A_2\} \subseteq T$ .  $\square$

Finally, we are in the position to characterize the closed lamination convex hull.

**Theorem 6.3** (Characterization of  $K^{\text{clc}}$ ). *Let  $K \subseteq \mathbb{M}^{2 \times 2}$  be compact and isotropic. Then its closed lamination convex hull is given by  $K^{\text{clc}} = K^{\text{lc},2}$ .*

**Proof.** Let  $T \subseteq \mathbb{M}^{2 \times 2}$  be as in Lemma 6.2. On the one hand, we know that  $K^{\text{clc}} = (K^{\text{lc},1})^{\text{clc}} \supseteq T$ . On the other hand, we have shown in Lemma 6.2 that the set  $T$  is lamination convex, compact and contains  $K$ . We conclude that  $K^{\text{clc}} = T$ .

Let  $Z \in \text{cc}(K^{\text{lc},1})$  be a connected component. Then  $Z^{\text{lc},1}$  is polyconvex as an application of Theorem 5.1. In particular, we have  $Z^{\text{clc}} = Z^{\text{lc},1}$  and, hence,  $K^{\text{clc}} \subseteq K^{\text{lc},2}$ . Since the other inclusion holds by definition, we conclude that  $K^{\text{clc}} = K^{\text{lc},2}$ .  $\square$

## 7 Quasiconvex hull

We show the equivalence of quasiconvexity and lamination convexity for isotropic compact subsets of  $\mathbb{M}^{2 \times 2}$ . We rely on a result by Faraco and Székelyhidi [FS08].

The next lemma deals with the case of two connected components.

**Lemma 7.1.** *Let  $S \subseteq \mathbb{M}^{2 \times 2}$  be a given compact set (not necessarily isotropic) and  $0 \leq \beta_1 < \alpha_2 \leq \beta_2$  real numbers. If  $S \subseteq \Delta_0(\beta_1) \cup \Delta_+(\alpha_2, \beta_2)$  holds and  $S^{\text{qc}}$  is connected, then one of the sets  $S \cap \Delta_0(\beta_1)$  and  $S \cap \Delta_+(\alpha_2, \beta_2)$  must be empty.*

**Proof.** By rescaling the matrix space  $\mathbb{M}^{2 \times 2}$ , if necessary, we can and we will assume that there exists a positive real number  $\epsilon > 0$  such that

$$\beta_1 \leq 1 - \epsilon < 1 + \epsilon \leq \alpha_2 \leq \beta_2. \quad (7)$$

Lemma 3.3 implies that  $f = \varphi_1^+ - 1$ , with  $f(A) = \lambda_1(A) + \lambda_2(A) - \det(A) - 1$ , is a polyconvex function. In particular, the set  $P = \{A \in \mathbb{M}^{2 \times 2} \mid f(A) \leq -\epsilon^2\}$  is polyconvex by definition. We are going to show that  $\Delta_0(\beta_1) \cup \Delta_+(\alpha_2, \beta_2)$  is a subset of  $P$ . Consider the matrices  $A_1 = \text{diag}(-\beta_1, \beta_1)$ ,  $A_2 = \text{diag}(\beta_1, \beta_1)$ ,  $A_3 = \text{diag}(\alpha_2, \alpha_2)$ ,  $A_4 = \text{diag}(\alpha_2, \beta_2)$  and  $A_5 = \text{diag}(\beta_2, \beta_2)$ . Since, in addition,  $P$  is isotropic, Lemma 4.2 implies that it is sufficient to show that  $A_i \in P$  for  $i = 1, \dots, 5$ . However this can be tested easily if we make use of (7) and the fact that for every  $A \in \mathbb{M}^{2 \times 2}$  we have  $f(A) = (1 - \lambda_1(A))(\lambda_2(A) - 1)$ .

We have shown that  $S \subseteq \Delta_0(\beta_1) \cup \Delta_+(\alpha_2, \beta_2)$  is a subset of  $P$ . Since the set  $P$  is polyconvex, the quasiconvex hull  $S^{\text{qc}}$  is also contained in  $P$ . Yet the identity matrix lies not in  $P$ . As a consequence of Remark 3.4, for every matrix  $A \in S^{\text{qc}}$  we must have  $\det(A) \neq 1$ . We know that  $S^{\text{qc}}$  is connected and, in addition,  $\det < 1$  holds in  $\Delta_0(\beta_1)$  and  $\det > 1$  in  $\Delta_+(\alpha_2, \beta_2)$ . Hence, one of the sets  $S \cap \Delta_0(\beta_1)$  and  $S \cap \Delta_+(\alpha_2, \beta_2)$  must be empty.  $\square$

Now we are going to prove our result about the equivalence of lamination convexity and quasiconvexity.

**Theorem 7.2 (Equivalence).** *Let  $K \subseteq \mathbb{M}^{2 \times 2}$  be a given compact and isotropic set. Then  $K$  is lamination convex if and only if  $K$  is quasiconvex.*

**Proof.** We only have to show one implication. Assume that  $K$  is lamination convex and let  $\nu \in \mathcal{P}^{\text{qc}}$  be a fixed homogenous gradient Young measure with support  $S = \text{supp}(\nu) \subseteq K$ . By Remark 2.1, we need to show that  $\bar{\nu} \in K$ . Let  $\mathcal{Z}$  be the set of all connected components  $Z \in \text{cc}(K)$  such that  $Z \cap S$  is non-empty. First, suppose that there exists only one such connected component, meaning  $\mathcal{Z} = \{Z\}$ . Since  $Z$  is isotropic, lamination convex, compact and connected, Theorem 5.1 implies that  $Z$  is quasiconvex (even polyconvex). Hence,  $\bar{\nu}$  must lie in  $Z \subseteq K$ .

Second, suppose that  $S$  is distributed over more than one connected component. By compactness arguments, we can fix  $Z_1, Z_2 \in \mathcal{Z}$  that are extremal in the following sense. For every

$Z \in \mathcal{Z}$  we have  $\sigma_2(Z_1) \leq \sigma_2(Z) \leq \sigma_2(Z_2)$ . Up to symmetry, there are only three different cases:  $(Z_2)^\Delta = \Delta_+(\alpha_2, \beta_2)$  and either  $(Z_1)^\Delta = \Delta_0(\beta_1)$ ,  $(Z_1)^\Delta = \Delta_+(\alpha_1, \beta_1)$  or  $(Z_1)^\Delta = \Delta_-(\alpha_1, \beta_1)$  for some reals  $0 \leq \alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2$ .

We fix real numbers  $\tilde{\beta}, \tilde{\alpha} \in \mathbb{R}$  such that  $\beta_1 < \tilde{\beta} < \tilde{\alpha} < \alpha_2$  holds as well as

$$S \subseteq \Delta_0(\tilde{\beta}) \cap \Delta_+(\tilde{\alpha}, \beta_2). \quad (8)$$

In order to see that this can be done, let  $\epsilon > 0$  be a given real number. Recall that  $K^{\text{lc},1} = K$  holds and, hence, Lemma 6.1 implies that elements in  $\mathcal{Z}$  are pairwise incompatible. We fix  $\tilde{\beta}(\epsilon), \tilde{\alpha}(\epsilon) \in \mathbb{R}$  such that  $\alpha_2 - \epsilon < \tilde{\beta}(\epsilon) < \tilde{\alpha}(\epsilon) < \alpha_2$  holds and, in addition, for every  $Z \in \mathcal{Z}$  we have either  $\sigma_2(Z) < \tilde{\beta}(\epsilon)$  or  $\sigma_1(Z) > \tilde{\alpha}(\epsilon)$ . Suppose that  $\tilde{\beta}(\epsilon)$  and  $\tilde{\alpha}(\epsilon)$  fail to fulfill (8) for every  $\epsilon > 0$ . Then there must be a sequence  $A_1, A_2, \dots$  in  $S$  such that  $\lambda_1(A_k) < 0$  holds for every  $k = 1, 2, \dots$  and  $\lambda_2(A_k) \rightarrow \alpha_2$ . By compactness of the set  $S$ , we can fix a cluster point  $A_0 \in S$  of this sequence. On the one hand, we know that  $\lambda_1(A_0) \leq 0$  and, hence,  $A_0 \notin (Z_2)^\Delta = \Delta_+(\alpha_2, \beta_2)$ . On the other hand,  $\lambda_2(A_0) = \alpha_2$  implies that  $\{A_0\}$  and  $(Z_2)^\Delta$  are compatible. By Lemma 6.1, we must have  $(Z_0)^\Delta = (Z_2)^\Delta$  where  $Z_0$  is given by  $A_0 \in Z_0 \in \mathcal{Z}$ . This is a contradiction.

A result by Faraco and Székelyhidi [FS08, Cor. 3] implies that  $S^{\text{qc}}$  is connected. As a consequence of Lemma 7.1, one of the sets  $S \cap \Delta_0(\tilde{\beta})$  and  $S \cap \Delta_+(\tilde{\alpha}, \beta_2)$  must be empty. This forms a contradiction. Hence, it is impossible that  $S$  is distributed over more than one connected component.  $\square$

Our next result can be used to compute the quasiconvex hull.

**Theorem 7.3** (Characterization of  $K^{\text{qc}}$ ). *Let  $K \subseteq \mathbb{M}^{2 \times 2}$  be compact and isotropic. Then its quasiconvex hull coincides with its lamination convex hull of order 2.*

**Proof.** Clearly, the set  $K^{\text{clc}}$  is compact, isotropic and lamination convex. Theorem 6.3 and Theorem 7.2 imply that  $K^{\text{lc},2} = K^{\text{clc}} = K^{\text{qc}}$  holds.  $\square$

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