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Primal-dual linear Monte Carlo algorithm for multiple stopping – an application to flexible caps

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Abstract

In this paper we consider the valuation of Bermudan callable derivatives with multiple exercise rights. We present in this context a new primal-dual *linear* Monte Carlo algorithm that allows for efficient simulation of lower and upper price bounds without using nested simulations (hence the terminology). The algorithm is essentially an extension of a primal–dual Monte Carlo algorithm for standard Bermudan options proposed in Schoenmakers et al. (2011), to the case of multiple exercise rights. In particular, the algorithm constructs upwardly a system of dual martingales to be plugged into the dual representation of Schoenmakers (2010). At each level the respective martingale is constructed via a backward regression procedure starting at the last exercise date. The thus constructed martingales are finally used to compute an upper price bound. At the same time, the algorithm also provides approximate continuation functions which may be used to construct a price lower bound. The algorithm is applied to the pricing of flexible caps in a Hull and White (1990) model setup. The simple model choice allows for comparison of the computed price bounds with the exact price which is obtained by means of a trinomial tree implementation. As a result, we obtain tight price bounds for the considered application. Moreover, the algorithm is generically designed for multi-dimensional problems and is tractable to implement.

1 Introduction

The goal of the paper is an efficient Monte Carlo algorithm for the pricing of Bermudan callable derivatives with multiple exercise rights. Such derivatives give the right to exercise a certain claim at a specific number of times within a given set of discrete exercise dates. These products are nowadays quite popular and frequently occur in various financial sectors, for example as flexible caps (also called chooser caps) in interest rate markets, or as swing options in energy markets. Further they can be found in the context of life insurance contracts, for instance as surrender and prepayment options embedded in mortgage backed securities and insurance contracts. From a mathematical point of view, pricing of a multiple exercise option comes down to solving a multiple stopping problem. Because in general a multiple exercise option may be specified with respect to a multi-dimensional underlying, we aim at developing an effective and generic Monte Carlo procedure, thus avoiding the curse of dimensionality typically connected with deterministic PDE solutions.

Monte Carlo procedures for single exercise Bermudan options may be somehow categorized in two groups. On the one hand there are the so called “primal” algorithms which aim at constructing a “good” stopping time leading to a lower biased price estimate. As some of the most popular methods in this category may be considered the regression based procedures in Carriere (1996), Longstaff and Schwartz (2001), and Tsitsiklis and Van Roy (2001). On the other hand there is the category of “dual” methods relying on the dual representation for the (standard) stopping problem developed by Rogers (2002) and independently Haugh and Kogan (2004), which involves an infimum over a set of martingales. In a dual method the goal is to find a “good” martingale which leads to an upper biased price estimate. A generic and popular Monte Carlo solution for the Bermudan pricing problem is proposed by Andersen and Broadie (2004). A drawback of this method is that it requires usually time consuming nested Monte Carlo simulations. In this respect Belomestny et al. (2009) propose a non-nested (linear) dual Monte Carlo algorithm in a Wiener environment based on the construction of a dual martingale via a regression estimate of a discretized Clark-Ocone derivative. Recently, as a particular result, in Schoenmakers et al. (2011) a new non-nested dual regression based algorithm is developed which is based on the idea of constructing "nearly optimal" or "low variance" dual martingales. As an additional feature, by this method one also obtains lower price bounds at the same time and one doesn't need a certain given "input" approximation to the Snell envelope.

The above mentioned primal regression based Monte Carlo procedures may be extended in a rather straightforward way to the multiple exercise case, using the reduction principle for multiple stopping. Further in Bender and Schoenmakers (2006) the iterative procedure of Kolodko and Schoenmakers (2006) is extended to the multiple stopping problem and analyzed regarding numerical stability. As a first extension of the dual representation for single exercise options, Meinshausen and Hambly (2004) developed a dual representation for multi exercise options in terms of an infimum over a family of martingales *and* a family of stopping times. Later on, Schoenmakers (2010) found an alternative dual representation for the multiple stopping problem in terms of an infimum over martingales *only.* The latter representation is recently generalized in Bender et al. (2011) to multiple stopping problems with respect to far more general pay-off structures, which may include volume constraints and refraction periods for example.

The main achievement in this paper is a linear, or non-nested (hence potentially efficient) Monte Carlo procedure for multiple exercise options that provides both price upper bounds and price lower bounds at the same time. Our new algorithm can be considered as a generalization of the particular regression based approach presented in Schoenmakers et al. (2011) using essentially the pure martingale dual representation in Schoenmakers (2010).

The proposed algorithm is implemented and tested for pricing flexible caps within a Hull and White (1990) model setup. This simple model is chosen in order to compare the price bounds computed by the new algorithm with the exact price obtained by means of a trinomial tree implementation along the lines of Hull and White (2000). In addition, we introduce the notion "$\epsilon-$relevance of the algorithm for solving the stopping problem" in order to assess whether a particular product under consideration involves a "real" stopping problem in the sense that the option price differs in a way measured by ϵ with respect to suitably specified lower and upper benchmark price bounds. To be more precise, the lower benchmark bound is due to the optimal *deterministic* exercise policy and the upper benchmark bound is due to the price in the view of a visionary. As a result, the proposed dual linear Monte Carlo algorithm gives tight price bounds for various versions of the flexible cap considered.

The outline of the paper is as follows. Section 2 states the pricing problem and gives a brief review of known prerequisites. The new linear Monte Carlo algorithm is developed in Section 3. In Section 4 we consider the payoff structure of flexible caps and introduce the notion of $\epsilon-$relevance of the algorithm, whereas in Section 5 we implement the new Monte Carlo algorithm and compare the simulated upper and lower price bounds with "exact" prices obtained from a trinomial tree implementation.

2 Multiple stopping problem, recap of former results

2.1 The problem of multiple stopping

The key problem is given by the multiple stopping problem which is implied a Bermudan Option with $\mathcal{L}$ exercise rights. A Bermudan option gives the right to exercise an option a specified number of times within a given discrete set of exercise dates $0 = t_0 < t_1 < \cdots < t_{\mathcal{J}} =: T$, identified henceforth with their respective indices $i \in \{0, 1, \ldots, \mathcal{J}\}$. In the case that the option is exercised at time i, the buyer of the Bermudan option instantaneously receives the payoff Z_i. In general, $(Z_i : i = 0, 1, \ldots, \mathcal{J})$ denotes a non–negative stochastic process in discrete time. Z is defined on a filtered probability space $(\Omega, \mathcal{F}, P)$ and is adapted to some filtration $F := (\mathcal{F}_i : 0 \le i \le \mathcal{J})$ and satisfies

$$\sum_{i=1}^{\mathcal{J}} E|Z_i| < \infty.$$

Throughout the following, we interpret Z as the discounted cash–flow, i.e. w.l.o.g. we set the interest rate equal to zero. The pricing of a Bermudan option with $\mathcal{L}$ exercise rights boils down to an optimal stopping problem w.r.t. Z. Let $\mathcal{S}_i(\mathcal{L})$ denote the set of F–stopping vectors $\tau := \left(\tau^{(1)}, \ldots, \tau^{(\mathcal{L})}\right)$ such that $i \le \tau^{(1)}$ and, for all l, $1 < l \le \mathcal{L}$, $\tau^{(l-1)} + 1 \le \tau^{(l)}$. Then, the (discounted) price $Y_i^{*\mathcal{L}}$ of the Bermudan option at i is given by

$$Y_i^{*\mathcal{L}} = \mathrm{sup}_{\tau \in \mathcal{S}_i} E_i \sum_{l=1}^{\mathcal{L}} Z_{\tau^{(l)}}$$

where $E_i := E_{\mathcal{F}_i}$ denotes the conditional expectation with respect to the σ–algebra $\mathcal{F}_i$ and $Z_i :\equiv 0$ and $\mathcal{F}_i :\equiv \mathcal{F}_{\mathcal{J}}$ for $i > \mathcal{J}$.

It is worth to emphasize that the multiple stopping problem can be reduced to $\mathcal{L}$ nested stopping problems with one exercise right, cf. for example Bender and Schoenmakers (2006). However, in the following, we derive a Monte Carlo simulation method which requires only one degree of nesting. First, we review some well known results which are needed to derive the regression based algorithm.

2.2 Review of former results

From Rogers (2002), Haugh and Kogan (2004) it is well known that

$$Y_i^* := Y_i^{*1} = \inf_{M \in \mathcal{M}} E_i \max_{i \le j \le \mathcal{J}} (Z_j + M_i - M_j) \tag{2.1}$$

$$= \max_{i \le j \le \mathcal{J}} (Z_j + M_i^* - M_j^*) \text{ a.s.} \tag{2.2}$$

where $\mathcal{M}$ is the set of all $\mathcal{F}$–martingales, and where M^* is the Doob martingale of the Snell envelope Y^* for one exercise right, which satisfies

$$Y_i^* = Y_0^* + M_i^* - A_i^*,$$

where A_i^* is predictable and nondecreasing, and $M_0^* = A_0^* = 0$. In particular, the price of a single exercise Bermudan option at time $i = 0$ is given by the pathwise maximum of the cash-flow minus the Doob martingale of the Snell envelope. Intuitively, a meaningful upper bound should now be obtained by replacing the Doob martingale by a *good* approximation to it.

As one corner stone of the method developed in this paper we consider the recently developed regression based non-nested Monte Carlo algorithm for solving the dual problem in the single exercise case (see Schoenmakers et al. (2011)). This algorithm is essentially based on minimizing the expected conditional variances

$$E\,\mathrm{Var}_i \vartheta_i(M), \quad i = 0, ..., \mathcal{J} \tag{2.3}$$

of the path-wise functional

$$\vartheta_i(M) := \max_{i \le j \le \mathcal{J}} (Z_j - M_j + M_i)$$

in a backward recursive way, by constructing a 'nearly optimal' martingale backwardly from $i = \mathcal{J}$ down to $i = 0$, cf. Theorem 12 in Schoenmakers et al. (2011). An 'optimal' dual martingale in the sense of Schoenmakers et al. (2011) is a martingale M° for which

$$\vartheta_i(M^\circ) = \max_{i \le j \le \mathcal{J}} (Z_j - M_j^\circ + M_i^\circ) \ \in \mathcal{F}_i, \quad i = 0, ..., \mathcal{J},$$

and due to Schoenmakers et al. (2011), Theorem 10, it then holds $Y_i^* = \vartheta_i^{(M^\circ)}$, $i = 0, ..., \mathcal{J}$. By observing that for $i < \mathcal{J}$,

$$\vartheta_i := \vartheta_i(M) = \max\left(Z_i,\ M_i - M_{i+1} + \vartheta_{i+1}\right),$$

and that trivially $Z_i \in \mathcal{F}_i$, Schoenmakers et al. (2011) point out a particular backward regression based approach to achieve minimization of (2.3) by (more strongly) minimizing

$$E\,\mathsf{Var}_i\left(M_i - M_{i+1} + \vartheta_{i+1}\right) \tag{2.4}$$

over a class of martingale increments $M_{i+1} - M_i$ represented by linear combinations of a suitable family of elementary martingales, assuming that the increments $M_j - M_{i+1}$ for $j \ge i+1$, and thus ϑ_{i+1}, are already constructed. Further details will be clear from the description of the multiple exercise version of the Schoenmakers et al. (2011) algorithm later on. For developing the latter algorithm we will need a next corner stone, namely a recently developed martingale representation for multiple stopping (Schoenmakers (2010)).
The dual martingale representation in the case of $\mathcal{L}$ exercise rights derived in Schoenmakers (2010), cf. Theorem 2.5, states that for $\mathcal{L} = 1, 2, ...$

$$Y_i^{*\mathcal{L}} = \inf_{M^{(1)},...,M^{(\mathcal{L})} \in \mathcal{M}} \max_{i \le j_1 < \cdots < j_{\mathcal{L}}} \sum_{k=1}^{\mathcal{L}} \left(Z_{j_k} + M_{j_{k-1}}^{(k)} - M_{j_k}^{(k)} \right),$$

$$Y_i^{*\mathcal{L}} = \max_{i \le j_1 < \cdots < j_{\mathcal{L}}} \sum_{k=1}^{\mathcal{L}} \left(Z_{j_k} + M_{j_{k-1}}^{*\mathcal{L}-k+1} - M_{j_k}^{*\mathcal{L}-k+1} \right), \tag{2.5}$$

$$E_i Y_{i+1}^{*\mathcal{L}} = \max_{i < j_1 < \cdots < j_{\mathcal{L}}} \sum_{k=1}^{\mathcal{L}} \left(Z_{j_k} + M_{j_{k-1}}^{*\mathcal{L}-k+1} - M_{j_k}^{*\mathcal{L}-k+1} \right) \tag{2.6}$$

almost surely with $j_0 := i$, and where M^{*k} is the Doob martingale of the Snell envelope Y^{*k} for k exercise rights.

Remark 1 *For formal reasons (in order to avoid maxima over empty domains in case the number of remaining exercise possibilities at time i is larger than $\mathcal{J} - i + 1$) we allow exercising beyond $\mathcal{J}$ yielding zero cash. Thus, since trivially $Y_j^{*k} = 0$ for $j > \mathcal{J}$, we have $M_{j+1}^{*k} - M_j^{*k} = Y_{j+1}^{*k} - E_j Y_{j+1}^{*k} = 0$ for $j \ge \mathcal{J}$, i.e. $M_j^{*k} = M_{\mathcal{J}}^{*k}$ for $j \ge \mathcal{J}$.*

At a first glance (2.5) requires the evaluation of a maximum that involves about $2^{\mathcal{J}}$ arguments in the case where $\mathcal{L} \approx \mathcal{J}/2$. However, as we will show later on (Remark 2), due to the very structure of the object to be maximized, it can be computed in a recursive way at a costs of $O(\mathcal{L}\mathcal{J})$, so $O(\mathcal{J}^2)$ in the worse case.

3 Primal-dual linear MC algorithm for multiple stopping

We are now ready for constructing a multiple exercise version of the regression based primal-dual algorithm for one exercise right proposed in Schoenmakers et al. (2011).

3.1 Backward procedure for multiple stopping

Let us fix $\mathcal{L}$ and consider $1 \leq l < \mathcal{L}$. As a well known fact, the Snell envelope Y_i^{*l+1} due to $l+1$ exercise rights may be equivalently considered as the Snell envelope under one exercise right due to the generalized cash-flow

$$Z_j^{*l+1} := Z_j + E_j Y_{j+1}^{*l}. \tag{3.1}$$

From (2.5) and (2.6) we observe that with $\widetilde{j}_0 := i+1$,

$$\begin{aligned} E_i Y_{i+1}^{*l} &= \max_{i<j_1<\cdots<j_L} \sum_{k=1}^{l} \left(Z_{j_k} + M_{j_{k-1}}^{*l-k+1} - M_{j_k}^{*l-k+1} \right) \\ &= M_i^{*l} - M_{i+1}^{*l} + \max_{i+1 \leq \widetilde{j}_1 < \cdots < \widetilde{j}_L} \sum_{k=1}^{l} \left(Z_{j_k} + M_{\widetilde{j}_{k-1}}^{*l-k+1} - M_{\widetilde{j}_k}^{*l-k+1} \right) \\ &= M_i^{*l} - M_{i+1}^{*l} + Y_{i+1}^{*l}, \end{aligned} \tag{3.2}$$

hence (3.1) may be written as

$$Z_j^{*l+1} := Z_j + M_i^{*l} - M_{i+1}^{*l} + Y_{i+1}^{*l}. \tag{3.3}$$

Now consider a given set of martingales $M^{(k)}$ satisfying $M_j^{(k)} = M_{\mathcal{J}}^{(k)}$, $j \geq \mathcal{J}$, $k = 1, ..., \mathcal{L}$ (cf. Remark 1) and define for $l < \mathcal{L}$ in view of (2.5),

$$Y_i^{*l+1} \approx \vartheta_i^{(l+1)} := \max_{i \leq j_1 < \cdots < j_{l+1}} \sum_{k=1}^{l+1} \left(Z_{j_k} + M_{j_{k-1}}^{(l+1-k+1)} - M_{j_k}^{(l+1-k+1)} \right) \tag{3.4}$$

with $j_0 := i$. It then holds with $\widetilde{j}_0 := \widehat{j}_0 := i+1$,

$$\begin{aligned} \vartheta_i^{(l+1)} &= \max \Bigg(Z_i + \max_{i<j_2\cdots<j_{l+1}} \sum_{k=2}^{l+1} \left(Z_{j_k} + M_{j_{k-1}}^{(l+1-k+1)} - M_{j_k}^{(l+1-k+1)} \right) \\ &\quad , \max_{i<j_1<\cdots<j_{l+1}} \sum_{k=1}^{l+1} \left(Z_{j_k} + M_{j_{k-1}}^{(l+1-k+1)} - M_{j_k}^{(l+1-k+1)} \right) \Bigg) \\ &= \max \Bigg(Z_i + M_i^{(l)} - M_{i+1}^{(l)} + \max_{i<\widetilde{j}_1\cdots<\widetilde{j}_l} \sum_{k=1}^{l} \left(Z_{\widetilde{j}_k} + M_{\widetilde{j}_{k-1}}^{(l-k+1)} - M_{\widetilde{j}_k}^{(l-k+1)} \right), \\ &\quad M_i^{(l+1)} - M_{i+1}^{(l+1)} + \max_{i<\widehat{j}_1<\cdots<\widehat{j}_{l+1}} \sum_{k=1}^{l+1} \left(Z_{j_k} + M_{\widehat{j}_{k-1}}^{(l+1-k+1)} - M_{\widehat{j}_k}^{(l+1-k+1)} \right) \Bigg) \\ &= \max \left(Z_i + M_i^{(l)} - M_{i+1}^{(l)} + \vartheta_{i+1}^{(l)},\, M_i^{(l+1)} - M_{i+1}^{(l+1)} + \vartheta_{i+1}^{(l+1)} \right) \\ &=: \max \left(Z_i^{(l+1)},\, M_i^{(l+1)} - M_{i+1}^{(l+1)} + \vartheta_{i+1}^{(l+1)} \right) \end{aligned} \tag{3.5}$$

in view of (3.3). Hence $Z_i^{(l+1)}$ may be considered an approximation to virtual cash-flow Z_i^{*l+1} in (3.1). We are now going to apply for $l = 0, 1, 2$ upwardly the regression method in Schoenmakers et al. (2011) to the approximation $Z_i^{(l+1)}$ of the virtual cash-flow process Z^{*l+1} in (3.1). Formally, in this upward construction it is assumed that at each step $1 \leq l < \mathcal{L}$, a martingale $M^{(l)}$, being an approximation to M^{*l}, and an approximation $\vartheta^{(l)}$ to Y^{*l} is constructed. Following Schoenmakers et al. (2011) we then construct a martingale $M^{(l+1)}$ and a process $\vartheta^{(l+1)}$ as approximations to M^{*l+1} and Y^{*l+1}, respectively, as explained in the next section. The upward construction may be naturally initialized with $M^{(0)} := \vartheta^{(0)} := 0$, and after $\mathcal{L}$ upward steps we end up with a set of martingales $M^{(1)}, ..., M^{(\mathcal{L})}$ and approximations $\vartheta^{(1)}, ..., \vartheta^{(\mathcal{L})}$ to the respective Snell envelopes $Y^{*1}, ..., Y^{*\mathcal{L}}$.

Remark 2 *(Complexity of the maximization problem (2.5)) Let us suppose that a family of martingales* $\left(M^{(k)}\right)$ *as above is available and that we are faced with the maximization problem (3.4) for* $i = 0$ *and* $l + 1 = \mathcal{L}$. *We may initialize* $M^{(0)} := \vartheta^{(0)} := 0$ *and then obtain* $\vartheta^{(l+1)}$ *from* $\vartheta^{(l)}$, $M^{(l)}$, *hence* $Z_i^{(l+1)}$, *and* $M^{(l+1)}$, *via (3.5) by backward induction. Indeed, after initializing* $\vartheta_{\mathcal{J}}^{(l+1)} = Z_{\mathcal{J}}$ *we can obtain for* $j < \mathcal{J}$ *inductively* $\vartheta_j^{(l+1)}$ *from* $\vartheta_{j+1}^{(l+1)}$ *by (3.5) and thus* $\vartheta_0^{(l+1)}$ *in* $\mathcal{J}$ *steps. We thus arrive at* $\vartheta_0^{(\mathcal{L})}$ *after* $\mathcal{L}\mathcal{J}$ *operations (rather than* $\mathcal{J}!/\left(\mathcal{L}!\left(\mathcal{J}-\mathcal{L}\right)!\right)$ *).*

3.2 Primal-dual linear MC algorithm

For the algorithm spelled out below we assume as in Schoenmakers et al. (2011) an underlying D-dimensional Markovian structure X with respect to a filtration generated by an m-dimensional Brownian motion W. Moreover, we assume that for $0 \leq j \leq \mathcal{J}$ and $0 \leq l \leq \mathcal{L}$ the martingales $M^{(l)}$ are of the form

$$M_j^{(l)} = \sum_{q=1}^{K} \xi_{l,q} \mathcal{E}_{q,j},$$

for certain suitably chosen "elementary" martingales $\mathcal{E}_{q,\cdot}$, $q = 1, ..., K$. For example,

$$\mathcal{E}_{q,j} = \int_0^{t_j} \varphi_q^{\mathsf{T}}(u, X_u) dW_u \tag{3.6}$$

for a set of basis functions $(\varphi_q(t, x))_{1 \leq q \leq K}$ with φ_q acting from $\mathbb{R} \times \mathbb{R}^D \to \mathbb{R}^m$, or, $\mathcal{E}_{q,j}$, $q = 1, ..., K$, may represent any set of discounted tradables at time j (hence t_j) available in a particular situation.

Let us initialize $M^{(0)} = \vartheta^{(0)} = 0$. Then, inductively, we are going to construct $M^{(l+1)}$ and $\vartheta^{(l+1)}$, assuming that $M^{(l)}$ and $\vartheta^{(l)}$ are constructed for l, $0 \leq l < \mathcal{L}$. The construction will be carried out on a sample of trajectories $X_{\mathcal{J}}^{(n)}$, $n = 1, ..., N$.

At $i = \mathcal{J}$ we trivially set $\vartheta_{\mathcal{J}}^{(l+1,n)} = Z_{\mathcal{J}}(X_{\mathcal{J}}^{(n)})$, $n = 1, ..., N$. Suppose we have constructed for fixed $i < \mathcal{J}$ the martingale increments $\left(M_j^{(l+1)} - M_{i+1}^{(l+1)}\right)_{i+1 \leq j \leq \mathcal{J}}$ (for $i = \mathcal{J} - 1$ this is trivially zero), and $\vartheta_{i+1}^{(l+1,n)}$, $n = 1, ..., N$ (as approximations to Y^{*l+1}), respectively, on each trajectory. We will then determine $\beta_1, ..., \beta_K$ such that for

$$M_{i+1}^{(l+1)} - M_i^{(l+1)} = \sum_{q=1}^{K} \beta_q \left(\mathcal{E}_{q,i+1} - \mathcal{E}_{q,i}\right)$$

the sample estimate of the expected conditional variance

$$E\,\mathsf{Var}_i(M_i^{(l+1)} - M_{i+1}^{(l+1)} + \vartheta_{i+1}^{l+1})$$

(cf. (2.4)) is minimized. This will be carried out by a regression procedure. As a candidate predictor for the $\mathcal{F}_i$-measurable conditional expectation

$$E_i(M_i^{(l+1)} - M_{i+1}^{(l+1)} + \vartheta_{i+1}^{l+1})$$

we will take an $\mathcal{F}_i$-measurable random variable of the form

$$\sum_{q=1}^{K} \gamma_q \psi_q(i, X_i),$$

where, for instance, $(\psi_q(t,x))_{1\le q\le K}$, with ψ_q acting from $\mathbb{R}^D \to \mathbb{R}$, is a second set of basis functions, or any other set of explanatory $\mathcal{F}_i$-measurable random variables suggested by the problem under consideration. We next consider the regression problem

$$(\beta_i^{(l+1)}, \gamma_i^{(l+1)}) := \arg\min_{\beta,\gamma} E\left[\sum_{q=1}^{K} \beta_q\left(\mathcal{E}_{q,i} - \mathcal{E}_{q,i+1}\right) + \vartheta_{i+1}^{(l+1)} - \sum_{q=1}^{K}\gamma_q\psi_q(i,X_i)\right]^2$$

which comes down to the following regression procedure on the Monte Carlo trajectories $(X_j^{(n)}, 0 \le j \le \mathcal{J}$, $n = 1, ..., N)$,

$$\begin{aligned}(\beta_i^{(l+1)}, \gamma_i^{(l+1)}) := \arg\min_{\beta,\gamma\in\mathbb{R}^K} \sum_{n=1}^{N} &\left[\beta_q\left(\mathcal{E}_{q,i}^{(n)} - \mathcal{E}_{q,i+1}^{(n)}\right) + \vartheta_{i+1}^{(l+1,n)}\right. \\ &\left. -\sum_{q=1}^{K}\gamma_q\psi_q(i, X_i^{(n)})\right]^2 . \end{aligned} \tag{3.7}$$

Next, we set

$$M_{i+1}^{(l+1)} - M_i^{(l+1)} = \sum_{q=1}^{K} \beta_{i,q}^{(l+1)}\left(\mathcal{E}_{q,i+1} - \mathcal{E}_{q,i}\right), \tag{3.8}$$

and then in view of (3.5) we proceed by setting

$$\begin{aligned}\vartheta_i^{(l+1,n)} &= \max\left(Z_i^{(n)} + M_i^{(l,n)} - M_{i+1}^{(l,n)} + \vartheta_{i+1}^{(l,n)}, M_i^{(l+1,n)} - M_{i+1}^{(l+1,n)} + \vartheta_{i+1}^{(l+1,n)}\right) \\ &= \max\left(Z_i^{(n)} + M_i^{(l,n)} - M_{i+1}^{(l,n)} + \vartheta_{i+1}^{(l,n)},\right. \\ &\left.\sum_{q=1}^{K}\beta_{i,q}^{(l+1)}\left(\mathcal{E}_{q,i}^{(n)} - \mathcal{E}_{q,i+1}^{(n)}\right) + \vartheta_{i+1}^{(l+1,n)}\right),\end{aligned}$$

where we note that the quantities indexed with level l are already determined at the previous level. For elementary martingales of the form (3.6), the respective Wiener integrals in (3.7) may be approximated as usual by

$$\int_{t_i}^{t_{i+1}} \varphi_q^{\mathsf{T}}(u, X_u^{(n)})dW_u^{(n)} \approx \sum_{l=0}^{L-1}\varphi_q^{\mathsf{T}}(t_i + l\delta), X_{t_i+l\delta}^{(n)})\left(W_{t_i+(l+1)\delta}^{(n)} - W_{t_i+l\delta}^{(n)}\right), \tag{3.9}$$

with $\delta := (t_{i+1} - t_i)/L$ for a large enough integer L. By working backward from $i = \mathcal{J}$ down to $i = 0$, the above regression procedure yields a martingale

$$M_j^{(l+1)} = \sum_{p=0}^{j-1} \sum_{q=1}^{K} \beta_{p,q}^{(l+1)} \left(\mathcal{E}_{q,p+1} - \mathcal{E}_{q,p} \right)$$

and, as a by-product, an additional system of approximations to the continuation value functions,

$$\left[E_i Y_{i+1}^{*l+1} \right](x) = \mathcal{C}_i^{(l+1)}(x) \approx \sum_{q=1}^{K} \gamma_{i,q}^{(l+1)} \psi_q(i,x), \quad i = 0, ..., \mathcal{J} - 1. \tag{3.10}$$

Finally, the martingales $M^{(1)}, ..., M^{(\mathcal{L})}$ may be used to compute a dual upper bound at $i = 0$, by starting a new simulation of trajectories $\widetilde{X}_i^{(\widetilde{n})}$, $\widetilde{n} = 1, ..., \widetilde{N}$, and computing

$$\begin{aligned} Y_0^{up,\mathcal{L}} \approx \frac{1}{\widetilde{N}} \sum_{\widetilde{n}=1}^{\widetilde{N}} \max_{0 \le j_1 < \cdots < j_L \le T} \sum_{k=1}^{L} \Big(Z_{j_k} (\widetilde{X}_{j_k}^{(\widetilde{n})}) \\ + \sum_{p=j_{k-1}}^{j_k - 1} \sum_{q=1}^{K} \beta_{p,q}^{(L-k+1)} \left(\tilde{\mathcal{E}}_{q,p}^{(\widetilde{n})} - \tilde{\mathcal{E}}_{q,p+1}^{(\widetilde{n})} \right) \Big). \end{aligned} \tag{3.11}$$

Notice that the upper bound is "true" in the sense that it is always an upper biased estimate, regardless the quality of the martingales $M^{(l)}$. Further note that the maximum in (3.11) may be computed efficiently along the lines explained in Remark 2.

On the other side, based on the approximate continuation functions (3.10), we may define an exercise policy $(\tau_0^{p,\mathcal{L}} : 1 \le p \le \mathcal{L})$ as follows. Define $\tau_0^{0,\mathcal{L}} := -1$ and for $0 < p \le \mathcal{L}$

$$\tau_0^{p,\mathcal{L}} := \inf\{ j : \tau_0^{p-1,\mathcal{L}} < j \le \mathcal{J}, \;\; Z_j(X_j) + \mathcal{C}_j^{(\mathcal{L}-p)}(X_j) \ge \mathcal{C}_j^{(\mathcal{L}-p+1)}(X_j) \},$$

and simulate a lower biased price estimate,

$$Y_0^{low,\mathcal{L}} \approx \frac{1}{\widetilde{N}} \sum_{\widetilde{n}=1}^{\widetilde{N}} \sum_{p=1}^{\mathcal{L}} Z_{\tau_0^{p,\mathcal{L},\widetilde{n}}} (\widetilde{X}_{\tau_0^{p,\mathcal{L},\widetilde{n}}}^{(\widetilde{n})}). \tag{3.12}$$

4 Application to Flexible Caps

Throughout the following, $D(t,T)$ denotes the $t-$ price of a zero coupon bond with maturity T. In addition, r denotes the simple compounded spot rate and L is the market LIBOR rate (EURIBOR rate, respectively), i.e.

$$L(t,T) = \frac{1}{\tau(t,T)} \left(\frac{1}{D(t,T)} - 1 \right), \tag{4.1}$$

where $\tau(t,T) = T - t$ is the time difference expressed in years.[1] For notational convenience, we consider an equidistant set of tenor dates

$$\underline{\underline{T}} = \{ T_0 = 0 < T_1 < \cdots < T_{\mathcal{J}} < T_{\mathcal{J}+1} \},$$

[1] Actual/360 day-count convention, respectively.

where $T_i = T_{i-1} + \delta$ for $i = 1, ..., \mathcal{J} + 1$ and $T_0 = 0$. In addition, we set

$$L_i(T_i) := \frac{1}{\delta}\left(\frac{1}{D(T_i, T_{i+1})} - 1\right).$$

Definition 3 (Payoff structure of caplets, caps, and flexible caps)

(i) The payoff of a caplet with settlement date T_i ($i \in \{1, \ldots \mathcal{J} + 1\}$) is given by the positive difference between the reference rate L (LIBOR rate) prevailing at T_{i-1} and the level κ, i.e.

$$\delta[L_{i-1}(T_{i-1}) - \kappa]^+. \tag{4.2}$$

(ii) The payoff structure of a cap with tenor structure $\underline{\underline{T}}$ is given by the payoff of a portfolio of caplets with settlement dates $T_1, \ldots, T_{\mathcal{J}+1}$.

(iii) A flexible cap with $\mathcal{L}$ exercise rights implies the right to exercise at most $\mathcal{L} \leq \mathcal{J} + 1$ of the caplets with payoffs at $T_1, \ldots, T_{\mathcal{J}+1}$.

Obviously, the value of the flexible cap is increasing in the number of exercise rights $\mathcal{L}$. In particular, for $\mathcal{L} \leq \mathcal{J} + 1$, a trivial upper bound for the flexible cap is given by the value of a flexible cap with $\mathcal{L} = \mathcal{J} + 1$, i.e. the value of the cap over the whole tenor structure.

For option pricing we consider a risk-neutral valuation framework with numeraire

$$B_t := e^{\int_0^t r_s ds}$$

and corresponding pricing measure P, i.e. for any $T > 0$, the discounted zero coupon bonds $D(t,T)/B_t$, $0 \leq t \leq T$, are P-martingales.

Thus, a caplet with settlement date T_{i+1}, and expiry T_i has at time t, $t \leq T_i$ the value

$$C_i(t) := B_t E_t\left[\delta\left(L_i(T_i) - \kappa\right)^+ / B_{T_{i+1}}\right].$$

The price of a cap starting at T_p and ranging over $[T_p, T_{q+1}]$ is for $t < T_p$,

$$Cap_{p,q}(t) := \sum_{i=p}^{q} C_i(t).$$

In order to specify the (discounted) cash flow which is relevant for the multiple stopping problem posed by a flexible cap, we observe that

$$\begin{aligned} C_i(T_i) &= B_{T_i} E_{T_i}\left[\delta\left(L_i(T_i) - \kappa\right)^+ / B_{T_{i+1}}\right] \\ &= \delta\left(L_i(T_i) - \kappa\right)^+ D(T_i, T_{i+1}). \end{aligned} \tag{4.3}$$

In particular, Equation (4.3) specifies the cashflow at T_i which is equivalent to a caplet with settlement date T_{i+1}. Thus, the multiple stopping problem corresponding to the flexible cap will be considered with respect to the discounted cash flow

$$Z_i := \frac{C_i(T_i)}{B_{T_i}} = \delta\left(L_i(T_i) - \kappa\right)^+ D(T_i, T_{i+1})/B_{T_i} = \frac{1}{B_{T_i}}(1 + \delta\kappa)\left[\frac{1}{1 + \delta\kappa} - D(T_i, T_{i+1})\right]^+,$$

at the exercise dates $T_0 \; \ldots, T_{\mathcal{J}}$. Notice that Z_i can also be interpreted the (discounted) payoff of $1+\delta\kappa$ put options with maturity T_i and strike $\frac{1}{1+\delta\kappa}$ which are written on a zero coupon bond with maturity T_{i+1}.
Recall that $\mathcal{S}_i(\mathcal{L})$ denotes the set of F–stopping vectors $\tau := \left(\tau^{(1)}, \ldots, \tau^{(\mathcal{L})}\right)$ such that $T_i \leq \tau^{(1)}$ and, for all l, $1 < l \leq \mathcal{L}$, $\tau^{(l-1)} + \delta \leq \tau^{(l)}$. Then, the price of the flexible cap with $\mathcal{L}$ exercise rights with exercise dates $T_0, ..., T_{\mathcal{J}}$ (corresponding to indices $0, ..., \mathcal{J}$) is given by

$$\begin{aligned} FlCap^{(\mathcal{L})}(T_0) &= \sup_{\tau \in \mathcal{S}_0} E \sum_{l=1}^{\mathcal{L}} Z_{\tau^{(l)}} \\ &= (1+\delta\kappa) \sup_{\tau \in \mathcal{S}_0} E \sum_{l=1}^{\mathcal{L}} \frac{1}{B_{\tau^{(l)}}} \left(\frac{1}{1+\delta\kappa} - D(\tau^{(l)}, \tau^{(l)} + \delta) \right)^+ \end{aligned} \tag{4.4}$$

where $Z_j := 0$ for $j > \mathcal{J}$. The Snell envelope due to $\mathcal{L}$ exercise rights is given by

$$Y_i^{*\mathcal{L}} = \sup_{\tau \in \mathcal{S}_i} E_i \sum_{l=1}^{\mathcal{L}} Z_{\tau^{(l)}}.$$

We now consider the question if the determination of the optimal stopping strategy has a substantial impact on the price of a Bermudan option. More precisely, we compare the (exact) price of the product based on an optimal stopping strategy with trivial benchmark price bounds which can be inferred from deterministic optimization procedures specified below. For a lower trivial benchmark price bound, we consider the following deterministic optimization problem. Let $\mathcal{T}_i(\mathcal{L})$ denote the set of vectors $t := \left(t^{(1)}, \ldots, t^{(\mathcal{L})}\right)$ such that $T_i \leq t^{(1)}$ and, for all l, $1 < l \leq \mathcal{L}$, $t^{(l-1)} + \delta \leq t^{(l)}$. Then, the trivial lower T_j–price bound $Y_j^{\text{triv, low, }\mathcal{L}}$ of a flexible cap with $\mathcal{L}$ exercise rights is given by

$$Y_j^{\text{triv, low, }\mathcal{L}} = \sup_{t \in \mathcal{T}_j} E_j \sum_{l=1}^{\mathcal{L}} Z_{t^{(l)}} \tag{4.5}$$

For an upper trivial benchmark price bound, we rely on a visionary, i.e. we consider the t_j-upper bound $Y_j^{\text{triv, up, }\mathcal{L}}$ of the flexible cap with $\mathcal{L}$ remaining exercise rights, which is given by is

$$Y_j^{\text{triv, up, }\mathcal{L}} = E_j \sup_{t \in \mathcal{T}_j} \sum_{l=1}^{\mathcal{L}} Z_{t^{(l)}}. \tag{4.6}$$

Loosely speaking we will asses the optimal stopping problem as relevant if there is a substantial difference of the exact price and the benchmark price bounds. In this respect we will exclude pricing scenarios where the exact price is equal (or close) to one of the above stated trivial price bounds. The motivation stems from the observation that, in both extreme cases, the exact price can be calculated without using a backward regression procedure, i.e. either by means of a Monte Carlo simulation of the *look back price* achieved by a visionary or by means of a simple optimization stemming from an optimal deterministic set of stopping times. Therefore we formulate a notion that expresses the relevance of the proposed dual linear Monte Carlo algorithm:

Definition 4 (ϵ–relevance of the algorithm for the stopping problem) *The algorithm is called ϵ–relevant for the stopping problem iff*

$$\min \left\{ \frac{Y_j^{\textit{triv, up, }\mathcal{L}}}{Y_j^{*\mathcal{L}}}, \frac{Y_j^{*\mathcal{L}}}{Y_j^{\textit{triv, low, }\mathcal{L}}} \right\} \geq 1 + \epsilon.$$

3M-EURIBOR-forward rates

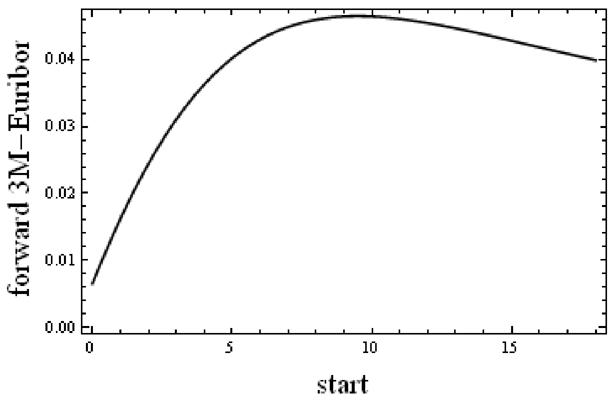

Figure 5.1: The initial term structure of interest rates along the lines of the Euro area yield curve from January 31, 2011. The 3M-EURIBOR-forward rates are obtained along the lines of Svensson (1994).

In particular, notice that the algorithm is, for example, not relevant (zero–relevant) in the case that the number of exercise rights coincides with the number of caplets, i.e. $\mathcal{L} = \mathcal{J}$. Obviously, we also have zero–relevance if the interest rate dynamics is deterministic.

5 Performance Test–Price Comparison (Hull White Model)

Throughout the following, we illustrate the prices and price bounds of flexible caps with a notional of $10,000$ EUR and where the reference rate is the 3–month–EURIBOR. If not mentioned otherwise, the caps range over 15 years, i.e. $\mathcal{J} = 60$, and the cap level is equal to $\kappa = 2\%$.[2] We consider exercise rights which vary from one to fifty, i.e. $\mathcal{L} \in \{1, \ldots, 50\}$. The initial term structure of interest rates is given by the Euro area yield curve from January 31, 2011.[3] The corresponding 3–month-forward rates are illustrated in Figure 5.

5.1 Hull White Model – Basics

The benchmark price values are obtained by assuming a Hull and White (1990) interest rate model which is calibrated to the initial term structure from January 31, 2011. For the sake of completeness, we review the Hull-White interest rate dynamics, and some well known results which are needed in further. However, the proofs are omitted. These can, for example, be found in the textbook of Brigo and Mercurio (2006). The Hull and White (1990) short rate dynamics are given by

$$d\,r_t \;=\; (\theta(t) - a\,r_t)\,dt + \sigma\,d\,W_t, \tag{5.1}$$

where a denotes the speed of mean reversion, $\frac{\theta(t)}{a}$ is the mean reversion level, and σ is the spot rate volatility. The time dependent variable θ allows to calibrate the model to the initial interest rate curve which is observed

[2] It is worth mentioning that all results can also be computed for other cap levels. In particular, one can apply the same basis functions as the ones which are used in the following.

[3] C.f. http://www.ecb.int/stats/money/yc/html/index.en.html.

at the market.

The model implies closed–form solutions for the zero coupon bond and option prices. The t–price of a zero coupon bond with maturity T is given by

$$D(t,T) = \exp\{\mathcal{A}(t,T) - \mathcal{B}(t,T)r_t\} \tag{5.2}$$

where

$$\begin{aligned} \mathcal{B}(t,T) &= \frac{1}{a}(1-\exp\{-a(T-t)\}) \\ \text{and } \mathcal{A}(t,T) &= \ln\frac{D(0,T)}{D(0,t)}\left(\mathcal{B}(t,T)f(0,t) - \frac{\sigma^2}{4a}(1-\exp\{-2at\})\mathcal{B}(t,T)^2\right). \end{aligned} \tag{5.3}$$

Let $f(0,t) = -\frac{\partial \ln D(0,t)}{\partial t}$ be the initial instantaneous forward rate prevailing at time t. Then for arbitrary parameters a and σ the model is calibrated to this initial forward rate curve by choosing

$$\theta(t) = -\frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}(1-e^{-2at}). \tag{5.4}$$

Throughout the following, we set $a = 0.1$ for the the speed of mean reversion and $\sigma = 0.02$ for the volatility. This can be viewed as consistent with swaption data.

Further, we recall the closed–form solution for the time t–price of a European put option with maturity T and strike κ on a zero coupon bond with maturity S ($S \geq T$) which is given by

$$Put(t,T,S,\kappa) = \kappa D(t,T)\mathcal{N}\left(-\frac{\ln\frac{D(t,S)}{\kappa D(t,T)} + \frac{1}{2}v^2}{v} + v\right) - D(t,S)\mathcal{N}\left(-\frac{\ln\frac{D(t,S)}{\kappa D(t,T)} + \frac{1}{2}v^2}{v}\right), \tag{5.5}$$

$$\text{where } v = \sigma\sqrt{\frac{1-\exp\{-2a(T-t)\}}{2a}}\mathcal{B}(T,S).$$

For Monte Carlo simulations later on, it is convenient to use the joined distribution of the spot rate and the interest rate integral. Let

$$\begin{aligned} \nu(t) &= f^M(0,t) + \frac{\sigma^2}{2a}(1-e^{-at})^2 \\ V(t,T) &= \frac{\sigma^2}{a^2}\left[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right]. \end{aligned}$$

Then, the joined distribution of r_t and $\int_s^t r_u du$ conditioned on the information at time s is given by

$$\left[\begin{array}{c|c} r_t & r_s \\ \int_s^t r_u\,du & \int_0^s r_u\,du \end{array}\right] \sim N\left(\begin{array}{c} \mu_1 \\ \mu_2 \end{array} \middle| \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\right), \tag{5.6}$$

where

$$\mu_1 = r_s e^{-a(t-s)} + \nu(t) - \nu(s)e^{-a(t-s)}$$
$$\mu_2 = \mathcal{B}(s,t)(r_s - \nu(s)) + \ln\frac{D^M(0,s)}{D^M(0,t)} + \frac{1}{2}(V(0,t) - V(0,s))$$
$$c_{11} = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right)$$
$$c_{12} = c_{21} = \frac{\sigma^2}{a^2}\left(1 - e^{-a(t-s)}\right) - \frac{\sigma^2}{2a^2}\left(1 - e^{-2a(t-s)}\right)$$
$$c_{22} = V(s,t).$$

5.2 Exact pricing and relevance of the algorithm for the stopping problems

We approximate the exact price of the flexible caps by means of a trinomial tree. The trinomial interest rate tree is implemented along the lines of Hull and White (2000). In particular, equidistant time steps with length $\frac{1}{52}$ are used, cf. for example Hull White (1996) or Brigo and Mercurio (2006). As such the tree is sufficiently high refined to consider the resulting prices as the “exact” ones. Recall (cf. Equation (4.4)) that the T_0–price of a flexible cap with $\mathcal{L}$ exercise rights is given by

$$\begin{aligned} FlCap^{(\mathcal{L})}(T_0) &= \sup_{\tau \in \mathcal{S}_0} E \sum_{l=1}^{\mathcal{L}} Z_{\tau^{(l)}} \\ &= (1+\delta\kappa) \sup_{\tau \in \mathcal{S}_0} E \sum_{l=1}^{\mathcal{L}} \frac{1}{B_{\tau^{(l)}}} \left(\frac{1}{1+\delta\kappa} - D(\tau^{(l)}, \tau^{(l)}+\delta) \right)^+ . \end{aligned}$$

We will compute “exact” prices of flexible caps on the tree by means of the Bellman principle. Notice in this respect that the discounted cash–flows

$$Z_i = (1+\delta\kappa)\frac{1}{B_{T_i}}\left(\frac{1}{1+\delta\kappa} - D(T_i, T_{i+1})\right)^+$$

are path–dependent in fact. However, this issue is easily avoided by considering the Bellman principle in terms of the un-discounted objects $\tilde{Z}_i := B_{T_i} Z_i$ and $\tilde{Y}_i := Y_i B_{T_i}$, which reads as follows.

Set $\tilde{Y}_i^{*,0} = 0$ for $i = 1, \ldots, \mathcal{J}$. At time $\mathcal{J}$, we have

$$\tilde{Y}_{\mathcal{J}}^{*,l} = \tilde{Z}_{\mathcal{J}} \text{ for all } l \geq 1,$$

and at $\mathcal{J} - i$ $(i = 1, \ldots, \mathcal{J})$, we then have backwardly

$$\tilde{Y}_{\mathcal{J}-i}^{*,l} = \max\left\{ \tilde{Z}_{\mathcal{J}-i} + E_{\mathcal{J}-i}\, e^{-\int_{T_{\mathcal{J}-i}}^{T_{\mathcal{J}-i+1}} r_u\, du}\, \tilde{Y}_{\mathcal{J}-i+1}^{*,l-1}, E_{\mathcal{J}-i}\, e^{-\int_{T_{\mathcal{J}-i}}^{\mathcal{J}-i+1} r_u\, du}\, \tilde{Y}_{\mathcal{J}-i+1}^{*l} \right\}.$$

In view of Definition 4, we can now asses the relevance of the algorithm for the multiple stopping problems. We compare the trivial upper and lower bounds of Equation (4.6) and (4.5) with the trinomial tree prices obtained by the Bellmann principle applied to the trinomial tree setup. Notice that the trivial lower bound (linked to the optimal deterministic exercise policy) can be calculated according to the closed–form put–pricing formula, cf.

Relevance of the algorithm for the multiple stopping problems

	trivial price bounds		exact price	percentage mispricing	
ex. rights	lower	upper	tri. tree	triv. lower	triv. upper
$\mathcal{L}$	$Y_0^{triv,low,\mathcal{L}}$	$Y_0^{triv,up,\mathcal{L}}$	$Y_0^{*\mathcal{L}}$	$\left(\frac{Y_0^{*\mathcal{L}}}{Y_0^{triv,low,\mathcal{L}}}-1\right)\cdot 100$	$\left(\frac{Y_0^{triv,up,\mathcal{L}}}{Y_0^{*\mathcal{L}}}-1\right)\cdot 100$
1	61.9212	124.897	93.582	**33.8321**	**33.4624**
2	123.842	241.967	185.871	33.3720	30.1799
3	185.674	353.302	276.780	32.9165	27.6472
4	247.498	459.925	366.253	32.4242	25.5757
5	309.159	562.499	454.258	31.9420	23.8281
6	370.782	661.425	540.764	31.4337	22.3131
7	432.198	756.929	625.758	30.9322	20.9619
8	493.505	849.280	709.201	30.4139	19.7516
9	554.608	938.611	791.078	29.8921	18.6496
10	615.478	1025.15	871.409	**29.3697**	**17.6423**

Table 5.1: The table summarizes the trivial upper $Y_0^{triv,up,\mathcal{L}}$ and lower bounds $Y_0^{triv,low,\mathcal{L}}$ given by Equation (4.6) and (4.5), the exact price $Y_0^{*\mathcal{L}}$ derived by means of the trinomial tree, and the percentage differences between the trivial price bounds and the trinomial prices

Equation (5.5). The trivial upper price bound is obtained by means of a Monte Carlo simulation with 10,000 paths. Observe that, for $\mathcal{L} \in \{1, \dots 10\}$, the exact price is at least 29.36% ($\mathcal{L} = 10$) above the trivial lower price bound derived by the optimal deterministic exercise policy. The maximal deviation is 33.83% for $\mathcal{L} = 1$. With respect to the upper price bound implied by a visionary, the highest (lowest) percentage value of the upper price bound and the exact prices is 33.46% for $\mathcal{L} = 1$ (17.64 for $\mathcal{L} = 10$). Thus, we conclude that, in view of Definition 4, the relevance of the algorithm for the stopping problems is at least $\epsilon = 17\%$.

5.3 Upper and lower pricing by the dual MC algorithm

Due to the stochastic interest rates, we need to consider pathwise discounted cash flow values Z_i. Recall that the zero coupon prices only depend on the spot rate r, cf. Equation (5.2). In particular, we have

$$Z_i = Z_i(r_{T_i}, B_{T_i}) = \frac{1}{B_{T_i}}(1+\delta\kappa)\left[\frac{1}{1+\delta\kappa} - D(T_i, T_{i+1})\right]^+ .$$

For each T_i ($i \in \{0, \dots, \mathcal{J}\}$), we simulate the short rate r_{T_i} and the accumulated short rate $\ln B_{T_i} = \int_0^{T_i} r_s ds$ according to the joined (conditional) distribution as given in Equation (5.6). The equidistant time grid is congruent to the exercise dates, i.e. $\delta = T_{i+1} - T_i = \frac{1}{4}$. In particular, the problem setup implies a two–dimensional Markovian structure $X_u = (r_u, B_u)$ with respect to the one–dimensional Brownian motion W. In view of the closed–form solutions for the zero bond and put option prices, cf. Equation (5.2) and (5.5), we use the martingale property of the discounted price processes of traded assets. We denote the discounted price of

Products used for the Monte carlo algorithm

q	Product q	maturity/settlement date T	level κ
1	Zerobond	15 years	
2	3M–caplet	15 years	2%
3	3M–caplet	7.5 years	2%
4	3M–caplet	3.75 years	2%
5	3M–caplet	1 year	2%

Table 5.2: The table summarizes the products which are used for the Monte Carlo algorithm, i.e. in terms of their discounted price processes.

the traded assets q by $\mathcal{E}_q$ and set

$$M_{i+1}^{(l+1)} - M_i^{(l+1)} = \sum_{q=1}^{K} \beta_{i,q}^{(l+1)} \left(\mathcal{E}_q(T_{i+1}, r_{T_{i+1}}, B_{T_{i+1}}) - \mathcal{E}_q(T_i, r_{T_i}, B_{T_i}) \right).$$

In particular, we use $K = 5$ and refer to the discounted prices of a zerobond with maturity in 15 years and 3M-caplets with settlement dates in 1, 3.75, 7.5, and 15 years such that

$$\begin{aligned}
\mathcal{E}_q(T_i, r_{T_i}, B_{T_i}) &:= \frac{D(T_i,T)}{B_{T_i}} \text{ for } q = 1 \\
\mathcal{E}_q(T_i, r_{T_i}, B_{T_i}) &:= \frac{1+\delta\kappa}{B_{T_i}} Put(T_i, T, T+\delta, \frac{1}{1+\delta\kappa}) \text{ for } q = 2,3,4,5.,
\end{aligned}$$

where $Put(\cdot)$ is given by Equation (5.5). The maturities (settlement dates, respectively) and cap levels are summarized in Table 5.2. In addition, we also use the prices of these products for the approximation of the continuation value, i.e. $\psi_q = \mathcal{E}_q$. The optimal weights γ_q and β_q ($q = 1, \ldots, 5$) are estimated by the backward regression along the lines of Equation (3.7). Since we work in a Markovian environment we can approximate the conditional expectations (continuation values with l exercise rights left) by

$$\mathcal{C}_i^{(l+1)} = \sum_{q=1}^{K} \gamma_{i,q}^{(l+1)} \mathcal{E}_q(T_i, r_{T_i}, B_{T_i}).$$

The martingales, i.e. the weights γ_q and β_q ($q = 1, \ldots, 5$), are determined using a a relatively small number of $1{,}000$ paths. By an additional larger simulation of $9{,}000$ paths the upper and lower price bounds according to (3.11) and (3.12) are computed.

For $\mathcal{L} \in \{1, \ldots, 10\}$, the resulting upper and lower prices obtained by the primal–dual Monte Carlo algorithm are summarized in Table 5.3. It is worth to emphasize that the maximal percentage difference of the upper price bound derived by our algorithm to the exact (trinomial tree) price is 0.56%.[4] In particular, the percentage differences vary only between 0.2% and 0.56%. The lower price bounds, which are actually obtained as by products, are also pretty good. Here, the percentage price differences to the exact price vary between 1.15% and 1.92%. The performance of the dual MC algorithm with regard to the price bounds is also illustrated in

[4] For the exact prices, we refer to Table 5.1.

Prices and price bounds of flexible caps

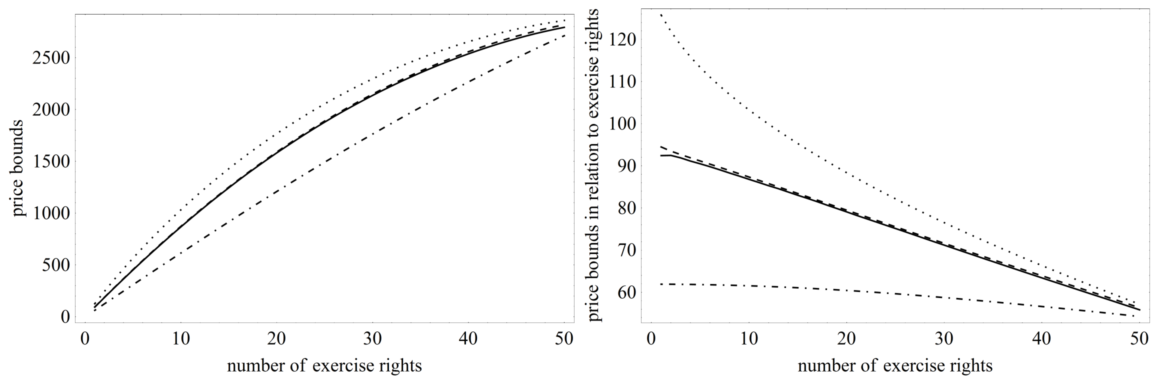

Figure 5.2: The figures illustrate the trivial price bounds (outer lines) and the price bounds given by the dual Monte Carlo algorithm (inner lines) for $\mathcal{L} = 1$ up to $\mathcal{L} = 50$ exercise rights. While the right figure gives a plot of the nominal prices, the right figure relates the price bounds to the number of exercise rights $\mathcal{L}$.

Figure 5.3, where the number of exercise rights varies from $\mathcal{L} = 1$ to $\mathcal{L} = 50$. In addition, the prices are plotted in relation to the number of exercise rights, i.e. we plot $\frac{Y_0^{*\mathcal{L}}}{L}$ ($\frac{Y_0^{up,\mathcal{L}}}{\mathcal{L}}$ and $\frac{Y_0^{down,\mathcal{L}}}{\mathcal{L}}$, respectively). Obviously, we have for any $\mathcal{L} \geq 1$, $Y^{*\mathcal{L}+1} - Y^{*\mathcal{L}} \leq Y^{*\mathcal{L}} - Y^{*\mathcal{L}-1}$ (with $Y^{*0} = 0$). From this we have immediately $Y^{*\mathcal{L}} \leq \mathcal{L}Y^{*1}$, and we may prove by induction that $\frac{Y^{*\mathcal{L}+1}}{\mathcal{L}+1} \leq \frac{Y^{*\mathcal{L}}}{\mathcal{L}}$. Notice that the latter inequality is also true for the upper price bound $Y_0^{up,\mathcal{L}}$ which is implied by the dual Monte Carlo algorithm. However, this feature may be violated in the case of a *non–dual* price approximation, in particular an approximation which relies on some reasonable but suboptimal exercise policy.

6 Conclusion

We propose a primal–dual Monte Carlo algorithm which gives an upper as well as a lower price bound. We implement the algorithm for the pricing of flexible caps. In order to compare the prices provided by our Monte Carlo algorithm with exact prices, we use a simple Hull and White model setup. We calibrate the model to market data and calculate the exact prices by means of a trinomial tree. In addition, we asses that the algorithm is relevant for solving the stopping problems under consideration. The relevance of the algorithm is captured by the differences of exact prices and some (trivial) price bounds which can be computed by a simple optimization (a simple Monte–Carlo simulation, respectively). The lower price bound is obtained by the optimal deterministic exercise policy, the upper price bound is linked to a visionary. Finally, we consider a set of multiple stopping problems (flexible cap products) where the mispricing caused by the trivial price bounds is at least 17% such that a sophisticated algorithm is essential. We illustrate that the proposed primal–dual linear Monte Carlo algorithm is not only tractable to implement but also gives tight price bounds. In particular, it turns out that for 15 year flexible caps, the upper MC price bound is less than 0.56% above the exact price. Finally, last but not least, we underline that the algorithm presented is designed for generic application to any, possibly high dimensional multiple stopping problem.

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Prices from primal-dual Monte Carlo algorithm

ex. rights	upper MC bound		perc. diff.
$\mathcal{L}$	$Y_0^{up\mathcal{L}}$	confidence interval	$\left(\frac{Y_0^{up,\mathcal{L}}}{Y_0^{*\mathcal{L}}} - 1\right) \cdot 100$
1	94.104	[93.767,94.440]	**0.557**
2	186.669	[186.027,187.311]	0.429
3	277.751	[276.820,278.682]	0.351
4	367.366	[366.194,368.539]	0.304
5	455.491	[454.108,456.874]	0.271
6	542.120	[540.553,543.687]	0.251
7	627.215	[625.492,628.938]	0.233
8	710.766	[708.921,712.612]	0.221
9	792.753	[790.829,794.677]	0.212
10	873.142	[871.185,875.100]	**0.199**

ex. rights	lower MC bound		perc. diff.
$\mathcal{L}$	$Y_0^{down\mathcal{L}}$	confidence interval	$\left(\frac{Y_0^{*\mathcal{L}}}{Y_0^{low,\mathcal{L}}} - 1\right) \cdot 100$
1	91.781	[90.690,92.872]	**1.925**
2	183.105	[181.029,185.181]	1.488
3	272.737	[269.689,275.784]	1.461
4	361.102	[357.099,365.105]	1.406
5	448.308	[443.361,453.255]	1.310
6	533.872	[528.000,539.744]	1.275
7	618.189	[611.405,624.973]	1.210
8	701.070	[693.390,708.751]	1.146
9	782.091	[773.527,790.655]	1.136
10	861.428	[851.994,870.862]	**1.145**

Table 5.3: The upper table summarizes the upper bound $Y_0^{up\mathcal{L}}$ for $\mathcal{L} = 1$ up to $\mathcal{L} = 10$ exercise rights and gives margin $\frac{Y_0^{up,\mathcal{L}}}{Y_0^{*\mathcal{L}}} - 100$ compared to the exact price $Y_0^{*\mathcal{L}}$ (computed by a trinomial tree procedure). The lower table gives the analogous results for the lower price bound $Y_0^{low\mathcal{L}}$ of the Monte Carlo algorithm.