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Existence, local uniqueness and asymptotic approximation of spike solutions to singularly perturbed elliptic problems

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Abstract. This paper concerns general singularly perturbed second order semilinear elliptic equations on bounded domains $\Omega \subset \mathbb{R}^n$ with nonlinear natural boundary conditions. The equations are not necessarily of variational type. We describe an algorithm to construct sequences of approximate spike solutions, we prove existence and local uniqueness of exact spike solutions close to the approximate ones (using an Implicit Function Theorem type result), and we estimate the distance between the approximate and the exact solutions. Here "spike solution" means that there exists a point in Ω such that the solution has a spike-like shape in a vicinity of such point and that the solution is approximately zero away from this point. The spike shape is not radially symmetric in general and may change sign.

1 Introduction

The aim of this paper is to study the existence, local uniqueness and asymptotic behaviour for $\varepsilon \to 0$ of spike solutions to singularly perturbed elliptic boundary value problems of the type

$$\left.\begin{array}{rcl} \varepsilon^2 \left(\sum\limits_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) + \sum\limits_{i=1}^n b_i(x)\partial_{x_i} u \right) &=& f(x,u,\varepsilon), \quad x \in \Omega, \\ \sum\limits_{i,j=1}^n a_{ij}(x)\nu_i(x)\partial_{x_j} u &=& g(x,u,\varepsilon), \quad x \in \partial\Omega. \end{array}\right\} \tag{1.1}$$

Here $\varepsilon > 0$ is a small parameter, $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, and ν_i are the components of the unit outer normal at $\partial\Omega$. The coefficients $a_{ij}, b_i : \overline{\Omega} \to \mathbb{R}$, and the right-hand sides $f : \overline{\Omega} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \times [0,1] \to \mathbb{R}$ are supposed to be sufficiently smooth. Further, the differential operator in (1.1) is supposed to be uniformly elliptic, i.e. $a_{ij} = a_{ji}$ and there exists a constant $c_0 > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x) y_i y_j \geq c_0 |y|^2 \quad \text{for all} \quad (x,y) \in \overline{\Omega} \times \mathbb{R}^n.$$

Roughly speaking, below we prove the existence, local uniqueness and asymptotic behaviour for $\varepsilon \to 0$ of solutions u to (1.1) with the following properties:

(i) There exists a point $\xi_0 \in \Omega$ such that u has a spike-like behaviour in the vicinity of ξ_0.

(ii) In all remaining points $x \in \Omega$ we have $u(x) \approx 0$.

Such solutions turn out to exist under a series of natural assumptions. The assumption, mainly implying property (II), is the following:

(A1) $f(x,0,0) = 0$ and $\partial_u f(x,0,0) > 0$ for all $x \in \overline{\Omega}$.

The rest three assumptions implying mainly property (I) we formulate as follows:

(A2) There exist a subdomain $\tilde{\Omega} \subseteq \Omega$ and a smooth map $(r,\xi) \in [0,\infty) \times \tilde{\Omega} \mapsto \phi_\xi(r) \in \mathbb{R}$ such that for every fixed $\xi \in \tilde{\Omega}$ the function $\phi = \phi_\xi$ solves the one-dimensional boundary value problem

$$\left.\begin{array}{l} \phi''(r) + \frac{n-1}{r}\phi'(r) = f(\xi,\phi(r),0), \quad 0 < r < \infty, \\ \phi'(0) = 0, \quad \phi(\infty) = 0, \quad \phi(0) \neq 0. \end{array}\right\} \tag{1.2}$$

(A3) There exists a non-degenerate solution $\xi_0 \in \tilde{\Omega}$ to the algebraic system

$$A^{-1}(\xi)b(\xi) + \nabla_\xi \log \left(\sqrt{\det A(\xi)} \int_{\mathbb{R}} \phi_\xi'(r)^2 r^{n-1} dr \right) = 0, \tag{1.3}$$

where

$$A(\xi) := [a_{ij}(\xi)]_{i,j=1}^n \quad \text{and} \quad b(\xi) := [b_i(\xi)]_{i=1}^n . \tag{1.4}$$

Each function ϕ_ξ from assumption (A1) corresponds, via $\Phi_\xi(y) := \phi_\xi(|y|)$, to a radially symmetric solution $v = \Phi_\xi$ of the following n-dimensional boundary value problem

$$\left. \begin{aligned} &\Delta_y v(y) = f(\xi, v(y), 0), \quad y \in \mathbb{R}^n, \\ &v(y) \to 0 \quad \text{for} \quad |y| \to \infty. \end{aligned} \right\} \tag{1.5}$$

In the scope of our consideration, such symmetric solutions Φ_ξ will be used to describe a scaled profile of the spike which may appear at point ξ. It is easy to show (see Remark 1.3) that the functions $v = \partial_{y_j} \Phi_{\xi_0}$ are solutions of the linearized problem

$$\left. \begin{aligned} &\Delta_y v(y) = \partial_u f(\xi_0, \Phi_{\xi_0}(y), 0) v(y), \quad y \in \mathbb{R}^n, \\ &v(y) \to 0 \quad \text{for} \quad |y| \to \infty. \end{aligned} \right\} \tag{1.6}$$

Our last assumption concerns the following non-degeneracy property:

(A4) For any solution v to (1.6) it holds $v \in \operatorname{span} \left\{ \partial_{y_j} \Phi_{\xi_0} \ : \ j = 1, \ldots, n \right\}$.

Our main result is of the following type:

For small $\varepsilon > 0$ and $m = 0, 1, \ldots$ we will construct smooth functions $\mathcal{W}_{\varepsilon,m} : \overline{\Omega} \to \mathbb{R}$ which have the properties (I) and (II) and which satisfy (1.1) approximately. Moreover, we will prove that for small $\varepsilon > 0$ there exists an exact solution $u = u_\varepsilon$ to (1.1) such that for any $\alpha \in (0, 1)$ and any m it holds

$$\|u_\varepsilon - \mathcal{W}_{\varepsilon,m}\|_{2+\alpha,\varepsilon;\Omega} = O(\varepsilon^{m+1}) \quad \text{for} \quad \varepsilon \to 0,$$

where

$$\|u\|_{2+\alpha,\varepsilon;\Omega} := \sum_{k=0}^{2} \varepsilon^k \sup_{|\mu|=k} \sup_{\Omega} |D^\mu u| + \varepsilon^{2+\alpha} \sup_{|\mu|=2} \sup_{\substack{x,y \in \Omega, \\ x \neq y}} \frac{|D^\mu u(x) - D^\mu u(y)|}{|x - y|^\alpha}$$

is an ε-dependent norm in the Hölder space $C^{2+\alpha}(\overline{\Omega})$. Finally, we will prove a local uniqueness assertion for u_ε: If $\varepsilon > 0$ is small and u is a solution to (1.1) which is close to $\mathcal{W}_{\varepsilon,0}$ (in a sense to be made precise) then $u = u_\varepsilon$.

In order to describe our results more exactly, let us consider the lowest approximation order case $m = 0$. Define

$$\mathcal{W}_\varepsilon(x) := \Phi_{\xi_0}(T_\varepsilon(x)).$$

Here $T_\varepsilon(x)$ are stretched coordinates defined as follows:

$$T_\varepsilon(x) := \frac{1}{\varepsilon} A(\xi_0 + \varepsilon x_1)^{-1/2} (x - \xi_0 - \varepsilon x_1) \quad \text{for} \quad x \in \Omega.$$

Further $A(\xi)^{-1/2}$ is the inverse square root of the positive definite matrix $A(\xi)$ (see notation (1.4)), and x_1 is the correction term of the first order to the spike's position determined from Eq. (2.58). Now our result for $m = 0$ reads as follows:

Theorem 1.1 *Suppose that assumptions (A1)–(A4) are fulfilled.*

Then for any $\alpha \in (0,1)$ there exist $\varepsilon_\alpha > 0$, $\delta_\alpha > 0$ and $c_\alpha > 0$ such that the following is true:

(i) For all $\varepsilon \in (0, \varepsilon_\alpha)$ there exists a solution $u = u_\varepsilon$ to (1.1) such that

$$\|u_\varepsilon - \mathcal{W}_\varepsilon\|_{2+\alpha,\varepsilon;\Omega} \le c_\alpha \varepsilon.$$

(ii) If u is a solution to (1.1) with $\varepsilon \in (0, \varepsilon_\alpha)$ and

$$\|u - \mathcal{W}_\varepsilon\|_{2+\alpha,\varepsilon;\Omega} < \delta_\alpha \varepsilon^2,$$

then $u = u_\varepsilon$.

Existence and multiplicity results for problem (1.1) have been objects of systematic investigation during last decades. This interest is, in particular, motivated by the study of standing waves in the nonlinear Schrödinger equation which leads typically to the consideration of concentrating solutions (so called bound states) of the following elliptic boundary value problem

$$\left.\begin{array}{rcll} \varepsilon^2 \Delta u &=& V(x)u - u^q, & x \in \Omega, \\ \partial_\nu u &=& 0, & x \in \partial\Omega, \end{array}\right\}$$

where $q > 1$, and $V : \overline{\Omega} \to \mathbb{R}$ is a smooth positive potential. Another source of applications for problem (1.1) is concerned with the study of pattern formation in chemical reaction-diffusion systems, including well-known Gierer-Meinchardt and FitzHugh-Nagumo models [26].

One can distinguish two main approaches used systematically in this field. A first one, initiated by Floer and Weinstein [11], relies on a finite dimensional Lyapunov-Schmidt reduction (see also [22, 23, 24]). A second one is based on variational methods jointly with a penalization technique (we recall, among many others, [34, 42, 28, 29, 30, 31], see also [1] for further references).

Our study differs from the above in several points. First, our elliptic equation does not have a divergence form, what makes impossible application of variational methods used, for example, for similar equations with $b_i(x) = 0$, see e.g. [37, 32]. Second, for arbitrary space dimension n we obtain a sequence of approximate solutions with pointwise asymptotic estimates in the L^∞-norm up to any power of ε. Note that in contrary to most of the previous studies concerned with (1.1), our approximate solutions, in general, comprise non-zero outer expansion parts. This fact leads to a more complicated formulas for the inner expansions of the spike and boundary layers, but simultaneously shows the universality of our approach. Third, the spike shapes are allowed to change sign. And finally, to prove our Theorem 4.6 we do not need eigenvalue estimates for the linearized (in the approximate solution) problem. Instead we use a lemma of R. Magnus [19, Lemma 1.3] which helps to verify the assumptions of a quite general implicit function theorem (see our Section 3).

Remark 1.2 *Various sufficient conditions for the existence of radially symmetric solutions of problem (1.5) can be found in literature (see, for example, [5, 6, 12, 39, 8]). Some of them [5, 6]* *were obtained with the help of variational methods, when instead of the solution to problem (1.5) one looks for a critical point of the energy functional*

$$\mathcal{E}_\xi(v) := \int\limits_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla_y v(y)|^2 dy + F(\xi, v(y), 0)\right) dy, \quad \textit{where} \quad F(\xi, v, \varepsilon) := \int\limits_0^v f(\xi, u, \varepsilon) du. \tag{1.7}$$

An important role in this analysis is played by the Pohozaev's identity (see [5, Section 2])

$$\frac{n-2}{2} \int\limits_{\mathbb{R}^n} |\nabla_y v(y)|^2 dy = -n \int\limits_{\mathbb{R}^n} F(\xi, v(y), 0) dy. \tag{1.8}$$

which is valid, in particular, for any radially symmetric solution $v \in W^{1,2}(\mathbb{R}^n)$ of problem (1.5). Remark, the identity (1.8) implies that for any radially symmetric solution of problem (1.5) holds

$$\mathcal{E}_\xi(v) = \frac{1}{n} \int\limits_{\mathbb{R}^n} |\nabla_y v(y)|^2 dy. \tag{1.9}$$

Another method to prove the existence of radially symmetric solutions of problem (1.5) is concerned with the direct analysis of corresponding one-dimensional problem (1.2). It was used, in particular, in [12, 39, 8].

Remark 1.3 *For the solution ϕ_ξ to problem (1.2), one can easily show (see [5, Lemma 4]) that*

$$\lim_{r \to 0} \frac{\phi'_\xi(r)}{r} = \lim_{r \to 0} \phi''_\xi(r) = \frac{1}{n} f(\xi, \phi_\xi(0), 0). \tag{1.10}$$

Since above we have assumed that $\phi_\xi(0) \neq 0$, limits (1.10) immediately imply that

$$f(\xi, \phi_\xi(0), 0) \neq 0. \tag{1.11}$$

Further, every solution $\phi = \phi_\xi$ to problem (1.2) corresponds to a solution $\theta = (\phi_\xi, \phi'_\xi)^T$ of the linear system

$$\theta'(r) = \Pi_\xi(r)\theta(r), \quad \textit{where} \quad \Pi_\xi(r) := \begin{bmatrix} 0 & 1 \\ \int\limits_0^1 \partial_u f(\xi, t\phi_\xi(r), 0) dt & -\frac{n-1}{r} \end{bmatrix}.$$

Hence, taking into account assumption (A1) and applying classical results of exponential dichotomy theory [7, Chapter 6, Proposition 1], we come to the conclusion that for every $\xi \in \tilde{\Omega}$ and every $\kappa \in (0, \sqrt{\partial_u f(\xi, 0, 0)})$ it holds

$$|\phi_\xi(r)|, |\phi'_\xi(r)|, |\phi''_\xi(r)| \leq C(\xi, \kappa) e^{-\kappa r} \quad \textit{for all} \quad r \in [0, \infty), \tag{1.12}$$

where $C(\xi, \kappa) > 0$ is a certain constant. Alternatively, one can get exponential estimates (1.12) from the determining system (1.5) for Φ_ξ (see [33]).

Moreover, it is easy to show that for each $\xi \in \tilde{\Omega}$ the partial derivatives $\partial_{\xi_j}\phi_\xi(r)$, $j = 1, \dots, n$, exist, that the corresponding functions $\partial_{\xi_j}\phi_\xi$ satisfy the linear inhomogeneous differential equation

$$\partial_{\xi_j}\phi_\xi''(r) + \frac{n-1}{r}\partial_{\xi_j}\phi_\xi'(r) - \partial_u f(\xi, \phi_\xi(r), 0)\partial_{\xi_j}\phi_\xi(r) = \partial_{\xi_j} f(\xi, \phi_\xi(r), 0), \quad 0 < r < \infty,$$

and, hence, that they satisfy estimates analogous to (1.12).

Remark 1.4 *Note that subdomain $\tilde{\Omega}$ in assumption (A2) plays a technical role only. In particular, if at the very beginning we know a point $\xi_0 \in \Omega$ and a corresponding solution ϕ_0 of problem (1.2), then a straightforward application of the Implicit Function Theorem guarantees the existence of a subdomain $\tilde{\Omega}$ containing ξ_0 and the existence of a smooth map $(r, \xi) \in [0, \infty) \times \tilde{\Omega} \mapsto \phi_\xi(r) \in \mathbb{R}$ such that (1.2) is satisfied for all $\xi \in \tilde{\Omega}$ and that $\phi_0 = \phi_{\xi_0}$.*

Remark 1.5 *One can easily check that in the case $b(x) = 0$ and $f(x, u, \varepsilon) = V(x)u - u^q$ with $q > 1$, $V(x) > 0$, Eq. (1.3) is equivalent to the equation for spike's position obtained in [32] by means of variational technique. Indeed, in this case, every solution $v = \Phi_\xi$ to problem (1.5) corresponds, via $\Phi_\xi(y) = V(\xi)^{1/(q-1)}U(\sqrt{V(\xi)}y)$, to a radially symmetric solution U of equation $\Delta U = U - U^q$ which decays to zero at infinity and does not depend on ξ. This implies, in particular, that*

$$\int_{\mathbb{R}^n} |\nabla_y \Phi_\xi(y)|^2 \, dy = V(\xi)^{\frac{q+1}{q-1} - \frac{n}{2}} \int_{\mathbb{R}^n} |\nabla_y U(y)|^2 \, dy,$$

hence our Eq. (1.3) determines the same spike's positions as the Theorem 1.3 in [32].

Note that, in contrary to the paper [32], we do not restrict our consideration to positive solutions only. Moreover, our method provides more accurate pointwise asymptotic estimates (in L^∞-norm) for the obtained solutions.

Remark 1.6 *Since functions Φ_ξ are assumed to be radially symmetric, a standard way to verify assumption (A4) is to find all bounded solutions of the problem (1.6) by the method of separation of variables. This scheme was previously used to demonstrate that assumption (A4) is fulfilled for any positive, radially symmetric solution of the problem (1.5) with the right-hand side $f(x, u, \varepsilon) = V(x)u - u^q$, $q > 1$, and $V(x) > 0$ (see [46, Appendix A] and [17]). Further generalizations of this result can be found in [21].*

Besides, assumption (A4) is always fulfilled in the case $n = 1$. This fact follows from assumption (A1) and well-known results on the exponential dichotomy [7, Chapter 6, Proposition 1].

Remark 1.7 *Below we prove existence of spike solutions to (1.1), where the spike shapes are approximately radially symmetric, but may change sign. Remark that, if the solution to (1.5), which approximately determines the spike shape, is positive, then it is necessarily radially symmetric (by the famous Gidas-Ni-Nirenberg theorem [15]).*

Remark 1.8 *Our results can be easily generalized on a broader class of singularly perturbed elliptic equations with non-variational structure. In particular, they are applicable to equations of the type*

$$\varepsilon^2 \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) = f(x, u, \varepsilon) + \varepsilon f_1(x, u, \varepsilon\nabla_x u, \varepsilon).$$

The proposed asymptotic analysis can also be used to generalize some known results about boundary spike solutions in singularly perturbed problems (see [20, 21, 14, 43, 44, 45, 4]).

Remark 1.9 *Our results can be easily generalized for the case of solution to problem (1.1) with a finite number of distinct spike's. The construction procedure and the technique of proof remain almost the same in this case.*

Our paper is organized as follows:

In Section 2 we describe the algorithm of the construction of our approximate solutions. In Section 3 we formulate and prove a generalized Implicit Function Theorem, and in Section 4 we derive from this existence, local uniqueness and estimates of exact solutions to (1.1) close to the approximate ones. Finally, some needed technical estimates are provided in Appendix.

2 Construction of the approximate solutions

In this section, we construct approximate solutions to problem (1.1). For this, we assume that the conditions (A1)–(A4) are satisfied and that the function f and the coefficients a_{ij} and b_i are sufficiently smooth to allow their representation via Taylor's formula with necessary number of terms.

Following standard scheme of singular perturbation theory [25, 40, 41], we look for approximate solutions of the type

$$\mathcal{W}_{\varepsilon,m}(x) = u_{\varepsilon,m}(x) + v_{\varepsilon,m}(x) + w_{\varepsilon,m}(x), \tag{2.1}$$

which consist of three different parts: the outer expansion $u_{\varepsilon,m}(x)$ (which is defined by the property $\mathcal{W}_{\varepsilon,m}(x) - u_{\varepsilon,m}(x) \approx 0$ for all x away from the spike center and from $\partial\Omega$), the inner expansion $v_{\varepsilon,m}(x)$ of the spike (which is defined by the property $\mathcal{W}_{\varepsilon,m}(x) - v_{\varepsilon,m}(x) \approx u_{\varepsilon,m}(x)$ for all x close to the spike center) and the inner expansion $w_{\varepsilon,m}(x)$ of the boundary layer (which is defined by the property $\mathcal{W}_{\varepsilon,m}(x) - w_{\varepsilon,m}(x) \approx u_{\varepsilon,m}(x)$ for all x close to $\partial\Omega$). The ansatz for the outer expansion and the inner expansion of the spike is

$$u_{\varepsilon,m}(x) = \sum_{k=0}^{m} \varepsilon^k u_k(x), \quad \text{and} \quad v_{\varepsilon,m}(x) = \sum_{k=0}^{m} \varepsilon^k v_k(T_{\varepsilon,m}(x)), \tag{2.2}$$

where $T_{\varepsilon,m}$ is a stretching transformation near the spike, given by

$$T_{\varepsilon,m}(x) = \frac{1}{\varepsilon} Q(x_{\varepsilon,m})(x - x_{\varepsilon,m}) \text{ with } x_{\varepsilon,m} = \sum_{k=0}^{m+1} \varepsilon^k x_k \text{ and } Q(x) := A(x)^{-1/2} \tag{2.3}$$

(cf. notation (1.4)). The ansatz for the inner expansion of the boundary layer is

$$w_{\varepsilon,m}(x) = \begin{cases} \chi\left(\delta^{-1}\operatorname{dist}(x,\partial\Omega)\right) \sum_{k=0}^{m} \varepsilon^k w_k(S_\varepsilon(x)) & \text{for } \operatorname{dist}(x,\partial\Omega) < 2\delta, \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

where $\chi : [0,\infty) \to \mathbb{R}$ is a non-increasing smooth cut-off function such that $\chi(r) = 1$ for $0 \le r \le 1$ and $\chi(r) = 0$ for $r \ge 2$. Further, $\delta > 0$ is a parameter, S_ε is a stretching transformation near the boundary given by

$$S_\varepsilon^{-1}(z,\zeta) := \zeta - \varepsilon z \nu(\zeta) \text{ with } \zeta \in \partial\Omega \text{ and } 0 \le z < \frac{2\delta}{\varepsilon}, \tag{2.5}$$

and $\nu(\zeta)$ is the unit normal vector of $\partial\Omega$ at $\zeta \in \partial\Omega$ pointing out of Ω. We fix δ sufficiently small such that the map $(z,\zeta) \mapsto \zeta - \varepsilon z \nu(\zeta)$ is bijective from $(0, 2\delta/\varepsilon) \times \partial\Omega$ onto the set of all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < 2\delta$, and, hence, the definitions (2.4) and (2.5) are correct.

In the ansatz (2.1)–(2.4) the functions $u_k : \overline{\Omega} \to \mathbb{R}$, $v_k : \mathbb{R}^n \to \mathbb{R}$ and $w_k : [0,\infty) \times \partial\Omega \to \mathbb{R}$ as well as the vectors $x_k \in \mathbb{R}^n$ are unknown and have to be determined by the algorithm described below.

For the sake of simplicity, in what follows we will use the notation

$$E_\varepsilon u := \varepsilon^2 \left(\sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) + \sum_{i=1}^n b_i(x)\partial_{x_i} u \right)$$

for the elliptic differential operator in problem (1.1).

Roughly speaking, the algorithm is as follows: First we determine the functions u_k such that the equation

$$E_\varepsilon u_{\varepsilon,m} - f(x, u_{\varepsilon,m}, \varepsilon) = 0 \tag{2.6}$$

is satisfied up to an error of order $O(\varepsilon^{m+1})$, this will be done in Subsection 2.1. Then we determine the functions v_k and the vectors x_k such that the system

$$\left.\begin{array}{rcl} E_\varepsilon v_{\varepsilon,m} - f(x, u_{\varepsilon,m} + v_{\varepsilon,m}, \varepsilon) + f(x, u_{\varepsilon,m}, \varepsilon) & = & 0, \\ \nabla_x \left(u_{\varepsilon,m} + v_{\varepsilon,m}\right)(x_{\varepsilon,m}) & = & 0 \end{array}\right\} \tag{2.7}$$

is satisfied up to an error of order $O(\varepsilon^{m+1})$, this will be done in Subsection 2.2. The requirement $\nabla_x \left(u_{\varepsilon,m} + v_{\varepsilon,m}\right)(x_{\varepsilon,m}) = 0$ means that the extremum of the approximate spike $u_{\varepsilon,m} + v_{\varepsilon,m}$ is located in the point $x_{\varepsilon,m}$, i.e. that $x_{\varepsilon,m}$ is approximately the extremum point of the exact spike. And finally we determine the functions w_k such that the boundary value problem

$$\left.\begin{array}{r} E_\varepsilon w_{\varepsilon,m} - f(x, u_{\varepsilon,m} + w_{\varepsilon,m}, \varepsilon) + f(x, u_{\varepsilon,m}, \varepsilon) = 0, \quad x \in \Omega, \\ \sum_{i,j=1}^n a_{ij}(x)\nu_i(x)\partial_{x_j}\left(u_{\varepsilon,m} + w_{\varepsilon,m}\right) - g(x, u_{\varepsilon,m} + w_{\varepsilon,m}, \varepsilon) = 0, \quad x \in \partial\Omega \end{array}\right\} \tag{2.8}$$

is satisfied up to an error of order $O(\varepsilon^{m+1})$, this will be done in Subsection 2.3. In summary, we are going to prove the following theorem.

Theorem 2.1 *Suppose that assumptions (A1)–(A4) are fulfilled.*

Then, following the algorithm described in Subsections 2.1–2.3 one can construct for any $\varepsilon \in (0,\infty)$ *and for any nonnegative integer* m *a smooth function* $\mathcal{W}_{\varepsilon,m} : \overline{\Omega} \to \mathbb{R}$ *such that for any* $\alpha \in (0,1)$ *it holds*

$$\left\| E_\varepsilon \mathcal{W}_{\varepsilon,m} - f(\cdot, \mathcal{W}_{\varepsilon,m}, \varepsilon) \right\|_{\alpha,\varepsilon;\Omega} = O(\varepsilon^{m+1}), \tag{2.9}$$

$$\left\| \sum_{i,j=1}^n a_{ij}(\cdot)\nu_i(\cdot)\partial_{x_j}\mathcal{W}_{\varepsilon,m} - g(\cdot, \mathcal{W}_{\varepsilon,m}, \varepsilon) \right\|_{1+\alpha,\varepsilon;\partial\Omega} = O(\varepsilon^m). \tag{2.10}$$

Moreover, the functions $\mathcal{W}_{\varepsilon,m}$ *have structure (2.1)-(2.5) with smooth functions* $u_k : \overline{\Omega} \to \mathbb{R}$, $v_k : \mathbb{R}^n \to \mathbb{R}$ *and* $w_k : [0,\infty) \times \partial\Omega \to \mathbb{R}$.

Finally, for any $\kappa \in (0, \kappa_0)$ *and* $\varkappa \in (0, \varkappa_0)$ *with*

$$\kappa_0 := \sqrt{\partial_u f(\xi_0, 0, 0)} \quad \text{and} \quad \varkappa_0 := \min_{\zeta \in \partial\Omega} \frac{\partial_u f(\zeta, 0, 0)}{\sum_{i,j=1}^{n} a_{ij}(\zeta)\nu_i(\zeta)\nu_j(\zeta)} \tag{2.11}$$

there exists $c > 0$ *such that for any* $k = 1, \dots, m$ *and* $|\mu| \leq 2$ *it holds*

$$|D^\mu v_k(y)| \leq c e^{-\kappa |y|} \quad \text{for all} \quad y \in \mathbb{R}^n, \tag{2.12}$$

$$|D^\mu w_k(z, \zeta)| \leq c e^{-\varkappa z} \quad \text{for all} \quad (z, \zeta) \in [0, \infty) \times \partial\Omega. \tag{2.13}$$

2.1 Outer expansion

We substitute the ansatz (2.2) for $u_{\varepsilon,m}$ into (2.6). Then we expand the left hand side of the resulting equation in the ε-power series. Equating to zero the coefficients of each power of ε, we obtain an array of algebraic equations. The lowest order equation is

$$f(x, u_0(x), 0) = 0.$$

According to (A1), we choose $u_0(x) \equiv 0$. Then the equations for u_k, $k \geq 1$ are given by

$$\partial_u f(x, 0, 0) u_1(x) + \partial_\varepsilon f(x, 0, 0) = 0, \tag{2.14}$$

$$\partial_u f(x, 0, 0) u_k(x) + (\text{function depending on } u_0, \dots, u_{k-1}) = 0, \quad k \geq 2.$$

Thanks to condition (A1) each u_k is uniquely determined successively for $k = 1, 2, \dots, m$. Moreover, we have

$$\|E_\varepsilon u_{\varepsilon,m} - f(\cdot, u_{\varepsilon,m}, \varepsilon)\|_{C^\alpha(\overline{\Omega})} = O(\varepsilon^{m+1}).$$

2.2 Inner expansion of the spike

Instead of variable x we will work with the stretched variable y given by (cf. (2.3))

$$y = T_{\varepsilon,m}(x) = \frac{1}{\varepsilon} Q(x_{\varepsilon,m})(x - x_{\varepsilon,m}), \quad \text{or} \quad x = T_{\varepsilon,m}^{-1}(y) = x_{\varepsilon,m} + \varepsilon Q(x_{\varepsilon,m})^{-1} y.$$

Obviously, for any smooth function $v : \mathbb{R}^n \to \mathbb{R}$ we have

$$\nabla_x (v \circ T_{\varepsilon,m}) = \frac{1}{\varepsilon} Q(x_{\varepsilon,m}) \nabla_y v \circ T_{\varepsilon,m}.$$

As usual, for vector functions $z : \Omega \to \mathbb{R}^n$ we denote by $z \cdot \nabla_x := \sum_{j=1}^{n} z_j \partial_{x_j}$ the first order differential operator, generated by z, and by $\nabla_x \cdot z := \sum_{j=1}^{n} \partial_{x_j} z_j$ the divergence of z.

Now we substitute the ansatz (2.2) for $v_{\varepsilon,m}$ and the ansatz (2.3) for $x_{\varepsilon,m}$ into (2.7). Further, we use that for any smooth function $v : \mathbb{R}^n \to \mathbb{R}$ it holds

$$E_\varepsilon(v \circ T_{\varepsilon,m})(T_{\varepsilon,m}^{-1}(y)) = \varepsilon^2 \left(\nabla_x \cdot A\nabla_x(v \circ T_{\varepsilon,m}) + (b \cdot \nabla_x)(v \circ T_{\varepsilon,m})\right)(T_{\varepsilon,m}^{-1}(y)) = \Delta_y v(y)$$

$$+\varepsilon \left(Q(x_{\varepsilon,m})\nabla_y \cdot \int_0^1 \left(Q(x_{\varepsilon,m})^{-1} y \cdot \nabla_x\right) A\left(x_{\varepsilon,m} + \varepsilon t\, Q(x_{\varepsilon,m})^{-1} y\right) dt\, Q(x_{\varepsilon,m})\nabla_y v(y) \right)$$

$$+\varepsilon \left(b\left(x_{\varepsilon,m} + \varepsilon\, Q(x_{\varepsilon,m})^{-1} y\right) \cdot Q(x_{\varepsilon,m})\nabla_y v(y) \right) \tag{2.15}$$

and

$$[f(\cdot, u_{\varepsilon,m} + v \circ T_{\varepsilon,m}, \varepsilon) - f(\cdot, u_{\varepsilon,m}, \varepsilon)]\,(T_{\varepsilon,m}^{-1}(y))$$

$$= \int_0^1 \partial_u f\left(x_{\varepsilon,m} + \varepsilon\, Q(x_{\varepsilon,m})^{-1} y, u_{\varepsilon,m}(x_{\varepsilon,m} + \varepsilon\, Q(x_{\varepsilon,m}) y, \varepsilon) + tv(y), \varepsilon\right) dt\; v(y). \tag{2.16}$$

This way we get

$$[E_\varepsilon v_{\varepsilon,m} - f(\cdot, u_{\varepsilon,m} + v_{\varepsilon,m}, \varepsilon) + f(\cdot, u_{\varepsilon,m}, \varepsilon)] \circ T_{\varepsilon,m}^{-1} = \Delta v_0 - f(x_0, v_0, 0)$$

$$+ \sum_{k=1}^m \varepsilon^k \left(\Delta_y v_k - \partial_u f(x_0, v_0, 0) v_k - F_k(y, x_0, \ldots, x_k, v_0, \ldots, v_{k-1})\right) + O(\varepsilon^{m+1}), \tag{2.17}$$

where the right hand sides $F_k(y, x_0, \ldots, x_k, v_0, \ldots, v_{k-1})$ depend on the functions $v_0, \ldots, v_{k-1}$ via the values in the point y of those functions and their first and second derivatives only. Moreover,

$$F_k(y, x_0, \ldots, x_k, 0, \ldots, 0) = 0.$$

Similarly, we get

$$Q(x_{\varepsilon,m})^{-1}\nabla_x \left(u_{\varepsilon,m} + v_{\varepsilon,m}\right)(x_{\varepsilon,m}) = \sum_{k=0}^m \varepsilon^{k-1} \left[\nabla_y v_k(0) + \varepsilon Q(x_{\varepsilon,m})^{-1}\nabla_x u_k(x_{\varepsilon,m})\right]$$

$$= \varepsilon^{-1}\nabla_y v_0(0) + \nabla_y v_1(0) + \sum_{k=2}^m \varepsilon^{k-1}\left(\nabla_y v_k(0) - d_k(x_0, \ldots, x_{k-2})\right) + O(\varepsilon^m),$$

where, because of the fact that $u_0(x) = 0$ (see Section 2.1), the right hand sides $d_k(x_0, \ldots, x_{k-2})$ do not depend on x_{k-1}.

We determine the functions v_k and the vectors x_k in the following order: In the step number zero we solve the problem

$$\left.\begin{array}{rcl} \Delta_y v_0(y) - f(x_0, v_0(y), 0) & = & 0, \\ \nabla_y v_0(0) & = & 0, \\ v_0(y) \to 0 \text{ for } |y| & \to & \infty \end{array}\right\} \tag{2.18}$$

with respect to v_0. In this step x_0 is still unknown, i.e. the solution v_0 depends on x_0.

In the step number one we solve the problem

$$\left.\begin{array}{rcl}\Delta_y v_1(y) - \partial_u f(x_0, v_0(y), 0) v_1(y) &=& F_1(y, x_0, x_1, v_0),\\ \nabla_y v_1(0) &=& 0,\\ v_1(y) \to 0 \;\text{ for }\; |y| &\to& \infty\end{array}\right\} \tag{2.19}$$

with respect to v_1. Because the differential equation is linear inhomogeneous and because of assumption (A4), the right hand side $F_1(y, x_0, x_1, v_0)$ has to be orthogonal to an n-dimensional subspace. This orthogonality condition gives a system of n nonlinear algebraic equations to be solved with respect to x_0. Thus, after this step v_1 and x_0 are determined, but x_1 is still unknown. Moreover, we show that x_0 does not depend on x_1, and v_1 depends on x_1 affinely.

In the step number two we solve the problem

$$\left.\begin{array}{rcl}\Delta_y v_2(y) - \partial_u f(x_0, v_0(y), 0) v_2(y) &=& F_2(y, x_0, x_1, x_2, v_0, v_1),\\ \nabla_y v_2(0) &=& d_2(x_0),\\ v_2(y) \to 0 \;\text{ for }\; |y| &\to& \infty\end{array}\right\} \tag{2.20}$$

with respect to v_2. For that the right hand side $F_2(y, x_0, x_1, x_2, v_0, v_1)$ has to be orthogonal to the n-dimensional subspace, again. Although the dependence of $F_2(y, x_0, x_1, x_2, v_0, v_1)$ on x_1 is not affine, the corresponding orthogonality condition produces a system of n inhomogeneous algebraic equations which are affine with respect to x_1 and can be uniquely solved with respect to x_1. Thus, after this step v_2 and x_1 are determined, but x_2 is still unknown, x_1 is independent on x_2, and v_2 depends affinely on x_2.

The next steps are as step number two: We have to solve

$$\left.\begin{array}{rcl}\Delta_y v_k(y) - \partial_u f(x_0, v_0(y), 0) v_k(y) &=& F_k(y, x_0, \ldots, x_k, v_0, \ldots, v_{k-1}),\\ \nabla_y v_k(0) &=& d_k(x_0, \ldots, x_{k-2}),\\ v_k(y) \to 0 \;\text{ for }\; |y| &\to& \infty\end{array}\right\} \tag{2.21}$$

with respect to v_k (linearly depending on x_k, which is still unknown) and to x_{k-1} (which does not depend on x_k). Remark that we have to work up to step number $m+2$ in order to determine all unknowns $v_0, \ldots, v_m$ and $x_0, \ldots, x_{m+1}$.

Straightforward calculations give the following representations for the right hand sides

$$F_1(y, x_0, x_1, v_0) = (x_1 \cdot \nabla_x) f(x_0, v_0(y), 0) + G(y, x_0, v_0(y)) - I(y, x_0, v_0) \tag{2.22}$$

and

$$\begin{aligned}&F_k(y, x_0, \ldots, x_k, v_0, \ldots, v_{k-1}) = (x_k \cdot \nabla_x) f(x_0, v_0(y), 0)\\ &+(x_{k-1} \cdot \nabla_x)\left(G(y, x_0, v_0(y)) - I(y, x_0, v_0)\right) + \partial_u G(y, x_0, v_0(y)) v_{k-1}(y) - I(y, x_0, v_{k-1})\\ &+\frac{2-\delta_{2k}}{2}\left((x_{k-1} \cdot \nabla_x) + v_{k-1}(y)\partial_u\right)\left((x_1 \cdot \nabla_x) + v_1(y)\partial_u\right) f(x_0, v_0(y), 0)\\ &+R_k(y, x_0, \ldots, x_{k-2}, v_0, \ldots, v_{k-2}) \;\text{ for }\; k \geq 2,\end{aligned} \tag{2.23}$$

where

$$I(y, x, v) := Q(x)\nabla_y \cdot \left[\left(Q(x)^{-1} y \cdot \nabla_x\right) A(x)\, Q(x) \nabla_y v(y)\right] + b(x) \cdot Q(x) \nabla_y v(y),$$

and

$$G(y,x,u) := \left(Q(x)^{-1}y \cdot \nabla_x\right) f(x,u,0) + \partial_u f(x,u,0)u_1(x) + \partial_\varepsilon f(x,u,0)$$

and each R_k is a certain function depending on y and x_j and v_j with $j \le k-2$ only. Remark that for $k \ge 1$ function F_k depends affinely on x_k. Moreover, for $k \ge 3$ it depends also affinely on x_{k-1} and v_{k-1}, but F_2 does not depend affinely on x_1 and v_1, in general.

Now let us show that all the steps of the algorithm can be done rigorously. Besides assumptions (A1)-(A4) we will need some properties of the linear operator

$$L_{\xi_0} := \Delta_y - \partial_u f(\xi_0, \Phi_{\xi_0}(y), 0),$$

which are formulated in the next two lemmas.

Lemma 2.2 *For any $\alpha \in (0,1)$ the linear operator $L_{\xi_0} : C^{2+\alpha}(\mathbb{R}^n) \to C^\alpha(\mathbb{R}^n)$ is Fredholm of index zero.*

Proof: The operator L_{ξ_0} is Fredholm of index zero because it can be represented as a sum of invertible and compact operators

$$L_{\xi_0} = \Delta_y - \partial_u f(\xi_0,0,0) + M(y), \quad \text{where} \quad M(y) := \partial_u f(\xi_0, \Phi_{\xi_0}(y), 0) - \partial_u f(\xi_0,0,0). \quad (2.24)$$

Indeed, since $\partial_u f(\xi_0,0,0) > 0$ (see assumption (A1)), the operator $\Delta_y - \partial_u f(\xi_0,0,0)$ acting from $C^{2+\alpha}(\mathbb{R}^n)$ to $C^\alpha(\mathbb{R}^n)$ is invertible (see for example [16, Theorem 3.4.3]). On the other hand, the fact that the multiplication by M is a compact operator from $C^{2+\alpha}(\mathbb{R}^n)$ to $C^\alpha(\mathbb{R}^n)$ can be verified as follows.

Let $\chi : [0,\infty) \to \mathbb{R}$ be a non-increasing smooth cut-off function such that $\chi(r) = 1$ for $0 \le r \le 1$ and $\chi(r) = 0$ for $r \ge 2$. Then for each $R > 0$ the function $\chi_R(y) := \chi(|y|^2/R^2)$ is smooth and has compact support. Hence the multiplication by $\chi_R M$ is a compact operator from $C^{2+\alpha}(\mathbb{R}^n)$ to $C^\alpha(\mathbb{R}^n)$. Now taking into account exponential estimates (1.12) for Φ_{ξ_0}, we easily see that the operator $\chi_R M$ tends to M in the operator norm of $L(C^{2+\alpha}(\mathbb{R}^n); C^\alpha(\mathbb{R}^n))$ when $R \to \infty$. However, the space of compact operators is closed in the operator norm, therefore the operator M is compact. $\Diamond$

Because of assumption (A4) we have

$$\operatorname{Ker} L_{\xi_0} = \operatorname{span}\left\{\partial_{y_j}\Phi_{\xi_0} : j = 1,\ldots,n\right\}.$$

Hence, Lemma 2.2 implies that

$$\operatorname{Ran} L_{\xi_0} = \left\{ F \in C^\alpha(\mathbb{R}^n) : \int\limits_{\mathbb{R}^n} F(y)\, \partial_{y_j}\Phi_{\xi_0}(y)\, dy = 0 \quad \text{for all} \quad j = 1,\ldots,n \right\},$$

and the restriction of L_{ξ_0} is an isomorphism from $C^{2+\alpha}(\mathbb{R}^n) \cap \operatorname{Ran} L_{\xi_0}$ onto $\operatorname{Ran} L_{\xi_0}$. The following lemma shows that the inverse of this isomorphism maps exponentially decaying functions onto exponentially decaying functions. To formulate our statement, let us define the family of exponentially decaying functions

$$\rho_\kappa(y) := e^{-\kappa(\sqrt{1+|y|^2}-1)} \quad \text{with} \quad y \in \mathbb{R}^n, \quad (2.25)$$

and recall the notation $\kappa_0 = \sqrt{\partial_u f(\xi_0,0,0)}$ from Theorem 2.1.

Lemma 2.3 *Suppose that $\alpha \in (0,1)$, $\kappa \in (0,\kappa_0)$, $F \in \operatorname{Ran} L_{\xi_0}$ such that $\rho_\kappa^{-1}F \in C^\alpha(\mathbb{R}^n)$, and $v \in C^{2+\alpha}(\mathbb{R}^n)$ such that $L_{\xi_0}v = F$. Then, $\rho_\kappa^{-1}v \in C^{2+\alpha}(\mathbb{R}^n)$.*

Proof: First, we take use of formula (2.24) and rewrite the equation $L_{\xi_0}v = F$ in the following form

$$\Delta_y v - \partial_u f(\xi_0,0,0)v = \tilde{F}(y) := M(y)v + F(y) \in C^\alpha(\mathbb{R}^n).$$

Here due to the exponential estimates (1.12) for Φ_{ξ_0} we have $\rho_\kappa^{-1}M \in C^\alpha(\mathbb{R}^n)$, and this together with the assumption $\rho_\kappa^{-1}F \in C^\alpha(\mathbb{R}^n)$ implies $\rho_\kappa^{-1}\tilde{F} \in C^\alpha(\mathbb{R}^n)$. Now we write function v as the Bessel potential (see [38, Chapter V, §3])

$$v(y) = -\kappa_0^{n-2}\int_{\mathbb{R}^n} G_2(\kappa_0(y-z))\tilde{F}(z)dz, \tag{2.26}$$

where G_2 is the Bessel kernel

$$G_2(x) = (2\pi)^{-n/2}K_{(n-2)/2}(|x|)|x|^{-(n-2)/2}$$

and K_ν is the modified Bessel function of the third kind. Regarding kernel G_2 we know that it is an analytic function of $|x|$, except at $x=0$. Moreover, for $x \to 0$ and for $|x| \to \infty$ one can write explicit asymptotic formulas describing the behaviour of kernel G_2 and of all its derivatives (see, for example, [3, Chapter II, §4]). In particular, for all $j,k = 1,\dots,n$ it holds

$$\|G_2\|_{L^1(\mathbb{R}^n)} < \infty, \quad \|\partial_k G_2\|_{L^1(\mathbb{R}^n)} < \infty, \tag{2.27}$$

$$|\partial_k\partial_j G_2(x)| \le \text{const}\, |x|^{-n} \quad \text{for} \quad |x| \to 0, \tag{2.28}$$

$$|G_2(x)|, |\partial_k G_2(x)|, |\partial_k\partial_j G_2(x)| \le \text{const}\, e^{-|x|} \quad \text{for} \quad |x| \to \infty, \tag{2.29}$$

where $\partial_k G_2(x)$ denotes the first partial derivative of $G_2(x)$ with respect to x_k, and $\partial_k\partial_j G_2(x)$ is the analogous notation for the second partial derivative with respect to x_k and x_j.

From (2.26) it follows

$$\left|\rho_\kappa^{-1}(y)v(y)\right| \le \kappa_0^{n-2}\left\|\rho_\kappa^{-1}\tilde{F}\right\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n}|G_2(\kappa_0(y-z))|\,\rho_\kappa^{-1}(y)\rho_\kappa(z)dz. \tag{2.30}$$

Let us show that the right-hand part of (2.30) is uniformly bounded for all $y \in \mathbb{R}^n$, i.e. that

$$\rho_\kappa^{-1}v \in L^\infty(\mathbb{R}^n). \tag{2.31}$$

Indeed, because of (2.27) the integrand in (2.30) is integrable over any compact region including those which contain point $z=y$. Hence, we need to consider the integrand's behaviour for $|y-z| \to \infty$ only. Taking into account that for every $x \in \mathbb{R}^n$ it holds $0 < \sqrt{1+|x|^2} - |x| \le 1$ we easily obtain

$$\rho_\kappa^{-1}(y)\rho_\kappa(z) \le e^\kappa e^{-\kappa(|z|-|y|)} \quad \text{for all} \quad y \in \mathbb{R}^n \quad \text{and} \quad z \in \mathbb{R}^n.$$

Then using asymptotic formula (2.29) we get

$$\begin{aligned}|G_2(\kappa_0(y-z))|\,\rho_\kappa^{-1}(y)\rho_\kappa(z) &\le e^{\kappa}e^{-\kappa(|z|-|y|)}\,|G_2(\kappa_0(y-z))| \\ &\le \text{const}\; e^{-\kappa(|z|-|y|+|y-z|)}e^{-(\kappa_0-\kappa)|y-z|} \quad \text{for} \quad |y-z|\to\infty.\end{aligned}$$

Now the triangle inequality $|y| \le |y-z| + |z|$ and the assumption $\kappa \in (0,\kappa_0)$ imply the boundedness of the right-hand part in (2.30). Hence, estimate (2.31) is true.

Next, we consider the partial derivatives $\partial_{y_k} v$. Because of the properties of Bessel potentials, they are given by integrals

$$\partial_{y_k} v(y) = -\kappa_0^{n-1}\int_{\mathbb{R}^n} \partial_k G_2(\kappa_0(y-z))\tilde{F}(z)dz, \qquad k=1,\dots,n. \tag{2.32}$$

Since each $\partial_k G_2$ obeys estimates (2.27) and (2.29), we apply arguments as above and obtain

$$\rho_\kappa^{-1}\partial_{y_k} v \in L^\infty(\mathbb{R}^n) \quad \text{for all} \quad k=1,\dots,n. \tag{2.33}$$

To show that $\rho_\kappa^{-1}\partial_{y_k}\partial_{y_j} v \in C^\alpha(\mathbb{R}^n)$ we need a more delicate analysis, since the corresponding derivatives are determined by the improper integral

$$\partial_{y_k}\partial_{y_j} v(y) = -\kappa_0^n \lim_{\mu\to+0}\int_{|z-y|\ge\mu} \partial_k\partial_j G_2(\kappa_0(y-z))\tilde{F}(z)dz, \tag{2.34}$$

which is not absolutely convergent (see asymptotics (2.28)). Nevertheless, according to the classical results of potential theory [38, Chapter V, §4] it is known that for every $\tilde{F} \in C^\alpha(\mathbb{R}^n)$ the singular integral (2.34) determines a function from $C^\alpha(\mathbb{R}^n)$.

On the other hand, from (2.34) it follows

$$\begin{aligned}\rho_\kappa^{-1}(y)\partial_{y_k}\partial_{y_j} v(y) &= -\kappa_0^n \lim_{\mu\to+0}\int_{|z-y|\ge\mu} \rho_\kappa^{-1}(y)\rho_\kappa(z)\partial_k\partial_j G_2(\kappa_0(y-z))\rho_\kappa^{-1}(z)\tilde{F}(z)dz \\ &= \hat{G}(y) - \kappa_0^n \lim_{\mu\to+0}\int_{|z-y|\ge\mu} \partial_k\partial_j G_2(\kappa_0(y-z))\rho_\kappa^{-1}(z)\tilde{F}(z)dz,\end{aligned} \tag{2.35}$$

where

$$\hat{G}(y) := -\kappa_0^n \lim_{\mu\to+0}\int_{|z-y|\ge\mu} \Big(\rho_\kappa^{-1}(y)\rho_\kappa(z)-1\Big)\,\partial_k\partial_j G_2(\kappa_0(z-y))\rho_\kappa^{-1}(z)\tilde{F}(z)dz. \tag{2.36}$$

In (2.36), the difference in parentheses can be rewritten as follows

$$\rho_\kappa^{-1}(y)\rho_\kappa(z) - 1 = e^{\kappa\left(\sqrt{1+|y|^2}-\sqrt{1+|z|^2}\right)} - 1 = -\kappa(z-y)\cdot\Theta(z-y,y),$$

where $\Theta : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\Theta(x,y) := \int_0^1 \frac{y+tx}{\sqrt{1+|y+tx|^2}} e^{\kappa\left(\sqrt{1+|y|^2}-\sqrt{1+|y+tx|^2}\right)}dt. \tag{2.37}$$

This identity together with estimates (2.28) and (2.29) implies that the improper integral (2.36) converges absolutely and it holds

$$\hat{G}(y) = \kappa\kappa_0^n \int\limits_{\mathbb{R}^n} (x \cdot \Theta(x,y))\, \partial_k\partial_j G_2(\kappa_0 x)\rho_\kappa^{-1}(x+y)\tilde{F}(x+y)dx.$$

Now we can demonstrate that the right-hand part of (2.35) belongs to $C^\alpha(\mathbb{R}^n)$. Indeed, since $\rho_\kappa^{-1}\tilde{F} \in C^\alpha(\mathbb{R}^n)$, the rightmost integral in (2.35) determines a $C^\alpha(\mathbb{R}^n)$-function (compare with formula (2.34)). Further, from (2.28), (2.29) and (2.37) we get the estimate

$$|\partial_k\partial_j G_2(\kappa_0 x)| \left(|\Theta(x,y)| + \frac{|\Theta(x,y) - \Theta(x,z)|}{|y-z|^\alpha} \right) \le \text{const } |x|^{-n} e^{-(\kappa_0-\kappa)|x|} \text{ for all } x,y,z \in \mathbb{R}^n.$$

Then, using $\rho_\kappa^{-1}\tilde{F} \in C^\alpha(\mathbb{R}^n)$ again, we easily verify that $\hat{G} \in C^\alpha(\mathbb{R}^n)$. $\Diamond$

After this preparation we are ready to formulate the construction algorithm.

Case $k = 0$. The problem to determine the leading term v_0 is (2.18). Due to assumption (A2), this problem is solved by

$$v_0(y) = \Phi_{x_0}(y).$$

Remember that at this step the value of x_0 is unknown, and we have obtained actually an x_0-parametric family of functions v_0. If we apply a differential operator $(c_1 \cdot \nabla_\xi)$ with any $c_1 \in \mathbb{R}^n$ to the differential equation in (1.5) we obtain

$$\Delta_y \left[(c_1 \cdot \nabla_\xi)\Phi_{x_0}\right] = \partial_u f(x_0, \Phi_{x_0}, 0) \left[(c_1 \cdot \nabla_\xi)\Phi_{x_0}\right] + (c_1 \cdot \nabla_x) f(x_0, \Phi_{x_0}, 0), \tag{2.38}$$

which implies

$$\int\limits_{\mathbb{R}^n} (c_1 \cdot \nabla_x) f(x_0, \Phi_{x_0}, 0)\partial_{y_j}\Phi_{x_0}(y)\, dy = 0, \quad j = 1, \ldots, n. \tag{2.39}$$

Now, we demonstrate that the problems (2.19), (2.20) and (2.21) determine recursively all unknown functions v_k and all unknown vectors x_k.

Case $k = 1$. Obviously, a necessary condition for solvability of problem (2.19) is

$$\int\limits_{\mathbb{R}^n} F_1(y, x_0, x_1, v_0)\partial_{y_j}\Phi_{x_0}(y)\, dy = 0, \quad j = 1, \ldots, n.$$

Notice that because of (2.22) and (2.39) this system of equations does not depend on the vector x_1. Actually it is equivalent to

$$\int\limits_{\mathbb{R}^n} \Big(G(y, x_0, v_0(y)) - I(y, x_0, v_0) \Big)\, \partial_{y_j}\Phi_{x_0}(y)\, dy = 0, \quad j = 1, \ldots, n, \tag{2.40}$$

which we are going to rewrite in terms of the data A, b, f and the spike's profile Φ_ξ only. For this, we use a series of relations collected in the lemma below.

Lemma 2.4 *We have*

$$\int_{\mathbb{R}^n} h(y)\, \partial_{y_j}\Phi_\xi(y)\, dy = 0 \text{ for any radially symmetric } h \in L^\infty(\mathbb{R}^n),$$

$$\int_{\mathbb{R}^n} \big(\partial_{y_j}\Phi_\xi(y)\big)\,(\partial_{y_k}\Phi_\xi(y))\, dy = \frac{\delta_{jk}}{n}\int_{\mathbb{R}^n} |\nabla_y\Phi_\xi(y)|^2 dy,$$

$$\int_{\mathbb{R}^n} y_j\, \partial_{x_l} f(\xi,\Phi_\xi(y),0)\, \partial_{y_k}\Phi_\xi(y)\, dy = -\partial_{\xi_l}\left(\frac{\delta_{jk}}{n}\int_{\mathbb{R}^n} |\nabla_y\Phi_\xi(y)|^2 dy\right),$$

$$\int_{\mathbb{R}^n} y_s\,(\partial_{y_k}\partial_{y_l}\Phi_\xi(y))\,\big(\partial_{y_j}\Phi_\xi(y)\big)\, dy = \frac{1}{2n}\,(\delta_{kl}\delta_{sj} - \delta_{ks}\delta_{lj} - \delta_{kj}\delta_{sl})\int_{\mathbb{R}^n} |\nabla_y\Phi_\xi(y)|^2 dy.$$

Proof: 1) Since all the derivatives $\partial_{y_j}\Phi_\xi(y)$ decay exponentially for $|y| \to \infty$ (see Remark 1.3), for any $h \in L^\infty(\mathbb{R}^n)$ it holds $h\partial_{y_j}\Phi_\xi \in L^1(\mathbb{R}^n)$. Moreover, because of $\Phi_\xi(y) = \phi_\xi(|y|)$ we have

$$\partial_{y_k}\Phi_\xi(y) = \frac{y_k}{|y|}\phi'_\xi(|y|), \tag{2.41}$$

and this implies the claimed identity.

2) Similarly because of (2.41) we obtain

$$\int_{\mathbb{R}^n} \big(\partial_{y_j}\Phi_\xi\big)\,(\partial_{y_k}\Phi_\xi)\, dy = \int_{\mathbb{R}^n} \frac{y_j y_k}{|y|^2}\phi'_\xi(|y|)dy = \delta_{jk}\int_{\mathbb{R}^n} (\partial_{y_1}\Phi_\xi)^2\, dy = \frac{\delta_{jk}}{n}\int_{\mathbb{R}^n} |\nabla_y\Phi_\xi|^2 dy. \tag{2.42}$$

3) Again, because of (2.41) we have

$$\begin{aligned} J_{jkl} \;&:=\; \int_{\mathbb{R}^n} y_j\, \partial_{x_l} f(\xi,\Phi_\xi(y),0)\, \partial_{y_k}\Phi_\xi(y)\, dy = \int_{\mathbb{R}^n} \frac{y_j y_k}{|y|}\, \partial_{x_l} f(\xi,\Phi_\xi(y),0)\, \phi'_\xi(|y|)\, dy \\ &=\; \delta_{jk}\int_{\mathbb{R}^n} y_1\, \partial_{y_1}\left(\int_0^{\Phi_\xi(y)} \partial_{x_l} f(\xi,u,0)du\right) dy. \end{aligned}$$

Then, integrating the latter expression by parts with respect to y_1 and taking into account the exponential decay property of Φ_ξ (see Remark 1.3), we obtain

$$J_{jkl} = -\delta_{jk}\int_{\mathbb{R}^n} dy \int_0^{\Phi_\xi(y)} \partial_{x_l} f(\xi,u,0)du.$$

On the other hand, due to the definition (1.7) we have

$$\partial_{\xi_l}\Big[F(\xi,\Phi_\xi(y),0)\Big] = \int_0^{\Phi_\xi(y)} \partial_{x_l} f(\xi,u,0)du + f(\xi,\Phi_\xi(y),0)\,\partial_{\xi_l}\Phi_\xi(y).$$

Moreover, since Φ_ξ solves problem (1.5) and decays exponentially at infinity together with its first derivatives (see Remark 1.3), the following identity holds

$$\int\limits_{\mathbb{R}^n} f(\xi,\Phi_\xi,0)\,\partial_{\xi_l}\Phi_\xi\,dy=\int\limits_{\mathbb{R}^n}\Delta_y\Phi_\xi\,\partial_{\xi_l}\Phi_\xi\,dy=-\int\limits_{\mathbb{R}^n}\nabla_y\Phi_\xi\cdot\nabla_y\partial_{\xi_l}\Phi_\xi\,dy=-\frac{1}{2}\partial_{\xi_l}\int\limits_{\mathbb{R}^n}|\nabla_y\Phi_\xi|^2dy.$$

Thus, collecting together the latter three formulas and applying identity (1.9), we finally obtain

$$J_{jkl}=-\delta_{jk}\partial_{\xi_l}\int\limits_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla_y\Phi_\xi|^2+F(\xi,\Phi_\xi,0)\right)dy=-\frac{\delta_{jk}}{n}\partial_{\xi_l}\int\limits_{\mathbb{R}^n}|\nabla_y\Phi_\xi|^2dy.$$

4) Differentiating formula (2.41) with respect to y_l, we obtain

$$\partial_{y_k}\partial_{y_l}\Phi_\xi(y)=\frac{\delta_{kl}}{|y|}\phi'_\xi(|y|)+\frac{y_ky_l}{|y|}\frac{d}{d|y|}\left(\frac{\phi'_\xi(|y|)}{|y|}\right). \tag{2.43}$$

This identity together with formulas (2.41) and (2.42) implies that

$$\int\limits_{\mathbb{R}^n} y_s\partial_{y_k}\partial_{y_l}\Phi_\xi(y)\partial_{y_j}\Phi_\xi(y)\,dy=\frac{1}{n}\delta_{kl}\delta_{sj}\int\limits_{\mathbb{R}^n}|\nabla_y\Phi_\xi(y)|^2dy+\int\limits_{\mathbb{R}^n}\frac{y_ky_ly_sy_j}{|y|^2}\phi'_\xi(|y|)\frac{d}{d|y|}\left(\frac{\phi'_\xi(|y|)}{|y|}\right)dy.$$

On the other hand, differentiating the left-hand side of previous relation by parts with respect to y_j, we obtain

$$\int\limits_{\mathbb{R}^n} y_s\partial_{y_k}\partial_{y_l}\Phi_\xi\partial_{y_j}\Phi_\xi\,dy=-\delta_{ks}\int\limits_{\mathbb{R}^n}\partial_{y_l}\Phi_\xi\partial_{y_j}\Phi_\xi\,dy-\int\limits_{\mathbb{R}^n} y_s\partial_{y_k}\partial_{y_j}\Phi_\xi\partial_{y_m}\Phi_\xi\,dy$$

$$=-\frac{1}{n}\left(\delta_{ks}\delta_{lj}+\delta_{kj}\delta_{sl}\right)\int\limits_{\mathbb{R}^n}|\nabla_y\Phi_\xi(y)|^2dy-\int\limits_{\mathbb{R}^n}\frac{y_ky_ly_sy_j}{|y|^2}\phi'_\xi(|y|)\frac{d}{d|y|}\left(\frac{\phi'_\xi(|y|)}{|y|}\right)dy.$$

Now, comparing the latter two formulas with each other, we easily find

$$\int\limits_{\mathbb{R}^n}\frac{y_ky_ly_sy_j}{|y|^2}\phi'_\xi(|y|)\frac{d}{d|y|}\left(\frac{\phi'_\xi(|y|)}{|y|}\right)dy=-\frac{1}{2n}\left(\delta_{kl}\delta_{sj}+\delta_{ks}\delta_{lj}+\delta_{kj}\delta_{sl}\right)\int\limits_{\mathbb{R}^n}|\nabla_y\Phi_\xi(y)|^2dy.$$

Hence,

$$\int\limits_{\mathbb{R}^n} y_s\partial_{y_k}\partial_{y_l}\Phi_\xi(y)\partial_{y_j}\Phi_\xi(y)\,dy=\frac{1}{2n}\left(\delta_{kl}\delta_{sj}-\delta_{ks}\delta_{lj}-\delta_{kj}\delta_{sl}\right)\int\limits_{\mathbb{R}^n}|\nabla_y\Phi_\xi(y)|^2dy.$$

And this ends the proof. ♢

Lemma 2.4 implies that the system of equations (2.40) can be written as follows

$$J_j(x_0):=\int\limits_{\mathbb{R}^n}|\nabla_y\Phi_{x_0}|^2dy\left(\frac{1}{2}\sum_{k,r,s=0}^{n}q^{-1}_{jk}(x_0)\,a^{-1}_{rs}(x_0)\,\partial_{x_k}a_{rs}(x_0)+\sum_{r=1}^{n}b_r(x_0)\,q_{rj}(x_0)\right)$$

$$+\sum_{k=1}^{n}q^{-1}_{jk}(x_0)\,\partial_{\xi_k}\int\limits_{\mathbb{R}^n}|\nabla_y\Phi_{x_0}|^2dy=0,\qquad j=0,\ldots,n. \tag{2.44}$$

Here we denote by $q_{jk}^{-1}(x_0)$ and $a_{rs}^{-1}(x_0)$ the components of the matrices $Q(x_0)^{-1} = A(x_0)^{1/2}$ (cf. (2.3)) and $A(x_0)^{-1}$ (cf. (1.4)), respectively. Next, transforming the first term in parenthesis with the help of Jacobi's formula

$$\partial_{x_k}(\det A) = \operatorname{tr}(A^{-1}\partial_{x_k} A),$$

we write equations (2.44) in a matrix form

$$\left(\frac{1}{2}Q(x_0)^{-1}\nabla_x(\log\det A(x_0)) + Q(x_0)b(x_0)\right)\int\limits_{\mathbb{R}^n} |\nabla_y \Phi_{x_0}|^2 dy + Q(x_0)^{-1}\nabla_\xi \int\limits_{\mathbb{R}^n} |\nabla_y \Phi_{x_0}|^2 dy = 0.$$

Multiplying the latter equation by the non-degenerate matrix $Q(x_0)$ and taking into account that $Q(x_0)^2 = A(x_0)^{-1}$, and

$$\int\limits_{\mathbb{R}^n} |\nabla_y \Phi_\xi|^2 dy = \frac{\Sigma_{n-1}}{n}\int\limits_0^\infty \phi_\xi'(r)^2 r^{n-1} dr,$$

where Σ_{n-1} is the surface area of the n-dimensional unit ball, we obtain (1.3) which, thus, is equivalent to the system (2.40). Hence, by assumption (A3) we can choose

$$x_0 = \xi_0, \text{ i.e. } v_0 = \Phi_{\xi_0}. \tag{2.45}$$

Now, we show that the problem (2.19) with x_0 and v_0 determined by (2.45) has, for any given $x_1 \in \mathbb{R}^n$, a unique solution v_1, and for any $\alpha \in (0,1)$ we have

$$\rho_\kappa^{-1} v_1 \in C^{2+\alpha}(\mathbb{R}^n) \text{ for all } \kappa \in (0,\kappa_0). \tag{2.46}$$

Indeed, due to equation (2.38) and the linear superposition principle any solution of problem (2.19) can be written in the following form

$$v_1(y) = \overline{v}_1(y) + (x_1 \cdot \nabla_\xi)\Phi_{\xi_0}(y), \tag{2.47}$$

where $\overline{v}_1$ solves the problem

$$\left.\begin{array}{rcl} \Delta_y \overline{v}_1(y) - \partial_u f(\xi_0, \Phi_{\xi_0}(y), 0)\overline{v}_1(y) & = & F_1(y, \xi_0, 0, \Phi_{\xi_0}) \\ & = & G(y, \xi_0, \Phi_{\xi_0}(y)) - I(y, \xi_0, \Phi_{\xi_0}), \\ \nabla_y \overline{v}_1(0) & = & 0, \\ \overline{v}_1(y) \to 0 \text{ for } |y| & \to & \infty. \end{array}\right\} \tag{2.48}$$

But, the latter problem does have a unique solution. To see this notice first that Lemma 2.3 implies the existence of a unique $\tilde{v}_1 \in C^{2+\alpha}(\mathbb{R}^n) \cap \operatorname{Ran} L_{\xi_0}$ such that $L_{\xi_0}\tilde{v}_1 = F_1(y, \xi_0, 0, \Phi_{\xi_0})$. This means that general solution of problem (2.48) reads

$$\overline{v}_1(y) = \tilde{v}_1(y) + (c_1 \cdot \nabla_y)\Phi_{\xi_0}(y), \quad c_1 \in \mathbb{R}^n, \tag{2.49}$$

where $c_1 \in \mathbb{R}^n$ is a free parameter. Then, substituting representation (2.49) into condition $\nabla_y \overline{v}_1(0) = 0$, we obtain

$$\nabla_y \tilde{v}_1(0) + \nabla_y (c_1 \cdot \nabla_y)\Phi_{\xi_0}(0) = 0. \tag{2.50}$$

This relation determines an n-dimensional linear system with respect to the unknown vector c_1. Since Φ_ξ is a radially symmetric solution of problem (2.18), direct calculation with the help of formulas (1.10) and (2.43) yields

$$\partial_{y_j}\partial_{y_k}\Phi_{\xi_0}(0) = \frac{\delta_{jk}}{n} f(\xi_0, \Phi_{\xi_0}(0), 0), \tag{2.51}$$

where $f(\xi_0, \Phi_{\xi_0}(0), 0) \neq 0$ due to (1.11). Formula (2.51) says that the matrix of n-dimensional linear system (2.50) is non-degenerate, hence (2.50) has a unique solution c_1.

Now, let us prove (2.46): From (1.12) it follows that $\rho_\kappa^{-1}\Phi_{\xi_0} \in C^{2+\alpha}(\mathbb{R}^n)$ for all $\kappa \in (0,\kappa_0)$. Therefore, from assumption (A1) and from (2.14) we obtain $\rho_\kappa^{-1}G(y,\xi_0,\Phi_{\xi_0}) \in C^\alpha(\mathbb{R}^n)$ for all $\kappa \in (0,\kappa_0)$. Similarly taking into account that for any $j = 1,\dots,n$ and any $\kappa \in (0,\kappa_0)$ it holds $y_j\rho_\kappa(y) \in C^\alpha(\mathbb{R}^n)$, we easily get that $\rho_\kappa^{-1}I(y,\xi_0,\Phi_{\xi_0}) \in C^\alpha(\mathbb{R}^n)$ for all $\kappa \in (0,\kappa_0)$. Hence, (2.22) yields

$$\rho_\kappa^{-1}F_1(y,\xi_0,x_1,\Phi_{\xi_0}) \in C^\alpha(\mathbb{R}^n) \ \text{ for all } \ \kappa \in (0,\kappa_0). \tag{2.52}$$

Therefore Lemma 2.3 implies (2.46).

Similarly to (2.52) one can show that, for any given functions $v_0,\dots,v_k \in C^{2+\alpha}(\mathbb{R}^n)$ such that $\rho_\kappa^{-1}v_0,\dots,\rho_\kappa^{-1}v_k \in C^{2+\alpha}(\mathbb{R}^n)$ for all $\kappa \in (0,\kappa_0)$, we have

$$\rho_\kappa^{-1}F_k(y,\xi_0,x_1,\dots,x_k,\Phi_{\xi_0},v_1,\dots,v_k) \in C^\alpha(\mathbb{R}^n) \ \text{ for all } \ \kappa \in (0,\kappa_0).$$

Case $k = 2$. We continue to construct the inner expansion of the spike and consider now the problem (2.21) with $k = 2$. First, we need to reveal exactly the dependence of the right-hand side F_2 on the unknown vector x_1. With this aim in view we substitute v_1 from (2.47) into the formula (2.23) for $k = 2$ and obtain

$$\begin{aligned} &F_2(y,\xi_0,x_1,x_2,\Phi_{\xi_0},\overline{v}_1 + (x_1\cdot\nabla_\xi)\Phi_{\xi_0}) \\ &= (x_2\cdot\nabla_x)f(\xi_0,\Phi_{\xi_0}(y),0) + (x_1\cdot\Psi(y)) + \frac{1}{2}\psi(y,x_1,x_1) + \overline{F}_2(y), \end{aligned} \tag{2.53}$$

where $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are functions defined by

$$\begin{aligned} (c_1\cdot\Psi(y)) \;&:=\; (c_1\cdot\nabla_x)(G(y,\xi_0,\Phi_{\xi_0}(y)) - I(y,\xi_0,\Phi_{\xi_0})) \\ &+\; \partial_u G(y,\xi_0,\Phi_{\xi_0}(y))\,[(c_1\cdot\nabla_\xi)\Phi_{\xi_0}] - I(y,\xi_0,(c_1\cdot\nabla_\xi)\Phi_{\xi_0}) \\ &+\; \Big((c_1\cdot\nabla_x)\partial_u f(\xi_0,\Phi_{\xi_0}(y),0) + \partial_u^2 f(\xi_0,\Phi_{\xi_0}(y),0)\ [(c_1\cdot\nabla_\xi)\Phi_{\xi_0}]\Big)\,\overline{v}_1, \end{aligned} \tag{2.54}$$

$$\psi(y,c_1,c_2) \;:=\; ((c_1\cdot\nabla_x) + [(c_1\cdot\nabla_\xi)\Phi_{\xi_0}]\,\partial_u)\,((c_2\cdot\nabla_x) + [(c_2\cdot\nabla_\xi)\Phi_{\xi_0}]\,\partial_u)\,f(\xi_0,\Phi_{\xi_0},0), \tag{2.55}$$

and $\overline{F}_2(y)$ is a function which depends neither on x_1 nor on x_2. Note that according to definitions (2.23), (2.53)–(2.55) and estimates (2.12), for any $\kappa \in (0,\kappa_0)$ it holds

$$\begin{aligned} |(c_1\cdot\Psi(y))| \le c(\kappa)|c_1|e^{-\kappa|y|}, \quad |\psi(y,c_1,c_2)| \le c(\kappa)|c_1||c_2|e^{-\kappa|y|} \quad \text{for all} \quad y \in \mathbb{R}^n, \\ |\overline{F}_2(y)| \le c(\kappa)e^{-\kappa|y|} \quad \text{for all} \quad y \in \mathbb{R}^n, \end{aligned} \tag{2.56}$$

where $c(\kappa)$ is a certain positive constant independent of c_1, c_2 and y.

Formula (2.53) shows that the dependence of the right-hand side F_2 on the vector x_1 is not affine. However, applying the differential operator $(c_1 \cdot \nabla_\xi)(c_2 \cdot \nabla_\xi)$ with any constant coefficients $c_1 \in \mathbb{R}^n$ and $c_2 \in \mathbb{R}^n$ to the differential equation in (1.5) and writing a consistency condition by analogy with (2.39) we get

$$\int\limits_{\mathbb{R}^n} \psi(y, c_1, c_2) \partial_{y_j} \Phi_{\xi_0} \, dy = 0, \quad j = 1, \ldots, n. \tag{2.57}$$

Hence, taking into account relations (2.39) and (2.57) we come to a necessary condition for solvability of problem (2.20) in the following form

$$\begin{aligned} 0 &= \int\limits_{\mathbb{R}^n} F_2(y, \xi_0, x_1, x_2, \Phi_{\xi_0}, \overline{v}_1 + (x_1 \cdot \nabla_\xi)\Phi_{\xi_0}) \, \partial_{y_j} \Phi_{\xi_0} \, dy \\ &= \int\limits_{\mathbb{R}^n} (x_1 \cdot \Psi(y)) \, \partial_{y_j} \Phi_{\xi_0} \, dy + \int\limits_{\mathbb{R}^n} \overline{F}_2(y) \, \partial_{y_j} \Phi_{\xi_0} \, dy, \quad j = 1, \ldots, n. \end{aligned} \tag{2.58}$$

Below we demonstrate that this system can be written as follows

$$(x_1 \cdot \nabla_x) J_j(x_0) = (\text{terms independent of } x_1). \tag{2.59}$$

For this, we apply the partial derivative operator ∂_{y_j} to both sides of (2.38) and get after simple transformations the identity

$$\begin{aligned} &\Delta_y \left[(x_1 \cdot \nabla_\xi)\partial_{y_j} \Phi_{\xi_0}\right] - \partial_u f(\xi_0, \Phi_{\xi_0}(y), 0) \left[(x_1 \cdot \nabla_\xi)\partial_{y_j} \Phi_{\xi_0}\right] \\ &= \Big((x_1 \cdot \nabla_x)\partial_u f(\xi_0, \Phi_{\xi_0}(y), 0) + \partial_u^2 f(\xi_0, \Phi_{\xi_0}(y), 0) \left[(x_1 \cdot \nabla_\xi)\Phi_{\xi_0}\right]\Big) \, \partial_{y_j} \Phi_{\xi_0}. \end{aligned} \tag{2.60}$$

Then, multiplying both sides of (2.60) by $\overline{v}_1$, integrating obtained equation by parts and taking into account the differential equation in (2.48), we obtain

$$\begin{aligned} &\int\limits_{\mathbb{R}^n} \Big((x_1 \cdot \nabla_x)\partial_u f(\xi_0, \Phi_{\xi_0}(y), 0) + \partial_u^2 f(\xi_0, \Phi_{\xi_0}(y), 0) \left[(x_1 \cdot \nabla_\xi)\Phi_{\xi_0}\right]\Big) \, \overline{v}_1 \, \partial_{y_j} \Phi_{\xi_0} \, dy \\ &= \int\limits_{\mathbb{R}^n} \Big(\Delta_y \left[(x_1 \cdot \nabla_\xi)\partial_{y_j} \Phi_{\xi_0}\right] - \partial_u f(\xi_0, \Phi_{\xi_0}(y), 0) \left[(x_1 \cdot \nabla_\xi)\partial_{y_j} \Phi_{\xi_0}\right]\Big) \, \overline{v}_1 \, dy \\ &= \int\limits_{\mathbb{R}^n} \Big(\Delta_y \overline{v}_1 - \partial_u f(\xi_0, \Phi_{\xi_0}(y), 0)\overline{v}_1\Big) \left[(x_1 \cdot \nabla_\xi)\partial_{y_j} \Phi_{\xi_0}\right] \, dy \\ &= \int\limits_{\mathbb{R}^n} \Big(G(y, \xi_0, \Phi_{\xi_0}(y)) - I(y, \xi_0, \Phi_{\xi_0})\Big) \left[(x_1 \cdot \nabla_\xi)\partial_{y_j} \Phi_{\xi_0}\right] \, dy. \end{aligned} \tag{2.61}$$

Combining (2.61) with (2.54), we get

$$\int_{\mathbb{R}^n} (x_1 \cdot \Psi(y))\, \partial_{y_j} \Phi_{\xi_0}\, dy = (x_1 \cdot \nabla_\xi) \left(\int_{\mathbb{R}^n} [G(y, \xi_0, \Phi_{\xi_0}(y)) - I(y, \xi_0, \Phi_{\xi_0})]\ \partial_{y_j} \Phi_{\xi_0}\, dy \right)$$
$$= (x_1 \cdot \nabla_\xi) J_j(\xi_0). \tag{2.62}$$

Hence, solvability condition (2.58) does have the form (2.59).

Since due to assumption (A3) the Jacobian matrix

$$H(\xi_0) := \{\partial_{x_k} J_j(\xi_0)\}_{j,k=1}^{n} \tag{2.63}$$

is non-degenerate, system (2.58) determines x_1 in a unique way. Knowing x_1 we proceed further as in the case $k = 1$. Due to the definition (2.23) and estimates (2.12) we have $\rho_\kappa^{-1} F_2(y, \xi_0, x_1, 0, \Phi_{\xi_0}, v_1) \in C^\alpha(\mathbb{R}^n)$ for any $\kappa \in (0, \kappa_0)$. Hence, Lemma 2.3 implies that the problem (2.21) with $k = 2$ and $x_2 = 0$ has a unique solution $\overline{v}_2$ such that $\rho_\kappa^{-1} \overline{v}_2 \in C^{2+\alpha}(\mathbb{R}^n)$ for all $\kappa \in (0, \kappa_0)$. Therefore the complete problem (2.21) with $k = 2$ has an x_2-dependent family of solutions

$$v_2(y) = \overline{v}_2(y) + (x_2 \cdot \nabla_\xi) \Phi_{\xi_0}(y), \tag{2.64}$$

and $\rho_\kappa^{-1} v_2 \in C^{2+\alpha}(\mathbb{R}^n)$ for all $\kappa \in (0, \kappa_0)$.

Case $k \geq 3$. By analogy with (2.47) and (2.64), we know at this step that

$$v_{k-1}(y) = \overline{v}_{k-1}(y) + (x_{k-1} \cdot \nabla_\xi) \Phi_{\xi_0}(y), \tag{2.65}$$

where the function $\overline{v}_{k-1}$ does not depend on x_{k-1}. Substituting this into the definition of F_k (see (2.23)) we separate again the terms depending on x_k and x_{k-1} as follows

$$\begin{aligned} &F_k(y, \xi_0, x_1, ..., x_k, \Phi_{\xi_0}, v_1, \ldots, \overline{v}_{k-1} + (x_{k-1} \cdot \nabla_\xi) \Phi_{\xi_0}) \\ &\quad = (x_k \cdot \nabla_x) f(\xi_0, \Phi_{\xi_0}(y), 0) + (x_{k-1} \cdot \nabla_x) \left(G(y, \xi_0, \Phi_{\xi_0}(y)) - I(y, \xi_0, \Phi_{\xi_0}) \right) \\ &\quad + \partial_u G(y, \xi_0, \Phi_{\xi_0}(y)) \left[(x_{k-1} \cdot \nabla_\xi) \Phi_{\xi_0} \right] - I(y, \xi_0, [(x_{k-1} \cdot \nabla_\xi) \Phi_{\xi_0}]) + \psi(y, x_{k-1}, x_1) \\ &\quad + \Big((x_{k-1} \cdot \nabla_x) \partial_u f(\xi_0, \Phi_{\xi_0}(y), 0) + \partial_u^2 f(\xi_0, \Phi_{\xi_0}(y), 0)\ [(x_{k-1} \cdot \nabla_\xi) \Phi_{\xi_0}] \Big) \overline{v}_1 + \overline{F}_k(y), \end{aligned} \tag{2.66}$$

where $\overline{F}_k(y)$ is a function collecting all the rest terms which are independent of x_{k-1} and x_k. Now, arguing in a similar way as in (2.61), we obtain

$$\int_{\mathbb{R}^n} \Big((x_{k-1} \cdot \nabla_x) \partial_u f(\xi_0, \Phi_{\xi_0}(y), 0) + \partial_u^2 f(\xi_0, \Phi_{\xi_0}(y), 0)\ [(x_{k-1} \cdot \nabla_\xi) \Phi_{\xi_0}] \Big) \overline{v}_1\, \partial_{y_j} \Phi_{\xi_0}\, dy$$

$$= \int_{\mathbb{R}^n} \Big(G(y, \xi_0, \Phi_{\xi_0}(y)) - I(y, \xi_0, \Phi_{\xi_0}) \Big) \left[(x_{k-1} \cdot \nabla_\xi) \partial_{y_j} \Phi_{\xi_0} \right]\, dy.$$

Using this identity and relations (2.57), we write a necessary condition for solvability of problem (2.21) in the following form

$$0 = \int\limits_{\mathbb{R}^n} F_k(y, \xi_0, x_1, ..., x_k, \Phi_{\xi_0}, v_1, \dots, \overline{v}_{k-1} + (x_{k-1} \cdot \nabla_\xi)\Phi_{\xi_0})\, \partial_{y_j} \Phi_{\xi_0}\, dy$$

$$= (x_{k-1} \cdot \nabla_\xi) \left(\int\limits_{\mathbb{R}^n} [G(y, \xi_0, \Phi_{\xi_0}(y)) - I(y, \xi_0, \Phi_{\xi_0})]\ \partial_{y_j} \Phi_{\xi_0}\, dy \right) + \int\limits_{\mathbb{R}^n} \overline{F}_k(y)\, \partial_{y_j} \Phi_{\xi_0}\, dy.$$

Hence, due to assumption (A3) the latter system determines a unique value of x_{k-1}. Then solving problem (2.21) we obtain an x_k-dependent family of functions v_k which also can be written in the form (2.65), and $\rho_\kappa^{-1} v_k \in C^{2+\alpha}(\mathbb{R}^n)$ for any $\kappa \in (0, \kappa_0)$.

It follows immediately from the above construction procedure that the inner expansion $v_{\varepsilon,m}$ satisfies

$$\|E_\varepsilon v_{\varepsilon,m} - f(\cdot, u_{\varepsilon,m} + v_{\varepsilon,m}, \varepsilon) + f(\cdot, u_{\varepsilon,m}, \varepsilon)\|_{C^\alpha(T_{\varepsilon,m}(\overline{\Omega}))} = O(\varepsilon^{m+1}).$$

2.3 Inner expansion for the boundary layer

The outer expansion $u_{\varepsilon,m}$ does not necessarily satisfy the boundary condition on $\partial\Omega$. In order to compensate this discrepancy, we correct our asymptotics adding to it a boundary layer term $w_{\varepsilon,m}$.

Recall that above (see (2.5)) we have introduced a local coordinate system near the boundary $\partial\Omega$. In this way every point $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < 2\delta$ is parameterized by the stretched distance to the boundary $z = \varepsilon^{-1} \operatorname{dist}(x, \partial\Omega)$ and the corresponding point $\zeta \in \partial\Omega$ for which this distance is attained, i.e. $\operatorname{dist}(x, \partial\Omega) = \operatorname{dist}(x, \zeta)$. Thus, substituting the ansatz (2.2) for $u_{\varepsilon,m}$ and the ansatz (2.4) for $w_{\varepsilon,m}$ into (2.8), and moving into the local coordinate system, we get

$$[E_\varepsilon w_{\varepsilon,m} - f(\cdot, u_{\varepsilon,m} + w_{\varepsilon,m}, \varepsilon) + f(\cdot, u_{\varepsilon,m}, \varepsilon)] \circ S_\varepsilon^{-1} = N(\zeta)\partial_z^2 w_0 - f(\zeta, w_0, 0)$$
$$+ \sum_{k=1}^{m} \varepsilon^k \left(N(\zeta)\partial_z^2 w_k - \partial_u f(\zeta, w_0, 0) w_k - H_k(z, \zeta, w_0, \dots, w_{k-1})\right) + O(\varepsilon^{m+1}), \quad (2.67)$$

where

$$N(\zeta) := \sum_{i,j=1}^{n} a_{ij}(\zeta)\nu_i(\zeta)\nu_j(\zeta),$$

and the right hand sides $H_k(z, \zeta, w_0, \dots, w_{k-1})$ depend on the functions $w_0, \dots, w_{k-1}$ via the values in the point (z, ζ) of those functions and their first and second derivatives. Moreover,

$$H_k(z, \zeta, 0, \dots, 0) = 0.$$

Similarly we rewrite the boundary condition of problem (1.1) in the local coordinates (z, ζ) and obtain

$$\left[\sum_{i,j=1}^{n} a_{ij}(x)\nu_i(x)\partial_{x_j}(u_{\varepsilon,m} + w_{\varepsilon,m}) - g(x, u_{\varepsilon,m} + w_{\varepsilon,m}, \varepsilon) \right] \circ S_\varepsilon^{-1} = -\varepsilon^{-1} N(\zeta)\partial_z w_0(0, \zeta)$$
$$- \sum_{k=1}^{m} \varepsilon^{k-1} \left(N(\zeta)\partial_z w_k(0, \zeta) + g_k(\zeta, w_0, \dots, w_{k-1})\right) + O(\varepsilon^m). \quad (2.68)$$

Here the right hand sides $g_k(\zeta, w_0, \ldots, w_{k-1})$ depend on the functions $w_0, \ldots, w_{k-1}$ via the values in the point $(0, \zeta)$ of those functions and their first derivatives.

Now, we proceed as follows. First, we solve the problem

$$\left.\begin{aligned} N(\zeta)\partial_z^2 w_0(z,\zeta) - f(\zeta, w_0(z,\zeta), 0) &= 0, \\ \partial_z w_0(0,\zeta) &= 0, \\ w_0(z,\zeta) \to 0 \text{ for } z &\to \infty, \end{aligned}\right\} \tag{2.69}$$

which is actually a one dimensional boundary value problem with respect to z, with variable ζ playing the role of parameter only. Due to assumption (A1), we can choose

$$w_0(z,\zeta) = 0.$$

Remark that problem (2.69) may have other, nonzero solutions. Those other solutions to (2.69) would produce other approximate solutions and, via the procedure of Section 4, other exact solutions to (1.1). Note that those exact solutions to (1.1) would not belong to the domains of local uniqueness, described by Theorems 1.1 and 4.1, of course.

After w_0 has been fixed, we solve in the next steps the linear boundary value problems which determine the functions w_k:

$$\left.\begin{aligned} N(\zeta)\partial_z^2 w_k(z,\zeta) - \partial_u f(\zeta,0,0) w_k &= H_k(z,\zeta,w_0,\ldots,w_{k-1}), \\ N(\zeta)\partial_z w_k(0,\zeta) &= -g_k(\zeta,w_0,\ldots,w_{k-1}), \\ w_k(z,\zeta) \to 0 \text{ for } z &\to \infty. \end{aligned}\right\} \tag{2.70}$$

Since the coefficients of corresponding homogeneous differential equation do not depend on z and because of assumption (A1), one can easily construct Green's function $G(z,z',\zeta)$ and write the unique solution to problem (2.70) in the following integral form

$$w_k(z,\zeta) = N(\zeta)^{-1}\mu(\zeta)^{-1} g_k(\zeta, w_0, \ldots, w_{k-1}) e^{-\mu(\zeta)z} + \int_0^\infty G(z,z',\zeta) H_k(\cdot) dz', \tag{2.71}$$

where

$$G(z,z',\zeta) := \begin{cases} -[\mu(\zeta)N(\zeta)]^{-1} e^{-\mu(\zeta)z'} \cosh(\mu(\zeta)z) & \text{for} \quad 0 \le z \le z', \\ -[\mu(\zeta)N(\zeta)]^{-1} \cosh(\mu(\zeta)z') e^{-\mu(\zeta)z} & \text{for} \quad z' < z, \end{cases}$$

and $\mu(\zeta) := [\partial_u f(\zeta,0,0)/N(\zeta)]^{1/2}$. Using formula (2.71) we easily derive the exponential estimates (2.13). Indeed, due to assumption (A1) we have $H_1(z,\zeta,0) = 0$. Hence, formula (2.71) for $k = 1$ determines w_1 which obviously satisfies estimate (2.13). Now, we proceed by induction. Suppose that all functions w_j, $j = 0, \ldots, k-1$, satisfy estimate (2.13). Then expansion formulas (2.67) and (2.68) implies that for all $\varkappa \in (0, \varkappa_0)$ there exists a constant $c > 0$ such that

$$|H_k(z,\zeta,0,\ldots,w_{k-1})| \le c e^{-\varkappa z} \text{ for all } (z,\zeta) \in [0,\infty) \times \partial\Omega.$$

This means, in particular, that integral formula (2.71) determines correctly a solution w_k to problem (2.70), and the exponential estimate (2.13) holds.

Now, we obtain immediately from the above construction procedure that the inner expansion $w_{\varepsilon,m}$ satisfies

$$\|E_\varepsilon w_{\varepsilon,m} - f(\cdot, u_{\varepsilon,m} + w_{\varepsilon,m}, \varepsilon) + f(\cdot, u_{\varepsilon,m}, \varepsilon)\|_{C^\alpha(S_\varepsilon(\overline{\Omega}))} = O(\varepsilon^{m+1}), \tag{2.72}$$
$$\left\| \sum_{i,j=1}^n a_{ij}(\cdot)\nu_i(\cdot)\partial_{x_j}(u_{\varepsilon,m} + w_{\varepsilon,m}) - g(\cdot, u_{\varepsilon,m} + w_{\varepsilon,m}, \varepsilon) \right\|_{C^{1+\alpha}(S_\varepsilon(\partial\Omega))} = O(\varepsilon^m).$$

Indeed, in the δ-vicinity of boundary $\partial\Omega$ the relation (2.72) is fulfilled because of the determining problems (2.69) and (2.70). In the rest of domain Ω this relation is satisfied since exponential estimates (2.13) hold.

3 A generalized Implicit Function Theorem

In this section we formulate and prove an implicit function theorem with minimal assumptions concerning continuity with respect to the control parameter.

Our implicit function theorem is very close to those of P. C. Fife and W. M. Greenlee [9, Theorem 4.2] and of R. Magnus [19, Theorem 1.2]. For other implicit function theorems with weak assumptions concerning continuity with respect to the control parameter see also [2, Theorem 7] and [10, Theorem 3.4]. For applications of our implicit function theorem to other singularly perturbed problems see [35, 27].

Theorem 3.1 *Let for any $\varepsilon \in (0, \varepsilon_0)$ be given Banach spaces U_ε and V_ε and maps $F_\varepsilon \in C^1(U_\varepsilon, V_\varepsilon)$ such that*

$$\|F_\varepsilon(0)\| \to 0 \quad \text{for} \quad \varepsilon \to +0, \tag{3.1}$$

$$\|F'_\varepsilon(u) - F'_\varepsilon(0)\| \to 0 \quad \text{for} \quad |\varepsilon| + \|u\| \to 0 \tag{3.2}$$

and

$$\left.\begin{array}{l}\text{there exist } \varepsilon_1 \in (0, \varepsilon_0] \text{ and } c > 0 \text{ such that for all } \varepsilon \in (0, \varepsilon_1)\\ \text{the operators } F'_\varepsilon(0) \text{ are invertible and } \|F'_\varepsilon(0)^{-1}\| \le c.\end{array}\right\} \tag{3.3}$$

Then there exist $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$ there exists exactly one $u = u_\varepsilon$ with $\|u\| < \delta$ and $F_\varepsilon(u) = 0$. Moreover,

$$\|u_\varepsilon\| \le 2c\,\|F_\varepsilon(0)\|. \tag{3.4}$$

Proof: For $\varepsilon \in (0, \varepsilon_1)$ we have $F_\varepsilon(u) = 0$ if and only if

$$G_\varepsilon(u) := u - F'_\varepsilon(0)^{-1}F_\varepsilon(u) = u. \tag{3.5}$$

Moreover, for such ε and all $u, v \in U_\varepsilon$ we have

$$\begin{aligned} G_\varepsilon(u) - G_\varepsilon(v) &= \int_0^1 G'_\varepsilon(su + (1-s)v)(u-v)ds \\ &= F'_\varepsilon(0)^{-1} \int_0^1 \left(F'_\varepsilon(0) - F'_\varepsilon(su + (1-s)v)\right)(u-v)ds. \end{aligned}$$

Hence, assumptions (3.2) and (3.3) imply that there exist $\varepsilon_2 \in (0, \varepsilon_1)$ and $\delta > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$

$$\|G_\varepsilon(u) - G_\varepsilon(v)\| \leq \frac{1}{2}\|u - v\| \quad \text{for all} \quad u, v \in K_\varepsilon^\delta := \{w \in U_\varepsilon : \ \|w\| \leq \delta\}.$$

Using this and (3.3) again, for all $\varepsilon \in (0, \varepsilon_2)$ we get

$$\|G_\varepsilon(u)\| \leq \|G_\varepsilon(u) - G_\varepsilon(0)\| + \|G_\varepsilon(0)\| \leq \frac{1}{2}\|u\| + c\|F_\varepsilon(0)\|. \tag{3.6}$$

Hence, assumption (3.1) yields that G_ε maps K_ε^δ into K_ε^δ for all $\varepsilon \in (0, \varepsilon_2)$, if ε_2 is chosen sufficiently small. Now, Banach's fixed point theorem gives a unique in K_ε^δ solution $u = u_\varepsilon$ to (3.5) for all $\varepsilon \in (0, \varepsilon_2)$. Moreover, inequality (3.6) yields $\|u_\varepsilon\| \leq 1/2\|u_\varepsilon\| + c\|F_\varepsilon(0)\|$, i.e. (3.4). $\diamondsuit$

The following lemma is [19, Lemma 1.3], translated to our setting. It gives a criterion how to verify the key assumption (3.3) of Theorem 3.1:

Lemma 3.2 *Let $F_\varepsilon'(0)$ be Fredholm of index zero for all $\varepsilon \in (0, \varepsilon_0)$. Suppose that there do not exist sequences $\varepsilon_1, \varepsilon_2 \ldots \in (0, \varepsilon_0)$ and $u_1 \in U_{\varepsilon_1}, u_2 \in U_{\varepsilon_2} \ldots$ with $\|u_k\| = 1$ for all $k \in \mathbb{N}$ and $|\varepsilon_k| + \|F_{\varepsilon_k}'(0)u_k\| \to 0$ for $k \to \infty$. Then (3.3) is satisfied.*

Proof: Suppose that proposition (3.3) is not true. Then there exists a sequence $\varepsilon_1, \varepsilon_2 \ldots \in (0, \varepsilon_0)$ with $\varepsilon_k \to 0$ for $k \to \infty$ such that either $F_{\varepsilon_k}'(0)$ is not invertible or it is but $\|F_{\varepsilon_k}'(0)^{-1}\| \geq k$ for all $k \in \mathbb{N}$. In the first case there exist $u_k \in U_{\varepsilon_k}$ with $\|u_k\| = 1$ and $F_{\varepsilon_k}'(0)u_k = 0$ (because $F_{\varepsilon_k}'(0)$ is Fredholm of index zero). In the second case there exist $v_k \in V_{\varepsilon_k}$ with $\|v_k\| = 1$ and $\|F_{\varepsilon_k}'(0)^{-1}v_k\| \geq k$, i.e.

$$\|F_{\varepsilon_k}'(0)u_k\| \leq \frac{1}{k} \quad \text{with} \quad u_k := \frac{F_{\varepsilon_k}'(0)^{-1}v_k}{\|F_{\varepsilon_k}'(0)^{-1}v_k\|}.$$

But this contradicts to the assumptions of the lemma. $\diamondsuit$

4 Existence and local uniqueness of exact solutions

In Section 2, we have constructed a sequence of formal approximate solutions $\mathcal{W}_{\varepsilon,m}$ to problem (1.1). Now we are going to prove the existence of a locally unique exact solution u_ε to problem (1.1) such that $\mathcal{W}_{\varepsilon,m}$ is close to u_ε for small ε. It will be shown that all $\mathcal{W}_{\varepsilon,m}$ approximate the same exact solution u_ε, and the larger is m the closer is $\mathcal{W}_{\varepsilon,m}$ to u_ε. In order to obtain such results we rewrite problem (1.1) in abstract form and then apply our generalized Implicit Function Theorem. As a result we obtain

Theorem 4.1 *Suppose that assumptions (A1)–(A4) are fulfilled. Then for any $m \geq 0$ and any $\alpha \in (0, 1)$ there exist $\varepsilon_{m,\alpha} > 0$, $\delta_{m,\alpha} > 0$ and $c_{m,\alpha} > 0$ such that the following is true:*

(i) For all $\varepsilon \in (0, \varepsilon_{m,\alpha})$ there exists a solution $u = u_\varepsilon$ to (1.1) such that

$$\|u_\varepsilon - \mathcal{W}_{\varepsilon,m}\|_{2+\alpha,\varepsilon;\Omega} \leq c_{m,\alpha}\varepsilon^{m+1}. \tag{4.1}$$

(ii) If u is a solution to (1.1) with $\varepsilon \in (0, \varepsilon_{m,\alpha})$ and

$$u \in B_{m,\alpha} := \left\{ u \in C^{2+\alpha}(\overline{\Omega}) \,:\, \|u - \mathcal{W}_{\varepsilon,m}\|_{2+\alpha,\varepsilon;\Omega} < \delta_{m,\alpha}\varepsilon^2 \right\}.$$

then $u = u_\varepsilon$.

We postpone the proof of Theorem 4.1 to the end of this section, since it is based on Theorem 4.6 to be formulated below.

Remark 4.2 *Theorem 1.1 is just Theorem 4.1 in the special case $m = 0$.*

Remark 4.3 *Suppose that the Hölder constant α is fixed. Then applying Theorem 4.1 with different $m = 0, \dots, k$ we obtain an array of solutions u_ε^m to problem (1.1), each of which is unique in the corresponding ball $B_{m,\alpha}$. Since $\min\limits_{m \le k} \delta_{m,\alpha} > 0$ and it holds*

$$\|\mathcal{W}_{\varepsilon,m} - \mathcal{W}_{\varepsilon,m+1}\|_{2+\alpha,\varepsilon;\Omega} = O(\varepsilon^{m+1}) \quad \textit{for} \quad \varepsilon \to 0, \tag{4.2}$$

one can choose $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ all the solutions u_ε^m coincide. In other words, for sufficiently small ε, Theorem 4.1 provides different asymptotics for the same solution to problem (1.1) which is unique in $\cup_{m=0}^k B_{m,\alpha}$.

In the rest of this section, we assume that the Hölder constant $\alpha \in (0,1)$ is a fixed number. Our main purpose is to reveal the ε-dependence of solution u_ε to problem (1.1). Therefore writing any estimate we will not monitor whether constants appearing there depend on α, although such a dependence is typically present.

Auxiliary family of approximate solutions $\mathcal{U}_{\varepsilon,m,\sigma}$. In Section 2, we have constructed a sequence of approximate solutions $\mathcal{W}_{\varepsilon,m}(x)$ consisting of three different parts: the outer expansion $u_{\varepsilon,m}(x)$, the inner expansion $w_{\varepsilon,m}(x)$ of the boundary layer and the inner expansion $v_{\varepsilon,m}(x)$ of the spike. Recall that the inner expansion of the spike is determined as the sum (2.2) of exponentially decaying functions v_k depending on the stretched variable $T_{\varepsilon,m}(x)$, and the latter is given by formula (2.3) which contains the approximate spike's position $x_{\varepsilon,m}$ as a parameter.

Keeping the outer expansion $u_{\varepsilon,m}$ and the inner expansion $w_{\varepsilon,m}$ of the boundary layer unchanged, we define the σ-parametric family of functions

$$\mathcal{U}_{\varepsilon,m,\sigma}(x) := u_{\varepsilon,m}(x) + w_{\varepsilon,m}(x) + v_{\varepsilon,m,\sigma}(x), \tag{4.3}$$

where

$$\begin{aligned} v_{\varepsilon,m,\sigma}(x) &:= \varepsilon(\sigma \cdot \nabla_\xi)\Phi_{\xi_0}(T_{\varepsilon,m,\sigma}(x)) + \sum_{k=0}^{m} \varepsilon^k v_k(T_{\varepsilon,m,\sigma}(x)), \\ T_{\varepsilon,m,\sigma}(x) &:= \frac{1}{\varepsilon}\, Q(x_{\varepsilon,m} + \varepsilon\sigma)(x - x_{\varepsilon,m} - \varepsilon\sigma), \end{aligned} \tag{4.4}$$

and $\sigma \in \mathbb{R}^n$ is a parameter. Compared with the approximate solution $\mathcal{W}_{\varepsilon,m}$, we performed the following modifications. To obtain $T_{\varepsilon,m,\sigma}$ from the definition of $T_{\varepsilon,m}$, we shifted the approximate spike's position $x_{\varepsilon,m}$ in the direction of vector $\varepsilon\sigma$. Respectively, we replaced $v_{\varepsilon,m}$ with $v_{\varepsilon,m,\sigma}$, where all the terms v_k are identical to those in definition of $v_{\varepsilon,m}$ (cf. (2.2)), but the stretched variable $T_{\varepsilon,m,\sigma}$ is different. Finally, in definition of $v_{\varepsilon,m,\sigma}$ we introduced the additional term $\varepsilon(\sigma\cdot\nabla_\xi)\Phi_{\xi_0}(T_{\varepsilon,m,\sigma}(x))$ which guarantees that the resulting function $\mathcal{U}_{\varepsilon,m,\sigma}$ satisfies the differential equation of problem (1.1) with a discrepancy of order $O(\varepsilon^2)$ for all σ on compact sets. Indeed, following the construction algorithm described in Subsection 2.2 (see, in particular, formulas (2.17), (2.22), (2.23) and (2.53)), we get

$$\Big(E_\varepsilon v_{\varepsilon,m,\sigma} - f(\cdot, u_{\varepsilon,m} + v_{\varepsilon,m,\sigma}, \varepsilon) + f(\cdot, u_{\varepsilon,m}, \varepsilon)\Big)\circ T^{-1}_{\varepsilon,m,\sigma}(y) =$$

$$-\varepsilon^2\left(\sigma\cdot\Psi(y) + \frac{1}{2}\psi(y, x_1+\sigma, x_1+\sigma) - \frac{1}{2}\psi(y, x_1, x_1)\right) + \varepsilon^3 r(y,\sigma,\varepsilon), \tag{4.5}$$

where the functions Ψ and ψ are defined in (2.54) and (2.55), and $r:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ is the remainder term in the corresponding Taylor formula. Taking into account exponential estimates (2.12) we easily verify that for any $\kappa\in(0,\kappa_0)$ and any multi-indices $|\mu_1|\le 2$ and $|\mu_2|\le 1$ it holds

$$|D^{\mu_1}_y D^{\mu_2}_\sigma r(y,\sigma,\varepsilon)| \le c(\kappa,\sigma_0,\varepsilon_0)\, e^{-\kappa|y|} \quad \text{for all} \quad y\in\mathbb{R}^n, \tag{4.6}$$

where $c(\kappa,\sigma_0,\varepsilon_0)$ is a positive constant independent of y, $|\sigma|<\sigma_0$ and $\varepsilon\in(0,\varepsilon_0)$.

Remark 4.4 *According to definition (5.6) from Appendix, for every non-negative integer k and every $\lambda\in(0,1)$ we have $\|u\|_{k+\lambda,\varepsilon;\Omega} = \|u\circ T_\varepsilon^{-1}\|_{C^{k+\lambda}(T_\varepsilon(\overline{\Omega}))}$. Since*

$$\left(T_{\varepsilon,m,\sigma}\circ T_\varepsilon^{-1}\right)(y) = Q(x_{\varepsilon,m}+\varepsilon\sigma)\left(y - \frac{x_{\varepsilon,m}}{\varepsilon} - \sigma\right) \quad \textit{for all} \quad y\in T_{\varepsilon,m,\sigma}(\overline{\Omega}),$$

and $u\circ T_\varepsilon^{-1} = \left(u\circ T^{-1}_{\varepsilon,m,\sigma}\right)\circ\left(T_{\varepsilon,m,\sigma}\circ T_\varepsilon^{-1}\right)$, it is easy to verify that there exist two positive constants c_1 and c_2 such that for any $\varepsilon\in(0,\varepsilon_0)$, any $|\sigma|<\sigma_0$ and all $u\in C^{k+\lambda}(\overline{\Omega})$ it holds

$$c_1\|u\circ T^{-1}_{\varepsilon,m,\sigma}\|_{C^{k+\lambda}(T_{\varepsilon,m,\sigma}(\overline{\Omega}))} \le \|u\|_{k+\lambda,\varepsilon;\Omega} \le c_2\|u\circ T^{-1}_{\varepsilon,m,\sigma}\|_{C^{k+\lambda}(T_{\varepsilon,m,\sigma}(\overline{\Omega}))}.$$

This means that norms $\|u\|_{k+\lambda,\varepsilon;\Omega}$ and $\|u\circ T^{-1}_{\varepsilon,m,\sigma}\|_{C^{k+\lambda}(T_{\varepsilon,m,\sigma}(\overline{\Omega}))}$ are equivalent uniformly with respect to ε and σ.

Estimates for approximate solutions $\mathcal{U}_{\varepsilon,m,\sigma}$. Below we are going to derive some estimates for approximate solutions $\mathcal{U}_{\varepsilon,m,\sigma}$. Our main tool will be the differentiation formula presented in the following

Remark 4.5 *For every smooth function $v(y,\sigma):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ and every $\overline{\sigma}\in\mathbb{R}^n$ it holds*

$$(\overline{\sigma}\cdot\nabla_\sigma)\Big(v(\cdot,\sigma)\circ T_{\varepsilon,m,\sigma}\Big)\circ T^{-1}_{\varepsilon,m,\sigma}(y) = (\overline{\sigma}\cdot\nabla_\sigma)v(y,\sigma) - \Big(\overline{\sigma}\cdot Q(x_{\varepsilon,m}+\varepsilon\sigma)\nabla_y\Big)v(y,\sigma)$$

$$+\varepsilon\Big((\overline{\sigma}\cdot\nabla_x)Q(x_{\varepsilon,m}+\varepsilon\sigma)Q(x_{\varepsilon,m}+\varepsilon\sigma)^{-1}y\cdot\nabla_y\Big)v(y,\sigma). \tag{4.7}$$

According to definition of $\mathcal{U}_{\varepsilon,m,\sigma}$ we have $\nabla_\sigma\mathcal{U}_{\varepsilon,m,\sigma} = \nabla_\sigma v_{\varepsilon,m,\sigma}$. Applying here formula (4.7) and taking into account exponential estimates (2.12) we conclude that for any $\kappa\in(0,\kappa_0)$ and any multi-index $|\mu|\le 3$ it holds

$$\left|D^\mu_y\left(\partial_{\sigma_j}\mathcal{U}_{\varepsilon,m,\sigma}\circ T^{-1}_{\varepsilon,m,\sigma}(y)\right)\right| \le c(\kappa,\sigma_0,\varepsilon_0)e^{-\kappa|y|} \quad \text{for all} \quad y\in\mathbb{R}^n, \quad j=1,\dots,n, \tag{4.8}$$

where $c(\kappa, \sigma_0, \varepsilon_0) > 0$ is a constant independent of y, $|\sigma| < \sigma_0$ and $\varepsilon \in (0, \varepsilon_0)$. The pointwise estimate (4.8) implies two corollaries formulated in terms of ε-dependent Hölder norms. Namely, for every $m \geq 0$ and every $\varepsilon \in (0, \varepsilon_0)$, $|\sigma| < \sigma_0$ it holds

$$\max_j \left\|\partial_{\sigma_j} \mathcal{U}_{\varepsilon,m,\sigma}\right\|_{2+\alpha,\varepsilon;\Omega} \leq c_0(\varepsilon_0, \sigma_0), \tag{4.9}$$

$$\max_j \left\|\partial_{\sigma_j} \mathcal{U}_{\varepsilon,m,\sigma}\right\|_{2+\alpha,\varepsilon;\partial\Omega} \leq c_0(\varepsilon_0, \sigma_0) e^{-c(\varepsilon_0,\sigma_0)/\varepsilon}, \tag{4.10}$$

where $c_0(\varepsilon_0, \sigma_0)$ and $c(\varepsilon_0, \sigma_0)$ are positive constants independent of ε, σ and Ω. Moreover, applying the mean value theorem and formulas (4.9) we get

$$\left\|\mathcal{U}_{\varepsilon,m,\sigma} - \mathcal{U}_{\varepsilon,m,0}\right\|_{2+\alpha,\varepsilon;\Omega} \leq c_0(\varepsilon_0, \sigma_0)|\sigma| \quad \text{for all} \quad |\sigma| \leq \sigma_0. \tag{4.11}$$

Remark also that in a similar way we obtain the estimate for the second derivative

$$\max_{i,j} \left\|\partial_{\sigma_i} \partial_{\sigma_j} \mathcal{U}_{\varepsilon,m,\sigma}\right\|_{2+\alpha,\varepsilon;\Omega} \leq \text{const} \tag{4.12}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and all $|\sigma| \leq \sigma_0$.

Finally we prove that

$$\left\|\varepsilon^{-2}(\overline{\sigma} \cdot \nabla_\sigma)\left(E_\varepsilon \mathcal{U}_{\varepsilon,m,\sigma} - f(\cdot, \mathcal{U}_{\varepsilon,m,\sigma}, \varepsilon)\right) + \overline{\sigma} \cdot \Psi(T_{\varepsilon,m,\sigma}) + \psi(T_{\varepsilon,m,\sigma}, x_1 + \sigma, \overline{\sigma})\right\|_{\alpha,\varepsilon;\Omega}$$
$$\leq c(\sigma_0, \varepsilon_0)|\overline{\sigma}|(|\sigma| + |\varepsilon|), \tag{4.13}$$

where $c(\sigma_0, \varepsilon_0)$ is a constant independent of $|\sigma| < \sigma_0$ and $\varepsilon \in (0, \varepsilon_0)$. For this, we differentiate formula (4.5) with the help of identity (4.7). Then, taking into account estimates (2.56), (4.6) and the identity

$$(\overline{\sigma} \cdot \nabla_\sigma)\psi(y, x_1 + \sigma, x_1 + \sigma) = 2\psi(y, x_1 + \sigma, \overline{\sigma})$$

following from definition (2.55), we obtain (4.13).

Reformulation of problem (1.1). For every $\varepsilon \in (0, \infty)$ let us define the pair of Banach spaces

$$U_\varepsilon := \left(C^{2+\alpha}(\overline{\Omega}), \|\cdot\|_{2+\alpha,\varepsilon;\Omega}\right) \times (\mathbb{R}^n, |\cdot|)$$

and

$$V_\varepsilon := \left(C^{\alpha}(\overline{\Omega}), \|\cdot\|_{\alpha,\varepsilon;\Omega}\right) \times \left(C^{1+\alpha}(\partial\Omega), \|\cdot\|_{1+\alpha,\varepsilon;\partial\Omega}\right) \times (\mathbb{R}^n, |\cdot|)\,,$$

where $|\cdot|$ denotes Euclidian norm in $\mathbb{R}^n$.

Now, instead of the original boundary value problem (1.1) we consider the following abstract equation

$$F_\varepsilon(v, \sigma) = 0, \tag{4.14}$$

where the operator $F_\varepsilon : U_\varepsilon \to V_\varepsilon$ reads

$$F_\varepsilon(v, \sigma) := \begin{pmatrix} \varepsilon^{-2}\left(E_\varepsilon(\varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma}) - f(\cdot, \varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma}, \varepsilon)\right) \\ \varepsilon^{-1}\left(\sum\limits_{i,j=1}^n a_{ij}(\cdot)\,\nu_i(\cdot)\,\partial_{x_j}(\varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma}) - g(\cdot, \varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma}, \varepsilon)\right) \\ \varepsilon^{-1}\left(\nabla_x(\varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma})\right)(x_{\varepsilon,m} + \varepsilon\sigma) \end{pmatrix},$$

and where solution u to problem (1.1) was represented via the following ansatz

$$u = \varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma} \quad \text{with} \quad (v,\sigma) \in U_\varepsilon. \tag{4.15}$$

In what follows we shall assume that $m \geq 2$. This restriction as well as the appearance of additional factors ε^2 and ε^{-2} in the definition of operator F_ε reflects, roughly speaking, the fact that to determine parameter σ during the construction of approximate solution one needs to consider the second order approximation equation (2.20) of the algorithm described in Section 2.

Definition of operator F_ε contains three components: the first and the second components coincide with the differential equation and boundary condition of problem (1.1), while the third component means that the point $x_{\varepsilon,m} + \varepsilon\sigma$ is an extremum of solution u. Hence, it is easy to see that every solution (v,σ) of augmented equation (4.14) determines via formula (4.15) a solution to problem (1.1). Further every $\|\cdot\|_{2+\alpha,\varepsilon;\Omega}$-vicinity of $\mathcal{W}_{\varepsilon,m}$ is naturally projected onto the vicinity of origin in U_ε, therefore proving the following theorem we simultaneously justify Theorem 4.1.

Theorem 4.6 *Suppose that assumptions (A1)–(A4) are fulfilled.*

Then there exist $\varepsilon_0 > 0$, $\delta > 0$ and $c > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ there exists exactly one solution $(v_\varepsilon, \sigma_\varepsilon)$ of equation $F_\varepsilon(v,\sigma) = 0$ with $\|(v_\varepsilon,\sigma_\varepsilon)\|_{U_\varepsilon} < \delta$. Moreover,

$$\|(v_\varepsilon,\sigma_\varepsilon)\|_{U_\varepsilon} \leq 2c\,\|F_\varepsilon(0,0)\|_{V_\varepsilon}.$$

Proof: We are going to apply Theorem 3.1, therefore we verify its assumptions.

Verification of assumption (3.1). The construction of function $\mathcal{U}_{\varepsilon,m,\sigma}$ implies that $\mathcal{U}_{\varepsilon,m,0} = \mathcal{W}_{\varepsilon,m}$ and $T_{\varepsilon,m,0} = T_{\varepsilon,m}$. Hence, we get

$$F_\varepsilon(0,0) = \begin{pmatrix} \varepsilon^{-2}\left(E_\varepsilon \mathcal{W}_{\varepsilon,m} - f(\cdot,\mathcal{W}_{\varepsilon,m},\varepsilon)\right) \\ \varepsilon^{-1}\left(\sum\limits_{i,j=1}^{n} a_{ij}(\cdot)\,\nu_i(\cdot)\,\partial_{x_j}\mathcal{W}_{\varepsilon,m} - g(\cdot,\mathcal{W}_{\varepsilon,m},\varepsilon)\right) \\ \varepsilon^{-1}(\nabla_x \mathcal{W}_{\varepsilon,m})(x_{\varepsilon,m}) \end{pmatrix}. \tag{4.16}$$

Now estimates (2.9) and (2.10) from Theorem 2.1 imply that for $\varepsilon \to 0$ it holds

$$\|F_\varepsilon(0,0)\|_{V_\varepsilon} \leq \text{const}\,\varepsilon^{m-1}. \tag{4.17}$$

In particular, $\|F_\varepsilon(0,0)\|_{V_\varepsilon} \to 0$ for $\varepsilon \to 0$ provided $m \geq 2$.

Verification of assumption (3.2). We calculate the derivative operator

$$F'_\varepsilon(v,\sigma)(\overline{v},\overline{\sigma}) = \begin{pmatrix} [F'_\varepsilon(v,\sigma)(\overline{v},\overline{\sigma})]_1 \\ [F'_\varepsilon(v,\sigma)(\overline{v},\overline{\sigma})]_2 \\ [F'_\varepsilon(v,\sigma)(\overline{v},\overline{\sigma})]_3 \end{pmatrix}.$$

Its first component reads as follows

$$[F'_\varepsilon(v,\sigma)(\overline{v},\overline{\sigma})]_1 \;=\; E_\varepsilon\overline{v} - \partial_u f(\cdot,\varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma},\varepsilon)\overline{v} + \varepsilon^{-2}(\overline{\sigma}\cdot\nabla_\sigma)\Big(E_\varepsilon\mathcal{U}_{\varepsilon,m,\sigma} - f(\cdot,\mathcal{U}_{\varepsilon,m,\sigma},\varepsilon)\Big)$$
$$+\;\varepsilon^{-2}(\overline{\sigma}\cdot\nabla_\sigma)\Big(f(\cdot,\mathcal{U}_{\varepsilon,m,\sigma},\varepsilon) - f(\cdot,\varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma},\varepsilon)\Big).$$

Similarly we calculate the second component

$$[F'_\varepsilon(v,\sigma)(\overline{v},\overline{\sigma})]_2 \;=\; \varepsilon\left(\sum_{i,j=1}^{n} a_{ij}(\cdot)\;\nu_i(\cdot)\;\partial_{x_j}\overline{v} - \partial_u g(\cdot,\varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma},\varepsilon)\overline{v}\right)$$
$$+\;\varepsilon^{-1}(\overline{\sigma}\cdot\nabla_\sigma)\left(\sum_{i,j=1}^{n} a_{ij}(\cdot)\;\nu_i(\cdot)\;\partial_{x_j}\mathcal{U}_{\varepsilon,m,\sigma} - g(\cdot,\mathcal{U}_{\varepsilon,m,\sigma},\varepsilon)\right)$$
$$+\;\varepsilon^{-1}(\overline{\sigma}\cdot\nabla_\sigma)\Big(g(\cdot,\mathcal{U}_{\varepsilon,m,\sigma},\varepsilon) - g(\cdot,\varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma},\varepsilon)\Big).$$

Finally, applying definition (4.3) we get

$$(\nabla_x\mathcal{U}_{\varepsilon,m,\sigma})(x_{\varepsilon,m}+\varepsilon\sigma) \;=\; (\nabla_x u_{\varepsilon,m})(x_{\varepsilon,m}+\varepsilon\sigma)$$
$$+\;\varepsilon^{-1}Q(x_{\varepsilon,m}+\varepsilon\sigma)\left(\varepsilon(\sigma\cdot\nabla_\xi)\nabla_y\Phi_{\xi_0}(0) + \sum_{k=0}^{m}\varepsilon^k\nabla_y v_k(0)\right),$$

and this together with the fact that $(\sigma\cdot\nabla_\xi)\nabla_y\Phi_{\xi_0}(0) = 0$ results in

$$[F'_\varepsilon(v,\sigma)(\overline{v},\overline{\sigma})]_3 = \varepsilon\,(\nabla_x\overline{v})\,(x_{\varepsilon,m}+\varepsilon\sigma) + \varepsilon^2\Big((\overline{\sigma}\cdot\nabla_x)\nabla_x v\Big)(x_{\varepsilon,m}+\varepsilon\sigma)$$
$$+\Big((\overline{\sigma}\cdot\nabla_x)\nabla_x u_{\varepsilon,m}\Big)(x_{\varepsilon,m}+\varepsilon\sigma) + \Big((\overline{\sigma}\cdot\nabla_x)Q(x_{\varepsilon,m}+\varepsilon\sigma)\Big)\left(\sum_{k=0}^{m}\varepsilon^{k-1}\nabla_y v_k(0)\right).$$

Using obtained formulas for components of the derivative operator $F'_\varepsilon(v,\sigma)$ we shall verify that $\|F'_\varepsilon(v,\sigma)(\overline{v},\overline{\sigma}) - F'_\varepsilon(0,0)(\overline{v},\overline{\sigma})\|_{V_\varepsilon} \to 0$ for $\varepsilon + \|(v,\sigma)\|_{U_\varepsilon} \to 0$, uniformly with respect to $\|(\overline{v},\overline{\sigma})\|_{U_\varepsilon} = 1$. In particular, for the first component we write the inequality

$$\Big\|\,[F'_\varepsilon(v,\sigma)(\overline{v},\overline{\sigma})]_1 - [F'_\varepsilon(0,0)(\overline{v},\overline{\sigma})]_1\Big\|_{\alpha,\varepsilon;\Omega}$$
$$\le \Big\|\,\big(\partial_u f(\cdot,\varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma},\varepsilon) - \partial_u f(\cdot,\mathcal{U}_{\varepsilon,m,0},\varepsilon)\big)\,\overline{v}\Big\|_{\alpha,\varepsilon;\Omega}$$
$$+\Big\|\varepsilon^{-2}(\overline{\sigma}\cdot\nabla_\sigma)\Big(E_\varepsilon\mathcal{U}_{\varepsilon,m,\sigma} - f(\cdot,\mathcal{U}_{\varepsilon,m,\sigma},\varepsilon) - E_\varepsilon\mathcal{U}_{\varepsilon,m,0} + f(\cdot,\mathcal{U}_{\varepsilon,m,0},\varepsilon)\Big)\Big\|_{\alpha,\varepsilon;\Omega}$$
$$+\Big\|\varepsilon^{-2}(\overline{\sigma}\cdot\nabla_\sigma)\big(f(\cdot,\varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma},\varepsilon) - f(\cdot,\mathcal{U}_{\varepsilon,m,\sigma},\varepsilon)\big)\Big\|_{\alpha,\varepsilon;\Omega} \tag{4.18}$$

and estimate separately each term in the right-hand part of (4.18). First, employing inequalities (4.11), (5.9) and (5.11), we easily get the following estimate

$$\Big\| \left(\partial_u f(\cdot, \varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma}, \varepsilon) - \partial_u f(\cdot, \mathcal{U}_{\varepsilon,m,0}, \varepsilon)\right) \overline{v} \Big\|_{\alpha,\varepsilon;\Omega}$$

$$= \left\| \int_0^1 \partial_u^2 f\left(\cdot, \varepsilon^2 t v + t\mathcal{U}_{\varepsilon,m,\sigma} + (1-t)\mathcal{U}_{\varepsilon,m,0}, \varepsilon\right) dt \cdot \left(\varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma} - \mathcal{U}_{\varepsilon,m,0}\right) \overline{v} \right\|_{\alpha,\varepsilon;\Omega}$$

$$\le \text{const } \|\overline{v}\|_{\alpha,\varepsilon;\Omega} \cdot \left\|\varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma} - \mathcal{U}_{\varepsilon,m,0}\right\|_{\alpha,\varepsilon;\Omega} \le \text{const } \|\overline{v}\|_{\alpha,\varepsilon;\Omega} \cdot \left\{\varepsilon^2 \|v\|_{\alpha,\varepsilon;\Omega} + |\sigma|\right\}.$$

In a similar way we consider the third term in the right-hand part of (4.18) and conclude that it obeys the inequality

$$\left\|\varepsilon^{-2}(\overline{\sigma}\cdot\nabla_\sigma)\left(f(\cdot, \varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma}, \varepsilon) - f(\cdot, \mathcal{U}_{\varepsilon,m,\sigma}, \varepsilon)\right)\right\|_{\alpha,\varepsilon;\Omega}$$

$$= \left\|(\overline{\sigma}\cdot\nabla_\sigma)\left(\int_0^1 \partial_u f(\cdot, t\varepsilon^2 v + \mathcal{U}_{\varepsilon,m,\sigma}, \varepsilon) dt\right) v\right\|_{\alpha,\varepsilon;\Omega} \le \text{const } |\overline{\sigma}| \cdot \|v\|_{\alpha,\varepsilon;\Omega}.$$

Finally, we apply formula (4.13) to estimate the second term in the right-hand part of (4.18), and considering the difference $\psi(y, x_1+\sigma, \overline{\sigma}) - \psi(y, x_1, \overline{\sigma})$ with the help of definition (2.55) and inequalities (2.56) we obtain

$$\left\|\varepsilon^{-2}(\overline{\sigma}\cdot\nabla_\sigma)\left(E_\varepsilon \mathcal{U}_{\varepsilon,m,\sigma} - f(\cdot, \mathcal{U}_{\varepsilon,m,\sigma}, \varepsilon) - E_\varepsilon \mathcal{U}_{\varepsilon,m,0} + f(\cdot, \mathcal{U}_{\varepsilon,m,0}, \varepsilon)\right)\right\|_{\alpha,\varepsilon;\Omega} \le \text{const } |\overline{\sigma}| \ (|\sigma| + |\varepsilon|).$$

The estimate for $\|[F'_\varepsilon(v,\sigma)(\overline{v},\overline{\sigma})]_2 - [F'_\varepsilon(0,0)(\overline{v},\overline{\sigma})]_2\|_{1+\alpha,\varepsilon;\partial\Omega}$ is even simpler to obtain, since the approximate solution $\mathcal{U}_{\varepsilon,m,\sigma}$ and all its partial derivatives involved into the definition of $[F'_\varepsilon(v,\sigma)(\overline{v},\overline{\sigma})]_2$ are exponentially small near the boundary $\partial\Omega$ (see inequalities (4.10)).

Finally, we analyze the third component of the derivative operator $F'_\varepsilon(v,\sigma)$. According to the construction procedure described in Section 2, we know that $u_0 = 0$, $\nabla_y v_0(0) = \nabla_y v_1(0) = 0$. Then taking into account definition (5.6), we easily obtain

$$\Big\| \ [F'_\varepsilon(v,\sigma)(\overline{v},\overline{\sigma})]_3 - [F'_\varepsilon(0,0)(\overline{v},\overline{\sigma})]_3 \Big\|_{\mathbb{R}^n} \le \text{const } (|\sigma| \ \|\overline{v}\|_{2+\alpha,\varepsilon;\Omega} + \|v\|_{2+\alpha,\varepsilon;\Omega} + \varepsilon) \to 0.$$

Hence, we have shown that assumption (3.2) is also satisfied.

Verification of assumption (3.3). We are going to apply Lemma 3.2. For this we first write operator $F'_\varepsilon(0,0)$ in the matrix form

$$F'_\varepsilon(0,0)(\overline{v},\overline{\sigma}) = \begin{pmatrix} \mathcal{F}_{11}\overline{v} & \mathcal{F}_{12}\overline{\sigma} \\ \mathcal{F}_{21}\overline{v} & \mathcal{F}_{22}\overline{\sigma} \\ \mathcal{F}_{31}\overline{v} & \mathcal{F}_{32}\overline{\sigma} \end{pmatrix},$$

where

$$\begin{pmatrix} \mathcal{F}_{11}\overline{v} \\ \mathcal{F}_{21}\overline{v} \\ \mathcal{F}_{31}\overline{v} \end{pmatrix} = \begin{pmatrix} E_\varepsilon\overline{v} - \partial_u f(\cdot, \mathcal{W}_{\varepsilon,m}, \varepsilon)\overline{v} \\ \varepsilon \left(\sum\limits_{i,j=1}^n a_{ij}(\cdot)\, \nu_i(\cdot)\, \partial_{x_j}\overline{v} - \partial_u g(\cdot, \mathcal{W}_{\varepsilon,m}, \varepsilon)\overline{v} \right) \\ \varepsilon \left(\nabla_x \overline{v} \right)(x_{\varepsilon,m}) \end{pmatrix}$$

and

$$\begin{pmatrix} \mathcal{F}_{12}\overline{\sigma} \\ \mathcal{F}_{22}\overline{\sigma} \\ \mathcal{F}_{32}\overline{\sigma} \end{pmatrix} = \begin{pmatrix} \varepsilon^{-2}(\overline{\sigma}\cdot\nabla_\sigma) \left(E_\varepsilon \mathcal{U}_{\varepsilon,m,\sigma} - f(\cdot, \mathcal{U}_{\varepsilon,m,\sigma}, \varepsilon) \right) \Big|_{\sigma=0} \\ \varepsilon^{-1}(\overline{\sigma}\cdot\nabla_\sigma) \left(\sum\limits_{i,j=1}^n a_{ij}(\cdot)\, \nu_i(\cdot)\, \partial_{x_j}\mathcal{U}_{\varepsilon,m,\sigma} - g(\cdot, \mathcal{U}_{\varepsilon,m,\sigma}, \varepsilon) \right) \Big|_{\sigma=0} \\ \left((\overline{\sigma}\cdot\nabla_x)\nabla_x u_{\varepsilon,m} \right)(x_{\varepsilon,m}) + \left((\overline{\sigma}\cdot\nabla_x) Q(x_{\varepsilon,m}) \right) \left(\sum\limits_{k=0}^m \varepsilon^{k-1} \nabla_y v_k(0) \right) \end{pmatrix}.$$

According to classical results on boundary value problems for linear elliptic equations (see for example [18]), the operator

$$\begin{pmatrix} \mathcal{F}_{11}\overline{v} \\ \mathcal{F}_{21}\overline{v} \end{pmatrix} : C^{2+\alpha}(\overline{\Omega}) \to C^\alpha(\overline{\Omega}) \times C^{1+\alpha}(\partial\Omega)$$

is a Fredholm operator of index zero. On the other hand all the rest components

$$\mathcal{F}_{31} : C^{2+\alpha}(\overline{\Omega}) \to \mathbb{R}^n,\ \ \mathcal{F}_{12} : \mathbb{R}^n \to C^\alpha(\overline{\Omega}),\ \ \mathcal{F}_{22} : \mathbb{R}^n \to C^{1+\alpha}(\partial\Omega)\ \text{ and }\ \mathcal{F}_{32} : \mathbb{R}^n \to \mathbb{R}^n$$

are operators with finite-dimensional ranges. Hence, the composite operator $F'_\varepsilon(0,0)$ is a Fredholm operator of index zero from U_ε to V_ε, and to apply Lemma 3.2 we yet need to verify its second assumption only.

We perform this verification by contradiction. For this we suppose that $\varepsilon_k \in (0,\infty)$ and $(u_k, \sigma_k) \in U_{\varepsilon_k}$ are two sequences with

$$\|(u_k,\sigma_k)\|_{U_{\varepsilon_k}} = \|u_k\|_{2+\alpha,\varepsilon_k;\Omega} + \|\sigma_k\|_{\mathbb{R}^n} = 1 \tag{4.19}$$

and

$$\varepsilon_k + \left\| F'_{\varepsilon_k}(0,0)(u_k,\sigma_k) \right\|_{V_{\varepsilon_k}} \to 0 \quad \text{for} \quad k\to\infty. \tag{4.20}$$

Then our strategy will be to demonstrate that assumptions (4.19) and (4.20) lead to the limit $\|(u_k,\sigma_k)\|_{U_{\varepsilon_k}} \to 0$ for $k\to\infty$, which obviously contradicts to (4.19).

Before we proceed further, let us write explicitely the meaning of limit (4.20) for each component of the operator $F'_{\varepsilon_k}(0,0)(u_k,\sigma_k)$. To simplify the resulting formulas we neglect in each of them all the terms that vanish for $\varepsilon\to 0$. Notice that because of (4.19) without loss of generality we may assume that there exists $\sigma_* \in \mathbb{R}^n$ such that

$$\sigma_k \to \sigma_* \quad \text{in} \quad \mathbb{R}^n \quad \text{for} \quad k\to\infty.$$

To this end, we consider the first component of operator $F'_{\varepsilon_k}(0,0)(u_k,\sigma_k)$ which reads

$$\begin{aligned}\left[F'_{\varepsilon_k}(0,0)(u_k,\sigma_k)\right]_1 &= E_{\varepsilon_k}u_k - \partial_u f(\cdot,\mathcal{W}_{\varepsilon_k,m},\varepsilon_k)u_k \\ &+ \varepsilon_k^{-2}(\sigma_k\cdot\nabla_\sigma)\left(E_{\varepsilon_k}\mathcal{U}_{\varepsilon_k,m,\sigma} - f(\cdot,\mathcal{U}_{\varepsilon_k,m,\sigma},\varepsilon_k)\right)\Bigg|_{\sigma=0}.\end{aligned}$$

Then taking into account assumptions (4.19) and (4.20), and simplifying the last term with the help of estimate (4.13), we get

$$\left\|E_{\varepsilon_k}u_k - \partial_u f(\cdot,\mathcal{W}_{\varepsilon_k,m},\varepsilon_k)u_k - \sigma_*\cdot\Psi(T_{\varepsilon_k,m}) - \psi(T_{\varepsilon_k,m},x_1,\sigma_*)\right\|_{\alpha,\varepsilon_k;\Omega}\to 0. \tag{4.21}$$

For the second component $\left[F'_{\varepsilon_k}(0,0)(u_k,\sigma_k)\right]_2$, we take use of the fact that function $\mathcal{U}_{\varepsilon,m,\sigma}$ and all its partial derivatives are exponentially small near boundary $\partial\Omega$ (see inequality (4.10)). Combining this with assumption (4.19) and neglecting in the limit

$$\left\|\left[F'_{\varepsilon_k}(0,0)(u_k,\sigma_k)\right]_2\right\|_{1+\alpha,\varepsilon_k;\partial\Omega}\to 0$$

all the terms vanishing for $\varepsilon\to 0$, we obtain

$$\left\|\varepsilon_k\sum_{i,j=1}^n a_{ij}(\cdot)\,\nu_i(\cdot)\,\partial_{x_j}u_k\right\|_{1+\alpha,\varepsilon_k;\partial\Omega}\to 0. \tag{4.22}$$

Finally, we consider the meaning of limit (4.20) for the third component $\left[F'_{\varepsilon_k}(0,0)(u_k,\sigma_k)\right]_3$. Here, since the outer expansion $u_{\varepsilon,m}$ starts with a term of order $O(\varepsilon)$ and because of identities $\nabla_y v_0(0)=\nabla_y v_1(0)=0$ (see construction procedure in Section 2), we easily get

$$\left\|\varepsilon_k\left(\nabla_x u_k\right)(x_{\varepsilon_k,m})\right\|_{\mathbb{R}^n}\to 0. \tag{4.23}$$

In the rest of proof we will show that as a consequence of assumptions (4.19) and (4.20) we have two limits

$$\sigma_k\to 0 \tag{4.24}$$

and

$$\left\|\varepsilon_k^2\sum_{i,j=1}^n\partial_{x_i}(a_{ij}(\cdot)\,\partial_{x_j}u_k) - \partial_u f(\cdot,0,0)u_k\right\|_{\alpha,\varepsilon_k;\Omega}\to 0. \tag{4.25}$$

Regarding the latter limit, we remark that in contrary to (4.21) it contains the positive coefficient $\partial_u f(x,0,0)$ (see assumption (A1)) instead of the sign-changing coefficient $\partial_u f(x,\mathcal{W}_{\varepsilon_k,m},\varepsilon_k)$. Therefore, as soon as we prove (4.25) we can apply the ε-dependent Schauder-type estimates from Appendix to conclude that $\|u_k\|_{2+\alpha,\varepsilon_k;\Omega}\to 0$ for $k\to\infty$. Then this limit together with (4.24) will constitute the necessary contradiction $\|(u_k,\sigma_k)\|_{U_{\varepsilon_k}}\to 0$ for $k\to\infty$.

For the sake of clearness we divide further argumentation into few steps.

Step 1. Operator $P_{\varepsilon,s}$. For every $s\in(0,\kappa_0)$, where κ_0 is given by (2.11), we define an operator

$$P_{\varepsilon,s}: C^\alpha(\overline{\Omega})\to C^\alpha(\mathbb{R}^n)\cap L^2(\mathbb{R}^n),$$

by

$$P_{\varepsilon,s}u := ((\chi_0 u)\circ T_{\varepsilon,m}^{-1})\rho_s. \tag{4.26}$$

Here

$$\rho_s(y) = e^{-s(\sqrt{1+|y|^2}-1)} \quad \text{with} \quad y\in\mathbb{R}^n$$

is the exponentially decaying function defined previously in (2.25), and $\chi_0 : \overline{\Omega}\to\mathbb{R}$ is a smooth cut-off function such that

$$\chi_0(x) = 1 \ \text{ for } \ |x-\xi_0|<\delta \quad \text{and} \quad \chi_0(x)=0 \ \text{ for } \ |x-\xi_0|>2\delta, \ \text{ where } \ \delta = \frac{1}{4}\operatorname{dist}(\xi_0,\partial\Omega).$$

Note, in definition (4.26) we assume that the product $\chi_0 u$ is extended by zero on the whole $\mathbb{R}^n$. Then the argument of resulting function is stretched according to the transformation $T_{\varepsilon,m}^{-1}$ and the obtained function is finally multiplied by the factor ρ_s.

Taking into account Remark 4.4 and inequalities (5.8), (5.9) from Appendix, we easily verify that for any $\varepsilon_0>0$ there exists $c_0(\varepsilon_0)>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and $u\in C^\alpha(\overline{\Omega})$ it holds

$$\left\|(\chi_0 u)\circ T_{\varepsilon,m}^{-1}\right\|_{C^\alpha(\mathbb{R}^n)} \le c_0(\varepsilon_0)\|u\|_{\alpha,\varepsilon;\Omega} \tag{4.27}$$

and

$$\|P_{\varepsilon,s}u\|_{C^\alpha(\mathbb{R}^n)} \le \|\rho_s\|_{C^\alpha(\mathbb{R}^n)}\left\|(\chi_0 u)\circ T_{\varepsilon,m}^{-1}\right\|_{C^\alpha(\mathbb{R}^n)} \le c_0(\varepsilon_0)\|\rho_s\|_{C^\alpha(\mathbb{R}^n)}\|u\|_{\alpha,\varepsilon;\Omega}.$$

Moreover, since definition (4.26) contains exponentially decaying factor $\rho_s\in L^2(\mathbb{R}^n)$ the estimate (4.27) implies

$$\|P_{\varepsilon,s}u\|_{L^2(\mathbb{R}^n)} \le \|\rho_s\|_{L^2(\mathbb{R}^n)}\left\|(\chi_0 u)\circ T_{\varepsilon,m}^{-1}\right\|_{L^\infty(\mathbb{R}^n)} \le c_0(\varepsilon_0)\|\rho_s\|_{L^2(\mathbb{R}^n)}\|u\|_{\alpha,\varepsilon;\Omega}. \tag{4.28}$$

Hence, for all $\varepsilon\in(0,\varepsilon_0)$ and all $u\in C^\alpha(\overline{\Omega})$ we have $P_{\varepsilon,s}u\in C^\alpha(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, provided $s>0$.

Similarly one shows that the operator $P_{\varepsilon,s}$ maps $C^{2+\alpha}(\overline{\Omega})$ into $C^{2+\alpha}(\mathbb{R}^n)\cap W^{2,2}(\mathbb{R}^n)$. In particular, for any $s>0$ and $\varepsilon_0>0$ there exists $c_1(s,\varepsilon_0)>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and $u\in C^{2+\alpha}(\overline{\Omega})$ it holds

$$\|P_{\varepsilon,s}u\|_{C^{2+\alpha}(\mathbb{R}^n)} + \|P_{\varepsilon,s}u\|_{W^{2,2}(\mathbb{R}^n)} \le c_1(s,\varepsilon_0)\|u\|_{2+\alpha,\varepsilon;\Omega}, \tag{4.29}$$

Now let us define the sequence

$$\hat{v}_k := P_{\varepsilon_k,s}u_k.$$

In fact each $\hat{v}_k$ depends also on s. But later on we will fix s independently of k, therefore we do not mention the s-dependence in the notation of $\hat{v}_k$ for the sake of simplicity.

Because of (4.19) and (4.29) the sequence $\hat{v}_k$ is bounded in the Hilbert space $W^{2,2}(\mathbb{R}^n)$. Without loss of generality we may assume that there exists $v_*\in W^{2,2}(\mathbb{R}^n)$ such that

$$\hat{v}_k \rightharpoonup v_* \quad \text{in} \quad W^{2,2}(\mathbb{R}^n) \quad \text{for} \quad k\to\infty. \tag{4.30}$$

Step 2. Derivation of equation for v_* and σ_*. From (2.15) it follows

$$\left|(E_{\varepsilon_k}u_k)\left(T_{\varepsilon_k,m}^{-1}(y)\right) - \Delta_y\left(u_k\circ T_{\varepsilon_k,m}^{-1}\right)(y)\right| \le \text{const}\ \varepsilon_k(1+|y|)\|u_k\|_{2+\alpha,\varepsilon_k;\Omega} \tag{4.31}$$

for all $y \in T_{\varepsilon_k,m}(\overline{\Omega})$. Further, according to the definitions of χ_0 and $T_{\varepsilon,m}$, for any $\varepsilon_0 > 0$ there exists $\hat{\delta} = \hat{\delta}(\varepsilon_0) > 0$ such that

$$\chi_0\left(T^{-1}_{\varepsilon,m}(y)\right) = 1 \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_0) \quad \text{and} \quad |y| \leq \hat{\delta}/\varepsilon.$$

Hence, assumption (4.19) implies for all $\eta \in L^2(\mathbb{R}^n)$

$$\int\limits_{|y|\leq\hat{\delta}/\varepsilon_k} \Big(P_{\varepsilon_k,s}\Big(E_{\varepsilon_k}u_k\Big) - \rho_s\Delta_y\left(u_k \circ T^{-1}_{\varepsilon_k,m}\right)\Big)\,\eta\,dy \to 0,$$

provided $s > 0$. Because of $u_k \circ T^{-1}_{\varepsilon_k,m} = \rho_s^{-1}\hat{v}_k$ this yields

$$\int\limits_{|y|\leq\hat{\delta}/\varepsilon_k} \Big(P_{\varepsilon_k,s}\Big(E_{\varepsilon_k}u_k\Big) - \Delta\hat{v}_k - 2(\rho_s\nabla\rho_s^{-1}\cdot\nabla\hat{v}_k) - \rho_s\hat{v}_k\Delta\rho_s^{-1}\Big)\,\eta\,dy \to 0. \tag{4.32}$$

But assumption (4.19) and the inequalities (5.8), (5.9) from the Appendix imply

$$\|E_{\varepsilon_k}u_k\|_{\alpha,\varepsilon_k;\Omega} \leq \text{const},$$

whereas the definition of ρ_s results in the inequalities

$$\|\rho_s\,\partial_{y_j}\rho_s^{-1}\|_{L^\infty(\mathbb{R}^n)} \leq s, \qquad \|\rho_s\Delta\rho_s^{-1}\|_{L^\infty(\mathbb{R}^n)} \leq s(s+2n-1).$$

Hence, in (4.32) the limits of integration may be extended to $\mathbb{R}^n$ and we get

$$\int\limits_{\mathbb{R}^n} \Big(P_{\varepsilon_k,s}\Big(E_{\varepsilon_k}u_k\Big) - \Delta\hat{v}_k - 2(\rho_s\nabla\rho_s^{-1}\cdot\nabla\hat{v}_k) - \rho_s\hat{v}_k\Delta\rho_s^{-1}\Big)\,\eta\,dy \to 0. \tag{4.33}$$

In other words, we have

$$P_{\varepsilon_k,s}\Big(E_{\varepsilon_k}u_k\Big) - \Delta\hat{v}_k - 2(\rho_s\nabla\rho_s^{-1}\cdot\nabla\hat{v}_k) - \rho_s\hat{v}_k\Delta\rho_s^{-1} \rightharpoonup 0 \quad \text{in} \quad L^2(\mathbb{R}^n). \tag{4.34}$$

Similarly one shows that

$$P_{\varepsilon_k,s}\Big(\partial_u f(\cdot,\mathcal{W}_{\varepsilon_k,m},\varepsilon_k)u_k\Big) - \partial_u f(\xi_0,\Phi_{\xi_0},0)\hat{v}_k \rightharpoonup 0 \quad \text{in} \quad L^2(\mathbb{R}^n). \tag{4.35}$$

Indeed, as above we can replace the integrals over $\mathbb{R}^n$ by integrals over $|y| \leq \hat{\delta}/\varepsilon_k$ because of

$$\left|\partial_u f(\cdot,\mathcal{W}_{\varepsilon_k,m},\varepsilon_k)\circ T^{-1}_{\varepsilon_k,m} - \partial_u f(\xi_0,\Phi_{\xi_0},0)\right| \leq \text{const}\;\varepsilon_k(1+|y|) \;\text{ for all }\; y \in T_{\varepsilon_k,m}(\overline{\Omega}).$$

The latter estimate follows directly from the structure of the formal asymptotics $\mathcal{W}_{\varepsilon,m}$.

Finally, we have

$$P_{\varepsilon_k,s}\Big[\sigma_*\cdot\Psi(T_{\varepsilon_k,m}) + \psi(T_{\varepsilon_k,m},x_1,\sigma_*)\Big] - \Big(\sigma_*\cdot\Psi + \psi(\cdot,x_1,\sigma_*)\Big)\rho_s \rightharpoonup 0 \quad \text{in} \quad L^2(\mathbb{R}^n), \tag{4.36}$$

where the functions Ψ and ψ are defined in (2.54) and (2.55), respectively. This weak convergence is true because the left hand side of (4.36) vanishes for $|y| \leq \hat{\delta}/\varepsilon_k$.

Collecting together the limits (4.34)–(4.36) and using (4.21) and (4.28) we get

$$\Delta\hat{v}_k + 2(\rho_s\nabla\rho_s^{-1}\cdot\nabla\hat{v}_k) - \left(\partial_u f(\xi_0,\Phi_{\xi_0},0) - \rho_s\Delta\rho_s^{-1}\right)\hat{v}_k$$
$$- \Big(\sigma_*\cdot\Psi + \psi(\cdot,x_1,\sigma_*)\Big)\rho_s \rightharpoonup 0 \quad \text{in} \quad L^2(\mathbb{R}^n).$$

This gives the desired equation for v_* and σ_*

$$D_s v_* := \Delta v_* + 2(\rho_s\nabla\rho_s^{-1}\cdot\nabla v_*) - \left(\partial_u f(\xi_0,\Phi_{\xi_0}(y),0) - \rho_s\Delta\rho_s^{-1}\right)v_*$$
$$= \Big(\sigma_*\cdot\Psi + \psi(\cdot,x_1,\sigma_*)\Big)\rho_s \quad \text{for almost all} \ \ y\in\mathbb{R}^n. \quad (4.37)$$

Step 3. Proof of the fact that $\sigma_* = 0$. Assumption (A1) and exponential estimate (2.12) imply that $f(\xi_0,\Phi_{\xi_0},0)\in C^\alpha(\mathbb{R}^n)$ and $(\sigma_*\cdot\Psi+\psi(\cdot,x_1,\sigma_*))\rho_s\in C^\alpha(\mathbb{R}^n)$, therefore every solution $v_*\in W^{2,2}(\mathbb{R}^n)$ to Eq. (4.37) belongs simultaneously to $C^{2+\alpha}(\mathbb{R}^n)$. Below we demonstrate that an appropriate choice of s guarantees that $\sigma_*=0$ and $v_*\in\operatorname{span}\{\rho_s\partial_{y_j}\Phi_{\xi_0} : j=1,\dots,n\}$. To this end, we use the following lemma.

Lemma 4.7 *There exists $s_0>0$ such that for every $s\in[0,s_0)$ the operator D_s (cf. (4.37)) mapping $C^{2+\alpha}(\mathbb{R}^n)$ into $C^\alpha(\mathbb{R}^n)$ is a Fredholm operator with* $\dim\operatorname{Ker}D_s = \operatorname{codim}\operatorname{Ran}D_s = n$. *Moreover,*

$$\operatorname{Ker}D_s = \operatorname{span}\left\{\rho_s\partial_{y_j}\Phi_{\xi_0} : j=1,\dots,n\right\},$$
$$\operatorname{Ran}D_s = \left\{ v\in C^\alpha(\mathbb{R}^n) \ : \int\limits_{\mathbb{R}^n} v(y)\rho_s^{-1}(y)\partial_{y_j}\Phi_{\xi_0}(y)dy = 0 \quad \textit{for all} \quad j=1,\dots,n \right\}.$$

Proof: Straightforward calculation yields

$$\lim_{s\to 0}\left\|2(\rho_s\nabla\rho_s^{-1}\cdot\nabla) - \rho_s\Delta\rho_s^{-1}\right\|_{L(C^{2+\alpha}(\mathbb{R}^n);C^\alpha(\mathbb{R}^n))} = 0. \quad (4.38)$$

Since small perturbations do not violate Fredholm property and do not increase the dimension of kernel and the codimension of range (see, for example, [36, Theorem 5.11]), estimate (4.38) together with Lemma 2.2 and assumption (A4) imply that for sufficiently small $s>0$ the operator D_s is Fredholm of index zero and $\dim\operatorname{Ker}D_s = \operatorname{codim}\operatorname{Ran}D_s \le n$.

Above we have assumed that $s\in(0,\kappa_0)$, where the constant κ_0 is given by (2.11). Therefore exponential estimates (2.12) guarantee that $\rho_s\partial_{y_j}\Phi_{\xi_0}\in C^{2+\alpha}(\mathbb{R}^n)$. Moreover, taking into account assumption (A4) we easily verify that $\rho_s\partial_{y_j}\Phi_{\xi_0}\in\operatorname{Ker}D_s$. The only remaining point regarding $\operatorname{Ker}D_s$ is to show that $\dim\operatorname{Ker}D_s = n$, i.e. that functions $\rho_s\partial_{y_j}\Phi_{\xi_0}$, $j=1,\dots,n$, are linearly independent. To check this we write the Gram matrix $\mathcal{G}(s)$ with elements

$$[\mathcal{G}(s)]_{jk} := \int\limits_{\mathbb{R}^n}\rho_s\partial_{y_j}\Phi_{\xi_0}\ \rho_s\partial_{y_k}\Phi_{\xi_0}\ dy.$$

It is clear that $\mathcal{G}(0)$ is non-degenerate (see assumption (A4)). On the other hand, simple calculation shows that the matrix derivative $\mathcal{G}'(0)$ with respect to s is bounded. Therefore for sufficiently small s matrix $\mathcal{G}(s)$ is non-degenerate too, hence, for such values s functions $\rho_s \partial_{y_j} \Phi_{\xi_0}$, $j = 1, \ldots, n$, are linearly independent.

Now let us prove the statement regarding $\operatorname{Ran} D_s$. For this we remark that due to exponential estimates (2.12), for any $s \in (0, \kappa_0)$ and any $v \in C^{2+\alpha}(\mathbb{R}^n)$ we can perform integration by parts in the following formula

$$\int_{\mathbb{R}^n} \left(\Delta v(y) + 2\left(\rho_s(y)\nabla\rho_s^{-1}(y) \cdot \nabla v(y)\right)\right) \rho_s^{-1}(y)\Phi_{\xi_0}(y)\, dy$$

$$= \int_{\mathbb{R}^n} v(y) \left(\Delta\left(\rho_s^{-1}\Phi_{\xi_0}\right)(y) - 2\left(\nabla\rho_s^{-1}(y) \cdot \nabla\Phi_{\xi_0}(y)\right) - 2\Phi_{\xi_0}(y)\Delta\rho_s^{-1}(y)\right) dy$$

$$= \int_{\mathbb{R}^n} v(y) \left(\rho_s^{-1}(y)\Delta\Phi_{\xi_0}(y) - \Phi_{\xi_0}(y)\Delta\rho_s^{-1}(y)\right) dy.$$

With the help of this identity we easily see that for any $s \in (0, \kappa_0)$ and any $v \in C^{2+\alpha}(\mathbb{R}^n)$ it holds

$$\int_{\mathbb{R}^n} (D_s v)(y)\ \rho_s^{-1}(y)\partial_{y_j}\Phi_{\xi_0}(y)\, dy = 0 \quad \text{for all} \quad j = 1, \ldots, n.$$

Moreover, in complete analogy with our consideration of functions $\rho_s \partial_{y_j} \Phi_{\xi_0}$ (see the Gram matrix argument above) we can show that for all $s > 0$ small enough functions $\rho_s^{-1}\partial_{y_j}\Phi_{\xi_0}$, $j = 1, \ldots, n$, are linearly independent. $\Diamond$

Let us assume that the parameter s of function ρ_s satisfies the inequality $0 < s < \min(\kappa_0, s_0)$. (Note that this is the only restriction that we impose on s in our proof!) Then regarding Eq. (4.37), Lemma 4.7 and the Fredholm alternative imply that

$$\int_{\mathbb{R}^n} (\sigma_* \cdot \Psi + \psi(\cdot, x_1, \sigma_*))\rho_s\ \rho_s^{-1}\partial_{y_j}\Phi_{\xi_0}\, dy = 0, \quad j = 1, \ldots, n. \tag{4.39}$$

These equations already appeared in Section 2, when we transformed system (2.58). Using identities (2.57) and (2.62) obtained there, we rewrite system (4.39) as follows

$$H(\xi_0)\sigma_* = 0,$$

where $H(\xi_0)$ is the Jacobian matrix of system (1.3) at point ξ_0 (see definition (2.63)). Due to assumption (A3) this matrix is non-degenerate. Hence, $\sigma_* = 0$ and $v_* \in \operatorname{Ker} D_s$, i.e.

$$v_* = \sum_{j=1}^{n} C_j\ \rho_s \partial_{y_j}\Phi_{\xi_0},$$

where $C_j \in \mathbb{R}$ are some constants.

Step 4. Proof of the fact that $v_* = 0$. With the help of limit (4.23), below we show that $v_* = 0$. To this end, we again define a non-increasing smooth cut-off function $\chi : [0, \infty) \to \mathbb{R}$

such that $\chi(r) = 1$ for $0 \le r \le 1$ and $\chi(r) = 0$ for $r \ge 2$. Then for every $R \in (0, \infty)$ we define the function $\chi_R(y) := \chi(|y|^2/R^2)$ that satisfies the inequality

$$\|\chi_R\|_{C^{2+\alpha}(\mathbb{R}^n)} \le \text{const} \quad \text{for all} \quad R \ge 1.$$

Since $\hat{v}_k \rightharpoonup v_*$ in $W^{2,2}(\mathbb{R}^n)$, for every $R > 0$ we also have $\chi_R \hat{v}_k \rightharpoonup \chi_R v_*$ in $W^{2,2}(\mathbb{R}^n)$. Then the compact imbedding $W^{2,2}(\mathbb{R}^n) \hookrightarrow L^2(\text{supp}(\chi_R))$ implies

$$\|\chi_R \hat{v}_k - \chi_R v_*\|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{for} \quad k \to \infty. \tag{4.40}$$

On the other hand, because of (4.19) and (4.29), for every $R \ge 1$ it holds

$$\|\chi_R \hat{v}_k\|_{C^{2+\alpha}(\mathbb{R}^n)} \le \text{const}. \tag{4.41}$$

Hence, from (4.40) and (4.41) we easily get

$$\|\chi_R \hat{v}_k - \chi_R v_*\|_{C^{1+\alpha}(\text{supp}(\chi_R))} \to 0 \quad \text{for} \quad k \to \infty. \tag{4.42}$$

Indeed, suppose that (4.42) is not true. Then there exists $c > 0$ and a subsequence $\chi_R \hat{v}_{k_j}$ of $\chi_R \hat{v}_k$ such that

$$\|\chi_R \hat{v}_{k_j} - \chi_R v_*\|_{C^{1+\alpha}(\text{supp}(\chi_R))} \ge c \quad \text{for all} \quad j = 1, 2, \ldots. \tag{4.43}$$

Taking into account the compact imbedding $C^{2+\alpha}(\text{supp}(\chi_R)) \hookrightarrow C^{1+\alpha}(\text{supp}(\chi_R))$ and the estimate (4.41), we derive from the sequence $\chi_R \hat{v}_{k_j}$ a subsequence converging in $C^{1+\alpha}(\text{supp}(\chi_R))$ to a certain function ω_R such that $\|\omega_R - \chi_R v_*\|_{C^{1+\alpha}(\text{supp}(\chi_R))} \ge c$ (cf. (4.43)). But this contradicts to the limit (4.40). Hence, the limit (4.42) holds true.

In particular, it implies

$$\nabla_y \hat{v}_k(0) \to \nabla_y v_*(0) = \sum_{j=1}^{n} C_j \nabla_y \partial_{y_j} \Phi_{\xi_0}(0) \quad \text{for} \quad k \to \infty, \tag{4.44}$$

where we have used the fact that $\rho_s(0) = 1$ and $\nabla_y \rho_s(0) = 0$. On the other hand, direct calculation with the help of definition (4.26) and limit (4.23) yields

$$\nabla_y \hat{v}_k(0) = \nabla_y \left(u_k \left(T^{-1}_{\varepsilon_k, m}(y) \right) \rho_s(y) \right) \Big|_{y=0} = \varepsilon_k Q(x_{\varepsilon_k, m})^{-1} \nabla_x u_k(x_{\varepsilon_k, m}) \to 0,$$

where we took into account that $Q(\xi_0)$ is a non-degenerate matrix and that $x_{\varepsilon,m} \to \xi_0$ for $\varepsilon \to 0$. Now comparing the latter limit with formula (4.44) we obtain $\nabla_y v_*(0) = 0$. Therefore considering the right-hand part of (4.44) as an n-dimensional linear system with respect to C_j, and taking into account that the $(n \times n)$-matrix $\partial_{y_j} \partial_{y_k} \Phi_{\xi_0}(0)$ is non-degenerate (see (2.51) and (1.11)) we come to the conclusion that $C_1 = \ldots = C_n = 0$, and hence $v_* = 0$.

The latter result has an important consequence: If we substitute $v_* = 0$ into limit (4.42) and apply definition (4.26), we easily get that for every fixed $R \ge 1$ it holds

$$\left\| \chi_R \left(u_k \circ T^{-1}_{\varepsilon_k, m} \right) \right\|_{C^{1+\alpha}(\text{supp}(\chi_R))} \le \text{const} \, \|\chi_R \hat{v}_k\|_{C^{1+\alpha}(\text{supp}(\chi_R))} \to 0 \quad \text{for} \quad k \to \infty. \tag{4.45}$$

This limit plays the crucial role in the next step.

Step 5. Construction of contradiction. Now we have all necessary ingredients to demonstrate that assumptions (4.19) and (4.20) do result in limit (4.25). In particular, above we have proved that $\sigma_* = 0$. Substituting this into formula (4.21) we obtain

$$\left\| E_{\varepsilon_k} u_k - \partial_u f(\cdot, \mathcal{W}_{\varepsilon_k,m}, \varepsilon_k) u_k \right\|_{\alpha,\varepsilon_k;\Omega} \to 0 \quad \text{for} \quad k \to \infty.$$

The latter limit can be further reduced to limit (4.25) if we show that the following two relations hold true

$$\left\| \varepsilon_k^2 \sum_{i=1}^{n} b_i(\cdot)\, \partial_{x_i} u_k \right\|_{\alpha,\varepsilon_k;\Omega} \to 0 \quad \text{for} \quad k \to \infty, \tag{4.46}$$

$$\left\| \Big(\partial_u f(\cdot, \mathcal{W}_{\varepsilon_k,m}, \varepsilon_k) - \partial_u f(\cdot, 0, 0) \Big) u_k \right\|_{\alpha,\varepsilon_k;\Omega} \to 0 \quad \text{for} \quad k \to \infty. \tag{4.47}$$

Limit (4.46) is trivial. Indeed, it follows from the estimate

$$\left\| \varepsilon_k^2 \sum_{i=1}^{n} b_i(\cdot)\, \partial_{x_i} u_k \right\|_{\alpha,\varepsilon_k;\Omega} \le \text{const}\ \varepsilon_k \max_i \left\| \varepsilon_k \partial_{x_i} u_k \right\|_{\alpha,\varepsilon_k;\Omega} \le \text{const}\ \varepsilon_k \left\| u_k \right\|_{2+\alpha,\varepsilon_k;\Omega},$$

because of assumption (4.19) and inequalities (5.8), (5.9) and (5.11) from Appendix.

To justify limit (4.47), we write the triangle inequality

$$\begin{aligned} &\left\| \Big(\partial_u f(\cdot, \mathcal{W}_{\varepsilon_k,m}, \varepsilon_k) - \partial_u f(\cdot, 0, 0) \Big) u_k \right\|_{\alpha,\varepsilon_k;\Omega} \\ &\le \left\| \Big(\partial_u f(\cdot, \mathcal{W}_{\varepsilon_k,m}, \varepsilon_k) - \partial_u f\left(\cdot, \Phi_{\xi_0} \circ T_{\varepsilon_k,m}, 0\right) \Big) u_k \right\|_{\alpha,\varepsilon_k;\Omega} \\ &+ \left\| \Big(\partial_u f\left(\cdot, \Phi_{\xi_0} \circ T_{\varepsilon_k,m}, 0\right) - \partial_u f(x, 0, 0) \Big) u_k \right\|_{\alpha,\varepsilon_k;\Omega}. \end{aligned} \tag{4.48}$$

Since the structure of formal asymptotics $\mathcal{W}_{\varepsilon,m}$ (see Theorem 2.1) implies that

$$\left\| \mathcal{W}_{\varepsilon,m} - \Phi_{\xi_0} \circ T_{\varepsilon,m} \right\|_{\alpha,\varepsilon;\Omega} = O(\varepsilon) \quad \text{for} \quad \varepsilon \to 0,$$

we easily get the estimate

$$\left\| \partial_u f(\cdot, \mathcal{W}_{\varepsilon,m}, \varepsilon) - \partial_u f\left(\cdot, \Phi_{\xi_0} \circ T_{\varepsilon,m}, 0\right) \right\|_{\alpha,\varepsilon;\Omega} = O(\varepsilon) \quad \text{for} \quad \varepsilon \to 0.$$

Hence, applying inequalities (5.9) and (5.11) and taking into account that $\|u_k\|_{2+\alpha,\varepsilon_k;\Omega} \le 1$, we see that the first term in the right-hand part of formula (4.48) vanishes for $k \to \infty$.

For the last term in the right-hand part of formula (4.48), we write the inequality

$$\begin{aligned} &\left\| \Big(\partial_u f\left(\cdot, \Phi_{\xi_0} \circ T_{\varepsilon_k,m}, 0\right) - \partial_u f(\cdot, 0, 0) \Big) u_k \right\|_{\alpha,\varepsilon_k;\Omega} \\ &= \left\| \int_0^1 \partial_u^2 f\left(\cdot, t\Phi_{\xi_0} \circ T_{\varepsilon_k,m}, 0\right) dt\ \left(\Phi_{\xi_0} \circ T_{\varepsilon_k,m}\right) u_k \right\|_{\alpha,\varepsilon_k;\Omega} \\ &\le \text{const}\ \left\| \left(\Phi_{\xi_0} \circ T_{\varepsilon_k,m}\right) u_k \right\|_{\alpha,\varepsilon_k;\Omega} \le \text{const}\ \left\| \Phi_{\xi_0} \left(u_k \circ T_{\varepsilon_k,m}^{-1} \right) \right\|_{C^\alpha(T_{\varepsilon_k,m}(\overline{\Omega}))}, \end{aligned}$$

where the norm $\|\cdot\|_{\alpha,\varepsilon;\Omega}$ was estimated by $\|\cdot\|_{C^\alpha(T_{\varepsilon_k,m}(\overline{\Omega}))}$ according to Remark 4.4. Now employing the notation of the cut-off function χ_R (see above), we get

$$\left\|\Phi_{\xi_0}\left(u_k\circ T^{-1}_{\varepsilon_k,m}\right)\right\|_{C^\alpha(T_{\varepsilon_k,m}(\overline{\Omega}))}\le\|\Phi_{\xi_0}\|_{C^\alpha(T_{\varepsilon_k,m}(\overline{\Omega}))}\left\|\chi_R\left(u_k\circ T^{-1}_{\varepsilon_k,m}\right)\right\|_{C^\alpha(T_{\varepsilon_k,m}(\overline{\Omega}))}$$

$$+\|(1-\chi_R)\Phi_{\xi_0}\|_{C^\alpha(T_{\varepsilon_k,m}(\overline{\Omega}))}\left\|u_k\circ T^{-1}_{\varepsilon_k,m}\right\|_{C^\alpha(T_{\varepsilon_k,m}(\overline{\Omega}))}$$

$$\le\|\Phi_{\xi_0}\|_{C^\alpha(\mathbb{R}^n)}\left\|\chi_R\left(u_k\circ T^{-1}_{\varepsilon_k,m}\right)\right\|_{C^\alpha(\mathrm{supp}(\chi_R))}+\|(1-\chi_R)\Phi_{\xi_0}\|_{C^\alpha(\mathrm{supp}(1-\chi_R))}\|u_k\|_{\alpha,\varepsilon_k;\Omega}\,,\quad(4.49)$$

The sum in the right-hand part of (4.49) tends to zero for $k\to\infty$ due to the following argument. Because of the exponential decay of Φ_{ξ_0} (see Remark 1.3), for arbitrarily small $\gamma>0$ we can first take R sufficiently large such that it holds

$$\|(1-\chi_R)\Phi_{\xi_0}\|_{C^\alpha(\mathrm{supp}(1-\chi_R))}\|u_k\|_{\alpha,\varepsilon_k;\Omega}\le\gamma\quad\text{for all}\quad k=1,2,\dots.$$

Then fixing this R and applying relation (4.45), we can choose sufficiently large k to obtain

$$\|\Phi_{\xi_0}\|_{C^\alpha(\mathbb{R}^n)}\left\|\chi_R\left(u_k\circ T^{-1}_{\varepsilon_k,m}\right)\right\|_{C^\alpha(\mathrm{supp}(\chi_R))}\le\gamma.$$

Thus we have justified limit (4.47).

Recall that obtained limits (4.46) and (4.47) guarantee that another limit (4.25) holds true. Therefore we can apply Theorem 5.2 from Appendix to relations (4.22) and (4.25). As a result we get $\|u_k\|_{2+\alpha,\varepsilon_k;\Omega}\to 0$ and this together with another limit $\sigma_k\to 0$ constitutes the necessary contradiction. Now, Lemma 3.2 provides us with the required estimate for the inverse operator $F'_\varepsilon(0,0)^{-1}$ and the claimed assertion follows from our generalized Implicit Function Theorem. ♢

Proof of Theorem 4.1: Translating the assertion of Theorem 4.6 into original settings we obtain the solution to problem (1.1)

$$u_\varepsilon=\varepsilon^2 v_\varepsilon+\mathcal{U}_{\varepsilon,m,\sigma_\varepsilon},$$

where $\|(v_\varepsilon,\sigma_\varepsilon)\|_{U_\varepsilon}=\|v_\varepsilon\|_{2+\alpha,\varepsilon;\Omega}+|\sigma_\varepsilon|=O(\varepsilon^{m-1})$ for $\varepsilon\to 0$ (see estimate (4.17)). Then recalling that $\mathcal{U}_{\varepsilon,m,0}=\mathcal{W}_{\varepsilon,m}$ and taking into account inequality (4.11) we derive the estimate

$$\|u_\varepsilon-\mathcal{W}_{\varepsilon,m}\|_{2+\alpha,\varepsilon;\Omega}\le\varepsilon^2\|v_\varepsilon\|_{2+\alpha,\varepsilon;\Omega}+\|\mathcal{U}_{\varepsilon,m,\sigma_\varepsilon}-\mathcal{U}_{\varepsilon,m,0}\|_{2+\alpha,\varepsilon;\Omega}=O(\varepsilon^{m-1}).$$

Note that the accuracy of difference $\mathcal{U}_{\varepsilon,m,\sigma_\varepsilon}-\mathcal{U}_{\varepsilon,m,0}$ is dominating in the latter expression. Now since $m\ge 2$, direct calculation with the help of relation (4.2) yields

$$\|u_\varepsilon-\mathcal{W}_{\varepsilon,m-2}\|_{2+\alpha,\varepsilon;\Omega}\le\|u_\varepsilon-\mathcal{W}_{\varepsilon,m}\|_{2+\alpha,\varepsilon;\Omega}+\|\mathcal{W}_{\varepsilon,m}-\mathcal{W}_{\varepsilon,m-2}\|_{2+\alpha,\varepsilon;\Omega}=O(\varepsilon^{m-1}),$$

and this after reindexing $m'=m-2$ gives the claimed result (4.1).

The second assertion of theorem is trivial, since for every $\varepsilon\in(0,\infty)$ and every $u\in C^{2+\alpha}(\overline{\Omega})$ we have

$$\left\|\left(\varepsilon^{-2}(u-\mathcal{W}_{\varepsilon,m}),0\right)\right\|_{U_\varepsilon}=\varepsilon^{-2}\|u-\mathcal{W}_{\varepsilon,m}\|_{2+\alpha,\varepsilon;\Omega}.$$

That ends the proof. ♢

5 Appendix: Schauder type estimates in Hölder spaces with ε-dependent norms

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and ε a scalar positive parameter. We consider the singularly perturbed linear elliptic operator

$$\mathcal{L}_\varepsilon u := \varepsilon^2 \sum_{i,j=1}^{n} \partial_{x_i}\left(a_{ij}(x,\varepsilon)\partial_{x_j} u\right) + c(x,\varepsilon)u \tag{5.1}$$

defined in Ω, which is equipped with the natural boundary operator

$$\mathcal{N}_\varepsilon u := \varepsilon \sum_{i,j=1}^{n} a_{ij}(x,\varepsilon)\nu_i(x)\partial_{x_j} u \tag{5.2}$$

defined on $\partial\Omega$, where ν_i are the components of the unit outer normal at $\partial\Omega$. Introducing weighted ε-dependent norms in Hölder spaces, we modify some well-known results of the Schauder theory for the composite operator $(\mathcal{L}_\varepsilon, \mathcal{N}_\varepsilon)$ in a way to produce the upper bound estimate for inverse operator $(\mathcal{L}_\varepsilon, \mathcal{N}_\varepsilon)^{-1}$, which is uniform with respect to $\varepsilon \to 0$.

For this we recall that for any $\lambda \in (0,1)$ function u is called Hölder continuous with exponent λ in Ω if the seminorm

$$[u]_{\lambda;\Omega} := \sup_{\substack{x,y\in\Omega,\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\lambda} \tag{5.3}$$

is finite. Respectively, for any integer $k \geq 0$ we define the Hölder space $C^{k+\lambda}(\overline{\Omega})$ as a subspace of $C^k(\overline{\Omega})$ consisting of all functions u with the finite norm

$$\|u\|_{k+\lambda;\Omega} := \|u\|_{k;\Omega} + \sup_{|\mu|=k} [D^\mu u]_{\lambda;\Omega}\,, \tag{5.4}$$

where

$$\|u\|_{k;\Omega} := \sum_{j=0}^{k} \sup_{|\mu|=j} \sup_{\Omega} |D^\mu u|$$

and a standard notation for multi-index μ was adopted.

If domain Ω belongs to a class $C^{k+\lambda}$ with $k \geq 1$ (see corresponding definition in [13, Sec. 6.3]), then one can naturally define a Banach space $C^{k+\lambda}(\partial\Omega)$ with the norm

$$\|u\|_{k;\partial\Omega} := \inf_{U} \|U\|_{k;\Omega}, \tag{5.5}$$

where U denotes a $C^{k+\lambda}(\overline{\Omega})$-extention of function u on $\overline{\Omega}$ and the infimum is taken over all possible extensions U. Since the set of such extensions U is nonempty (see Lemma 6.38 in [13]), definition (5.5) is always correct.

To eliminate the singularity occurring for $\varepsilon \to 0$ in operators $\mathcal{L}_\varepsilon$ and $\mathcal{N}_\varepsilon$, one might employ a simple coordinate transformation $T_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ defined for all $\varepsilon \in (0,\infty)$ with the formula

$T_\varepsilon x := x/\varepsilon$. Indeed, in the new coordinates these differential operators have regular coefficients and read as follows

$$\begin{aligned} \tilde{\mathcal{L}}_\varepsilon v &:= \sum_{i,j=1}^n \partial_{y_i}\left(a_{ij}(\varepsilon y,\varepsilon)\partial_{y_j} v\right) + c(\varepsilon y,\varepsilon)v, \\ \tilde{\mathcal{N}}_\varepsilon v &:= \sum_{i=1}^n a_{ij}(\varepsilon y,\varepsilon)\nu_i(\varepsilon y)\partial_{y_j} v. \end{aligned}$$

However, the former acts now in the ε-dependent domain $\Omega/\varepsilon := T_\varepsilon(\Omega)$, whereas the latter acts on the ε-dependent surface $\partial\Omega/\varepsilon := T_\varepsilon(\partial\Omega)$. Taking this into account we define the new ε-dependent norms

$$\|u\|_{k+\lambda,\varepsilon;\Omega} := \|u \circ T_\varepsilon^{-1}\|_{k+\alpha;\Omega/\varepsilon} = \sum_{j=0}^k \varepsilon^j \sup_{|\mu|=j} \sup_{\Omega} |D^\mu u| + \varepsilon^{k+\lambda} \sup_{|\mu|=k} [D^\mu u]_{\lambda;\Omega} \tag{5.6}$$

and

$$\|u\|_{k+\lambda,\varepsilon;\partial\Omega} := \|u \circ T_\varepsilon^{-1}\|_{k+\lambda;\partial\Omega/\varepsilon}, \tag{5.7}$$

in Hölder spaces $C^{k+\lambda}(\overline{\Omega})$ and $C^{k+\lambda}(\partial\Omega)$, respectively. Such norms turn out to be a natural setting for analysis of singularly perturbed composite operator $(\mathcal{L}_\varepsilon, \mathcal{N}_\varepsilon)$. In particular, they satisfy a series of inequalities with a simple explicit dependence on parameter ε. We present these inequalities in the following lemma.

Lemma 5.1 *Let $k \geq 0$ be an integer and $\lambda \in (0,1)$. Then for any $\varepsilon \in (0,\infty)$ it holds:*

$$\min(1,\varepsilon^{k+\lambda})\|u\|_{k+\lambda;\Omega} \leq \|u\|_{k+\lambda,\varepsilon;\Omega} \leq \max(1,\varepsilon^{k+\lambda})\|u\|_{k+\lambda;\Omega} \quad \textit{for all} \quad u \in C^{k+\lambda}(\overline{\Omega}), \tag{5.8}$$

$$\|uv\|_{\lambda,\varepsilon;\Omega} \leq \|u\|_{\lambda,\varepsilon;\Omega}\, \|v\|_{\lambda,\varepsilon;\Omega} \quad \textit{for all} \quad u,v \in C^{\lambda}(\overline{\Omega}). \tag{5.9}$$

Moreover, if $k \geq 1$ then it holds:

$$\min(1,\varepsilon^{k+\lambda})\|u\|_{k+\lambda;\partial\Omega} \leq \|u\|_{k+\lambda,\varepsilon;\partial\Omega} \leq \max(1,\varepsilon^{k+\lambda})\|u\|_{k+\lambda;\partial\Omega} \quad \textit{for all} \quad u \in C^{k+\lambda}(\partial\Omega), \tag{5.10}$$

$$\|u\|_{k-1+\lambda,\varepsilon;\Omega} \leq C(n,k,\lambda)\|u\|_{k+\lambda,\varepsilon;\Omega} \quad \textit{for all} \quad u \in C^{k+\lambda}(\overline{\Omega}), \tag{5.11}$$

where $C(n,k,\lambda)$ is a constant independent of ε and Ω.

Proof: Inequalities (5.8)-(5.10) follow directly from definitions (5.5)-(5.7).
To verify the inequality (5.11) we first write the estimate

$$\varepsilon^{k-1+\lambda} \sup_{|\mu|=k-1} [D^\mu u]_{\lambda;\Omega} \leq \varepsilon^{k-1+\lambda} \sup_{|\mu|=k-1} \left(\sup_{\Omega} \left(2|D^\mu u|\right)^{1-\lambda} \sup_{\substack{x,y\in\Omega,\\ x\neq y}} \frac{|D^\mu u(x) - D^\mu u(y)|^\lambda}{|x-y|^\lambda} \right)$$

$$\leq \sup_{|\mu|=k-1} \left(\sup_{\Omega} \left(2\varepsilon^{k-1}|D^\mu u|\right)^{1-\lambda} \right) \sup_{|\mu|=k} \left(\sup_{\Omega} \left(n\varepsilon^k |D^\mu u|\right)^\lambda \right) \leq C_*(n,k,\lambda)\|u\|_{k+\lambda,\varepsilon;\Omega},$$

where $C_*(n,k,\lambda) > 0$ is a constant independent of ε and Ω. Denoting $C(n,k,\lambda) = 1+C_*(n,k,\lambda)$ and taking into account definition (5.6) we obtain the claimed inequality (5.11). $\diamondsuit$

Now we are ready to formulate and prove the main statement concerning the upper bound estimate of inverse operator $(\mathcal{L}_\varepsilon, \mathcal{N}_\varepsilon)^{-1}$.

Theorem 5.2 *Let Ω be a bounded domain in $\mathbb{R}^n$ of class $C^{2+\alpha}$ with $\alpha \in (0,1)$. Suppose that the following assumptions hold:*

(i) For every $\varepsilon > 0$ it holds $a_{ij}(\cdot,\varepsilon) \in C^{1+\alpha}(\overline{\Omega})$ and $c(\cdot,\varepsilon) \in C^{\alpha}(\overline{\Omega})$. Furthermore, there exists a constant $M > 0$ such that

$$\|a_{ij}(\cdot,\varepsilon)\|_{1+\alpha;\Omega}\ ,\ \|c(\cdot,\varepsilon)\|_{\alpha;\Omega} \le M \quad \text{for all} \quad \varepsilon \in (0,\infty). \tag{5.12}$$

(ii) There exist constants $\kappa > 0$ and $c_0 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,\varepsilon)\xi_i\xi_j \ge \kappa|\xi|^2 \quad \text{for all} \quad (x,\varepsilon,\xi) \in \Omega \times (0,\infty) \times \mathbb{R}^n, \tag{5.13}$$

and

$$c(x,\varepsilon) \le -c_0 \quad \text{for all} \quad (x,\varepsilon) \in \Omega \times (0,\infty). \tag{5.14}$$

Then there exist $\varepsilon_0 > 0$ and $C_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and all $u \in C^{2+\alpha}(\overline{\Omega})$ it holds

$$\|u\|_{2+\alpha,\varepsilon;\Omega} \le C_0(\|\mathcal{L}_\varepsilon u\|_{\alpha,\varepsilon;\Omega} + \|\mathcal{N}_\varepsilon u\|_{1+\alpha,\varepsilon;\partial\Omega}).$$

Proof: We base our proof on Lemma 3.2. First we remark that inequality (5.8) implies the equivalence of norms $\|\cdot\|_{k+\alpha,\varepsilon;\Omega}$ and $\|\cdot\|_{k+\alpha;\Omega}$ for any $k \ge 0$. Similarly, from inequality (5.10) follows the equivalence of norms $\|\cdot\|_{k+\lambda,\varepsilon;\partial\Omega}$ and $\|\cdot\|_{k+\lambda;\partial\Omega}$ with $k \ge 1$. Hence, taking into account classical results of the theory of linear elliptic operators (see Theorem 3.2 in [18]), we easily see that the composite operator

$$(\mathcal{L}_\varepsilon, \mathcal{N}_\varepsilon)\ :\ \left(C^{2+\alpha}(\overline{\Omega}), \|\cdot\|_{2+\alpha,\varepsilon;\Omega}\right) \to \left(C^{\alpha}(\overline{\Omega}), \|\cdot\|_{\alpha,\varepsilon;\Omega}\right) \times \left(C^{1+\alpha}(\partial\Omega), \|\cdot\|_{1+\alpha,\varepsilon;\partial\Omega}\right)$$

is a Fredholm operator of index zero.

Let $\varepsilon_k \in (0,\infty)$ and $u_k \in C^{2+\alpha}(\overline{\Omega})$ be sequences with

$$\|u_k\|_{2+\alpha,\varepsilon_k;\Omega} = 1 \tag{5.15}$$

and

$$\varepsilon_k + \|\mathcal{L}_{\varepsilon_k} u_k\|_{\alpha,\varepsilon_k;\Omega} + \|\mathcal{N}_{\varepsilon_k} u_k\|_{1+\alpha,\varepsilon_k;\partial\Omega} \to 0 \quad \text{for} \quad k \to \infty. \tag{5.16}$$

We are going to demonstrate that these two assumptions actually imply

$$\|u_k\|_{2+\alpha,\varepsilon_k;\Omega} \to 0 \quad \text{for} \quad k \to \infty \tag{5.17}$$

what is the necessary contradiction.

First we will show that assumption (5.16) together with properties (5.12)–(5.14) results in the uniform estimate

$$\|u_k\|_{0;\Omega} \to 0 \quad \text{for} \quad k \to \infty. \tag{5.18}$$

Indeed, for each u_k we can construct two functions of the following form

$$
\begin{aligned}
u_k^{\pm}(x) := u_k(x) \quad &\pm \quad K\left(\|\mathcal{L}_{\varepsilon_k} u_k\|_{\alpha,\varepsilon_k;\Omega} + \|\mathcal{N}_{\varepsilon_k} u_k\|_{1+\alpha,\varepsilon_k;\partial\Omega}\right) \\
&\pm \quad \|\mathcal{N}_{\varepsilon_k} u_k\|_{1+\alpha,\varepsilon_k;\partial\Omega}\, \chi(x) \exp\left(-\frac{1}{\varepsilon\kappa} \operatorname{dist}(x,\partial\Omega)\right),
\end{aligned}
$$

where K is a positive constant to be chosen later, and $\chi : [0,\infty) \to \mathbb{R}$ is a smooth cut-off function such that

$$
\chi(r) = 1 \quad \text{for} \quad 0 \le r \le \delta \quad \text{and} \quad \chi(r) = 0 \quad \text{for} \quad r \ge 2\delta,
$$

with $\delta > 0$ being a fixed number, small enough to guarantee that for every $x \in \Omega$ satisfying $\operatorname{dist}(x,\partial\Omega) < 2\delta$ there exists the only point $\zeta \in \partial\Omega$ such that $\operatorname{dist}(x,\zeta) = \operatorname{dist}(x,\partial\Omega)$. Then simple calculation and estimate (5.13) yield

$$
\pm\mathcal{N}_{\varepsilon_k} u_k^{\pm}(x) = \pm\mathcal{N}_{\varepsilon_k} u_k(x) + \|\mathcal{N}_{\varepsilon_k} u_k\|_{1+\alpha,\varepsilon_k;\partial\Omega} \frac{1}{\kappa} \sum_{i,j=1}^{n} a_{ij}(x,\varepsilon_k)\nu_i(x)\nu_j(x) \ge 0 \quad \text{for all} \quad x \in \partial\Omega.
$$

On the other hand, using assumption (5.12) we easily check that

$$
\left\| \varepsilon^2 \sum_{i,j=1}^{n} \partial_{x_i}\left(a_{ij}(x,\varepsilon)\partial_{x_j}\left(\chi(x)\exp\left(-\frac{1}{\varepsilon\kappa}\operatorname{dist}(x,\partial\Omega)\right)\right)\right)\right\|_{0;\Omega} \le \text{const} \quad \text{for all} \quad \varepsilon \in (0,1).
$$

Hence, assumption (5.14) allows us to choose $K > 0$ such that for all k with $\varepsilon_k \in (0,1)$ it holds

$$
\mathcal{L}_{\varepsilon_k} u_k^{+}(x) \le 0 \quad \text{and} \quad \mathcal{L}_{\varepsilon_k} u_k^{-}(x) \ge 0 \quad \text{for all} \quad x \in \Omega.
$$

Now Strong Maximum Principle for linear elliptic operators (see [13, Theorem 3.5]) implies

$$
u_k^{+}(x) \ge 0 \quad \text{and} \quad u_k^{-}(x) \le 0 \quad \text{for all} \quad x \in \Omega,
$$

and this gives (5.18). The latter limit can be easily transformed into a stronger one. Indeed, since the following inequality holds

$$
\varepsilon_k^{\alpha}[u_k]_{\alpha;\Omega} \le \varepsilon_k^{\alpha} \sup_{\Omega}(2|u_k|)^{1-\alpha} \sup_{\substack{x,y\in\Omega,\\ x\neq y}} \frac{|u_k(x)-u_k(y)|^{\alpha}}{|x-y|^{\alpha}} \le (2\|u_k\|_{0;\Omega})^{1-\alpha}\left(n\varepsilon_k \sup_{|\mu|=1}\sup_{\Omega}|D^{\mu}u_k|\right)^{\alpha},
$$

assumption (5.15) and limit (5.18) guarantee that

$$
\|u_k\|_{\alpha,\varepsilon_k;\Omega} \to 0 \quad \text{for} \quad k \to \infty. \tag{5.19}
$$

To proceed further we remark that for every $\varepsilon \in (0,1)$ estimates (5.8) and (5.10) imply

$$
\begin{aligned}
&\varepsilon^{\alpha}\|u\|_{\alpha;\Omega} \le \|u\|_{\alpha,\varepsilon;\Omega} \le \|u\|_{\alpha;\Omega} \quad \text{for all} \quad u \in C^{\alpha}(\overline{\Omega}), \\
&\varepsilon^{1+\alpha}\|u\|_{1+\alpha;\partial\Omega} \le \|u\|_{1+\alpha,\varepsilon;\partial\Omega} \le \|u\|_{1+\alpha;\partial\Omega} \quad \text{for all} \quad u \in C^{1+\alpha}(\partial\Omega),
\end{aligned} \tag{5.20}
$$

respectively. Hence, assuming without loss of generality that $\varepsilon_k < 1$, and applying inequality (5.9), we get the limit

$$\varepsilon_k^{2+\alpha} \left\| \sum_{i,j=1}^{n} a_{ij}(\cdot,\varepsilon_k)\partial_{x_i}\partial_{x_j} u_k \right\|_{\alpha;\Omega} \leq \left\| \varepsilon_k^2 \sum_{i,j=1}^{n} a_{ij}(\cdot,\varepsilon_k)\partial_{x_i}\partial_{x_j} u_k \right\|_{\alpha,\varepsilon_k;\Omega} \leq \|\mathcal{L}_{\varepsilon_k} u_k\|_{\alpha,\varepsilon_k;\Omega}$$

$$+ \left\| \varepsilon_k^2 \sum_{i,j=1}^{n} \partial_{x_i} a_{ij}(\cdot,\varepsilon_k)\partial_{x_j} u_k \right\|_{\alpha,\varepsilon_k;\Omega} + \|c(\cdot,\varepsilon_k)\|_{\alpha;\Omega} \|u_k\|_{\alpha,\varepsilon_k;\Omega} \to 0 \quad \text{for} \quad k \to \infty, \quad (5.21)$$

where all the terms in the right hand part of (5.21) vanish because of assumptions (5.12), (5.15) and limits (5.16), (5.19).

According to classical Schauder estimates for linear elliptic operators (see for example Theorem 6.30 in [13]), there exists a constant $C_1 = C_1(n, \alpha, \kappa, M, \Omega) > 0$ which is independent of ε such that for every $u \in C^{2+\alpha}(\overline{\Omega})$ it holds

$$\|u\|_{2+\alpha;\Omega} \leq C_1 \left(\left\| \sum_{i,j=1}^{n} a_{ij}(\cdot,\varepsilon)\partial_{x_i}\partial_{x_j} u \right\|_{\alpha;\Omega} + \left\| \sum_{i,j=1}^{n} a_{ij}(\cdot,\varepsilon)\nu_i(\cdot)\partial_{x_j} u \right\|_{1+\alpha;\partial\Omega} + \|u\|_{0;\Omega} \right).$$

Multiplying both sides of this inequality with $\varepsilon^{2+\alpha}$ we get

$$\begin{aligned} \varepsilon^{2+\alpha}\|u\|_{2+\alpha;\Omega} \quad \leq \quad & C_1 \left(\varepsilon^{2+\alpha} \left\| \sum_{i,j=1}^{n} a_{ij}(\cdot,\varepsilon)\partial_{x_i}\partial_{x_j} u \right\|_{\alpha;\Omega} \right. \\ + \quad & \left. \varepsilon^{1+\alpha} \left\| \varepsilon \sum_{i,j=1}^{n} a_{ij}(\cdot,\varepsilon)\nu_i(\cdot)\partial_{x_j} u \right\|_{1+\alpha;\partial\Omega} + \varepsilon^{2+\alpha}\|u\|_{0;\Omega} \right). \end{aligned} \quad (5.22)$$

Hence, taking into account previously obtained estimate (5.21), assumptions (5.15), (5.16) and inequality (5.20) we obtain from (5.22) that

$$\varepsilon_k^{2+\alpha}\|u_k\|_{2+\alpha;\Omega} \to 0 \quad \text{for} \quad k \to \infty. \quad (5.23)$$

Now, the last step is to derive from limits (5.19) and (5.23) the necessary contradiction (5.17). For this we employ the interpolation inequality (see Lemma 6.3.1 in [16])

$$\varepsilon^s \|u\|_{s;\Omega} \leq C_2 \left(\varepsilon^{2+\alpha}\|u\|_{2+\alpha;\Omega} + (\varepsilon^s + 1)\|u\|_{0;\Omega} \right)$$

that holds true for all $0 \leq s \leq 2+\alpha$ and $\varepsilon \in (0, \infty)$ with the constant $C_2 = C_2(n, \alpha, s, \Omega)$ which is independent of ε. Indeed, due to limits (5.19) and (5.23) we easily get

$$\varepsilon_k\|u_k\|_{1;\Omega} \to 0 \quad \text{and} \quad \varepsilon_k^2\|u_k\|_{2;\Omega} \to 0 \quad \text{for} \quad k \to \infty.$$

Thus, all terms in the definition of norm $\|u_k\|_{2+\alpha,\varepsilon_k;\Omega}$ vanish when $k \to \infty$ and limit (5.17) does hold. This means that Lemma 3.2 works and this ends the proof. $\Diamond$

Remark 5.3 *The prove of Theorem 5.2 can be easily modified to cover the case of Dirichlet boundary conditions. In result we obtain the following statement.*

Suppose that all assumptions of Theorem 5.2 are fulfilled. Then there exist $\varepsilon_0 > 0$ *and* $C_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *and all* $u \in C^{2+\alpha}(\overline{\Omega})$ *it holds*

$$\|u\|_{2+\alpha,\varepsilon;\Omega} \leq C_0(\|\mathcal{L}_\varepsilon u\|_{\alpha,\varepsilon;\Omega} + \|u\|_{2+\alpha,\varepsilon;\partial\Omega}).$$

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