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Optimal elliptic regularity at the crossing of a material interface and a Neumann boundary edge

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Abstract

We investigate optimal elliptic regularity of anisotropic div–grad operators in three dimensions at the crossing of a material interface and an edge of the spatial domain on the Neumann boundary part within the scale of Sobolev spaces.

1 Formulation of the problem

In this paper we prove optimal regularity of elliptic div–grad operators

$$-\nabla\cdot\mu\nabla : \; W^{1,p}_\Gamma(\Omega) \to W^{-1,p}_\Gamma(\Omega) \tag{1.1}$$

for the spatio–material model constellations of Figure 1 and Figures 2,3 in three–dimensional real space. The elliptic coefficient function μ on Ω takes its values in the set of real, symmetric, positive definite 3×3 matrices and $W^{1,p}_\Gamma(\Omega)$ is the Sobolev space with a homogeneous Dirichlet condition on $\partial\Omega \setminus \Gamma$, see Section 2 for details. The crucial point is that two phenomena, each possibly causing singularities in the solution of the elliptic equation do appear simultaneously in this setting: a geometric edge of the spatial domain on the Neumann boundary part and a material heterogeneity. It should be noted that we allow concave edges. More precisely, we get the following theorems.

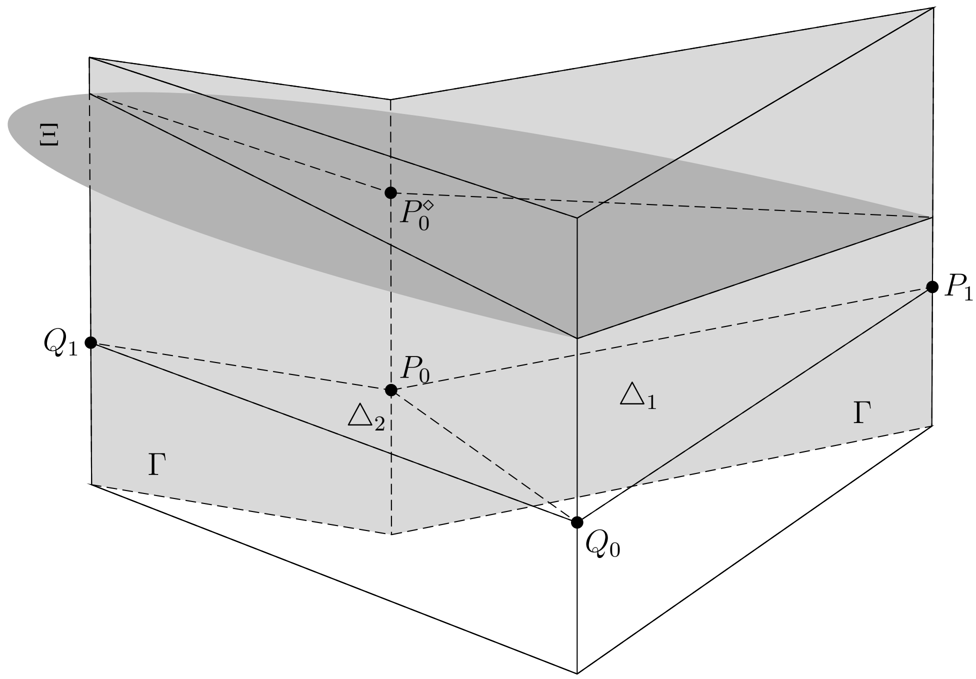

Figure 1: Right prism $\Omega = (\triangle_1 \cup \triangle_2 \cup \overline{P_0Q_0}) \times \{h_- < z < h_+\}$ of Theorem 1.1 with Neumann boundary part $\Gamma = (\overline{Q_1P_0} \cup \overline{P_1P_0} \cup \{P_0\}) \times \{h_- < z < h_+\}$ (fair–grey) and material interface Ξ (dark–grey). $P_0^\diamond$ is the point where the line $\{P_0\}\times\{h_- < z < h_+\}$ and Ξ intersect.

Theorem 1.1. *(Constellation of Figure 1.) Let $\triangle_1$, $\triangle_2$ be two open triangles in the $z = 0$ plane of $\mathbb{R}^3$ which share one edge with vertices P_0 and Q_0. We define the open right prism*

$$\Omega \overset{\text{def}}{=} (\triangle_1 \cup \triangle_2 \cup \overline{P_0Q_0}) \times \{h_- < z < h_+\}, \quad h_- < 0 < h_+.$$

If P_1, Q_1 are the other vertices of $\triangle_1$ and $\triangle_2$ which share an edge with P_0, then the Neumann boundary part shall be

$$\Gamma \overset{\text{def}}{=} (\overline{Q_1P_0} \cup \overline{P_1P_0} \cup \{P_0\}) \times \{h_- < z < h_+\}.$$

Furthermore Ξ is a plane in $\mathbb{R}^3$ which intersects Ω, but avoids its ground and upper plate. We assume that the elliptic coefficient function μ takes its values in the set of real, symmetric, positive definite 3×3 matrices and is constant on both components of $\Omega \setminus \Xi$. Then there is a $p > 3$ such that (1.1) *is a topological isomorphism.*

Theorem 1.2. *(Constellation of Figures 2, 3.) Let $\triangle_1$, $\triangle_2$, be two open triangles in the $z = 0$ plane of $\mathbb{R}^3$ with vertices P_0, P_1, P_2 and P_0, Q_1, Q_2, respectively. The triangles $\triangle_1$ and $\triangle_2$ share the vertex P_0, and the vertex P_1 of $\triangle_1$ is either on the open edge $\overline{P_0Q_1}$ of $\triangle_2$ or $P_1 = Q_1$. We define the open right prism*

$$\Omega \stackrel{\text{def}}{=} (\triangle_1 \cup \triangle_2 \cup \overline{P_0P_1}) \times \{h_- < z < h_+\}.$$

The Neumann boundary part shall be

$$\Gamma \stackrel{\text{def}}{=} \begin{cases} (\overline{Q_1P_1} \cup \{P_1\} \cup \overline{P_1P_2}) \times \{h_- < z < h_+\}, & \text{if } P_1 \neq Q_1, \\ (\overline{P_2P_1} \cup \{P_1\} \cup \overline{P_1Q_2}) \times \{h_- < z < h_+\}, & \text{if } P_1 = Q_1. \end{cases}$$

Furthermore, we assume that the elliptic coefficient function μ takes its values in the set of real, symmetric, positive definite 3×3 matrices and is constant on each of the open subsets $\triangle_1 \times \{h_- < z < h_+\}$ and $\triangle_2 \times \{h_- < z < h_+\}$ of Ω. Then there is a $p > 3$ such that (1.1) *is a topological isomorphism.*

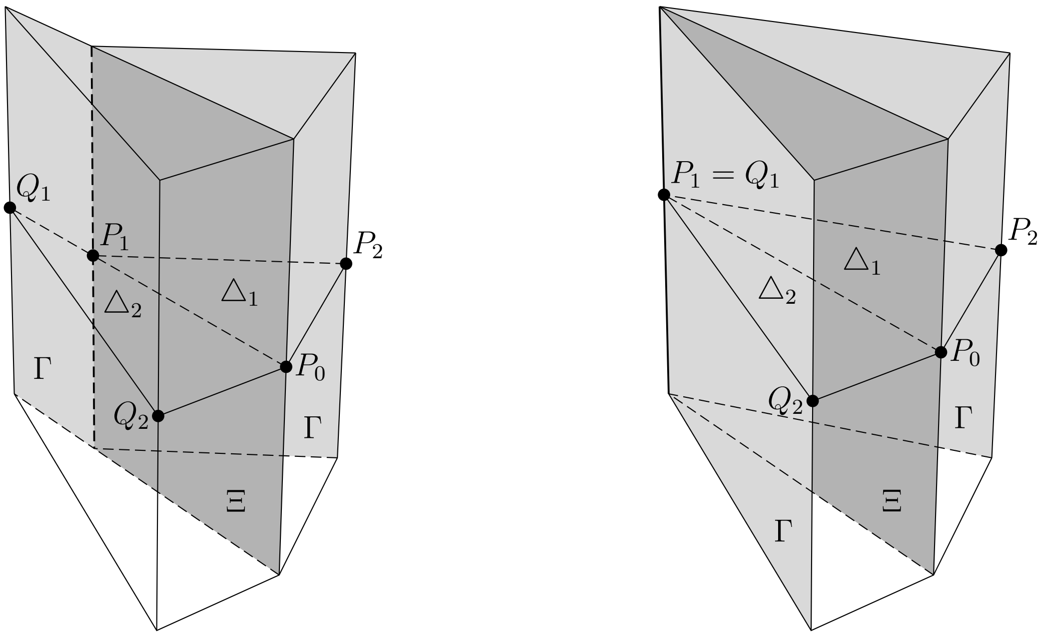

Figure 2: Right prism $\Omega = (\triangle_1 \cup \triangle_2 \cup \overline{P_0P_1}) \times \{h_- < z < h_+\}$ of Theorem 1.2 with a spatio–material edge (boldly dashed line) on the Neumann boundary part Γ (fair–grey) and material interface Ξ (dark–grey) in the two cases $P_1 \neq Q_1$ (left) and $P_1 = Q_1$ (right).

The constellation of Theorem 1.2 covers the benchmark problem of the three–dimensional L–shape of different materials which was regarded in [7, Fig. 2], [35, Fig. 1], and [20,

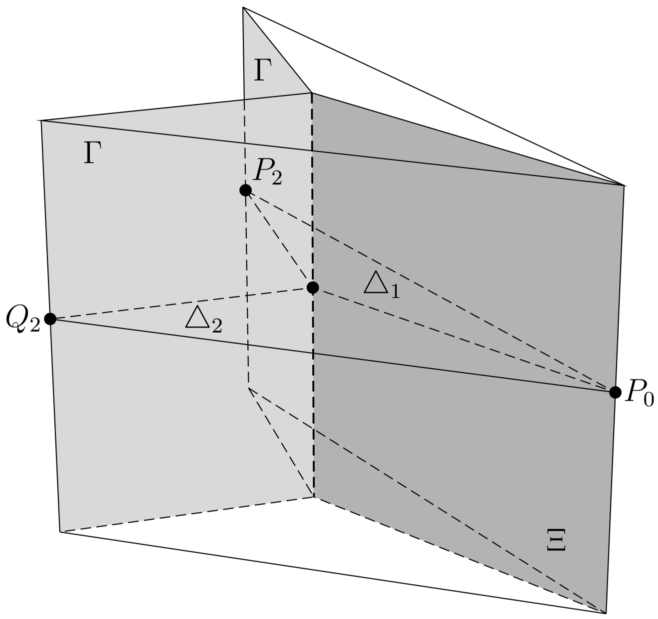

Figure 3: Constellation of Theorem 1.2 in the case $P_1 = Q_1$ (unnamed bullet) with a concave spatio–material edge (boldly dashed line) on the Neumann boundary part Γ (fair–grey) of the right prism $\Omega = (\triangle_1 \cup \triangle_2 \cup \overline{P_0P_1}) \times \{h_- < z < h_+\}$ with material interface Ξ (dark–grey).

Fig. 7]. This kind of structure really plays a role in technology: Just have a look at a ridge waveguide multiple quantum well laser, see for instance [3].

The proof of Theorems 1.1 and 1.2 is based upon the following fundamental result by Maz′ya [31]: Let (1.1) be an elliptic div–grad operator with Dirichlet boundary conditions ($\Gamma = \emptyset$) on a polyhedron Ω and a coefficient function μ which is constant on each sub–polyhedron of a polyhedral partition of Ω. Then the maximal regularity of (1.1) is delimitated by its edge singularities, see Proposition 3.3 below for details.

The main steps in proving Theorems 1.1, 1.2 are: First we transform the operators (1.1) of Theorems 1.1, 1.2 into operators to which Maz′ya's result applies. Next, we check that regularity is invariant under this transformation. Then we delimitate the edge singularities of the transformed problem. Thus, we get the maximal regularity of the transformed and the original operator by Maz′ya's result.

$W^{1,p}$ regularity with p larger than the space dimension d has the following advantages: Firstly, such $W^{1,p>d}$ spaces have powerful multiplier properties, see [34], which can be used in the treatment of quasilinear equations, see [22] and [21]. Secondly, $W^{1,p>d}$ regularity plays a role in the analysis of general nonlinear systems. Applications are for — among others — mathematical models in electrochemistry, see for instance [5], [15]; in semiconductor device operation, see [12], [28]; in thermoelectrics, see [8], [23]; in porous media, see [11]; in nonequilibrium thermodynamics, see [27]; and in inverse scattering, see [30]. For a review of mathematical models in physics and technology where optimal regularity of operators (1.1) is important, see for instance [19], [9], [10], [20]. In particular, $W^{1,p>d}$ regularity often is the decisive instrument to show uniqueness of solutions, see [13], [14].

Furthermore, the $W^{-1,p}$ calculus allows for jumps in the conormal derivative of solutions across internal interfaces. Thus, one can deal on the right hand side of the elliptic equation with distributions which are concentrated on internal interfaces. In electrostatics, for instance, a charge density on an interface causes a jump in the normal component of the dielectric displacement, see for instance [37, Ch. 1].

Up to now there are optimal $W^{1,p}$ regularity results for the following constellations with nonsmooth data: The Laplacian on strong Lipschitz domains with Dirichlet and Neumann boundary conditions has been treated in [26] and [39], respectively. The papers [31], [9], and [10] deal with div–grad operators with discontinuous coefficients and Dirichlet boundary conditions. Operators with mixed Dirichlet and Neumann boundary conditions have been regarded in [6], [18], and [36] — and if there are, additionally, discontinuous coefficients, in [19]. For more results concerning optimal regularity in the $W^{1,q} \to W^{-1,q}$ scale see for instance [33] with further references.

A numerical approach to multi–material elliptic problems has been taken in [1], [24], [4], [38] and the references cited there.

The outline of the paper is as follows: In the next section we define the function spaces and operators under consideration. Then we establish in Section 3 the bonding between the regularity of (1.1) on a multi–material polyhedral compound and its edge singularities. In Section 4 we deal with transformations of the problem. In particular, the regularity of (1.1) is invariant under bi–Lipschitz transformations of the spatio–material constellation, see Proposition 4.1. Generalising ideas from [25] and [9], we prove that the set of singular values of an operator pencil remains unchanged, if a bijective, piecewise linear transformation is applied to the two–dimensional spatio–material constellation, see Theorem 4.3. Moreover, we justify the reflection argument which is crucial for passing from a problem with mixed boundary conditions to a Dirichlet problem, see Proposition 4.5. The Sections 5 and 6 are devoted to the proof of Theorem 1.1 and Theorem 1.2, respectively.

2 Function spaces and basic operators

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $\Gamma \subset \partial\Omega$ an open part of its boundary. $W^{1,p}(\Omega)$ denotes the (complex) Sobolev space on Ω consisting of those $L^p(\Omega)$ functions whose first order distributional derivatives also belong to $L^p(\Omega)$, see for instance [17] or [32]. We use the symbol $W^{1,p}_\Gamma(\Omega)$ for the closure of the set

$$\{\ v|_\Omega : v \in C^\infty(\mathbb{R}^d),\ \operatorname{supp} v \cap (\partial\Omega \setminus \Gamma) = \emptyset\ \}$$

in the space $W^{1,p}(\Omega)$. Furthermore, $W^{-1,p'}_\Gamma(\Omega)$ denotes the space of continuous antilinear forms on $W^{1,p}_\Gamma(\Omega)$, where $1/p + 1/p' = 1$. As usual we write $W^{1,p}_0(\Omega)$ instead of $W^{1,p}_\emptyset(\Omega)$ and $W^{-1,p}(\Omega)$ instead of $W^{-1,p}_\emptyset(\Omega)$ when there are homogeneous Dirichlet conditions on the whole boundary of Ω. If Ω is understood, we abbreviate L^p, $W^{1,p}$,..., dropping the reference to Ω. Moreover, in the case of an interval $\Omega =]a,b[$ we write $L^p(a,b)$, $W^{1,p}(a,b)$, $W^{1,p}_0(a,b)$, and $W^{-1,p}(a,b)$.

The sesquilinear pairing between a Banach space X and the space of antilinear forms on X is written $\langle \cdot, \cdot \rangle_X$; in the case $X = \mathbb{C}^d$ we write $\langle \cdot, \cdot \rangle$ for short.

If μ is a Lebesgue measurable, essentially bounded function on Ω taking its values in the set of real, symmetric $d \times d$ matrices, then we define the div–grad operator $-\nabla \cdot \mu \nabla : W^{1,2}_\Gamma(\Omega) \to W^{-1,2}_\Gamma(\Omega)$ by

$$\langle -\nabla \cdot \mu \nabla \psi_1, \psi_2 \rangle_{W^{1,2}_\Gamma} \stackrel{\text{def}}{=} \int_\Omega \langle \mu \nabla \psi_1, \nabla \psi_2 \rangle \, d\mathfrak{r}, \quad \psi_1, \psi_2 \in W^{1,2}_\Gamma(\Omega).$$

The maximal restriction of $-\nabla \cdot \mu \nabla$ to any of the spaces $W^{-1,p}_\Gamma(\Omega)$, $p > 2$ is denoted by the same symbol.

We now define the basic operators to be used in the investigation of edge singularities.

Definition 2.1. Let real numbers $\theta_0 < \theta_1 < \ldots < \theta_n \leq \theta_0 + 2\pi$ be given and, additionally, real, symmetric, positive definite 2×2 matrices $\rho^1, \ldots, \rho^n$. We introduce on $]\theta_0, \theta_n[\setminus \{\theta_1, \ldots, \theta_{n-1}\}$ the matrix valued function ρ and three real valued coefficient functions b_0, b_1 and b_2, whose restrictions to each of the intervals $]\theta_{j-1}, \theta_j[$ are given by

$$\begin{aligned} \rho(\theta) &\stackrel{\text{def}}{=} \rho^j, \\ b_0(\theta) &\stackrel{\text{def}}{=} \rho^j_{11} \cos^2 \theta + 2\rho^j_{12} \sin \theta \cos \theta + \rho^j_{22} \sin^2 \theta, \\ b_1(\theta) &\stackrel{\text{def}}{=} (\rho^j_{22} - \rho^j_{11}) \sin \theta \cos \theta + \rho^j_{12} (\cos^2 \theta - \sin^2 \theta), \\ b_2(\theta) &\stackrel{\text{def}}{=} \rho^j_{11} \sin^2 \theta - 2\rho^j_{12} \sin \theta \cos \theta + \rho^j_{22} \cos^2 \theta, \end{aligned}$$

for $\theta_{j-1} < \theta < \theta_j$, $j = 1, \ldots, n$. By

$$\mathfrak{t}_\lambda[v] \stackrel{\text{def}}{=} \int_{\theta_0}^{\theta_n} b_2 \, v' \, \bar{v}' + \lambda b_1 v \, \bar{v}' - \lambda b_1 v' \, \bar{v} - \lambda^2 b_0 v \bar{v} \, d\theta$$

we define for every $\lambda \in \mathbb{C}$ a quadratic form on each of the following Sobolev spaces:

$$\begin{aligned} W^{1,2}(\theta_0, \theta_n) \quad \text{or} \quad W^{1,2}_0(\theta_0, \theta_n) \quad &\text{if } \theta_n < \theta_0 + 2\pi, \\ W^{1,2}(\theta_0, \theta_0 + 2\pi) \cap \{v : v(\theta_0) = v(\theta_0 + 2\pi)\} \quad &\text{if } \theta_n = \theta_0 + 2\pi. \end{aligned}$$

The operators on $L^2(\theta_0, \theta_n)$ which are induced by these forms are $\mathcal{A}^{\text{N}}_\lambda$, $\mathcal{A}^{\text{D}}_\lambda$, and $\mathcal{A}^{\pi}_\lambda$, respectively. This corresponds to Neumann, Dirichlet, and periodic boundary conditions.

Remark 2.2. One can check that

$$b_2 \geq \frac{\rho_{11} + \rho_{22}}{2} - \sqrt{\frac{(\rho_{11} - \rho_{22})^2}{4} + \rho_{12}^2} > 0.$$

Hence, each of the forms $\mathfrak{t}_\lambda$ is sectorial for every $\lambda \in \mathbb{C}$. A fortiori the induced operators $\mathcal{A}_\lambda$ are sectorial, see [29, Ch. VI], where $\mathcal{A}_\lambda$ is any of the operators $\mathcal{A}^{\text{N}}_\lambda$, $\mathcal{A}^{\text{D}}_\lambda$, or $\mathcal{A}^{\pi}_\lambda$. Moreover, all values λ from the right halfplane, for which the kernel of $\mathcal{A}_\lambda$ is nontrivial, are located in the sector

$$\mathcal{S} = \left\{ \lambda_1 + i\lambda_2 : |\lambda_2| \leq \max_j \frac{\rho^j_{11} + \rho^j_{22}}{2\sqrt{D_j}} \lambda_1 \right\},$$

where D_j is the determinant of the matrix ρ^j. In particular, the kernel of $\mathcal{A}_\lambda$ is trivial for all λ on the imaginary axis, see [20, Thm. 2.9].

3 Edge singularities

In this section we recall an optimal regularity result by MAZ′YA et alii, see [31], for the Dirichlet problem on a polyhedral multi–material compound. First we briefly describe the connection between elliptic three–dimensional edge singularities and two–dimensional vertex singularities for such a compound.

Definition 3.1. Let $\Omega \subset \mathbb{R}^3$ be a polyhedral Lipschitz domain and let there be a finite set $\{\Omega_k\}_k$ of polyhedral domains such that Ω is the interior of the closure of the disjoint union of all Ω_k. Moreover, the faces of the sub-polyhedra Ω_k, which are open sets with respect to the surface measure, shall be either in the interior of Ω or carry Neumann boundary conditions or carry Dirichlet boundary conditions. We define Γ as the interior of the closure of the union of all faces carrying Neumann boundary conditions. Furthermore, μ shall be a matrix function on Ω which is constant on each Ω_k and takes real, symmetric, positive definite 3×3 matrices as values. We regard now the edges of the sub-polyhedra Ω_k, which are open sets with respect to the curve measure, thus do not contain any vertex of Ω_k. Let E be such an edge and let P be an arbitrary interior point of this edge. Choosing a new orthogonal coordinate system (x, y, z) with origin at the point P such that the direction of E coincides with the z axis, we denote by ω the corresponding orthogonal transformation matrix and by μ_ω the piecewise constant matrix function which satisfies in a neighbourhood $\mathcal{O}$ of the origin

$$\mu_\omega(x,y,z) = \omega\mu\big(\omega^{-1}(x,y,z)\big)\omega^{-1} \quad \text{for all } (x,y,z) \in \mathcal{O},$$
$$\mu_\omega(tx,ty,z) = \mu_\omega(x,y,0) \quad \text{for all } (x,y,0) \in \mathcal{O},\ z \in \mathbb{R}, \text{ and } t > 0.$$

Finally, we denote by ρ the upper left 2×2 block of μ_ω at $z = 0$.

Remark 3.2. Obviously, ρ in Definition 3.1 does not depend on the choice of the point P. The matrix ρ is given on a sector

$$K_a^b \stackrel{\text{def}}{=} \{(r\cos\theta, r\sin\theta) : r > 0,\ \theta \in]a,b[\}, \quad a < b \le a + 2\pi$$

which coincides near P with the intersection of the x–y plane with the ω image of the domain Ω. There exist uniquely determined angles

$$a = \theta_0 < \theta_1 < \ldots < \theta_n = b \le \theta_0 + 2\pi \tag{3.1}$$

such that each $K_{\theta_j}^{\theta_{j+1}}$ coincides near P with the intersection of the x–y plane with the ω image of exactly one domain Ω_k. Thus, ρ is constant on each of the sectors $K_{\theta_j}^{\theta_{j+1}}$ and takes real, symmetric, positive definite 2×2 matrices ρ^j as values. Indeed there are real numbers $\tilde\mu_{13}^j$, $\tilde\mu_{23}^j$, and $\tilde\mu_{33}^j$ such that

$$\mu_\omega(x,y,z) = \mu_\omega^j = \begin{pmatrix} \rho^j & & \tilde\mu_{13}^j \\ & & \tilde\mu_{23}^j \\ \tilde\mu_{13}^j & \tilde\mu_{23}^j & \tilde\mu_{33}^j \end{pmatrix} \quad \text{if } z \in \mathbb{R} \text{ and } (x,y) \in K_{\theta_j}^{\theta_{j+1}}.$$

For an interior edge $E \subset \Omega$ the sector K_a^b is the full plane that means $\theta_n = \theta_0 + 2\pi$. On the other hand for an outer edge $E \subset \partial\Omega$ there is $\theta_n < \theta_0 + 2\pi$. If in particular $E \subset \partial\Omega$

belongs to the closure of exactly one sub-polyhedron Ω_k, then $n = 1$ and we call E a *mono–material outer edge* of Ω. Analogously, we call E a *bi–material outer edge* of Ω, if $E \subset \partial\Omega$ and E belongs to the closure of exactly two sub-polyhedra. Mutatis mutandis we term a *multi–material outer edge* of Ω. Thus, for an n–material outer edge there are n angles in (3.1). However, this does not mean that necessarily n different materials meet at an n–material outer edge since even for adjacent sectors the corresponding matrices ρ^j can be equal.

We proceed by quoting MAZ′YA's central optimal regularity result for the Dirichlet problem on a polyhedral compound conforming to Definition 3.1.

Proposition 3.3. *[31, Thm. 2.3], [9, Remark 2.2]. Let us regard the constellation of Definition 3.1. If for every edge E the induced operator*

$$\begin{array}{ll} \mathcal{A}^{\pi}_{\lambda} & \textit{in case of an interior edge } E \subset \Omega, \\ \mathcal{A}^{\mathrm{D}}_{\lambda} & \textit{in case of an outer edge } E \subset \partial\Omega, \end{array}$$

see Definition 2.1, has a trivial kernel for all λ *with* $0 < \Re\lambda < 1/3 + \epsilon$ *and some* $\epsilon > 0$, *then there is a* $p > 3$ *such that*

$$-\nabla\cdot\mu\nabla : \ W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$$

is a topological isomorphism.

For a detailed discussion of how to find the parameters λ for which the operators $\mathcal{A}^{\pi}_{\lambda}$ and $\mathcal{A}^{\mathrm{D}}_{\lambda}$ have only a trivial kernel we refer to [19].

At a mono–material outer edge on the Dirichlet boundary we have the following result about the corresponding operator $\mathcal{A}^{\mathrm{D}}_{\lambda}$.

Proposition 3.4. *[19, Thm. 24], [9, Cor. 3.1]. For any mono–material outer edge E, see Remark 3.2 and Definition 3.1, the kernels of the associated operators* $\mathcal{A}^{\mathrm{D}}_{\lambda}$, *see Definition 2.1, are trivial in each of the following two cases:*

$$\begin{array}{lll} 0 < \theta_1 - \theta_0 \le \pi & \textit{and} & 0 < \Re\lambda < 1, \\ \pi < \theta_1 - \theta_0 < 2\pi & \textit{and} & 0 < \Re\lambda \le 1/2. \end{array}$$

For bi–material outer edges we know the following.

Proposition 3.5. *[19, Thm. 25], [9, Lem. 2.3]. For any bi–material outer edge E, see Remark 3.2 and Definition 3.1, the kernels of the associated operators* $\mathcal{A}^{\mathrm{D}}_{\lambda}$ *and* $\mathcal{A}^{\mathrm{N}}_{\lambda}$ *(with Dirichlet and with Neumann boundary conditions, respectively, see Definition 2.1) are trivial if*

$$\theta_1 - \theta_0 \le \pi, \quad \theta_2 - \theta_1 \le \pi, \quad \theta_2 - \theta_0 < 2\pi, \quad \textit{and } 0 < \Re\lambda \le 1/2.$$

4 Transformations

We now collect results about the invariance of the regularity of an elliptic problem under certain transformations of the spatio–material constellation. In particular we look at bi–Lipschitz transformations and investigate the reflection of the problem at a singular Neumann face.

Proposition 4.1. *(Bi–Lipschitz transformation, see [19, Prop. 16] and [16, Thm. 2.10].) Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let Γ be an open subset of its boundary. Further assume that ϕ is a mapping from a neighbourhood of $\operatorname{cl}\Omega$ into $\mathbb{R}^d$ which is bi–Lipschitz. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then:*

1. *The mapping ϕ induces a linear, topological isomorphism*
$$\Psi_p : W^{1,p}_{\phi(\Gamma)}(\phi(\Omega)) \to W^{1,p}_{\Gamma}(\Omega) \quad \text{given by} \quad (\Psi_p f)(\mathfrak{r}) = (f \circ \phi)(\mathfrak{r}) = f(\phi(\mathfrak{r})).$$

2. *The mapping $\Psi^*_{p'}$ is a topological isomorphism between $W^{-1,p}_{\Gamma}(\Omega)$ and $W^{-1,p}_{\phi(\Gamma)}(\phi(\Omega))$.*

3. *If μ is a bounded measurable function on Ω, taking its values in the set of $d\times d$ matrices, then*
$$\Psi^*_{p'} \left(-\nabla\cdot\mu\nabla\right) \Psi_p = -\nabla\cdot\mu_\phi\nabla$$
with
$$\mu_\phi(\phi(\mathfrak{r})) = J_\phi(\mathfrak{r})\mu(\mathfrak{r})J^T_\phi(\mathfrak{r})\frac{1}{|\det J_\phi(\mathfrak{r})|}, \qquad \mathfrak{r} \in \Omega, \tag{4.1}$$
where J_ϕ denotes the Jacobian of ϕ. If, in particular, $-\nabla\cdot\mu\nabla$ is a topological isomorphism from $W^{1,p}_{\Gamma}(\Omega)$ to $W^{-1,p}_{\Gamma}(\Omega)$, then $-\nabla\cdot\mu_\phi\nabla$ is a topological isomorphism from $W^{1,p}_{\phi(\Gamma)}(\phi(\Omega))$ to $W^{-1,p}_{\phi(\Gamma)}(\phi(\Omega))$ and vice versa.

Remark 4.2. From the proof of Theorem 2.10 in [16] one can conclude that the assertion of Proposition 4.1 mutatis mutandis holds true in the whole space case $\Omega = \mathbb{R}^d$.

The following theorem specifically concerns the piecewise linear, bijective transformation of an operator pencil.

Theorem 4.3. *With respect to the constellation of Definition 2.1 let θ_n and γ be such angles that*
$$0 < \gamma - \theta_0 < \pi \quad \text{and} \quad 0 < \theta_n - \gamma < \pi,$$
and L the line through $0 \in \mathbb{R}^2$ with slope γ, see Figure 4. We denote by H_A and H_B the two (closed) halfspaces lying to the sides of L. Furthermore, let A and B be two real, nonsingular 2×2 matrices with the property $A(cos\gamma, sin\gamma)^T = B(cos\gamma, sin\gamma)^T$ and ϕ the mapping which acts as A on H_A and as B on H_B. If ϕ is additionally bijective, then holds true:

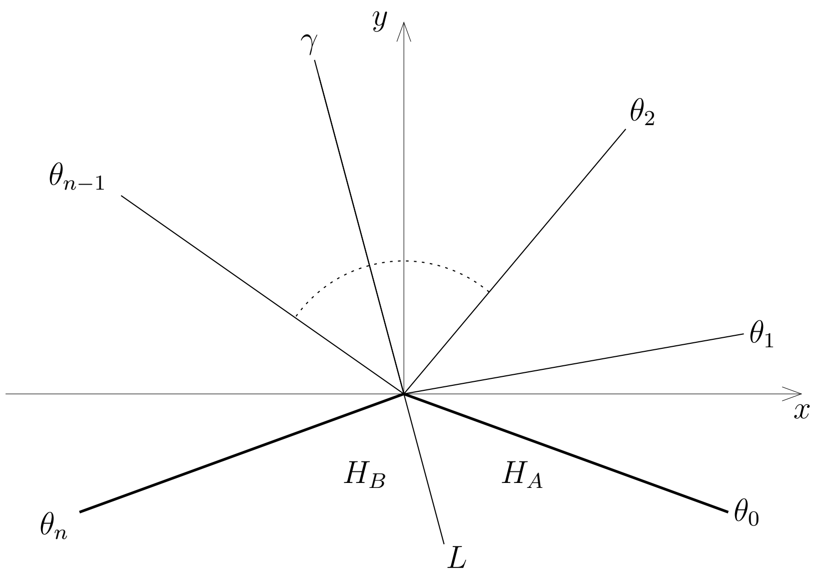

Figure 4: Spatial constellation of Theorem 4.3 where the line L with slope γ splits the plane into the halfspaces H_A and H_B

1. *The mapping ϕ induces a sector partition of the sector $\phi(K_{\theta_0}^{\theta_n})$ such that the corresponding coefficient function ρ_ϕ, see Proposition 4.1, is constant on each of these sectors. More precisely, the nonempty elements of*

$$\left\{\phi(H_A \cap K_{\theta_j}^{\theta_{j+1}})\right\}_{j=0}^{n-1} \cup \left\{\phi(K_{\theta_j}^{\theta_{j+1}} \cap H_B)\right\}_{j=0}^{n-1}$$

   are a sector partition of $\phi(K_{\theta_0}^{\theta_n})$ with angles $\theta_0^\phi < \ldots < \gamma^\phi < \ldots < \theta_n^\phi$ and ρ_ϕ is constant on each of these sectors. Thus, after transformation by ϕ we end up again with a constellation as described in Definition 2.1 with possibly one more sector than in the initial constellation.

2. *For every $\lambda \in \mathbb{C}$ with $\Re\lambda > 0$, the operators $\mathcal{A}_\lambda^{\mathrm{N}}$ and $\mathcal{A}_\lambda^{\mathrm{D}}$, associated with the coefficient function ρ, have a nontrivial kernel iff the kernel of the corresponding operator associated with the coefficient function ρ_ϕ is nontrivial.*

Proof. 1. The nonempty elements of $\{H_A \cap K_{\theta_j}^{\theta_{j+1}}\}_{j=0}^{n-1} \cup \{K_{\theta_j}^{\theta_{j+1}} \cap H_B\}_{j=0}^{n-1}$ form a sector partition of $K_{\theta_0}^{\theta_n}$ such that the function ρ is constant in the interior of each sector of this partition. Moreover, the function ϕ is constant on each sector, given either by A or by B. Thus, the image under ϕ of this sector partition is a sector partition of $\phi(K_{\theta_0}^{\theta_n})$ such that, due to (4.1), ρ_ϕ is constant on each of the transformed sectors.

2. According to (4.1) the coefficient function ρ_ϕ of the ϕ transformed problem is almost everywhere given by

$$\rho_\phi(x,y) = \begin{cases} \frac{A\rho\left(A^{-1}\binom{x}{y}\right)A^T}{\det A} & \text{if } \binom{x}{y} \in A(H_A \cap K_{\theta_0}^{\theta_n}), \\ \frac{B\rho\left(B^{-1}\binom{x}{y}\right)B^T}{\det B} & \text{if } \binom{x}{y} \in A(H_B \cap K_{\theta_0}^{\theta_n}). \end{cases} \tag{4.2}$$

If w is a continuous function on an interval $]a,b[$, with $b < a + 2\pi$, then the function

$$\mathrm{W}(r\cos\theta, r\sin\theta) \stackrel{\text{def}}{=} w(\theta), \quad r > 0,\ a < \theta < b \tag{4.3}$$

is a continuous function on the sector $K_a^b \stackrel{\text{def}}{=} \{(r\cos\theta, r\sin\theta) : r > 0,\ a < \theta < b\}$. If in particular $w \in W^{1,2}(a,b)$ and $\lambda \in \mathbb{R}$, then

$$\nabla \left((x^2+y^2)^{\lambda/2}\mathrm{W}(x,y)\right)\Big|_{(x,y)=(r\cos\theta, r\sin\theta)} = \frac{r^\lambda}{r}\begin{pmatrix} \lambda w(\theta)\cos\theta - w'(\theta)\sin\theta \\ \lambda w(\theta)\sin\theta + w'(\theta)\cos\theta \end{pmatrix}.$$

Thus, for all $u, v \in W^{1,2}(\theta_0, \theta_n)$ and the functions U and V associated by (4.3) holds true

$$\begin{aligned} &\left[\rho(x,y)\nabla\left((x^2+y^2)^{\lambda/2}\mathrm{U}(x,y)\right)\right]\cdot\nabla\left((x^2+y^2)^{-\lambda/2}\overline{\mathrm{V}}(x,y)\right)\Big|_{(x,y)=(r\cos\theta, r\sin\theta)} \\ &= \frac{1}{r^2}\left(b_2(\theta)u'(\theta)\bar{v}'(\theta) - \lambda b_1(\theta)u'(\theta)\bar{v}(\theta) + \lambda b_1(\theta)u(\theta)\bar{v}'(\theta) - \lambda^2 b_0(\theta)u(\theta)\bar{v}(\theta)\right), \end{aligned} \tag{4.4}$$

where b_0, b_1, b_2 are the functions from Definition 2.1 for the initial constellation. Integrating (4.4) over the set $K \stackrel{\text{def}}{=} \{(r\cos\theta, r\sin\theta) : 1 \le r \le 2,\ \theta_0 < \theta < \theta_n\}$ one obtains

$$\begin{aligned} &\log 2 \int_{\theta_0}^{\theta_n} b_2(\theta)u'(\theta)\bar{v}'(\theta) - \lambda b_1(\theta)u'(\theta)\bar{v}(\theta) + \lambda b_1(\theta)u(\theta)\bar{v}'(\theta) - \lambda^2 b_0(\theta)u(\theta)\bar{v}(\theta)\, d\theta \\ &\qquad = \int_K \left[\rho(x,y)\nabla\left((x^2+y^2)^{\lambda/2}\mathrm{U}(x,y)\right)\right]\cdot\nabla\left((x^2+y^2)^{-\lambda/2}\overline{\mathrm{V}}(x,y)\right)\, dx\, dy \end{aligned} \tag{4.5}$$

We now transform the integral on the right hand side by means of the mapping ϕ which is A on $K \cap H_A$ and B on $K \cap H_B$:

$$\begin{aligned} &\int_K \left[\rho(x,y)\nabla\left((x^2+y^2)^{\lambda/2}\mathrm{U}(x,y)\right)\right]\cdot\nabla\left((x^2+y^2)^{-\lambda/2}\overline{\mathrm{V}}(x,y)\right) dx\, dy \\ &= \int_{A(K\cap H_A)} \left[\left(\frac{A\rho(\cdot)A^T}{\det A}\nabla\left(\|\cdot\|_{\mathbb{R}^2}^{\lambda}\mathrm{U}(\cdot)\right)\right)\cdot\nabla\left(\|\cdot\|_{\mathbb{R}^2}^{-\lambda}\overline{\mathrm{V}}(\cdot)\right)\right]\circ A^{-1}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) dx\, dy \\ &\quad + \int_{B(K\cap H_B)} \left[\left(\frac{B\rho(\cdot)B^T}{\det B}\nabla\left(\|\cdot\|_{\mathbb{R}^2}^{\lambda}\mathrm{U}(\cdot)\right)\right)\cdot\nabla\left(\|\cdot\|_{\mathbb{R}^2}^{-\lambda}\overline{\mathrm{V}}(\cdot)\right)\right]\circ B^{-1}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) dx\, dy \end{aligned} \tag{4.6}$$

Using (4.2) this can be written as

$$\begin{aligned} &= \int_{A(K\cap H_A)} \left[\rho_\phi(x,y)\nabla\left((x^2+y^2)^{\lambda/2}\mathrm{U}_A(x,y)\right)\right]\cdot\nabla\left((x^2+y^2)^{-\lambda/2}\overline{\mathrm{V}}_A(x,y)\right) dx\, dy \\ &\quad + \int_{B(K\cap H_B)} \left[\rho_\phi(x,y)\nabla\left((x^2+y^2)^{\lambda/2}\mathrm{U}_B(x,y)\right)\right]\cdot\nabla\left((x^2+y^2)^{-\lambda/2}\mathrm{V}_B(x,y)\right) dx\, dy, \end{aligned}$$

where U_A is the complex function defined on $\phi(K \cap H_A)$ by

$$\mathrm{U}_A(x,y) \stackrel{\text{def}}{=} \frac{\|A^{-1}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\|_{\mathbb{R}^2}^{\lambda}}{\|\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\|_{\mathbb{R}^2}^{\lambda}}\,\mathrm{U}\left(A^{-1}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right).$$

Mutatis mutandis u_B v_A v_B. The functions u_A, u_B, v_A, v_B, are all homogeneous of degree 0 since u and v are homogeneous of degree 0. Thus, there are uniquely determined functions $u_A, v_A :]\theta_0^\phi, \gamma^\phi[\to \mathbb{C}$ and $u_B, v_B :]\gamma^\phi, \theta_n^\phi[\to \mathbb{C}$ such that

$$\begin{aligned} \mathrm{u}_A(r\cos\theta, r\sin\theta) &= u_A(\theta), & \mathrm{u}_B(r\cos\theta, r\sin\theta) &= u_B(\theta), \\ \mathrm{v}_A(r\cos\theta, r\sin\theta) &= v_A(\theta), & \mathrm{v}_B(r\cos\theta, r\sin\theta) &= v_B(\theta). \end{aligned}$$

The continuity of u and v and the relation $A(\cos\gamma, \sin\gamma)^T = B(\cos\gamma, \sin\gamma)^T$ entail

$$u_A(\gamma^\phi) = u_B(\gamma^\phi) \quad \text{and} \quad v_A(\gamma^\phi) = v_B(\gamma^\phi). \tag{4.7}$$

We define functions

$$u_\phi(\theta) \stackrel{\text{def}}{=} \begin{cases} u_A(\theta) & \text{if } \theta_0^\phi < \theta < \gamma^\phi, \\ u_B(\theta) & \text{if } \gamma^\phi < \theta < \theta_n^\phi, \end{cases} \qquad v_\phi(\theta) \stackrel{\text{def}}{=} \begin{cases} v_A(\theta) & \text{if } \theta_0^\phi < \theta < \gamma^\phi, \\ v_B(\theta) & \text{if } \gamma^\phi < \theta < \theta_n^\phi, \end{cases}$$

which, due to (4.7), are from the space $W^{1,2}(\theta_0^\phi, \theta_n^\phi)$. Moreover, the mapping

$$W^{1,2}(\theta_0, \theta_n) \ni u \mapsto u_\phi \in W^{1,2}(\theta_0^\phi, \theta_n^\phi)$$

is a topological isomorphism, and its restriction to $W_0^{1,2}(\theta_0, \theta_n)$ is a topological isomorphism onto $W_0^{1,2}(\theta_0^\phi, \theta_n^\phi)$.

In terms of the functions u_ϕ and v_ϕ associated to u_ϕ and v_ϕ by (4.3) one now can write (4.6) in the form

$$\begin{aligned} &\int_K \left[\rho(x,y)\nabla\left((x^2+y^2)^{\lambda/2}\mathrm{u}(x,y)\right)\right] \cdot \nabla\left((x^2+y^2)^{-\lambda/2}\overline{\mathrm{v}}(x,y)\right) dx\,dy \\ &\quad = \int_{\phi(K)} \left[\rho_\phi(x,y)\nabla\left((x^2+y^2)^{\lambda/2}\mathrm{u}_\phi(x,y)\right)\right] \cdot \nabla\left((x^2+y^2)^{-\lambda/2}\overline{\mathrm{v}}_\phi(x,y)\right) dx\,dy. \end{aligned}$$

Analogously to (4.4) the integrand on the right hand side of this equation can be expressed in terms of the functions b_0^ϕ, b_1^ϕ, b_2^ϕ, see Definition 2.1, of the transformed constellation. Please note that ρ_ϕ enters as determined in (4.2). Thus, one obtains for the last integral

$$\begin{aligned} \log 2 \int_{\theta_0^\phi}^{\theta_n^\phi} & b_2^\phi(\theta)u_\phi'(\theta)\bar{v}_\phi'(\theta) - \lambda b_1^\phi(\theta)u_\phi'(\theta)\bar{v}_\phi(\theta) \\ &+ \lambda b_1^\phi(\theta)u_\phi(\theta)\bar{v}_\phi'(\theta) - \lambda^2 b_0^\phi(\theta)u_\phi(\theta)\bar{v}_\phi(\theta)d\theta. \end{aligned} \tag{4.8}$$

Please note that due to the linearity of A and B the integral over the radial variable is $\int_{r(\theta)}^{2r(\theta)} \frac{dr}{r} = \log 2$ for each θ.

One concludes that, if there is a function u from $W^{1,2}(\theta_0, \theta_n)$ such that (4.5) vanishes for all v from $W^{1,2}(\theta_0, \theta_n)$, then there is a function u_ϕ from the space $W^{1,2}(\theta_0^\phi, \theta_n^\phi)$ such that (4.8) vanishes for all $v_\phi \in W^{1,2}(\theta_0^\phi, \theta_n^\phi)$ and vice versa. That means the operator $\mathcal{A}_\lambda^{\mathrm{N}}$ has a nontrivial kernel iff the corresponding operator of the transformed constellation has. The same argument holds for the Dirichlet case with the function spaces $W_0^{1,2}(\theta_0, \theta_n)$ and $W_0^{1,2}(\theta_0^\phi, \theta_n^\phi)$. □

Remark 4.4. The result of Theorem 4.3 holds true also in the case of mixed boundary conditions with the function spaces $W_\Gamma^{1,2}(\theta_0, \theta_n)$ and $W_{\Gamma^\phi}^{1,2}(\theta_0^\phi, \theta_n^\phi)$, where $\Gamma = \{\theta_0\}$ and $\Gamma^\phi = \{\theta_0^\phi\}$ or $\Gamma = \{\theta_n\}$ and $\Gamma^\phi = \{\theta_n^\phi\}$. Furthermore, one can apply Theorem 4.3 iteratively to get a similar result for a piecewise linear homeomorphism ϕ.

Now we regard even reflection of the three–dimensional problem at a (singular) Neumann face.

Proposition 4.5. *(Material reflection at a Neumann face, see [19, Prop. 17].) Let $\Omega \subset \mathbb{R}^3$ be a bounded, polyhedral Lipschitz domain and let Γ be an open subset of $\partial\Omega$ such that*

$$\operatorname{cl}\Omega \cap \{(x, 0, z) : x, z \in \mathbb{R}\} = \operatorname{cl}\Gamma.$$

We define a new domain $\widehat{\Omega}$ by reflecting Ω at the $y = 0$ plane:

$$\widehat{\Omega} \stackrel{\text{def}}{=} \operatorname{int}\left(\Omega \cup \{(x, y, z) : (x, -y, z) \in \Omega\} \cup \operatorname{cl}\Gamma\right).$$

If μ is a bounded, measurable function on Ω taking its values in the set of real, symmetric $3{\times}3$ matrices, then we define a coefficient function $\hat{\mu}$ on $\widehat{\Omega}$ by μ on Ω, by zero on Γ, and if $(x, -y, z) \in \Omega$ by

$$\hat{\mu}(x, y, z) \stackrel{\text{def}}{=} \begin{pmatrix} \mu_{11}(x, -y, z) & -\mu_{12}(x, -y, z) & \mu_{13}(x, -y, z) \\ -\mu_{12}(x, -y, z) & \mu_{22}(x, -y, z) & -\mu_{23}(x, -y, z) \\ \mu_{13}(x, -y, z) & -\mu_{23}(x, -y, z) & \mu_{33}(x, -y, z) \end{pmatrix}.$$

If $\psi \in W_\Gamma^{1,2}(\Omega)$ satisfies the equation $-\nabla\cdot\mu\nabla\psi = f$ in $W_\Gamma^{-1,2}(\Omega)$, then the equation $-\nabla\cdot\hat{\mu}\nabla\widehat{\psi} = \hat{f}$ holds in $W^{-1,2}(\widehat{\Omega})$ with

$$\widehat{\psi}(x, y, z) \stackrel{\text{def}}{=} \begin{cases} \psi(x, y, z), & \text{if } (x, y, z) \in \Omega, \\ \psi(x, -y, z), & \text{if } (x, -y, z) \in \Omega, \end{cases}$$

$$\langle \hat{f}, \varphi \rangle_{W^{1,2}(\widehat{\Omega})} \stackrel{\text{def}}{=} \langle f, \check{\varphi} \rangle_{W_\Gamma^{1,2}(\Omega)}, \quad \varphi \in W_0^{1,2}(\widehat{\Omega}),$$

where the function $\check{\varphi} \in W_\Gamma^{1,2}(\Omega)$ is defined by

$$\check{\varphi}(x, y, z) \stackrel{\text{def}}{=} \varphi(x, y, z) + \varphi(x, -y, z), \quad (x, y, z) \in \Omega.$$

Moreover, if $f \in W_\Gamma^{-1,p}(\Omega)$, then $\hat{f} \in W^{-1,p}(\widehat{\Omega})$ and if $-\nabla\cdot\hat{\mu}\nabla : W_0^{1,p}(\widehat{\Omega}) \to W^{-1,p}(\widehat{\Omega})$ is a topological isomorphism, then $-\nabla\cdot\mu\nabla : W_\Gamma^{1,p}(\Omega) \to W_\Gamma^{-1,p}(\Omega)$ also is a topological isomorphism.

Next we look at how even reflection of the problem acts on edges in the mirror Neumann face.

Lemma 4.6. *(Material reflection of the operator pencil) Let us specify the constellation of Definition 2.1 to the halfplane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ that means $\theta_0 = 0$ and $\theta_n = \pi$ and* *let $\mathcal{A}_\lambda^{\mathrm{D}}$ and $\mathcal{A}_\lambda^{\mathrm{N}}$ be the associated operator pencils with Dirichlet and Neumann boundary conditions, respectively. By material reflection*

$$\rho(x,y) \stackrel{\text{def}}{=} \begin{cases} \begin{pmatrix} \rho_{11}^j & \rho_{12}^j \\ \rho_{12}^j & \rho_{22}^j \end{pmatrix} & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} \in K_{\theta_{j-1}}^{\theta_j}, \\ \begin{pmatrix} \rho_{11}^j & -\rho_{12}^j \\ -\rho_{12}^j & \rho_{22}^j \end{pmatrix} & \text{if } \begin{pmatrix} x \\ -y \end{pmatrix} \in K_{\theta_{j-1}}^{\theta_j} \end{cases}$$

one obtains a corresponding constellation in the full plane. Let $\mathcal{A}_\lambda^{\pi}$ denote the operator according to Definition 2.1 which corresponds to the latter constellation. Then for any number λ with $\Re\lambda > 0$ the kernel of this operator $\mathcal{A}_\lambda^{\pi}$ is trivial, if for this same λ the kernels of $\mathcal{A}_\lambda^{\mathrm{D}}$ and *$\mathcal{A}_\lambda^{\mathrm{N}}$ are trivial.*

Proof. In [19, Lemma 22] a proof is given for the case of two sectors in the halfspace. Mutatis mutandis, the proof can be carried over to the case of a finite number of sectors. □

The next proposition in a way reverses the construction of an edge pencil by associating a div–grad operator on $\mathbb{R}^3$ to such an operator on a sector partition of $\mathbb{R}^2$.

Proposition 4.7. *[19, Lem. 14] With respect to the constellation of Definition 2.1 let λ with $\Re\lambda \in]0,1[$ be a number such that there exists a (nontrivial) function v_λ from the kernel of $\mathcal{A}_\lambda^{\pi}$. Let μ_{13}, μ_{23}, and μ_{33} be real valued, bounded, measurable functions on $\mathbb{R}^2$ and let us define the coefficient function μ on $\mathbb{R}^3$ by*

$$\mu(x,y,z) \stackrel{\text{def}}{=} \begin{pmatrix} \rho_{11}^j & \rho_{12}^j & \mu_{13}(x,y) \\ \rho_{12}^j & \rho_{22}^j & \mu_{23}(x,y) \\ \mu_{13}(x,y) & \mu_{23}(x,y) & \mu_{33}(x,y) \end{pmatrix} \quad \text{if } (x,y) \in K_{\theta_j}^{\theta_{j+1}}. \tag{4.9}$$

Then the function ψ, given by

$$\psi(x,y,z) = (x^2+y^2)^{\lambda/2} v_\lambda(\arg(x+iy)), \quad (x,y,z) \in \mathbb{R}^3 \tag{4.10}$$

satisfies $-\nabla\cdot\mu\nabla\psi = 0$ on $\mathbb{R}^3$ in the sense of distributions. Consequently, for every compactly supported function η from $W^{1,\infty}(\mathbb{R}^3)$ one has $-\nabla\cdot\mu\nabla(\eta\psi) \in W^{-1,6}(\mathbb{R}^3)$. Moreover, if $\eta \equiv 1$ in a neighbourhood of $0 \in \mathbb{R}^3$, then $\eta\psi$ does not belong to $W^{1,2/(1-\Re\lambda)}(\mathbb{R}^3)$.

Remark 4.8. In particular Proposition 4.7 applies to the matrix μ_ω of Definition 3.1, see also Remark 3.2.

5 Proof of Theorem 1.1

For the proof of Theorem 1.1 we first transform the spatio–material constellation under consideration there while keeping the regularity of the elliptic div–grad operator invariant. Then we discuss the edge singularities of the transformed problem to obtain the regularity of (1.1) according to Proposition 3.3.

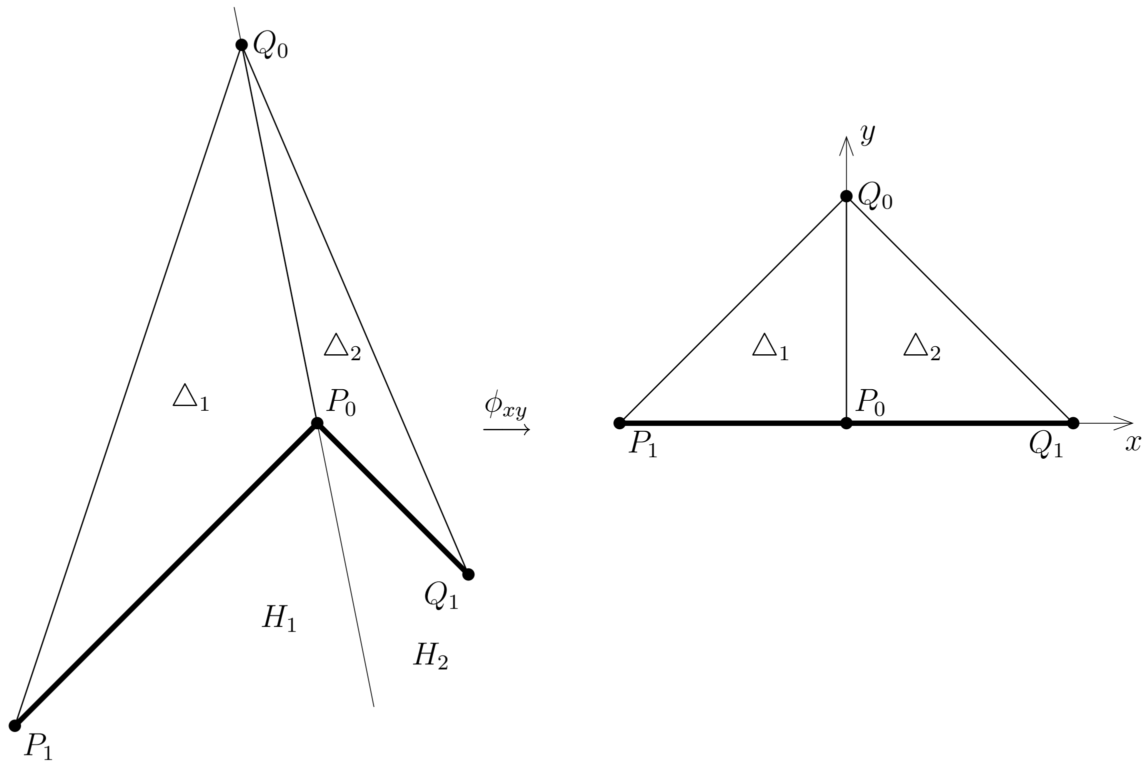

Figure 5: Two–dimensional constellation which generates the right prism of Figure 1 under consideration in Theorem 1.1, originally (left) and after piecewise linear transformation by ϕ_{xy} (right). For the sake of simplicity the transformed objects bear the same name as the originals. The trace of the Neumann boundary face is indicated by bold lines.

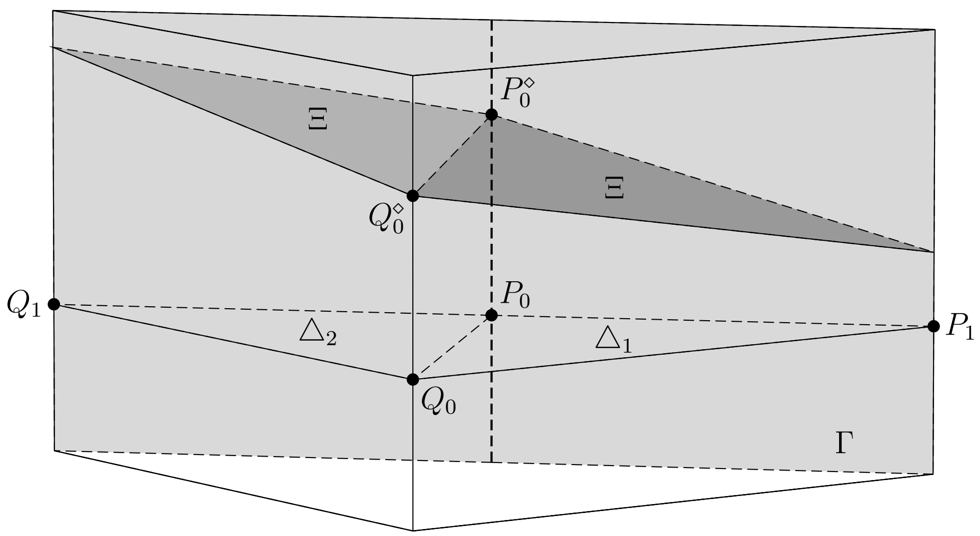

Figure 6: Right prism of Figure 1 after transformation by ϕ (all objects keep their name after transformation). The Neumann boundary part Γ (fair–grey) is planar after transformation while the material interface Ξ (dark–grey) now is broken along the line through $P_0^\diamond$ and $Q_0^\diamond$. Thus, $\overline{P_0^\diamond Q_0^\diamond}$ is a four–material edge.

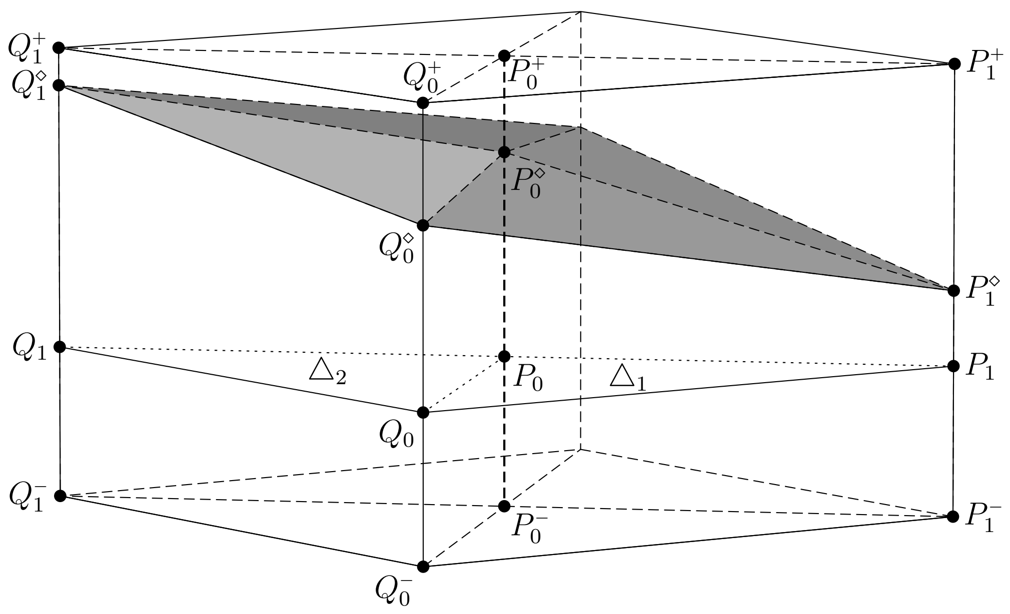

Figure 7: Right prism of Figure 1 after transformation by ϕ and even reflection. The triangles $\triangle_1$ and $\triangle_2$ are the image of the generating two–dimensional constellation. All other lines in the figure represent edges to be discussed with respect to the regularity of the associated edge pencil. The boldly dashed line $\overline{P_0^- P_0^+}$ is the image of the edge originally on the Neumann boundary part Γ. The shaded areas are the image of the original material interface.

5.1 Transformation of the problem

Let us first regard the two–dimensional constellation which generates the right prism under consideration in Theorem 1.1. The left hand side of Figure 5 shows the constellation which generates the prism of Figure 1. Without loss of generality we can assume that the point P_0 has the coordinates $x = 0$ and $y = 0$ in the plane. The line through P_0 and Q_0 divides the plane into a halfplane H_1 containing the triangle $\triangle_1$ and a halfplane H_2. By

$$P_1 \mapsto \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right), \quad Q_0 \mapsto \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \quad \text{and} \quad Q_0 \mapsto \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \quad Q_1 \mapsto \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$$

two linear mappings of the plane are defined, respectively. Let ϕ_{xy} be the mapping which is given on H_1 by the first of these linear mappings and on H_2 by the second one, see Figure 5. Since the two linear mappings are identical on the line through P_0 and Q_0 the mapping ϕ_{xy} is a bi–Lipschitz transformation of the plane. Now we define

$$\phi : \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{by} \quad \phi(x, y, z) \stackrel{\text{def}}{=} \begin{pmatrix} \phi_{xy}(x, y) \\ z \end{pmatrix}. \tag{5.1}$$

Thus, ϕ is a bi–Lipschitz transformation of the space $\mathbb{R}^3$. Hence, Proposition 4.1 applies.

Next, we reflect the problem as in Proposition 4.5 at the x–z plane that means at the Neumann boundary, see Figure 7. We end up with an elliptic operator $-\nabla\cdot\hat{\mu}\nabla : W_0^{1,p}(\widehat{\Omega}) \to W^{-1,p}(\widehat{\Omega})$ with Dirichlet boundary conditions. According to Proposition 4.5, in order to prove Theorem 1.1, it suffices that this operator is a topological isomorphism for some $p > 3$. To this end we apply Proposition 3.3 and show that for each edge E of the transformed constellation the edge pencil of Definition 3.1 induces by Definition 2.1 an operator $\mathcal{A}_\lambda^\pi$ (interior edges) or $\mathcal{A}_\lambda^{\mathrm{D}}$ (outer edges) which has a trivial kernel for all λ with $0 < \Re\lambda \leq 1/2$. The occurring edges are the following, see also Figure 7:

- Mono–material outer edges like for instance $\overline{Q_1^- Q_0^-}$.
- Bi–material outer edges like for instance $\overline{Q_0^\diamond Q_0^+}$, $\overline{Q_0^+ P_0^+}$, $\overline{P_0^+ P_1^+}$, $\overline{P_1^+ P_1^\diamond}$, and $\overline{P_1^\diamond Q_0^\diamond}$.
- The four–material interior edges $\overline{P_0^\diamond P_0^+}$ and $\overline{P_0^- P_0^\diamond}$, the image of the edge on the Neumann part Γ of the boundary.
- The four–material interior edges $\overline{P_0^\diamond P_1^\diamond}$ and $\overline{P_0^\diamond Q_1^\diamond}$, the image of the intersection of the original Neumann face and the original material interface.
- The four–material interior edge $\overline{P_0^\diamond Q_0^\diamond}$ and its reflection at the ϕ image of the original Neumann face.

5.2 Discussion of the edge singularities

Due to the symmetry relations of the transformed problem, Proposition 3.4 and Proposition 3.5 provide for mono–material outer edges like for instance $\overline{Q_1^- Q_0^-}$ and for bi–material outer edges like for instance $\overline{Q_0^\diamond Q_0^+}$ that $\mathcal{A}_\lambda^{\mathrm{D}}$ has a trivial kernel if $\Re\lambda \in]0, 1/2]$.

Next we consider the image of the edge on the Neumann part Γ of the boundary. Let us first regard $\overline{P_0^\diamond P_0^+}$ and make use of Lemma 4.6. Thus, we only have to show that the operators $\mathcal{A}_\lambda^{\mathrm{D}}$ and $\mathcal{A}_\lambda^{\mathrm{N}}$ of the bi–material edge pencil both have a trivial kernel if $\Re\lambda \in]0, 1/2]$. This is provided again by Proposition 3.5. The same disquisition applies to the four–material edge $\overline{P_0^- P_0^\diamond}$.

The image of the intersection of the original Neumann face with the original material interface is made up of the four–material interior edges $\overline{P_0^\diamond P_1^\diamond}$ and $\overline{P_0^\diamond Q_1^\diamond}$. To get the edge pencil of $\overline{P_0^\diamond P_1^\diamond}$ we transform the problem according to Definition 3.1. The orthogonal transformation matrix ω is given here by

$$\omega = \begin{pmatrix} \cos\varsigma & 0 & -\sin\varsigma \\ 0 & 1 & 0 \\ \sin\varsigma & 0 & \cos\varsigma \end{pmatrix} \quad \text{with } \varsigma = \sphericalangle P_1^\diamond P_0^\diamond P_0^+.$$

A straightforward calculation shows that the matrix function ρ has the following structure:

$$\rho(\theta) = \begin{cases} \begin{pmatrix} s_{11} & -s_{12} \\ -s_{12} & s_{22} \end{pmatrix} & \text{if } -\theta \in]\zeta, \pi[, \\ \begin{pmatrix} t_{11} & -t_{12} \\ -t_{12} & t_{22} \end{pmatrix} & \text{if } -\theta \in]0, \zeta[, \\ \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} & \text{if } \theta \in]0, \zeta[, \\ \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} & \text{if } \theta \in]\zeta, \pi[\end{cases}$$

for some $\zeta \in]0, \pi[$, Thus, one gets the result as for the edge $\overline{P_0^\diamond P_0^+}$, using again Lemma 4.6 and Proposition 3.5.

It remains to delimitate the singularities at the four–material interior edge $\overline{P_0^\diamond Q_0^\diamond}$ and, correspondingly, at its reflected counterpart. Again we transform the problem according to Definition 3.1 to get the edge pencil of $\overline{P_0^\diamond Q_0^\diamond}$ and we show that the operator $\mathcal{A}_\lambda^\pi$, see Definition 2.1, has a trivial kernel if $\Re\lambda \in]0, 1/2]$.

Suppose that for some λ with $\Re\lambda \in]0, 1/2]$ there is a (nontrivial) function v_λ in the kernel of $\mathcal{A}_\lambda^\pi$. Then, by Proposition 4.7, there is a compactly supported element $f \in W^{-1,6}(\mathbb{R}^3)$ such that the — also compactly supported — variational solution $\psi \in W^{1,2}(\mathbb{R}^3)$ of the equation $-\nabla\cdot\mu\nabla\psi = f$ does not belong to $W^{1,4}(\mathbb{R}^3)$. Here μ is according to (4.9). Since the support of ψ is compact, ψ cannot belong to $W^{1,6}(\mathbb{R}^3)$. Additionally, we may assume that both the support of f and ψ are arbitrarily small, hence $\operatorname{supp}\psi \subset \omega\phi(\Omega)$. Now we revoke the transformations ω and ϕ. By Proposition 4.1, applied with the bi–Lipschitz transformation $\tilde{\phi} = \phi^{-1}\circ\omega^{-1}$ one obtains a distribution $f_{\tilde{\phi}} \in W^{-1,6}(\mathbb{R}^3)$ and a function $\psi_{\tilde{\phi}} \in W^{1,2}(\mathbb{R}^3) \setminus W^{1,4}(\mathbb{R}^3)$ satisfying $-\nabla\cdot\mu_{\tilde{\phi}}\nabla\psi_{\tilde{\phi}} = f_{\tilde{\phi}}$. By making the support of ψ and f sufficiently small one can arrange that the matrix valued function $\mu_{\tilde{\phi}}$ equals in a neighbourhood of $\operatorname{supp}\psi \cup \operatorname{supp} f$ the original coefficient function μ which is constant above and below the plane Ξ. (Thus, the "artificial" edge we investigate here has "disappeared" again.) By [10, Thm. 3.11], see also [2, Ch. 4.5], the operator $-\nabla\cdot\mu_{\tilde{\phi}}\nabla + 1 : \ W^{1,p}(\mathbb{R}^3) \to W^{-1,p}(\mathbb{R}^3)$ is a topological isomorphism for all $p \in]1, \infty[$ if $\mu_{\tilde{\phi}}$ is constant on two complementing halfspaces. But, this contradicts the above supposition that the kernel of $\mathcal{A}_\lambda^\pi$ is not trivial.

The proof for the reflected counterpart of $\overline{P_0^\diamond Q_0^\diamond}$ is similar to that for $\overline{P_0^\diamond Q_0^\diamond}$.

6 Proof of Theorem 1.2

As in the proof of Theorem 1.1 we first regularity invariant transform the spatio–material constellation such that we meet the premise of Proposition 3.3. Then we discuss the edge singularities of the transformed problem to obtain the regularity of (1.1) according to Proposition 3.3.

With respect to the prisms under consideration inTheorem 1.2 (see also Figures 2&3) we can assume without loss of generality that the generating two–dimensional domain $\triangle_1 \cup \triangle_2 \cup \overline{P_0P_1}$ is in the x–y plane and that P_1 is the origin in $\mathbb{R}^3$. This always can be achieved by a shift and a rotation without changing the regularity properties, see Proposition 4.1. Now we further transform the prism, treating the cases $P_1 \neq Q_1$ and $P_1 = Q_1$ separately.

Let us first regard the case $Q_1 = P_1$ as exemplified on the right hand side of Figure 2 and in Figure 3. By rotation around the z axis we ensure that P_0 is on the positive y axis. If necessary, we achieve by reflection at the y–z plane that $\triangle_1$ is in the halfplane $x < 0$ and $\triangle_2$ consequently in the halfplane $x > 0$, see also the left hand side of Figure 8. Furthermore, we define a piecewise linear mapping ϕ_{xy} of the x–y plane onto itself which is the identity on the y axis and is on $x \leq 0$ such that $P_2 \mapsto \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right)$ while on $x \geq 0$ it is such that $Q_2 \mapsto \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, see the right hand side of Figure 8. Thus, by (5.1) we get a bi–Lipschitz transformation from $\mathbb{R}^3$ onto $\mathbb{R}^3$ which maps the originally right prism Ω onto the direct product of the open triangle $\triangle P_0P_2Q_2$ with some (open) interval on

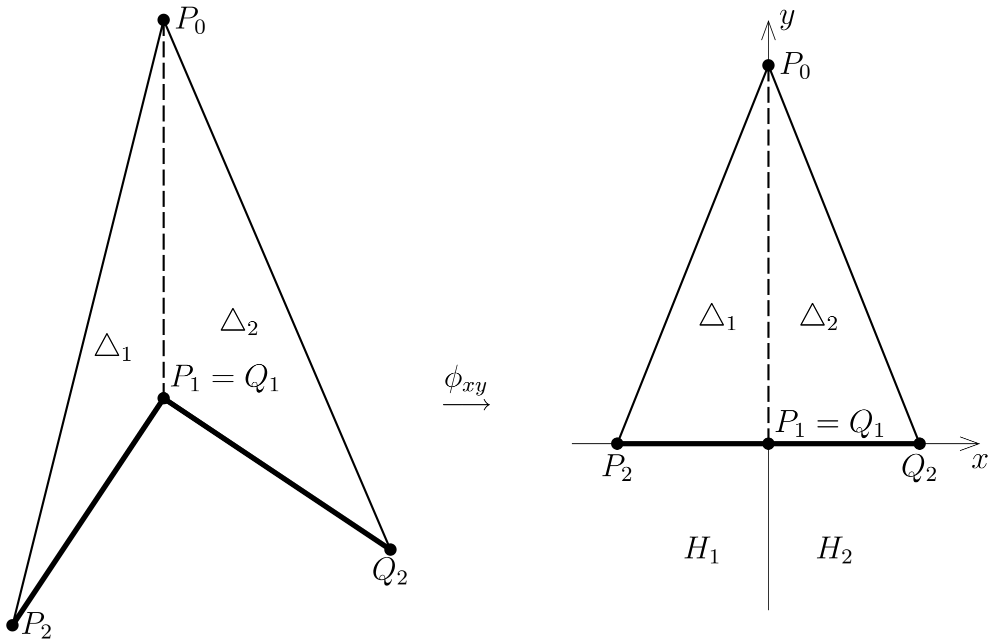

Figure 8: Two–dimensional domain which generates the right prism $\Omega = (\triangle_1 \cup \triangle_2 \cup \overline{P_0 P_1}) \times \{h_- < z < h_+\}$ of Theorem 1.2 (case $P_1 = Q_1$) before (left) and after transformation (right) by ϕ_{xy}. Without loss of generality one can assume that the domain $\triangle_1 \cup \triangle_2 \cup \overline{P_0 P_1}$ is in the x–y plane and that P_1 is the origin of $\mathbb{R}^3$ and that P_0 lies on the positive y axis. The transformation ϕ_{xy} maps the halfplane $x \leq 0$ linearly onto itself such that it is the identity on the y axis and maps P_2 onto the negative x axis; it maps the halfplane $x \geq 0$ linearly onto itself such that it is the identity on the y axis and maps Q_2 onto the positive x axis. The bold line is the trace of the Neumann boundary part Γ and the dashed line is the trace of the material interface Ξ. — Transformed objects bear the same name as before.

the z axis; note that we keep names through the various transformations. Hence, the Neumann boundary part is now a domain in the x–z plane and Proposition 4.5 applies to the whole constellation. After material reflection at the Neumann boundary we end up with a polyhedral multi–material compound with homogeneous Dirichlet conditions on the whole boundary which conforms to Proposition 3.3.

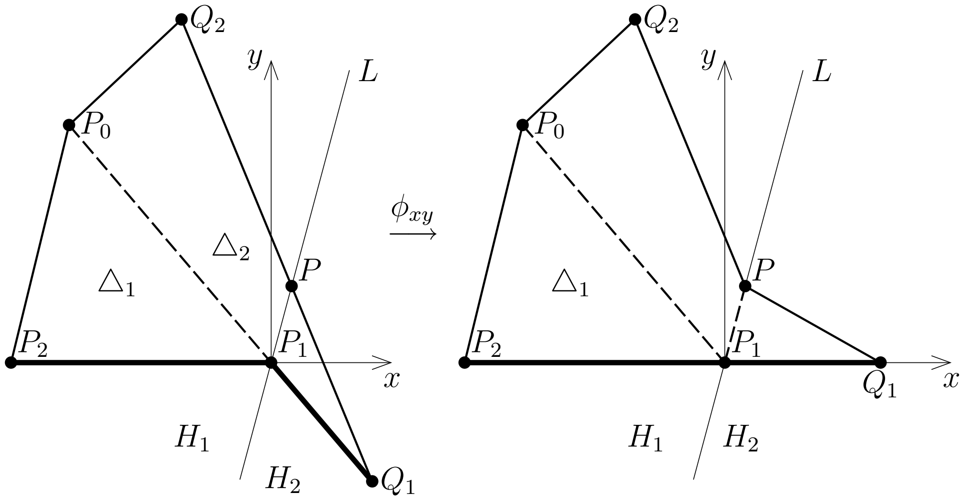

Figure 9: Two–dimensional domain which generates the right prism $\Omega = (\triangle_1 \cup \triangle_2 \cup \overline{P_0P_1}) \times \{h_- < z < h_+\}$ of Theorem 1.2 (case $P_1 \neq Q_1$) before (left) and after transformation (right) by ϕ_{xy}. Without loss of generality one can assume that the domain $\triangle_1 \cup \triangle_2 \cup \overline{P_0P_1}$ is in the x–y plane and that P_1 is the origin of $\mathbb{R}^3$. The line L which separates the halfspaces H_1 and H_2 has been chosen such that L cuts the triangle Δ_2 but not Δ_1. The piecewise linear transformation ϕ_{xy} is the identity on H_1 and maps H_2 onto itself such that L is invariant and the image of Q_1 is on the positive x axis. The bold line is the trace of the Neumann boundary part Γ and the dashed lines are traces of material interfaces. Before transformation there is only the trace of the original material interface Ξ, while after transformation by ϕ_{xy} there is another interface $\overline{P_1P}$. More precisely, the ϕ_{xy} image of $\triangle_2 = \triangle P_0Q_1Q_2$ is the tetragon $\square P_0P_1PQ_2$ (which has the original material properties of $\triangle_2$) plus the triangle $\triangle PP_1P_2$ (which has different material properties after transformation). — Transformed objects bear the same name as before.

Next we consider the case $Q_1 \neq P_1$ as exemplified on the left hand side of Figure 2. By rotation around the z axis we ensure that P_2 is on the negative x axis. If necessary, we achieve by reflection at the x–z plane that $\triangle_1$ is in the halfplane $y > 0$, see also the left hand side of Figure 9. We now choose a line L through the origin which splits the x–y plane into halfspaces H_1 and H_2 such that L cuts the triangle Δ_2 but not Δ_1. Furthermore, we define a piecewise linear mapping ϕ_{xy} of the x–y plane onto itself which is the identity on H_1 and maps H_2 onto itself such that L is invariant and the image of Q_1 is on the positive x axis, see the right hand side of Figure 9. The line L cuts $\overline{Q_1Q_2}$ in a point P. The ϕ_{xy} image of $\triangle_2 = \triangle P_0Q_1Q_2$ is the tetragon $\square P_0P_1PQ_2$ (which has the original material properties of $\triangle_2$) plus the triangle $\triangle PP_1P_2$ (which has different material properties after transformation). Thus, by (5.1) we get a bi–Lipschitz transformation from $\mathbb{R}^3$ onto $\mathbb{R}^3$ which maps the originally right prism Ω onto the direct product of an open polygon (defined by the successional vertices P, Q_1, P_2, P_0, and Q_2) with some (open) interval on the z axis; note that we keep names through the various transformations. Hence, as in the previous case the Neumann boundary part is now a domain in the x–z plane and Proposition 4.5 applies to the whole constellation. After material reflection at the Neumann boundary we end up again with a polyhedral multi–material compound with homogeneous Dirichlet conditions which conforms to Proposition 3.3.

6.1 Discussion of the edge singularities

We now discuss the edge singularities of the transformed constellation according to Proposition 3.3 for both cases $P_1 \neq Q_1$ and $P_1 = Q_1$. First there are mono–material outer edges, then bi–material outer edges, and finally an interior edge on the z axis which is a four-material edge in the case $P_1 = Q_1$ and a six-material edge in the case $P_1 \neq Q_1$. For the outer edges we know by Propositions 3.4&3.5 that the kernels of the associated edge pencil operators $\mathcal{A}_\lambda^{\mathrm{D}}$ with Dirichlet boundary conditions, see Definition 3.1 and Definition 2.1, are trivial for all λ with $0 < \Re\lambda \leq 1/2$. Note that the angle conditions are fulfilled by construction.

It remains to show for the multi–material interior edge on the z axis that the induced operator $\mathcal{A}_\lambda^{\pi}$, see Definition 2.1, has a trivial kernel for all λ with $0 < \Re\lambda < 1/3 + \epsilon$ and some $\epsilon > 0$.

It follows directly from Proposition 4.5 that the left upper block ρ of the coefficient matrix μ_ω, see Definition 3.1, corresponding to the edge on the z axis is as in Lemma 4.6.

If $P_1 = Q_1$, in each halfplane $y > 0$ and $y < 0$ of the x–y plane there are two sectors. Thus, according to Lemma 4.6 and Proposition 3.5 the kernels of the associated edge pencil operators $\mathcal{A}_\lambda^{\mathrm{D}}$ and $\mathcal{A}_\lambda^{\mathrm{N}}$, see Definition 2.1, are trivial for all λ with $0 < \Re\lambda \leq 1/2$.

If $P_1 \neq Q_1$ we have to deal with three sectors in each of the halfplanes $y > 0$ and $y < 0$. However, these three sectors, for instance in the halfplane $y > 0$, have been obtained by a bi–Lipschitz transformation ϕ_{xy} from originally a two sector constellation. Thus, by Lemma 4.3 and Proposition 3.5 we have $\ker \mathcal{A}_\lambda^{\mathrm{D}} = \ker \mathcal{A}_\lambda^{\mathrm{N}} = \{0\}$ for all λ with $0 < \Re\lambda \leq 1/2$. Lemma 4.6 now ensures that $\ker \mathcal{A}_\lambda^{\pi}$ is also trivial for these λ.

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