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Minimum return guarantees with funds switching rights– An optimal stopping problem

Antje Mahayni[1], John Schoenmakers[2]

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[1] Mercator School of Management
Universität Duisburg-Essen
Lotharstr. 65
D-47057 Duisburg
Germany,
E-Mail: antje.mahayni@uni-due.de
john.schoenmakers@wias-berlin.de

[2] Weierstrass Institute
for Applied Analysis and Stochastics
Mohrenstr. 39
10117 Berlin
Germany

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: +49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

Recently, there is a growing trend to offer guarantee products where the investor is allowed to shift her account/investment value between multiple funds. The switching right is granted a finite number per year, i.e. it is American style with multiple exercise possibilities. In consequence, the pricing and the risk management is based on the switching strategy which maximizes the value of the guarantee put option. We analyze the optimal stopping problem in the case of one switching right within different model classes and compare the exact price with the lower price bound implied by the optimal deterministic switching time. We show that, within the class of log-price processes with independent increments, the stopping problem is solved by a deterministic stopping time if (and only if) the price process is in addition continuous. Thus, in a sense, the Black & Scholes model is the only (meaningful) pricing model where the lower price bound gives the exact price. It turns out that even moderate deviations from the Black & Scholes model assumptions give a lower price bound which is really below the exact price. This is illustrated by means of a stylized stochastic volatility model setup.

1 Introduction

Recently, there is a growing trend towards products which, in addition to some minimum return rate guarantee, provide the right to shift the current account value between multiple funds. A prominent example is a so called guaranteed minimum accumulation benefit where the switching right is offered as an additional rider.[1] The payoff of the products under consideration is given by the maximum of a guaranteed amount and the value of an investment strategy which is chosen by the buyer/investor. Alternatively, the payoff can be stated as the terminal value of the investment strategy plus the payoff of a put option. The underlying of the put is a synthetic asset defined by the investment strategy. The guaranteed amount gives the strike of the option. Thus, there is a close link between the contract price and the investment strategy. While the investment decisions are influenced by the preferences of the investor, the pricing and the risk management of the embedded put option are based on the risk neutral measure. In particular, the provider must take into account for the *worst case* strategy which maximizes the value of the embedded put option. Although the main motivation of this paper stems from the analysis of guaranteed minimum accumulation benefits, the results are valid for products where the underlying of a put–option (call–option) is the investment strategy of a retail investor who faces short–selling and borrowing constraints.

The main focus of the paper is on the value maximizing strategy, or on the riskiness of (pseudo) assets underlying an option, and its impact on the option price.[2] We consider a given contract maturity.[3] The guarantee put is a plain vanilla European put option but the underlying is exotic, i.e it is an investment/switching strategy. Normally, the rider to switch between multiple funds is granted four times per year. In this article we consider the case of one switching right, i.e. the switching right is of American type. The investor is allowed to shift her whole portfolio value to another asset once during the contract horizon. The switching time which maximizes the expected discounted payoff

[1]These products belong to the class of Variable Annuities. Products which are currently traded on the market are for example AXA Twinstar, Allianz Invest4Life, R+V Premium GarantRente and Swiss Life Champion.

[2]A discussion how the riskiness of assets (distributions) is measured already dates back to Rothschild and Stiglitz (1970). The relation between the value of a call–option and the riskiness of its underlying stock is firstly discussed in Jaganathan (1984). While diversification aspects do play an important role under the *real world measure*, this is not the case under the risk neutral (pricing) measure.

[3]Any mortality risk is assumed to be perfectly diversifiable, i.e. we do not take into account for a stochastic mortality rate.

gives the price setting of the provider. The resulting optimal stopping problem is our key problem.[4] We also consider the *optimal* deterministic stopping time. The associated lower price bound can be interpreted in terms of freezing the contract. In addition, the difference in the associated option prices is interesting because of the following reasoning. It is realistic to assume that the investor herself does not use the optimal (price maximizing) stopping strategy. To name two prominent explanations: First, the optimal strategy is the most risky strategy. However, the investor is risk–averse. She maximizes her expected utility under the real world measure instead of the risk neutral one.[5] Second, the investor faces model risk, i.e. she does not know the true data generating process. In summary, the investor pays, in terms of the embedded option, for the most expensive strategy she is not going to implement anyway.[6] Thus, it is also interesting to consider the difference of the price setting according to the overall optimal stopping strategy and the price w.r.t. some benchmark strategy (here the optimal deterministic solution).

We focus on different model classes. Besides a simple Black and Scholes (1973) model setup we also consider the impact of jumps and stochastic volatility. Thus, we do not aim at a sophisticated numerical pricing algorithm within one state of the art model.[7] Instead, we are merely interested in the effects of jumps and stochastic volatility on the difference between the exact price and the (tractable) lower price bounds.

Our main contributions are as follows. In a Black & Scholes type model setup, the overall optimal solution and the deterministic stopping solution coincide. The most risky strategy maximizes the cumulated deterministic volatility (all–in volatility). Thus, the exact price coincides with the lower price bound and is given in closed–form. However, we show that, within the class of log-price processes with independent increments, the stopping problem is solved by a deterministic stopping time if (and only if) the price process is in addition continuous. Thus, the Black & Scholes model may be regarded as the only (meaningful) pricing model where the lower price bound gives the exact price. We also consider the additional problem which is posed by taking into account for stochastic volatility. It is safe to say that the risk structure of the economy itself, i.e. the riskiness of the underlying assets, changes randomly over time. We consider a *simple* regime switching volatility model. We state some meaningful (tractable) lower and upper price bounds which are motivated by different stopping rules. W.r.t. a discrete set of switching times, we also determine the optimal stopping price via multiple integration. In a numerical example we compare the resulting prices and price bounds. It turns out that the relative pricing error which is due to the usage of a simplified stopping rule, in particular the one which is implied by the optimal deterministic stopping time, can be significant. Surprisingly, this is even true for slight deviations from the Black & Scholes model setup. Finally, we hint at the impacts on the *fair* contract design of minimum accumulation benefits and its consequences for the buyer.

Without claiming completeness, we mention some related literature. First, we consider the one dealing with the fair pricing of options which are embedded into minimum return contracts. Pricing embedded options by no arbitrage already dates back to Brennan and Schwartz (1976). We also refer to the works of Boyle and Schwartz (1977), Delbaen (1986), Bacinello and Ortu (1993a,b), Ekern and Persson (1996), Boyle and Hardy (1997), Bacinello (2001), Miltersen and Persson (2003), Coleman et al. (2006), Coleman et al. (2007) and Nielsen et al. (2009). A description of the different

[4] Notice that in the case of continuous–time switching, the worst case strategy can *easily* be found by well known results from the literature. For example, in a stochastic volatility model, the most risky (risk neutral) distribution is generated by a worst case asset price dynamics where the volatility process is equal to the maximum process of all single asset volatilities, cf. Bergman et al. (1996).

[5] In this paper, we take an emphasis on the stopping problem, i.e. the pricing. We do not consider the expected utility maximization problem (under short sale and borrowing constraints) of the investor further. The interested reader is, for example, referred to the works of El Karoui et al. (2005), Tepla (2000) and Tepla (2001).

[6] We do not analyze the investor's choice of funds. This is an interesting and important topic, but beyond the scope of this paper. For example, we refer to Goldbaum and Mizrach (2008) who analyze the intensity of choice in an agent based financial optimization problem.

[7] The numerical evaluation techniques which are needed in our context is similar to the ones for American compound options. The numerical problem is an interesting problem for its own. References on the existing literature are given at the end of this introduction.

product groups which are subsumed under the class of variable annuities and a universal pricing framework are given in Bauer et al. (2008). Benhamou and Gauthier (2009) price these products under stochastic volatility and interest rates.

We also refer to the literature on Passport options, cf. Andersen et al. (1998), Henderson and Hobson (1998), Delbaen and Yor (1998), Nagayama (1999), and Shreve and Vecer (2000). To some extend, these options are similar to our problem formulation. Passport options are options on an investment strategy (traded account, respectively), too. Their pricing also relies on the value maximizing strategy. Nevertheless, the pricing problem is different to ours. The strategies underlying a Passport option are restricted by their number of assets. Normally, the investor is allowed to take positions between -1 (short) and 1 (long) in one asset, i.e. a Passport option refers to *position limits*. In a Black & Scholes model setup, the value maximizing strategy is locally characterized by an extremal position of 1 (if behind) or -1 (if ahead). In contrast, the (implicit) restriction which the investor of a guaranteed minimum accumulation benefit faces concerns the fractions of the account value which she can invest in multiple assets, i.e. we have *limits on the investment fractions*. Besides the self–financing condition, all weights are restricted to the interval $[0, 1]$. In particular, we already take the extremal situation/strategy where the investor shifts 100% of the account value from one asset to another asset as the starting point of our analysis.

The additional rider to switch between different funds is firstly analyzed in Mahayni and Schneider (2010). By means of expected utility maximization, the authors focus on the trade–off between the *optimal* diversified strategy and the value maximizing strategy. However, they do not restrict the number of possible funds switches which gives our pricing problem. To the best of our knowledge, there is no work on this particular stopping problem until now. To some extend, our problem can be linked to the one which is posed by an American compound option. Taking into account for stochastic volatility and/or jumps makes the pricing problem numerically demanding. Recent (numerical) literature on the pricing of American Call options includes Chiarella and Ziogas (2005), Chiarella and Ziogas (2009) or, under stochastic volatility and jump diffusion, Chiarella et al. (2009).[8] The evaluation of American compound options is analyzed in Chiarella and Kang (2009).

The paper is structured as follows. In Sec. 2, we motivate and formulate the key stopping problem and its deterministic counterpart, i.e. the optimization problem w.r.t. deterministic switching times. We give the problem solution w.r.t. the Black & Scholes model and show that this is the only meaningful model which implies that the two solutions (overall and deterministic) coincide. In particular, we show that in a time inhomogeneous Levy framework (the framework which essentially characterizes all processes with independent increments) the Black & Scholes model is the only pricing model which implies a deterministic stopping time. In Sec. 3, we also take into account random changes in the risk structure of the economy, i.e. we consider a stochastic volatility setup. For a regime switching volatility model, we determine upper and lower price bounds on the embedded option price. Sec 4. gives the exact price by multiple integration for a discrete set of switching times. Based on a numerical example, we discuss and compare the exact prices and price bounds. Sec. 5 then gives the application to the *fair* contract design of guaranteed minimum accumulation benefits. Sec. 6 concludes the paper.

2 Key Problem and Log–Processes with independent increments

First, we state the key problem. We omit a detailed discussion of the design of minimum accumulation benefits. An application to minimum accumulation benefits is dedicated to Section 5. The nature of the problem is much more general. We consider a put option on an investment strategy. The put option is a plain vanilla put, but the underlying is exotic. To simplify the expositions, we consider one–time switching strategies where the whole account value is shifted from one asset/funds into another asset.

[8] Cf. also the literature given herein.

We assume an economy which lives on an initial stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}_0)$, which satisfies the usual hypotheses. It is assumed that trading terminates at time T. With respect to this basis we consider arbitrage-free asset processes S^1 and S^2 which are described by strictly positive semi-martingales, and in addition a risk-less asset (bond) $B_t := e^{rt}$. Absence of arbitrage requires that there exists a nonempty set of measures $\mathcal{Q}$ such that for each $\mathbb{P} \in \mathcal{Q}$, $\mathbb{P} \sim \mathbb{P}_0$ and S^i/B is a $\mathbb{P}$-martingale, for $i = 1, 2$.[9] Henceforth we sort out from this collection a fixed pricing measure $\mathbb{P}^* \in \mathcal{Q}$ and consider all dynamics under this measure $\mathbb{P}^*$. In practice, this pricing measure is typically obtained implicitly by calibrating the underlying asset price processes to liquid products quoted in the market. Since we will only work within this measure we may next drop the * for notational convenience.

DEFINITION 2.1 (Trading Strategy, switching strategy). *A trading strategy ϕ in the assets S^1, S^2 is given by a $\mathbf{R}^2-$ valued, predictable process on $[0,T]$ which is integrable with respect to S on $[0,T]$. The value process $V(\phi)$ associated with ϕ is defined by*

$$V_t(\phi) = \sum_{i=1}^{2} \phi_t^i S_t^i \qquad 0 \le t \le T.$$

A (one–time) switching strategy $\phi = (\phi^1, \phi^2)$ is defined by a stopping time τ with $\tau \in [0,T]$ a.s., where

$$\phi_t^1 = 1_{t\le\tau} \text{ and } \phi_t^2 = \frac{S_\tau^1}{S_\tau^2} 1_{t>\tau}. \tag{1}$$

For given τ the value process of this switching strategy is given by

$$V_t^{(\tau)}(\phi) = S_t^1 1_{t\le\tau} + \frac{S_\tau^1}{S_\tau^2} S_t^2 1_{t>\tau}.$$

Along the lines of Equation (1), a (one–time) switching strategy characterizes an investor who is initially (100%) invested in a single asset (called S^1). The strategy is a buy–and–hold strategy on the interval $[0,\tau]$. Immediately after τ, the investor shifts her whole account value to another asset (called asset S^2). Obviously, a single switching strategy is self–financing, i.e., after an initial investment, there are no further in– or outflows.[10] Thus, the value process $V^{(\tau)}$ can also be interpreted as the price process of a synthetic asset.

Now, consider a European put–option with maturity T and strike K on the (pseudo) asset $V^{(\tau)}$. Using standard theory of no–arbitrage, the value of the put (with respect to the chosen pricing measure $\mathbb{P}$) is given by the expectation of the discounted payoff, that is

$$E(K - V_T^{(\tau)})^+ e^{-rT} = E(Ke^{-rT} - \frac{S_\tau^1}{S_\tau^2} S_T^2 e^{-rT})^+ =: E(\widehat{K} - \frac{\widehat{S}_\tau^1}{\widehat{S}_\tau^2} \widehat{S}_T^2)^+.$$

Hence, to simplify the notation, we may set w.l.o.g. the interest rate equal to zero, that is, we think in terms of discounted assets $\widehat{S^i}$ and a discounted strike $\widehat{K}$. In particular, if the investor receives the above put, her terminal wealth is floored by the strike K.[11] In the case that the buyer of the contract is allowed to decide on the switching strategy τ herself, the provider must take into account for the value maximizing switching strategy τ. The pricing problem is

$$Y_0^* := \sup_{\tau, 0\le\tau\le T} E(K - \frac{S_\tau^1}{S_\tau^2} S_T^2)^+,$$

where strike and assets are now considered to be discounted.

[9] The martingale pricing theory is based on the works of Harrison and Kreps (1979) and Harrison and Pliska (1983).

[10] To be more precise, ϕ is self–financing iff $V_t(\phi) = V_0(\phi) + \sum_{i=1}^2 \int_0^t \phi_u^i \, dS_u^i$.

[11] Obviously, to finance the put option, the initial investment must be reduced by the price of the associated put option. A detailed discussion of this is postponed to Section 5 which applies the results to guaranteed minimum accumulation benefits.

DEFINITION 2.2 (Optimization problems). *The key problem is given by*

$$\tau^* := \underset{\tau, 0\le\tau\le T}{\arg\sup}\, E\left(K - \frac{S^1_\tau}{S^2_\tau} S^2_T\right)^+ \tag{2}$$

where τ is some stopping time and E denotes the expectation under the risk neutral (pricing) measure $\mathbb{P}$. [12] *In addition, we also consider the simplified problem which is implied by deterministic switching times $t \in [0,T]$, i.e.*

$$t^* := \underset{t, 0\le t\le T}{\arg\sup}\, E\left(K - \frac{S^1_t}{S^2_t} S^2_T\right)^+ \tag{3}$$

(where a similar remark as Footnote 12 applies).

Notice that the first optimization problem seeks for the single switching strategy which maximizes the value of the put option. It is exclusively specified in terms of the optimal stopping time τ^*. In contrast, the second optimization problem restricts the set of switching strategies to the ones where the switching time is already fixed today, i.e. the optimization is only due to deterministic switching times t. In general, the optimal deterministic time t^* gives only a lower price bound of the claim. Problem (3) can be interpreted as freezing the contract. Alternatively, it can be used as a (tractable) approximation of the price or as a benchmark strategy which is used by the investor. In a first step, we consider a model setup where both solutions coincide such that the lower price bound gives the exact price.

LEMMA 2.3 (Black & Scholes model). *Let the (discounted) asset price processes are given by*

$$dS^i_t = S^i_t \sigma_i^\top(t)\, dW_t,\, S^i_0 = s^i_0 \quad (i = 1,2) \tag{4}$$

where the σ_i are two dimensional bounded volatility vector functions, and where W denotes a two–dimensional Brownian motion under $\mathbb{P}$. Then, the put price implied by a deterministic switching time $t \in [0,T]$ is given by

$$E\left(K - \frac{S^1_t}{S^2_t} S^2_T\right)^+ = \mathcal{B}(S^1_0, K, v(0,t,T)). \tag{5}$$

The right-hand-side in (5) *is Black's 77 formula*

$$\mathcal{B}(S,K,v) = -S\mathcal{N}\left(\frac{\log\frac{K}{S} - \frac{1}{2}v^2}{v}\right) + K\mathcal{N}\left(\frac{\log\frac{K}{S} + \frac{1}{2}v^2}{v}\right), \tag{6}$$

where $v =: \sigma^{Black}\sqrt{T}$ is usually referred to as an "all-in" volatility, $\mathcal{N}$ denotes the cumulated distribution function of the standard normal distribution, and

$$v(0,t,T) := \sqrt{v_1^2(0,t) + v_2^2(t,T)} \tag{7}$$

$$\text{where } v_i(t,T) := \sqrt{\int_t^T \|\sigma_i(u)\|^2\, du}, \qquad i = 1,2, \tag{8}$$

denote 'all-inn' volatilities with respect to the σ_i.

In particular, the solution of the simplified problem (3) is given by

$$t^* = \underset{t,\ 0\le t\le T}{\arg\sup}\, v^2(0,t,T) = \underset{t,\ 0\le t\le T}{\arg\sup}\, v_1^2(0,t) + v_2^2(t,T) = \underset{t,\ 0\le t\le T}{\arg\sup}\, v_2^2(t,T) - v_1^2(t,T). \tag{9}$$

[12] If there are more than one optimal stopping times, τ^* is to be understood as the infimum of them.

PROOF: The proof follows immediately by the observation that the asset price increments are log-normally distributed, i.e.

$$\ln \frac{S_t^1}{S_t^2} S_T^2 \sim N\left(., v_1^2(0,t) + v_2^2(t,T)\right)$$

and that the Black (& Scholes) pricing formula is monotonously increasing in the all-in volatility v.

THEOREM 2.4 (Optimal stopping time versus optimal deterministic time). *(i) In a Black & Scholes type model where the asset price dynamics are given by Equation (4), the solutions of the problems (2) and (3) coincide. In particular, it holds $\tau^* = t^*$ where t^* is given by Equation (9).*
(ii) The result (i) is impeded by the introduction of jumps. In particular, let us assume that the asset price dynamics are given by exponential Levy processes S^1 and S^2 driven by

$$\frac{dS_t^i}{S_t^i} = \sigma_i^\top(t) dW_t + \int (e^{u_i} - 1)(\mu - \nu)(dt, du) \tag{10}$$

where μ is a random measure on $[0,T] \times \mathbb{R}^2$, with compensator measure of the form $\nu(dt,du) := \nu(t,du)dt$ which satisfies certain integrability conditions (for details see Cont and Tankov (2004)). If $\nu \neq 0$, then, the optimal stopping time τ^* does not coincide with the optimal deterministic stopping time t^* (except possibly in the degenerate cases $\tau^* = 0$ or $\tau^* = T$).

PROOF: The proof is given in the appendix, cf. Appendix A.1 for part (i) and Appendix A.2 for part (ii) .

REMARK 2.5. It should be noted that any martingale with independent log increments is essentially an exponential Levy process which may be represented in the form of (10).[13] So, Theorem 2.4 basically states that within the class of log–price processes with independent increments, the stopping problem (2) is solved by a deterministic stopping time if (and only if) the price process is in addition *continuous*.

The intuition behind the Black & Scholes result, where the only risk parameter is a deterministic volatility function, is as follows. The optimal deterministic time t^* gives the maximal cumulated (all–in) volatility v up to T which is feasible. Then it can be easily seen that, as a consequence of independent (log-)increments, any stopping rule τ which prescribes to stop before t^* is suboptimal. Indeed, conditioning on the information $\mathcal{F}_\tau$ we compare Black & Scholes prices which are increasing in v. To validate that stopping after t^* is also suboptimal is more demanding however. [14]

Now, consider part (ii) where the result (i) breaks down. At a first glance, this is surprising as we do not introduce a new state variable here. Although we introduce jumps into the model, the risk structure of the asset price model is still deterministic and the logarithmic stock prices have still independent increments. However, in contrast to the Black & Scholes case, in the jump-diffusion model the optimal deterministic stopping time turns out to depend on the moneyness or strike of the option, which is stochastic in time. Hence the optimal deterministic time seen at time zero will generally not coincide with the optimal deterministic time seen (a little bit) later. Thus, the solutions of the optimization problems, overall and deterministic times, only coincide in the special case where $\nu = 0$, i.e. in the case of the Black & Scholes model.

Recall that the buyer of the put can decide on the single switching strategy τ. The put price according to the strategy τ^* defines the price setting of the provider. In contrast, the solution t^* gives a lower price bound. If the buyer/investor of the contract deviates from the strategy τ^*, she pays too much for her option.[15] To some extend, the difference of the exact price and the lower price bound can

[13] This may be easily inferred from Skorohod (1991) and Cont and Tankov (2004) .

[14] For details on this, cf. Appendix A.1.

[15] As mentioned in the introduction, there are various reasons which explain a deviation from the stopping strategy τ^*. For example, assume that the investor maximizes her expected utility of terminal wealth under the real world measure.

be interpreted as *sunk costs*. According to the above theorem, these costs can be explained by the introduction of jumps. Obviously, the magnitude of the effect depends on the jump size and jump intensity. However, there is another source which makes the additional contract feature to switch the funds expensive. Until now, we did not allow for random changes in the risk structure of the economy.

3 Stochastic volatility

Our key stopping problem is motivated by products which give a minimum return guarantee on the investment which, in terms of the stopping (switching time, respectively) can be chosen by the buyer. Typically, these are long term contracts such that the European option on the investment strategy is also a long term contract. Concerning the long time to maturity, it becomes important to take into account for random changes in the economy, in particular to take into account for stochastic volatility.

Intuitively, the impacts on the key problem are as follows. In addition to the dependence on the moneyness of the put option, which we already observed in a Levy–type model framework, we also need to tackle with the volatility of volatility. In a first step, we formulate a merely general stochastic volatility framework. A further step concerns a stylized version which is more tractable to explain the points and effects we are after. Along the ways, we highlight the similarity of our problem formulation to the one posed by an American compound option.

For now, let us assume that on our initial probability space the (discounted) asset price dynamics of asset S^1 and asset S^2 are given by

$$\frac{dS^i_s}{S^i_s} = \sigma_i^{\mathsf{T}}(s,V_s)dW \tag{11}$$

where V denotes some driving scalar volatility Markov process, and W is a two dimensional Brownian motion. Equation (11) immediately implies that for any $0 \le s \le t \le T$ it holds

$$S^i_t = S^i_s e^{-\frac{1}{2}\int_s^t \|\sigma_i(u,V_u)\|^2 du + \int_s^t \sigma_i(u,V_u)^\top dW_u}.$$

In addition, the arbitrage–free t–price of a European put option with maturity T on the asset S^i can be represented by

$$E^{\mathcal{F}_t}\left(\kappa - S^i_T\right)^+ = E^{\mathcal{F}_t}\left(\kappa - S^i_t e^{-\frac{1}{2}\int_t^T \|\sigma_i(u,V_u)\|^2 du + \int_t^T \sigma_i(u,V_u)^\top dW_u}\right)^+ =: P_i(t,T,S^i_t,V_t,\kappa)$$

where the function P_i, is smooth on its domain for $i = 1,2$. Moreover, the functions P_i have obviously the following degree-1 homogeneity property.

$$P_i(t,T,\beta S,V,\beta\kappa) = \beta P_i(t,T,S,V,\kappa), \quad \beta > 0, \quad i = 1,2. \tag{12}$$

Recall that the put option which is written on a single switching strategy τ is a plain vanilla option with an *exotic underlying*. Formally, for any stopping time τ, the (strong) Markov property together with the homogeneity property (12) implies that

$$\begin{aligned} E\left(K - \frac{S^1_\tau}{S^2_\tau} S^2_T\right)^+ &= EE^{\mathcal{F}_\tau}\left(K - S^1_\tau \frac{S^2_T}{S^2_\tau}\right)^+ = E\left[\frac{S^1_\tau}{S^2_\tau} E^{\mathcal{F}_\tau}\left(K\frac{S^2_\tau}{S^1_\tau} - S^2_T\right)^+\right] \\ &= E\left[\frac{S^1_\tau}{S^2_\tau} P_2(\tau,T,S^2_\tau,V_\tau,K\frac{S^2_\tau}{S^1_\tau})\right] = EP_2(\tau,T,S^1_\tau,V_\tau,K) \\ &= EE^{\mathcal{F}_\tau}\left(K - S^1_\tau e^{-\frac{1}{2}\int_\tau^T \|\sigma_2(u,V_u)\|^2 du + \int_\tau^T \sigma_2(u,V_u)^{\mathsf{T}} dW_u}\right)^+ =: EZ_\tau. \end{aligned} \tag{13}$$

Thus, the key problem (2) is equivalent to an American option on a cash-flow Z_t where

$$Z_t = E^{\mathcal{F}_t}\left(K - S^1_t e^{-\frac{1}{2}\int_t^T \|\sigma_2(u,V_u)\|^2 du + \int_t^T \sigma_2(u,V_u)^{\mathsf{T}} dW_u}\right)^+.$$

Notice that Z_t can be interpreted as the t–price of a European put option on S^1_T with strike K, but, where $\left(S^1_u, t \le u \le T\right)$ is driven by the *wrong* volatility process, namely the one actually driving the asset S^2 instead of S^1.

PROPOSITION 3.1 (Value of switching option). *Let Y^*_0 denote the t_0–price of the put on the value maximizing single switching strategy, then it holds*

$$Y^*_0 = E(K - S^1_T)^+ + \sup_{\tau,\ 0 \le \tau \le T} E\left(P_2(\tau, T, S^1_\tau, V_\tau, K) - P_1(\tau, T, S^1_\tau, V_\tau, K)\right).$$

In particular, the value of the switching right is given by the difference of Y^*_0 and the t_0–price of the single asset option on S^1 (i.e. staying with asset S^1).

PROOF:

$$\begin{aligned} Y^*_0 &= \sup_{\tau,\ 0 \le \tau \le T} EZ_\tau = \sup_{\tau,\ 0 \le \tau \le T} EP_2(\tau, T, S^1_\tau, V_\tau, K) \\ &= \sup_{\tau,\ 0 \le \tau \le T} \left\{E\left(P_2(\tau, T, S^1_\tau, V_\tau, K) - P_1(\tau, T, S^1_\tau, V_\tau, K)\right) + EP_1(\tau, T, S^1_\tau, V_\tau, K)\right\} \\ &= P_1(0, T, S^1_0, V_0, K) + \sup_{\tau,\ 0 \le \tau \le T} E\left(P_2(\tau, T, S^1_\tau, V_\tau, K) - P_1(\tau, T, S^1_\tau, V_\tau, K)\right) \\ &= E(K - S^1_T)^+ + \sup_{\tau,\ 0 \le \tau \le T} E\left(P_2(\tau, T, S^1_\tau, V_\tau, K) - P_1(\tau, T, S^1_\tau, V_\tau, K)\right). \end{aligned}$$

Obviously, the value of the switching right is non–negative, i.e. it holds (take $\tau = T$)

$$\sup_{\tau,\ 0 \le \tau \le T} E\left(P_2(\tau, T, S^1_\tau, V_\tau, K) - P_1(\tau, T, S^1_\tau, V_\tau, K)\right) \ge 0.$$

Hence $Y^*_0 \ge E(K - S^1_T)^+$ a.s. and it is never optimal to exercise at time t if

$$P_2(t, T, S^1_t, V_t, K) < P_1(t, T, S^1_t, V_t, K).$$

It is worth to emphasize that Proposition 3.1 together with the above remarks states that the value of the switching right is given by an American option on an option, i.e. we have, in addition to the American feature, to tackle a compound option.[16]

As a simplifying assumption we assume now that V and W are independent which allows a quasi closed–form solution for the simplified problem concerning the optimal deterministic stopping time. It is well known that the independence assumption implies that option prices can be represented as the average of Black & Scholes prices. In particular, if $\mathcal{F}_t = \mathcal{F}^W_t \vee \mathcal{F}^V_t$, where $\left(\mathcal{F}^V_t\right)$ and $\left(\mathcal{F}^W_t\right)$ denote the filtration generated by V and W respectively, then we have the following representation.

PROPOSITION 3.2 (Put price, optimal deterministic stopping time).

$$E^{\mathcal{F}_0}\left(K - \frac{S^1_t}{S^2_t} S^2_T\right)^+ = E\mathcal{B}(S_0, K, \sqrt{\int_0^t \|\sigma_1(u, V_u)\|^2\, du + \int_t^T \|\sigma_2(u, V_u)\|^2\, du})$$

where $\mathcal{B}(S, K, \sigma^{Black}\sqrt{T})$ is defined as in Equation (6). In particular,

$$t^* = \arg\sup_{0 \le t \le T} E\mathcal{B}(S_0, K, \sqrt{\int_0^t \|\sigma_1(u, V_u)\|^2\, du + \int_t^T \|\sigma_2(u, V_u)\|^2\, du}).$$

PROOF: For the sake of completeness, the proof is given in Appendix B.1.

Again, the above proposition emphasizes that the solution of the key problem (2) does, in general, not coincide with the optimal deterministic solution. For a deterministic switching time, the price of the put depends on the strike K (moneyness, respectively) as well as on the distribution of V. Therefore, the optimal deterministic stopping time changes over time which impedes its overall optimality.

[16] For example, Chiarella and Kang (2009) evaluate American compound option prices where mother and daughter options are America style under stochastic volatility using the sparse grid approach.

3.1 Single regime switch

In order to analyze the impacts of mild deviations from the Black & Scholes model where the solution of the key problem and its simplified version coincide, we consider a single regime switching model. In particular, there is only one (random) time η where the economy changes its risk structure. Within the regimes, the risk structure is summarized by a (deterministic) volatility. We now consider two correlated scalar Brownian motions W^1, W^2 with $d\langle W^1, W^2\rangle_t = \rho dt$ for some constant ρ, and a random variable η independent of $W^{1,2}$ on $[0,\infty)$ with distribution p, driving the asset price dynamics

$$\frac{dS_t^1}{S_t^1} = {}_\eta\sigma_1(t)dW^1 := (\sigma_{11}(t)1_{t\le\eta} + \sigma_{12}(t)1_{t>\eta})\, dW_t^1, \tag{14}$$

$$\frac{dS_t^2}{S_t^2} = {}_\eta\sigma_2(t)dW^2 := (\sigma_{21}(t)1_{t\le\eta} + \sigma_{22}(t)1_{t>\eta})\, dW_t^2, \tag{15}$$

where the $\sigma_{ij}(\cdot)$ are deterministic volatility functions describing the volatility of asset i in regime j, and the deterministic functions ${}_\cdot\sigma_i(\cdot)$ are defined accordingly. In fact, (14), (15) may be seen as a stochastic volatility model fitting into the framework (11), where $V_t = 1_{\eta>t}$ and $\mathcal{F}_t^V = \sigma\{\eta > t\}$ (with obvious definition of $\sigma_i(\cdot,\cdot)$ there).

We next consider a lower price bound which is implied by the optimal deterministic stopping time and then introduce an improved lower price bound which can be explained by a clever but still suboptimal investor. In addition, we consider an upper price bound of a *visionary* who knows when the regime switch occurs.

PROPOSITION 3.3 (Lower price bounds). *Let the asset price dynamics be given by Equations (14) and (15). It then holds*

(i) The lower price bound $\underline{Y}_0^{det}$ *implied by the optimal deterministic stopping time* t^0 *is given by*

$$\underline{Y}_0^{det} = \int_0^T P(S_0^1, K, t^\circ; u)\, p(u) du,$$

where

$$P(S_0^1,K,t;u) := E\left[\left(K - \frac{S_t^1}{S_t^2}S_T^2\right)^+ \middle|\, \eta = u\right] = \mathcal{B}\left(S_0^1, K, \sqrt{\int_0^t {}_u\sigma_1^2 ds + \int_t^T {}_u\sigma_2^2 ds}\right)$$

and $t^\circ = \underset{0\le t\le T}{\arg\sup} \int_0^T P(S_0^1,K,t;u)\, p(u) du.$

(ii) An improved lower price bound $\underline{Y}_0^{detmix}$ *(*$\underline{Y}_0^{detmix} \ge \underline{Y}_0^{det}$*) is given by*

$$\underline{Y}_0^{detmix} = \sup_{0\le s\le T}\left\{\int_0^s P(S_0^1,K,s;u)\, p(u) du + \int_s^T P(S_0^1,K,t_s(u);u)\, p(u) du\right\}$$

In particular, the above price bound is due to a 'good' nondeterministic stopping time $\tau_{s_0} = s_0 1_{s_0<\eta} + t_{s_0}(\eta) 1_{s_0>\eta}$ *where*

$$t_r(u) := \underset{s:\, r\le s\le T}{\arg\sup}\left\{\int_r^s {}_u\sigma_1^2 du + \int_s^T {}_u\sigma_2^2 du\right\} =: \underset{s:\, r\le s\le T}{\arg\sup}\, g_u(s) \tag{16}$$

$$\text{and } \underline{s}_0 = \underset{0\le s\le T}{\arg\sup}\left\{\int_0^s P(S_0^1,K,s;u)\, p(u) du + \int_s^T P(S_0^1,K,t_s(u);u)\, p(u) du\right\}.$$

PROOF: The proof is given in the appendix, cf. Appendix B.2.

$\underline{Y}_0^{\text{det}}$ denotes the put value according to the optimal deterministic switching date t°. In contrast, the investor associated with the value $\underline{Y}_0^{\text{detmix}}$ uses a semi–stochastic stopping strategy. She achieves a higher put value by the following reasonings. She also chooses a fixed switching date s at t_0. However, she does so with having in mind that the economy can change only once. Thus, after the regime switch, the model is a simple Black & Scholes model. In consequence, she optimizes according to a deterministic time s but considers the possibility that the regime switch may also occur before s. In this case, she switches the assets according to the optimal *lognormal rule* given in Theorem 2.4, part (i). In the opposite case, i.e. $s < \eta$, she switches at the time point s which she has fixed today. In analogy to her stopping strategy τ_{s_0}, which can be interpreted w.r.t. the deterministic time s_0 as well as the stochastic part introduced by η, we call the resulting lower price bound $\underline{Y}_0^{\text{detmix}}$. Now, consider the upper price bound.

PROPOSITION 3.4 (Upper price bound). *Let the model assumptions be given as in Proposition 3.3. Then, we have the following upper price bound* $\overline{Y}_0^{\text{vis}}$ *where*

$$\overline{Y}_0^{\text{vis}} = EE\left[\left(K - \frac{S^1_{t_0(\eta)}}{S^2_{t_0(\eta)}} S^2_T\right)^+ \middle| \eta\right] = \int_0^T P(S^1_0, K, t_0(u); u)\, p(u) du$$

and where $t_0(\eta)$ *is defined via* (16).

PROOF: Notice that Theorem 2.4 implies that $t_r(\eta)$ is the optimal stopping time prevailing at time r in a model where the regime switching date η is known a.s.. Thus, we immediately have

$$\sup_{\tau:\, 0 \le \tau \le T} E\left(K - \frac{S^1_\tau}{S^2_\tau} S^2_T\right)^+ \le EE\left[\left(K - \frac{S^1_{t_0(\eta)}}{S^2_{t_0(\eta)}} S^2_T\right)^+ \middle| \eta\right] = \int_0^T P(S^1_0, K, t_0(u); u)\, p(u) du.$$

Obviously, the question which presents itself concerns the comparison of the price bounds with the exact price Y_0^*. In particular, we are interested in the *sunk costs* an investor faces if she uses the optimal deterministic switching rule or, more cleverly, the improved version.

4 Comparison of price bounds

In the following, we consider numerical examples which are based on a discrete set of switching dates. The calculation of the exact price Y_0^* is based on the following proposition.

PROPOSITION 4.1. *Consider a discrete set of switching dates* $\{t_0 = 0,\ t_1, ...,\ t_n = T\}$ *and assume that the random regime switching date* η *is concentrated on the interval* $[0, T]$ *with distribution* $p(du)$. *Let* $Y^*_{t_k} = Y^*_{t_k}(\ln S^1_{t_k}, \varepsilon_{t_k})$ *where* $\varepsilon_t = 1_{\eta \le t}$. *Then,* $Y_0^* = Y_0^*(\ln S^1_{t_0}, 0)$ *is recursively defined by using the terminal condition*

$$Y_T^*(\ln S^1_T, \varepsilon_T) = \left(K - S^1_T\right)^+$$

and for $k < n$,

$$Y^*_{t_k}(y, \varepsilon) = \max(Z_{t_k}, E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}}) = \max\left(Z_{t_k}(y, \varepsilon), \left[E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}}\right](y, \varepsilon)\right),$$

where

$$
\begin{aligned}
Z_{t_k}(y,\varepsilon) &= \varepsilon \mathcal{B}\left(e^y, K, \sqrt{\int_{t_k}^T \sigma_{22}^2 ds}\right) \\
&\quad + (1-\varepsilon) \int p(du|\eta > t_k) \mathcal{B}\left(e^y, K, \sqrt{\int_{t_k}^T {}_u\sigma_2^2 ds}\right) \\
\left[E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}}\right](y,\varepsilon) &= \varepsilon \int \phi(\zeta) d\zeta\, Y^*_{t_{k+1}}(y - \frac{1}{2}\int_{t_k}^{t_{k+1}} \sigma_{12}^2 ds + \zeta \sqrt{\int_{t_k}^{t_{k+1}} \sigma_{12}^2 ds}, 1) \\
&\quad + (1-\varepsilon)\, \pi_k \int \phi(\zeta) d\zeta\, Y^*_{t_{k+1}}(y - \frac{1}{2}\int_{t_k}^{t_{k+1}} \sigma_{11}^2 ds + \zeta \sqrt{\int_{t_k}^{t_{k+1}} \sigma_{11}^2 ds}, 0) \\
&\quad + (1-\varepsilon) \int_{(t_k, t_{k+1}]} p(du|\eta > t_k) \int \phi(\zeta) d\zeta \cdot \\
&\quad \cdot Y^*_{t_{k+1}}(y - \frac{1}{2}\int_{t_k}^{u} \sigma_{11}^2 ds - \frac{1}{2}\int_{u}^{t_{k+1}} \sigma_{12}^2 ds + \zeta \sqrt{\int_{t_k}^{u} \sigma_{11}^2 ds + \int_{u}^{t_{k+1}} \sigma_{12}^2 ds}, 1).
\end{aligned}
$$

ϕ denotes the density of the standard normal distribution and $\pi_k := P(\eta > t_{k+1} | \eta > t_k)$.

PROOF: A detailed proof is given in the appendix, cf. Appendix C.

The above algorithm gives the exact price by multiple integration. Notice that the first step is

$$
Y^*_{t_{n-1}}(y,\varepsilon) = \max\left[Z_{t_{n-1}}(y,\varepsilon), \left[E^{\mathcal{F}_{t_{n-1}}} Y^*_T\right](y,\varepsilon)\right],
$$

i.e. it requires, for each y, ε, two one-dimensional integrations due to p. Given the function $Y^*_{t_{n-1}}(y,\varepsilon)$, the next step requires a two-fold integration. Hence, $Y^*_{t_{n-2}}(y,\varepsilon)$ requires three-fold integration, $Y^*_{t_{n-3}}(y,\varepsilon)$ five-fold and so on.

4.1 Numerical case study

Let us consider the case $n=3$, $p(\{t_1\}) = p_1$, $p(\{t_2\}) = p_2$, $p(\{t_3\}) = p(\{T\}) = p_3$, with $p_1 + p_2 + p_3 = 1$, and σ_{ij} to be constant. For notational convenience, we use $\sigma_{11} = \sigma_-$, $\sigma_{21} = \sigma_+$, $\sigma_{12} = \sigma_+$, $\sigma_{22} = \sigma_-$. The adjustments to the slightly more general case $\sigma_{ij} \neq \sigma_{ji}$ are straightforward. With the above proposition, we immediately have

$$
\begin{aligned}
\left[E^{\mathcal{F}_{t_2}} Y^*_T\right](y,\varepsilon) &= \varepsilon \mathcal{B}\left(e^y, K, \sigma_+\sqrt{T-t_2}\right) + (1-\varepsilon)\mathcal{B}\left(e^y, K, \sigma_-\sqrt{T-t_2}\right), \\
Z_{t_2}(y,\varepsilon) &= \varepsilon \mathcal{B}\left(e^y, K, \sigma_-\sqrt{T-t_2}\right) + (1-\varepsilon)\mathcal{B}\left(e^y, K, \sigma_+\sqrt{T-t_2}\right)
\end{aligned}
$$

and so

$$
Y^*_{t_2}(y,\varepsilon) = \max(Z_{t_2}(y,\varepsilon), \left[E^{\mathcal{F}_{t_2}} Y^*_T\right](y,\varepsilon)).
$$

With

$$
\max\left\{\mathcal{B}\left(e^y, K, \sigma_+\sqrt{T-t_2}\right), \mathcal{B}\left(e^y, K, \sigma_-\sqrt{T-t_2}\right)\right\} = \mathcal{B}\left(e^y, K, \sigma_+\sqrt{T-t_2}\right),
$$

we have

$$
Y^*_{t_2}(y,0) = Y^*_{t_2}(y,1) = \mathcal{B}\left(e^y, K, \sigma_+\sqrt{T-t_2}\right),
$$

i.e. one step before T, we have no uncertainty about the regime until T. Thus, we can obtain the volatility level σ_+, independent of ε. Next,

$$
\left[E^{\mathcal{F}_{t_1}} Y^*_{t_2}\right](y,\varepsilon)
$$

$$
\begin{aligned}
&= \varepsilon \int \phi(\zeta) d\zeta \, Y^*_{t_2}(y - \frac{1}{2}\sigma^2_+ (t_2 - t_1) + \zeta \sigma_+ \sqrt{t_2 - t_1}, 1) \\
&+ (1-\varepsilon) \frac{p_3}{p_2 + p_3} \int \phi(\zeta) d\zeta \, Y^*_{t_2}(y - \frac{1}{2}\sigma^2_- (t_2 - t_1) + \zeta \sigma_- \sqrt{t_2 - t_1}, 0) \\
&+ (1-\varepsilon) \frac{p_2}{p_2 + p_3} \int \phi(\zeta) d\zeta \, Y^*_{t_2}(y - \frac{1}{2}\sigma^2_- (t_2 - t_1) + \zeta \sigma_- \sqrt{t_2 - t_1}, 1) \\
&= \varepsilon \mathcal{B}\left(e^y, K, \sigma_+ \sqrt{T - t_1}\right) + (1-\varepsilon) \mathcal{B}\left(e^y, K, \sqrt{\sigma^2_-(t_2 - t_1) + \sigma^2_+(T - t_2)}\right),
\end{aligned}
$$

$$
\begin{aligned}
Z_{t_1}(y, \varepsilon) &= \varepsilon \mathcal{B}\left(e^y, K, \sigma_- \sqrt{T - t_1}\right) \\
&+ \frac{(1-\varepsilon)}{p_2 + p_3} \left[p_2 \mathcal{B}\left(e^y, K, \sqrt{\sigma^2_+(t_2 - t_1) + \sigma^2_-(T - t_2)}\right) + p_3 \mathcal{B}\left(e^y, K, \sigma_+ \sqrt{T - t_1}\right) \right].
\end{aligned}
$$

So,

$$
Y^*_{t_1}(y, \varepsilon) = \max(Z_{t_1}(y, \varepsilon), \left[E^{\mathcal{F}_{t_1}} Y^*_{t_2}\right](y, \varepsilon))
$$

and

$$
\begin{aligned}
Y^*_{t_1}(y, 0) &= \max \Bigg\{ \mathcal{B}\left(e^y, K, \sqrt{\sigma^2_-(t_2 - t_1) + \sigma^2_+(T - t_2)}\right), \\
&\frac{1}{p_2 + p_3} \left[p_2 \mathcal{B}\left(e^y, K, \sqrt{\sigma^2_+(t_2 - t_1) + \sigma^2_-(T - t_2)}\right) + p_3 \mathcal{B}\left(e^y, K, \sigma_+ \sqrt{T - t_1}\right) \right] \Bigg\} \\
Y^*_{t_1}(y, 1) &= \mathcal{B}\left(e^y, K, \sigma_+ \sqrt{T - t_1}\right).
\end{aligned}
$$

In particular, for $\varepsilon = 1$ we can achieve the volatility level σ_+ by sticking with asset 1, i.e. we simply have the Black & Scholes price according to σ_+. Further, ($\varepsilon_0 = 0$ a.s.)

$$
\left[E^{\mathcal{F}_0} Y^*_{t_1}\right](y, 0) =
$$

$$
\begin{aligned}
&= (p_2 + p_3) \int \phi(\zeta) d\zeta \, Y^*_{t_1}(y - \frac{1}{2}\sigma^2_- t_1 + \zeta \sigma_- \sqrt{t_1}, 0) + p_1 \int \phi(\zeta) d\zeta \, Y^*_{t_1}(y - \frac{1}{2}\sigma^2_- t_1 + \zeta \sigma_- \sqrt{t_1}, 1) \\
&= (p_2 + p_3) \int \phi(\zeta) d\zeta \, Y^*_{t_1}(y - \frac{1}{2}\sigma^2_- t_1 + \zeta \sigma_- \sqrt{t_1}, 0) + p_1 \mathcal{B}\left(e^y, K, \sqrt{\sigma^2_- t_1 + \sigma^2_+(T - t_1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
Z_0(y, 0) &= p_1 \mathcal{B}\left(e^y, K, \sqrt{\sigma^2_+ t_1 + \sigma^2_-(T - t_1)}\right) \\
&+ p_2 \mathcal{B}\left(e^y, K, \sqrt{\sigma^2_+ t_2 + \sigma^2_-(T - t_2)}\right) \\
&+ p_3 \mathcal{B}\left(e^y, K, \sigma_+ \sqrt{T}\right)
\end{aligned}
$$

and

$$
Y^*_{0_1}(y, 0) = \max(Z_0(y, 0), \left[E^{\mathcal{F}_0} Y^*_{t_1}\right](y, 0)).
$$

For a numerical study we consider the case where $T = 1$, $\{t_0 = 0, t_1 = \frac{1}{3}, t_2 = \frac{2}{3}, t_3 = 1\}$, and $p_0(\frac{i}{3}) = P(\eta = t_i) = \frac{1}{3}$ for $i = 1, 2, 3$. The put is at the money, the initial asset prices are normalized to one, i.e. $K = S^1_0 = S^2_0 = 1$. Throughout the example, we assume $\sigma_{11} < \sigma_{21}$ (in the original state "1", the volatility of asset 2 is higher than the one of asset 1) but $\sigma_{22} < \sigma_{12}$ (the opposite is true in regime "2"). Recall that we start in asset 1, i.e. the asset with the lower volatility in the original regime "1". Intuitively, it is clear that the optimal deterministic stopping rule either prescribes an immediate switch (is) to asset 2 or no switch (ns) at all. An immediate switch implies that the price effect from starting in the

Exact price versus price bounds

σ_{21}	$\overline{Y}_0^{\text{vis}}$	Y_0^*	$\underline{Y}_0^{\text{detmix}}$	$\underline{Y}_0^{\text{det}}$	$\frac{\overline{Y}_0^{\text{vis}}}{Y_0^*}$	$\frac{Y_0^*}{Y_0^*}$	$\frac{\underline{Y}_0^{\text{detmix}}}{Y_0^*}$	$\frac{\underline{Y}_0^{\text{det}}}{Y_0^*}$
0.20	0.0854	0.0842	0.0776	0.0721 (ns)	1.0137	1.0000	**0.9216**	**0.8563**
0.21	0.0867	0.0842	0.0782	0.0721 (ns)	1.0290	1.0000	0.9287	0.8559
0.22	0.0880	0.0843	0.0789	0.0743 (is)	1.0448	1.0000	0.9363	0.8823
0.23	0.0900	0.0843	0.0802	0.0773 (is)	1.0677	1.0000	0.9512	0.9174
0.24	0.0923	0.0843	0.0819	0.0803 (is)	1.0958	1.0000	0.9715	0.9532
0.25	0.0947	0.0843	0.0836	0.0833 (is)	1.1235	1.0000	0.9914	0.9888

Table 1: Exact price and price bounds for $\sigma_{12} = 0.3$ and $\sigma_{11} = \sigma_{22} = 0.1$. W.r.t. the lower price bounds implied by the deterministic switching time, (ns) indicates no switch and (is) an immediate switch.

Implied volatilities

σ_{21}	$\sigma_{\text{imp}}^{\text{vis}}$	σ_{imp}^*	$\sigma_{\text{imp}}^{\text{detmix}}$	$\sigma_{\text{imp}}^{\text{det}}$	$\frac{\sigma_{\text{imp}}^{\text{vis}}}{\sigma_{\text{imp}}^*}$	$\frac{\sigma_{\text{imp}}^*}{\sigma_{\text{imp}}^*}$	$\frac{\sigma_{\text{imp}}^{\text{detmix}}}{\sigma_{\text{imp}}^*}$	$\frac{\sigma_{\text{imp}}^{\text{det}}}{\sigma_{\text{imp}}^*}$
0.20	0.2143	0.2114	0.1948	0.1810	1.0136	1.0000	0.9212	0.8557
0.21	0.2177	0.2115	0.1964	0.1810	1.0291	1.0000	0.9284	0.8555
0.22	0.2210	0.2115	0.1980	0.1865	1.0450	1.0000	0.9361	0.8819
0.23	0.2260	0.2116	0.2012	0.1941	1.0680	1.0000	0.9511	0.9171
0.24	0.2320	0.2116	0.2056	0.2017	1.0962	1.0000	0.9714	0.9531
0.25	0.2379	0.2117	0.2098	0.2093	1.1240	1.0000	0.9914	0.9888

Table 2: The parameters are given as in Table 1.

higher volatility (but taking into account for a lower volatility in the case of a regime switch) dominates the price effect of starting at a lower volatility and *hoping* for a higher volatility because of a regime switch. However, the optimal deterministic switching time depends on the moneyness m_t (where $m_t = \frac{K}{S_t^1}$) and the state of the economy. In particular, the optimal deterministic switching time prevailing today may be different from the one at future times. Obviously, a deterministic stopping time which is not *robust* as time goes by causes a price deviation if one compares the lower price bound with the exact price. Similar reasonings are true with respect to the improved semi–deterministic stopping time proposed in Proposition 3.3.

For $\sigma_{12} = 0.3$, $\sigma_{11} = \sigma_{22} = 0.1$ and varying σ_{21} ($\sigma_{21} \in [0.1, 0.3]$), the exact price Y_0^* as well as the price bounds of Propositions 3.3 and 3.4 are summarized in Table 1. The corresponding implied volatilities are given in Table 2. Recall that $\overline{Y}_0^{\text{vis}} \geq Y_0^* \geq \underline{Y}_0^{\text{detmix}} \geq \underline{Y}_0^{\text{det}}$. Intuitively it is clear that an early asset switch is the better the higher the volatility level σ_{21} is, i.e. the volatility level of asset 2 in the initial regime "1". Thus, there is a sufficiently large σ_{21} which implies that Y_0^* is given in terms of an immediate switch towards asset 2. In this case, all prices (besides the upper price bound in terms of the visionary) coincide such that the relative underestimation $\frac{\underline{Y}}{Y^*}$ implied by the lower price bounds converges to one.

In contrast, if the optimal deterministic stopping time implies no switching at all (ns), i.e. $t^* = T$, the lower bounds are well below the exact price. Consider for example the scenario where $\sigma_{21} = 0.2$. Here, the lower price bound Y_0^{det} underestimates the exact price by 14%. To some extend, this is mitigated by the optimal "detmixßtopping time (the improved lower price bound). Here, the underestimation is about 8%. Compared to the 14%, this is rather low. However, in terms of the (overall) optimal stopping time, the price effect is still to be considered as high. Intuitively, the effect of "laterßtopping is twofold. The improvement which is achieved by the "detmix-stopping rule is obvious: In the case of a regime switch prior to s^0, we switch according to the (then) optimal stopping rule. In particular, the probability of this event is the higher the later the fixed time s^0 (t^0, respectively) is.[17] However, on the other hand, the investor *looses* something compared to the overall optimal stopping time because of a similar argument. The later the time s^0 (t^0, respectively) is, the higher is the probability that we achieve a moneyness where the optimal stopping strategy prescribes to switch.

Thus, we can summarize that even in this stylized setup where the deviation from the Black & Scholes model can be viewed as moderate, the pricing errors resulting from a suboptimal stopping strategy are substantial. In addition, we can also interpret the fraction $\frac{Y_0}{Y_0^*}$ as the percentage of the product value which is obtained by an investor who deviates from the optimal stopping strategy. An illustrative example where such *sunk costs* are already implemented into the product design are so–called guaranteed minimum accumulation benefits which are briefly introduced in the next section.

5 Guaranteed minimum accumulation benefits

The basic structure of a guaranteed minimum accumulation benefit ($GMAB$) is as follows. Assume that the the insured pays his contributions in terms of a single up front premium P.[18] For a fixed maturity T, the contract payoff is given by the maximum of the account value V_T and a guaranteed amount K, i.e.

$$GMAB_T := \max\{V_T, K\} = V_T + [K - V_T]^+ .$$

Normally, K is calculated by granting an interest rate g ($g < r$) on the whole premium P, i.e. $K = Pe^{gT}$. $V_T = V_T(\phi)$ is the terminal value of a self–financing strategy ϕ. The initial investment V_0 is to be interpreted as the premium P reduced by the hedging costs from the embedded put, i.e. $V_0 = \alpha P$ ($\alpha \in]0,1[$). Along the lines of the conventions used in practice, αP is called the *risk capital* and $(1-\alpha)P$ denotes the *guarantee costs* (hedging costs, respectively). Obviously, the hedging costs depend on the riskiness of the investment strategy ϕ. Assume that ϕ is initially invested in asset S^1. Without the additional rider to switch it holds

$$(1-\alpha)P = e^{-rT} E\left(K - \alpha P \frac{S_T^1}{S_{t_0}^1}\right)^+$$

or using $K = Pe^{gT}$

$$(1-\alpha) = e^{-rT} E\left(e^{gT} - \alpha \frac{S_T^1}{S_{t_0}^1}\right)^+ .$$

In contrast, the risk management of a $GMAB$ which also provides the additional rider to switch the investments must take into account for the worst case strategy, i.e.

$$(1-\bar{\alpha}) = \sup_{\tau, 0 \le \tau \le T} e^{-rT} E\left(e^{gT} - \bar{\alpha} \frac{S_\tau^1}{S_{t_0}^1} \frac{S_T^2}{S_\tau^2}\right)^+ .$$

Thus, the funds switching right gives a lower risk capital $\bar{\alpha}$. In addition, one can argue that the investor looses risk capital if she does not want or is not able to apply the optimal stopping strategy.

For a simple illustration, take the Black & Scholes model (4) with only one Brownian motion, take $\sigma_1(t) = \sigma_1$, a constant, and

$$\sigma_2^2(t) = \begin{cases} \frac{\sigma_1^2}{2} & 0 \le t \le \frac{T}{2} \\ \frac{3\sigma_1^2}{2} & \frac{T}{2} < t \le T \end{cases} .$$

[17] Notice that $s^0 \ge t^0$.

[18] We abstract from periodic premia. In fact, a periodic premium introduces also an Asian option feature.

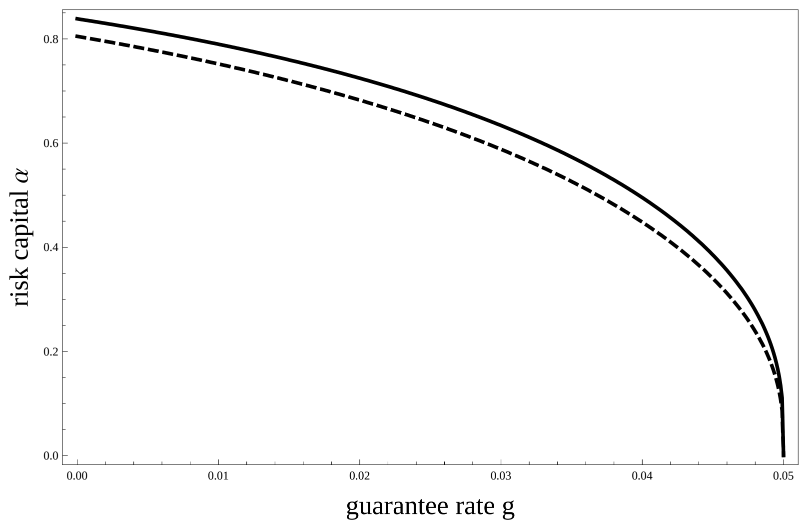

Figure 1: Risk capital

The figure gives the risk capital α (thick line) and $\bar{\alpha}$ (dashed line) for varying guarantee rates g for $\sigma_1 = 0.3$, $T = 10$ and $r = 0.05$.

In particular, $v_1(0,T) = v_2(0,T)$, $\tau^* = \frac{1}{2}$ and $v^* = \sqrt{\frac{5}{4}} v_1(0,T)$.[19] Figure 1 illustrates the risk capital α ($\bar{\alpha}$, respectively) of a contract without (or not used) switching right and the one with switching right.

6 Conclusion

Recently, there is a growing trend to introduce funds switching rights to minimum return rate guarantee products. At a first glance, this feature is very attractive. It grants the buyer an additional flexibility on the investment decisions. However, it does not come for free. The provider must take into account for the *worst case* strategy, i.e. the strategy which maximizes the value of the embedded option. For a given number of switching times per year, the relevant pricing strategy is given in terms of an optimal stopping problem. Besides the simple Black & Scholes model, there is no other meaningful pricing model which yields an analytically tractable, and in particular deterministic, optimal stopping time. Even in a Levy–model framework, i.e. independent asset price increments, the optimal solution is to be determined numerically. While the risk structure of the economy is deterministic, this is not true for its impact on the value of the switching right.

Due to the long time investment horizons, it is also essential to take into account for random changes in the risk structure of the economy, i.e. stochastic volatility. It turns out that the lower price bound, achieved by the simplified problem concerning the optimal deterministic stopping time, gives rise to a pronounced underestimation of the exact price. This is even true in the case of moderate deviations from the Black & Scholes model. Thus, the pricing and risk management of the switching option must indeed rely on the optimal stopping time.

This is bad news for the investor. It is safe to say that the investor herself will not implement the optimal stopping strategy. On the one hand, even if she wants to, she has to know the true model. On the other hand, if she is risk averse, she does not want the value maximizing strategy anyway. Instead of maximizing the put value (under the risk neutral measure), she is going to maximize her expected utility (under the real world measure). However, any deviation from the optimal stopping strategy give rise to (pronounced) sunk costs.

[19] The cumulated volatilities are defined as Lemma 2.3. According to Theorem 2.4 it holds $\tau^* = t^*$ where t^* is given by Equation (9).

A Proofs of Section 2

A.1 Proof of Theorem 2.4, part (i)

We first consider a Bermudan version of the optimal stopping problem. For each fixed $N = 1, 2, \ldots$ we consider the grid $t_n := \frac{n}{N}T$, $n = 0, \ldots, N$, and $Y_t^{*,N}$ is considered to be the Snell envelope of the stopping problem where exercise is only allowed at the dates t_n, $n = 0, \ldots, N$. Thus, we have

$$Y_t^{*,N} = \sup_{\tau,\ \tau \in \{t_{m(t)}, \ldots, t_N = T\}} E^{\mathcal{F}_t}\left(K - \frac{S_\tau^1}{S_\tau^2} S_T^2\right)^+ \quad \text{where } m(t) := \inf\{m : t_m \geq t\}.$$

Further, let

$$\begin{aligned} t_n^{*,N} &= \operatorname*{arg\,sup}_{t_m:\ t_n \leq t_m \leq T} \left\{ v_2^2(t_m, T) - v_1^2(t_m, T) \right\} \\ &= \operatorname*{arg\,sup}_{t_m:\ t_n \leq t_m \leq T} \left\{ v_1^2(t_n, t_m) + v_2^2(t_m, T) \right\} \end{aligned}$$

with the convention of Footnote 12.

LEMMA A.1 (Bermudan counterpart). *(a) Any stopping time τ with $\tau \in \{t_n, \ldots, t_n^{*,N}\}$ is suboptimal w.r.t. $t_n^{*,N}$ for $Y_{t_n}^{*,N}$ (τ_1 sub optimal w.r.t. τ_2 for $Y_t^* = \sup_{\tau \geq t} E^{\mathcal{F}_t} Z_\tau$ means $E^{\mathcal{F}_t} Z_{\tau_1} \leq E^{\mathcal{F}_t} Z_{\tau_2}$).*

(b) Any $\mathcal{F}_{t_n^*}$ measurable stopping time τ with $\tau \in \{t_n^*, \ldots, T\}$ is suboptimal w.r.t. $t_n^{*,N}$ for $Y_{t_n}^{*,N}$.

PROOF: (a) For any stopping time τ with $\tau \in \{t_n, \ldots, t_n^{*,N}\}$ we have

$$\begin{aligned} & v_1^2(t_n, t_n^*) + v_2^2(t_n^*, T) \geq v_1^2(t_n, \tau) + v_2^2(\tau, T) \\ \text{i.e } & v_1^2(t_n, \tau) + v_1^2(\tau, t_n^*) + v_2^2(t_n^*, T) \geq v_1^2(t_n, \tau) + v_2^2(\tau, t_n^*) + v_2^2(t_n^*, T) \\ & \text{which implies } v_1^2(\tau, t_n^*) \geq v_2^2(\tau, t_n^*). \end{aligned}$$

So, it holds

$$\begin{aligned} E^{\mathcal{F}_{t_n}}\left(K - \frac{S_\tau^1}{S_\tau^2} S_T^2\right)^+ &= E^{\mathcal{F}_{t_n}}\left(K - \frac{S_\tau^1}{S_{t_n^*}^2} S_{t_n^*}^2 \frac{S_T^2}{S_\tau^2}\right)^+ \\ &= E^{\mathcal{F}_{t_n}}\left[S_\tau^1 E^{\mathcal{F}_\tau}\left(\frac{K}{S_\tau^1} - \frac{S_{t_n^*}^2}{S_\tau^2}\frac{S_T^2}{S_{t_n^*}^2}\right)^+\right] \\ &\leq E^{\mathcal{F}_{t_n}}\left[S_\tau^1 E^{\mathcal{F}_\tau}\left(\frac{K}{S_\tau^1} - \frac{S_{t_n^*}^1}{S_\tau^1}\frac{S_T^2}{S_{t_n^*}^2}\right)^+\right] = E^{\mathcal{F}_{t_n}}\left(K - \frac{S_{t_n^*}^1}{S_{t_n^*}^2} S_T^2\right)^+. \end{aligned}$$

Now, consider part (b). In a similar way we now obtain $v_1^2(t^*, \tau) \leq v_2^2(t^*, \tau)$. Thus, for any $\mathcal{F}_{t_n^*}$ measurable stopping time τ with $\tau \in \{t_n^*, \ldots, T\}$ we have

$$\begin{aligned} E^{\mathcal{F}_{t_n}}\left(K - \frac{S_\tau^1}{S_\tau^2} S_T^2\right)^+ &= E^{\mathcal{F}_{t_n}}\left(K - \frac{S_\tau^1}{S_{t_n^*}^1} S_{t_n^*}^1 \frac{S_T^2}{S_\tau^2}\right)^+ \\ &= E^{\mathcal{F}_{t_n}}\left[S_{t_n^*}^1 E^{\mathcal{F}_{t_n^*}}\left(\frac{K}{S_{t_n^*}^1} - \frac{S_\tau^1}{S_{t_n^*}^1}\frac{S_T^2}{S_\tau^2}\right)^+\right] \\ &\leq E^{\mathcal{F}_{t_n}}\left[S_{t_n^*}^1 E^{\mathcal{F}_{t_n^*}}\left(\frac{K}{S_{t_n^*}^1} - \frac{S_\tau^2}{S_{t_n^*}^2}\frac{S_T^2}{S_\tau^2}\right)^+\right] = E^{\mathcal{F}_{t_n}}\left(K - \frac{S_{t_n^*}^1}{S_{t_n^*}^2} S_T^2\right)^+. \end{aligned}$$

We are now ready to prove part (i) of the theorem in a Bermudan setting:

PROPOSITION A.2.

$$Y_{t_n}^{*,N} = \sup_{\tau,\ \tau\in\{t_n,\dots,t_N=T\}} E^{\mathcal{F}_{t_n}}\left(K - \frac{S_\tau^1}{S_\tau^2}S_T^2\right)^+ = E^{\mathcal{F}_{t_n}}\left(K - \frac{S^1_{t_n^{*,N}}}{S^2_{t_n^{*,N}}}S_T^2\right)^+,\ \ 0\le n\le N. \tag{17}$$

PROOF: We use backward induction. The case $n=N$ yields $Y_T^{*,N} = \left(K - S_T^1\right)^+$ which is obvious. Let the statement (17) be true for $0<n\le N$. Let $\tau_{n-1}^{*,N}$ satisfy

$$Y_{t_{n-1}}^{*,N} = E^{\mathcal{F}_{t_{n-1}}}\left(K - \frac{S^1_{\tau_{n-1}^{*,N}}}{S^2_{\tau_{n-1}^{*,N}}}S_T^2\right)^+. \tag{18}$$

Using the Bellman principle and the induction hypothesis then gives

$$\begin{aligned}
Y_{t_{n-1}}^{*,N} &= 1_{\tau_{n-1}^{*,N}=t_{n-1}}Y_{t_{n-1}}^{*,N} + 1_{\tau_{n-1}^{*,N}>t_{n-1}}E^{\mathcal{F}_{t_{n-1}}}Y_{t_n}^{*,N} \\
&= 1_{\tau_{n-1}^{*,N}=t_{n-1}}Y_{t_{n-1}}^{*,N} + 1_{\tau_{n-1}^{*,N}>t_{n-1}}E^{\mathcal{F}_{t_{n-1}}}E^{\mathcal{F}_{t_n}}\left(K - \frac{S^1_{t_n^{*,N}}}{S^2_{t_n^{*,N}}}S_T^2\right)^+ \\
&= 1_{\tau_{n-1}^{*,N}=t_{n-1}}Y_{t_{n-1}}^{*,N} + 1_{\tau_{n-1}^{*,N}>t_{n-1}}E^{\mathcal{F}_{t_{n-1}}}\left(K - \frac{S^1_{t_n^{*,N}}}{S^2_{t_n^{*,N}}}S_T^2\right)^+,
\end{aligned}$$

i.e. $\tau_{n-1}^{*,N} = t_{n-1}$ or $\tau_{n-1}^{*,N} = t_n^{*,N}$.

Now, if $t_{n-1} < t_{n-1}^*$ then $t_{n-1}^* = t_n^*$, and so due to part (a) of Lemma A.1 we must have $\tau_{n-1}^{*,N} \ge t_n^* > t_{n-1}$ a.s.. But, by the induction hypothesis this means $\tau_{n-1}^{*,N} = \tau_n^{*,N} = t_n^{*,N} = t_{n-1}^*$.

Else, if $t_{n-1} = t_{n-1}^*$ then due to the Lemma A.1-(b), $t_n^{*,N}$ is suboptimal for w.r.t. t_{n-1}^* for (18). That means by the induction hypothesis $\tau_n^{*,N} = t_n^{*,N}$ that $\tau_n^{*,N}$ is suboptimal w.r.t. t_{n-1}^* for (18). This implies $\tau_{n-1}^{*,N} = t_{n-1} = t_{n-1}^*$. Thus in either case $\tau_{n-1}^{*,N} = t_{n-1}^*$.

PROPOSITION A.3. *In the continuous setting, it holds*

$$Y_0^* = \sup_{\tau,\ \tau\in[0,T]} E^{\mathcal{F}_{t_0}}\left(K - \frac{S_\tau^1}{S_\tau^2}S_T^2\right)^+ = E^{\mathcal{F}_{t_0}}\left(K - \frac{S_{t^*}^1}{S_{t^*}^2}S_T^2\right)^+$$

were t^* is defined as in Equation (9).

PROOF: Notice that the function $f(t) := v_1(0,t) + v_2(t,T)$ is continuous (hence uniformly) on $[0,T]$. Let $V_N := \{\frac{n}{N} : n = 0,\dots,N\}$ and consider the nested sequence V_{2^m}, $m = 1,2,..$, i.e $V_{2^1} \subset V_{2^2} \subset$ and so on. Clearly, $f(t_0^{*,2^m})$ is non-decreasing and $f(t_0^{*,2^m}) \to f(t^*)$. So there exists a convergent subsequence $t_0^{*,2^{m_k}}$, $k = 1,2,$ with (m_k) increasing such that $t_0^{*,2^{m_k}} \to \widetilde{t}^*$ (say) and $f(t_0^{*,2^{m_k}}) \to f(t^*) = f(\widetilde{t}^*)$. Thus, combining Proposition A.2, (5), and using dominated convergence gives

$$\begin{aligned}
Y_0^* &= \lim_{k\uparrow\infty} Y_0^{*,2^{m_k}} = \lim_{k\uparrow\infty} E^{\mathcal{F}_{t_0}}\left(K - \frac{S^1_{t_0^{*,2^{m_k}}}}{S^2_{t_0^{*,N}}}S_T^2\right)^+ = E^{\mathcal{F}_{t_0}}\left(K - \frac{S^1_{\widetilde{t}^*}}{S^2_{\widetilde{t}^*}}S_T^2\right)^+ \\
&= E^{\mathcal{F}_{t_0}}\left(K - \frac{S^1_{t^*}}{S^2_{t^*}}S_T^2\right)^+.
\end{aligned}$$

A.2 Proof of Theorem 2.4, part (ii)

First, we consider the representation of the put–price in a Levy model setup.

LEMMA A.4 (Price representation). *Let the asset price dynamics be given by* (10) *and let* $U_{r,t,T} := E^{\mathcal{F}_r}\left(K - \frac{S_t^1}{S_t^2} S_T^2\right)^+$ *for* $r < t \leq T$. *It then holds*

$$U_{r,t,T} = U_{r,t,T}^{\mathcal{B}} + \frac{S_r^1}{2\pi}\int_{-\infty}^{\infty} \frac{\Phi^{\mathcal{B}}(z-\mathrm{i},r,t,T) - \Phi(z-\mathrm{i},r,t,T)}{z(z-\mathrm{i})} e^{-\mathrm{i}z\ln\frac{K}{S_r^1}} dz \tag{19}$$

where $\Phi(z,r,t,T) := E^{\mathcal{F}_r}\exp\left[\mathrm{i}z\mathcal{X}_{r,t,T}\right]$ *with* $\mathcal{X}_{r,t,T} = \ln\frac{S_t^1}{S_r^1}\frac{S_T^2}{S_t^2}$ *and* $\Phi^{\mathcal{B}}(z,r,t,T)$, *due to a Black & Scholes model obtained by setting* $\nu = 0$, *are specified in the proof below.*

PROOF: From (10), we may derive straightforwardly the following dynamics for the logarithm of the asset price process,

$$d\ln S^i = -\frac{1}{2}||\sigma_i||^2(t)dt + \sigma_i^\top(t)dW_t + \int u_i\,(\mu - \nu)\,(dt,du) - dt\int\left(e^{u_i} - u_i - 1\right)\nu(t,du),$$

and the characteristic exponents of $X_t^i := \ln S_t^i - \ln s_0^i$ are given by

$$\begin{aligned}\psi_i(z,t) &:= -\frac{z^2}{2}\int_0^t ||\sigma_i||^2(s)ds - \mathrm{i}z\int_0^t ds\left[\frac{1}{2}||\sigma_i||^2(s) + \int\left(e^{u_i} - u_i - 1\right)\nu(s,du)\right] \\ &+ \int_0^t ds\int\left(e^{\mathrm{i}zu_i} - \mathrm{i}zu_i - 1\right)\nu(s,du),\end{aligned} \tag{20}$$

i.e. $\Phi_i(z,t) := E\exp\left[\mathrm{i}zX_t^i\right] = \exp\psi_i(z,t)$.
For the put–price at time r according to a deterministic switching time t ($t > r$) we have

$$\begin{aligned}U_{r,t,T} &:= E^{\mathcal{F}_r}\left(K - \frac{S_t^1}{S_t^2}S_T^2\right)^+ = S_r^1 E^{\mathcal{F}_r}\left(\frac{K}{S_r^1} - \frac{S_t^1}{S_r^1}\frac{S_T^2}{S_t^2}\right)^+ \\ &=: S_r^1 E^{\mathcal{F}_r}\left(\frac{K}{S_r^1} - e^{X_{r,t}^1 + X_{t,T}^2}\right)^+ =: E^{\mathcal{F}_r}\left(K - S_r^1 e^{\mathcal{X}_{r,t,T}}\right)^+.\end{aligned}$$

Since $X_{r,t}^1$ and $X_{t,T}^2$ are independent, the characteristic exponent of $\mathcal{X}_{r,t,T}$ is obviously given by

$$\psi^{\mathcal{X}}(z,r,t,T) := \psi_1^{\mathcal{X}}(z,r,t) + \psi_2^{\mathcal{X}}(z,t,T),$$

where in view of (20),

$$\begin{aligned}&\psi_1^{\mathcal{X}}(z,r,t) := -\frac{z^2}{2}\int_r^t ||\sigma_1||^2(s)ds - \mathrm{i}z\int_r^t ds\left[\frac{1}{2}||\sigma_1||^2(s) + \int\left(e^{u_1} - u_1 - 1\right)\nu(s,du)\right] \\ &+ \int_r^t ds\int\left(e^{\mathrm{i}zu_1} - \mathrm{i}zu_1 - 1\right)\nu(s,du), \\ &\psi_2^{\mathcal{X}}(z,t,T) - \frac{z^2}{2}\int_t^T ||\sigma_2||^2(s)ds - \mathrm{i}z\int_t^T ds\left[\frac{1}{2}||\sigma_2||^2(s) + \int\left(e^{u_2} - u_2 - 1\right)\nu(s,du)\right] \\ &+ \int_t^T ds\int\left(e^{\mathrm{i}zu_2} - \mathrm{i}zu_2 - 1\right)\nu(s,du)\end{aligned}$$

(Note that $\psi^{\mathcal{X}}(-\mathrm{i},r,t,T) = 0$). Thus, we have

$$\Phi(z,r,t,T) := E^{\mathcal{F}_r}\exp\left[\mathrm{i}z\mathcal{X}_{r,t,T}\right] = \exp\left[\psi^{\mathcal{X}}(z,r,t,T)\right].$$

and then along the lines of Carr and Madan (1999), we get the put price

$$U_{r,t,T} = \left(K - S_r^1\right)^+ + \frac{S_r^1}{2\pi}\int_{-\infty}^{\infty}\frac{1 - \Phi(z-\mathrm{i},r,t,T)}{z(z-\mathrm{i})}e^{-\mathrm{i}z\ln\frac{K}{S_r^1}}dz. \tag{21}$$

A numerically more efficient representation is obtained by considering the Black & Scholes model setup (obtained by $v \equiv 0$)

$$U^{\mathcal{B}}_{r,t,T} = \left(K - S^1_r\right)^+ + \frac{S^1_r}{2\pi}\int_{-\infty}^{\infty} \frac{1-\Phi^{\mathcal{B}}(z-\mathrm{i},r,t,T)}{z(z-\mathrm{i})} e^{-\mathrm{i}z\ln\frac{K}{S^1_r}}dz \tag{22}$$

with

$$\begin{aligned}
\Phi^{\mathcal{B}}(z,r,t,T) &:= \exp\left[-\frac{z^2}{2}\int_r^T \sigma_t^2(s)ds\right]\\
U^{\mathcal{B}}_{r,t,T} &= -S^1_r\mathcal{N}\Big(\frac{\log\frac{K}{S^1_r} - \frac{1}{2}\int_r^T \sigma_t^2(s)ds}{\sqrt{\int_r^T \sigma_t^2(s)ds}}\Big) + K\mathcal{N}\Big(\frac{\log\frac{K}{S^1_r} + \frac{1}{2}\int_r^T \sigma_t^2(s)ds}{\sqrt{\int_r^T \sigma_t^2(s)ds}}\Big),
\end{aligned}$$

where $\sigma_t(s) := ||\sigma_1(s)||1_{s\le t} + ||\sigma_2(s)||1_{s>t}$. Next, by subtracting (21) and (22) we obtain (19).

PROPOSITION A.5. *Consider the stopping problem*

$$\begin{aligned}
Y_0^* &:= \sup_{\tau,\ 0\le\tau\le T} E\left(K - \frac{S^1_\tau}{S^2_\tau}S^2_T\right)^+ = \sup_{\tau,\ 0\le\tau\le T} EE^{\mathcal{F}_\tau}\left(K - \frac{S^1_\tau}{S^2_\tau}S^2_T\right)^+\\
&= \sup_{\tau,\ 0\le\tau\le T} EU_{\tau,\tau,T}
\end{aligned}$$

where $U_{r,t,T}$ is defined as in Lemma A.4. In the case $v \neq 0$, there exist no optimal deterministic stopping time except possibly $t_0^* = 0$ or $t_0^* = T$.

PROOF: For $r < t < T$, Lemma A.4 immediately implies

$$\frac{\partial}{\partial t}U_{r,t,T} = \frac{\partial}{\partial t}U^{\mathcal{B}}_{r,t,T} + \frac{S^1_r}{2\pi}\int_{-\infty}^{\infty} \frac{\frac{\partial}{\partial t}\Phi^{\mathcal{B}}(z-\mathrm{i},r,t,T) - \frac{\partial}{\partial t}\Phi(z-\mathrm{i},r,t,T)}{z(z-\mathrm{i})} e^{-\mathrm{i}z\ln\frac{K}{S^1_r}}dz = 0 \tag{23}$$

where

$$\begin{aligned}
U^{\mathcal{B}}_{r,t,T} &= -S^1_r\mathcal{N}\Big(\frac{\log\frac{K}{S^1_r} - \frac{1}{2}\int_r^T \sigma_t^2(s)ds}{\sqrt{\int_r^T \sigma_t^2(s)ds}}\Big) + K\mathcal{N}\Big(\frac{\log\frac{K}{S^1_r} + \frac{1}{2}\int_r^T \sigma_t^2(s)ds}{\sqrt{\int_r^T \sigma_t^2(s)ds}}\Big)\\
&=: -S^1_r\mathcal{N}(d_-) + K\mathcal{N}(d_+).
\end{aligned}$$

Let ϕ denote the density of the standard normal distribution. Using the identity $S^1_r\phi(d_-) = K\phi(d_+)$ gives

$$\begin{aligned}
\frac{\partial}{\partial t}U^{\mathcal{B}}_{r,t,T} &= -S^1_r\phi(d_-)\frac{\partial}{\partial t}\left(\frac{\log\frac{K}{S^1_r} - \frac{1}{2}\int_r^T \sigma_t^2(s)ds}{\sqrt{\int_r^T \sigma_t^2(s)ds}}\right) + K\phi(d_+)\frac{\partial}{\partial t}\left(\frac{\log\frac{K}{S^1_r} + \frac{1}{2}\int_r^T \sigma_t^2(s)ds}{\sqrt{\int_r^T \sigma_t^2(s)ds}}\right)\\
&= S^1_r\phi(d_-)\frac{\partial}{\partial t}\sqrt{\int_r^T \sigma_t^2(s)ds}.
\end{aligned}$$

The above together with Equation (23) yields

$$\begin{gathered}
\phi(d_-)\frac{\partial}{\partial t}\sqrt{\int_r^T \sigma_t^2(s)ds} +\\
\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\frac{\partial}{\partial t}\Phi^{\mathcal{B}}(z-\mathrm{i},r,t,T) - \frac{\partial}{\partial t}\Phi(z-\mathrm{i},r,t,T)}{z(z-\mathrm{i})} e^{-\mathrm{i}z\ln\frac{K}{S^1_r}}dz = 0
\end{gathered}$$

which solution is independent of S^1_r and K (only) if $\Phi^{\mathcal{B}} = \Phi$. This means that for $v \neq 0$ there can be no deterministic stopping time other than 0 or T. Indeed, suppose there was a deterministic (first) optimal stopping time $0 < t_0^* < T$. By consistency of the optimal stopping time we must then have $t_0^* = t_r^*$ for any $0 < r < t_0^*$ where

$$t_0^* = t_r^* = \underset{\tau,\ r \le \tau \le T}{\arg\sup} E^{\mathcal{F}_r} U_{\tau,\tau,T} = \underset{t,\ r \le t \le T}{\arg\sup} E^{\mathcal{F}_r} \left(K - \frac{S_\tau^1}{S_\tau^2} S_T^2 \right)^+ = \underset{t,\ r \le t \le T}{\arg\sup} U_{r,t,T},$$

but, due to the above analysis t_r^* will depend on S_r^1, which is in contradiction with the fact that t_0^* is deterministic.

B Proofs of section 3

B.1 Proof of Proposition 3.2

Notice that for any fixed time t

$$E^{\mathcal{F}_0} \left(K - \frac{S_t^1}{S_t^2} S_T^2 \right)^+ = E^{\mathcal{F}_0} E^{\mathcal{F}_T^V} \left(K - \frac{S_t^1}{S_t^2} S_T^2 \right)^+,$$

and then observe that, given the information $\mathcal{F}_T^V$, i.e. given V_u, $0 \le u \le T$, $\ln \frac{S_t^1}{S_t^2} S_T^2 = \ln S_t^1 + \ln \frac{S_T^2}{S_t^2}$ is normally distributed. In particular, given $\mathcal{F}_T^V$, $\ln S_t^1$ and $\ln \frac{S_T^2}{S_t^2}$ are independent Gaussian $\mathcal{N}(\mu, \sigma^2)$ where

$$\ln S_t^1 \sim \mathcal{N}\left(\ln S_0^1 - \frac{1}{2} \int_0^t \|\sigma_1(u, V_u)\|^2 du, \int_0^t \|\sigma_1(u, V_u)\|^2 du \right), \text{ and}$$
$$\ln \frac{S_T^2}{S_t^2} \sim \mathcal{N}\left(-\frac{1}{2} \int_t^T \|\sigma_2(u, V_u)\|^2 du, \int_t^T \|\sigma_2(u, V_u)\|^2 du \right)$$

respectively. Hence

$$\ln \frac{S_t^1}{S_t^2} S_T^2 \sim \mathcal{N}\Bigg(\ln S_0^1 - \frac{1}{2} \int_0^t \|\sigma_1(u, V_u)\|^2 du - \frac{1}{2} \int_t^T \|\sigma_2(u, V_u)\|^2 du,$$
$$\int_0^t \|\sigma_1(u, V_u)\|^2 du + \int_t^T \|\sigma_2(u, V_u)\|^2 du \Bigg)$$

and so

$$\begin{aligned} E^{\mathcal{F}_0} \left(K - \frac{S_t^1}{S_t^2} S_T^2 \right)^+ &= E^{\mathcal{F}_0} E^{\mathcal{F}_T^V} \left(K - \frac{S_t^1}{S_t^2} S_T^2 \right)^+ \\ &= E \mathcal{B}(S_0, K, \sqrt{\int_0^t \|\sigma_1(u, V_u)\|^2 du + \int_t^T \|\sigma_2(u, V_u)\|^2 du}) \end{aligned}$$

where $\mathcal{B}$ is defined as in Equation (6).

B.2 Proof of Proposition 3.3

Analogously to the proof of Proposition 3.2 it follows that

$$\ln \frac{S_t^1}{S_t^2} S_T^2 = \ln S_t^1 + \ln \frac{S_T^2}{S_t^2}$$

where, conditioned on η, $\ln S_t^1$ and $\ln \frac{S_T^2}{S_t^2}$ are independent Gaussian with distributions $\mathcal{N}\left(\mu,\sigma^2\right)$ given by

$$\ln S_t^1 \sim \mathcal{N}\left(\ln S_0^1 - \frac{1}{2}\int_0^t {}_\eta\sigma_1^2 ds, \int_0^t {}_\eta\sigma_1^2 ds\right), \text{ and}$$
$$\ln \frac{S_T^2}{S_t^2} \sim \mathcal{N}\left(-\frac{1}{2}\int_t^T {}_\eta\sigma_2^2 ds, \int_t^T {}_\eta\sigma_2^2 ds\right)$$

respectively. Hence,

$$\ln \frac{S_t^1}{S_t^2} S_T^2 \sim \mathcal{N}\left(\ln S_0^1 - \frac{1}{2}\int_0^t {}_\eta\sigma_1^2 ds - \frac{1}{2}\int_t^T {}_\eta\sigma_2^2 ds, \int_0^t {}_\eta\sigma_1^2 ds + \int_t^T {}_\eta\sigma_2^2 ds\right).$$

By introducing

$$P(S_0^1, K, t; \eta) := E\left[\left(K - \frac{S_t^1}{S_t^2} S_T^2\right)^+ \middle| \eta\right]$$
$$= \mathcal{B}\left(S_0^1, K, \sqrt{\int_0^t {}_\eta\sigma_1^2 ds + \int_t^T {}_\eta\sigma_2^2 ds}\right),$$

part (i) of the proposition follows immediately by taking the best deterministic stopping time t^0, i.e.

$$t^\circ = \arg\sup_{0\le t\le T} E\left(K - \frac{S_t^1}{S_t^2} S_T^2\right)^+ = \arg\sup_{0\le t\le T} EE\left[\left(K - \frac{S_t^1}{S_t^2} S_T^2\right)^+ \middle| \eta\right]$$
$$= \arg\sup_{0\le t\le T} \int_0^T P(S_0^1, K, t; u)\, p(u) du$$

which yields the lower bound

$$\int_0^T P(S_0^1, K, t^\circ; u)\, p(u) du.$$

Now, consider part (ii), i.e. we construct a stopping rule which leads to an improved lower bound. Recall that the process

$$Y_s^* := \sup_{\tau:\ s\le\tau\le T} E^{\mathcal{F}_s}\left(K - \frac{S_\tau^1}{S_\tau^2} S_T^2\right)^+ = E^{\mathcal{F}_s}\left(K - \frac{S_{\tau_s^*}^1}{S_{\tau_s^*}^2} S_T^2\right)^+$$

may be considered as a Snell envelope of the cash-flow

$$Z_t := E^{\mathcal{F}_t}\left(K - S_t^1 e^{-\frac{1}{2}\int_t^T {}_\eta\sigma_2^2 du + \int_t^T {}_\eta\sigma_2 dW^2}\right)^+$$

so that $Y_s^* = E^{\mathcal{F}_s} Z_{\tau_s^*}$. As a sensible approximation for τ_s^* let us take

$$\tau_s = s 1_{s<\eta} + t_s(\eta) 1_{s>\eta}$$

where τ_s is motivated as follows. Although we somehow think in terms of a deterministic time s, we are aware that we know the optimal stopping rule at the regime switching date η. Notice that in our model there is only one regime switch possible, i.e. we are in a lognormal setup after the regime switch occurred and can apply the results of Theorem 2.4 part (i). In consequence, a generalization of deterministic stopping times is possible by setting

$$\tau_0^{(s)} = \tau_s = s 1_{s<\eta} + t_s(\eta) 1_{s\ge\eta}.$$

Now, we consider the parameter s which maximizes the put–price due to the stopping time $\tau_0^{(s)}$, i.e. we consider

$$\begin{aligned}
s^\circ &= \arg\max_{0\le s\le T} E\left(K - \frac{S^1_{\tau_0^{(s)}}}{S^2_{\tau_0^{(s)}}} S^2_T\right)^+ \\
&= \arg\max_{0\le s\le T} E\left[\left(K - \frac{S^1_s}{S^2_s} S^2_T\right)^+ 1_{s<\eta} + \left(K - \frac{S^1_{t_s(\eta)}}{S^2_{t_s(\eta)}} S^2_T\right)^+ 1_{s\ge\eta}\right] \\
&= \arg\max_{0\le s\le T} \left\{E 1_{s<\eta} E\left[\left(K - \frac{S^1_s}{S^2_s} S^2_T\right)^+ \middle| \eta\right] + E 1_{s\ge\eta} E\left[\left(K - \frac{S^1_{t_s(\eta)}}{S^2_{t_s(\eta)}} S^2_T\right)^+ \middle| \eta\right]\right\} \\
&= \arg\max_{0\le s\le T} \left\{\int_0^s P(S^1_0, K, s; u)\, p(u) du + \int_s^T P(S^1_0, K, t_s(u); u)\, p(u) du\right\}.
\end{aligned}$$

C Proof of Proposition 4.1

Let us define

$$Y^*_s := \sup_{\tau:\, s\le\tau\le T} E^{\mathcal{F}_s}\left(K - \frac{S^1_\tau}{S^2_\tau} S^2_T\right)^+ = E^{\mathcal{F}_s} E^{\mathcal{F}_\tau}\left(K - \frac{S^1_\tau}{S^2_\tau} S^2_T\right)^+ = E^{\mathcal{F}_s} Z_\tau$$

with

$$\begin{aligned}
Z_t &= E^{\mathcal{F}_t}\left(K - \frac{S^1_t}{S^2_t} S^2_T\right)^+ = 1_{\eta\le t} E^{\mathcal{F}_t}\left(K - \frac{S^1_t}{S^2_t} S^2_T\right)^+ + 1_{\eta>t} E^{\mathcal{F}_t}\left(K - \frac{S^1_t}{S^2_t} S^2_T\right)^+ \\
&= 1_{\eta\le t} E^{\mathcal{F}_t}\left(K - S^1_t e^{-\frac{1}{2}\int_t^T \sigma_{22}^2 ds + \int_t^T \sigma_{22} dW^2}\right)^+ + 1_{\eta>t} \int p(\eta\in du|\eta>t) P(S^1_r, K, r, t; u) \\
&= 1_{\eta\le t} \mathcal{B}\left(S^1_t, K, \sqrt{\int_t^T \sigma_{22}^2 ds}\right) + 1_{\eta>t} \int p(\eta\in du|\eta>t) \mathcal{B}\left(S^1_t, K, \sqrt{\int_t^T {}_u\sigma_2^2 ds}\right).
\end{aligned}$$

For any $1\le k\le n$ we have $Y^*_{t_k} = Y^*_{t_k}(\ln S^1_{t_k}, \varepsilon_{t_k})$ where $\varepsilon_t = 1_{\eta\le t}$. At time T we obviously have

$$Y^*_T(\ln S^1_T, \varepsilon_T) = Y^*_T(\ln S^1_T) = \left(K - S^1_T\right)^+$$

(hence $Y^*_T(y,\varepsilon) = (K - e^y)^+$). Suppose $Y^*_{t_{k+1}} = Y^*_{t_{k+1}}(\ln S^1_{t_{k+1}}, \varepsilon_{t_{k+1}})$ is known $k<n$. Then,

$$Y^*_{t_k} = \max(Z_{t_k}, E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}}), \qquad \text{where}$$

$$\begin{aligned}
E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}} &= E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}}(\ln S^1_{t_{k+1}}, \varepsilon_{t_{k+1}}) = 1_{\eta\le t_k} E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}}(\ln S^1_{t_{k+1}}, 1) \\
&+ 1_{\eta>t_k} E^{\mathcal{F}_{t_k}} E^{\mathcal{F}_{t_k}\vee\varepsilon_{t_{k+1}}} Y^*_{t_{k+1}}(\ln S^1_{t_{k+1}}, \varepsilon_{t_{k+1}}) \\
&= 1_{\eta\le t_k} E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}}(\ln S^1_{t_{k+1}}, 1) \\
&+ 1_{\eta>t_k} E^{\mathcal{F}_{t_k}} \Big[P(t_k<\eta\le t_{k+1}|\eta>t_k) Y^*_{t_{k+1}}(\ln S^1_{t_{k+1}}, 1) \\
&+ P(\eta>t_{k+1}|\eta>t_k) Y^*_{t_{k+1}}(\ln S^1_{t_{k+1}}, 0)\Big] \\
&= \varepsilon_{t_k} E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}}(\ln S^1_{t_{k+1}}, 1) + (1-\varepsilon_{t_k})\, \pi_k E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}}(\ln S^1_{t_{k+1}}, 0) \\
&+ (1-\varepsilon_{t_k})\, E^{\mathcal{F}_{t_k}} \int_{(t_k, t_{k+1}]} p(du|\eta>t_k)\cdot \\
&\cdot Y^*_{t_{k+1}}\left(\ln S^1_{t_k} - \frac{1}{2}\int_{t_k}^u \sigma_{11}^2 ds - \frac{1}{2}\int_u^{t_{k+1}} \sigma_{12}^2 ds + \int_{t_k}^u \sigma_{11} dW + \int_u^{t_{k+1}} \sigma_{12} dW, 1\right)
\end{aligned}$$

with $\pi_k := P(\eta > t_{k+1} | \eta > t_k)$. Let ϕ denote the standard normal density, then

$$
\begin{aligned}
E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}} &= \left[E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}} \right] (y, \varepsilon) \\
&= \varepsilon E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}} (y - \frac{1}{2} \int_{t_k}^{t_{k+1}} \sigma_{12}^2 ds + \int_{t_k}^{t_{k+1}} \sigma_{12} dW, 1) \\
&+ (1-\varepsilon) \pi_k E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}} (y - \frac{1}{2} \int_{t_k}^{t_{k+1}} \sigma_{11}^2 ds + \int_{t_k}^{t_{k+1}} \sigma_{11} dW, 0) \\
&+ (1-\varepsilon) E^{\mathcal{F}_{t_k}} \int_{(t_k, t_{k+1}]} p(du | \eta > t_k) \cdot \\
&\cdot Y^*_{t_{k+1}} (y - \frac{1}{2} \int_{t_k}^{u} \sigma_{11}^2 ds - \frac{1}{2} \int_{u}^{t_{k+1}} \sigma_{12}^2 ds + \int_{t_k}^{u} \sigma_{11} dW + \int_{u}^{t_{k+1}} \sigma_{12} dW, 1)
\end{aligned}
$$

$$
\begin{aligned}
&= \varepsilon \int \phi(\zeta) d\zeta \, Y^*_{t_{k+1}} (y - \frac{1}{2} \int_{t_k}^{t_{k+1}} \sigma_{12}^2 ds + \zeta \sqrt{\int_{t_k}^{t_{k+1}} \sigma_{12}^2 ds}, 1) \\
&+ (1-\varepsilon) \pi_k \int \phi(\zeta) d\zeta \, Y^*_{t_{k+1}} (y - \frac{1}{2} \int_{t_k}^{t_{k+1}} \sigma_{11}^2 ds + \zeta \sqrt{\int_{t_k}^{t_{k+1}} \sigma_{11}^2 ds}, 0) \\
&+ (1-\varepsilon) \int_{(t_k, t_{k+1}]} p(du | \eta > t_k) \int \phi(\zeta) d\zeta \cdot \\
&\cdot Y^*_{t_{k+1}} (y - \frac{1}{2} \int_{t_k}^{u} \sigma_{11}^2 ds - \frac{1}{2} \int_{u}^{t_{k+1}} \sigma_{12}^2 ds + \zeta \sqrt{\int_{t_k}^{u} \sigma_{11}^2 ds + \int_{u}^{t_{k+1}} \sigma_{12}^2 ds}, 1).
\end{aligned}
$$

Further, we have

$$
Z_{t_k}(y, \varepsilon) = \varepsilon \mathcal{B} \left(e^y, K, \sqrt{\int_{t_k}^{T} \sigma_{22}^2 ds} \right) + (1-\varepsilon) \int p(du | \eta > t_k) \mathcal{B} \left(e^y, K, \sqrt{\int_{t_k}^{T} {}_u\sigma_2^2 ds} \right)
$$

and so

$$
Y^*_{t_k}(y, \varepsilon) = \max(Z_{t_k}, E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}}) = \max \left(Z_{t_k}(y, \varepsilon), \left[E^{\mathcal{F}_{t_k}} Y^*_{t_{k+1}} \right] (y, \varepsilon) \right).
$$

Step $k = n-1$: Note that $\pi_{n-1} = 0$, hence

$$
\begin{aligned}
\left[E^{\mathcal{F}_{t_{n-1}}} Y^*_T \right] (y, \varepsilon) &= \varepsilon \int \phi(\zeta) d\zeta \, Y^*_T (y - \frac{1}{2} \int_{t_{n-1}}^{T} \sigma_{12}^2 ds + \zeta \sqrt{\int_{t_{n-1}}^{T} \sigma_{12}^2 ds}, 1) \\
&+ (1-\varepsilon) \int_{(t_{n-1}, T]} p(du | \eta > t_{n-1}) \int \phi(\zeta) d\zeta \cdot \\
&\cdot Y^*_T (y - \frac{1}{2} \int_{t_{n-1}}^{u} \sigma_{11}^2 ds - \frac{1}{2} \int_{u}^{T} \sigma_{12}^2 ds + \zeta \sqrt{\int_{t_{n-1}}^{u} \sigma_{11}^2 ds + \int_{u}^{T} \sigma_{12}^2 ds}, 1)
\end{aligned}
$$

$$
\begin{aligned}
&= \varepsilon \int \phi(\zeta) d\zeta \left(K - e^{y - \frac{1}{2} \int_{t_{n-1}}^{T} \sigma_{12}^2 ds + \zeta \sqrt{\int_{t_{n-1}}^{T} \sigma_{12}^2 ds}} \right)^+ + (1-\varepsilon) \\
&\int_{(t_{n-1}, T]} p(du | \eta > t_{n-1}) \int \phi(\zeta) d\zeta \left(K - e^{y - \frac{1}{2} \int_{t_{n-1}}^{u} \sigma_{11}^2 ds - \frac{1}{2} \int_{u}^{T} \sigma_{12}^2 ds + \zeta \sqrt{\int_{t_{n-1}}^{u} \sigma_{11}^2 ds + \int_{u}^{T} \sigma_{12}^2 ds}} \right)^+ \\
&= \varepsilon \, \mathcal{B} \left(e^y, K, \sqrt{\int_{t_{n-1}}^{T} \sigma_{12}^2 ds} \right) \\
&+ (1-\varepsilon) \int_{(t_{n-1}, T]} p(du | \eta > t_{n-1}) \mathcal{B} \left(e^y, K, \sqrt{\int_{t_{n-1}}^{u} \sigma_{11}^2 ds + \int_{u}^{T} \sigma_{12}^2 ds} \right).
\end{aligned}
$$

We further have

$$Z_{t_{n-1}}(y,\varepsilon) = \varepsilon \mathcal{B}\left(e^y, K, \sqrt{\int_{t_{n-1}}^{T} \sigma_{22}^2 ds}\right) + (1-\varepsilon)\int p(du|\eta > t_{n-1})\mathcal{B}\left(e^y, K, \sqrt{\int_{t_{n-1}}^{T} {}_u\sigma_2^2 ds}\right)$$

and

$$Y^*_{t_{n-1}}(y,\varepsilon) = \max\left[Z_{t_{n-1}}(y,\varepsilon), \left[E^{\mathcal{F}_{t_{n-1}}} Y_T^*\right](y,\varepsilon)\right].$$

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