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A substitute for the maximum principle for singularly perturbed time-dependent semilinear reaction-diffusion problems – part I

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Abstract

As the maximum principle does not hold true for nonlinear problems unless global, restrictive and often nonphysical assumptions are imposed over the whole domain, we introduce less restrictive, viable assumptions and show that the theory of upper and lower solutions is an appropriate substitute for the maximum principle in the case of singularly perturbed time-dependent semilinear reaction-diffusion problems with Dirichlet boundary conditions. The upper and lower solutions capture the boundary, interior and corner layers and the boundary conditions.

Introduction

Natural phenomena in biomathematics [25, §14.7], materials science, electrochemistry are often described by singularly perturbed, nonlinear, time-dependent reaction-diffusion problems [13, §2.3] Most of the analyses on time-dependent singularly perturbed problems rely on a maximum or comparison principle to establish existence of solutions and uniform stability, thus global, restrictive, even unrealistic assumptions are often imposed for the maximum principle to hold true, at the expense of the feasibility of the models. Reactiondiffusion equations, frequently semilinear, describe population dynamics [17], e.g., predator-pray interaction, where the inhomogeneous term reflects a local population density and may be positive or negative, to model, say, death or birth, cell lysis by toxicity [25], as well as dielectric properties of heterogeneous materials, ground-water percolation, processes in inhomogeneous media [36], homogenisation [27]. An asymptotic analysis of a singularly perturbed time-dependent initial-value problem with Neumann boundary conditions is given in [35, §3.2.3]; asymptotic analyses of singularly perturbed steady-state problems feature in [28, 26]. Numerical methods for general singularly perturbed differential equations feature in [23, 31, 8]. Computing solutions of equations that are singularly perturbed, i.e., characterized by singularities of boundary-layer type induced by a small parameter multiplying their highest derivative, require layer-adapted meshes, e.g., Bakhvalov (logarithmic) meshes [2, 3] or Shishkin (piecewise) meshes [24, 9, 10, 11]. In the sequel we review the use of the maximum principle, the assumptions and results in a couple of papers on singularly perturbed problems.

We refer to [18] for the numerical analysis of the time-dependent equation

$$\frac{\partial}{\partial t}u - \varepsilon\frac{\partial^2}{\partial x^2}u - \frac{\partial}{\partial x}[p(x)u] = f(x,t), \quad 0 < x < 1,\ 0 < t \leq 1,$$

subject to initial and Dirichlet boundary conditions and to the assumption $p(x) \geq \text{const} > 0$. If N is the number of mesh points in space, a boundary layer of width $C(N^{-2}\ln^2 N)$ on a Shishkin mesh and $C(N^{-2})$ on a Bakhvalov mesh exists at $x = 0$. The difference schemes use a four-point upwind space difference operator. The analysis technique does not involve maximum/comparison principles. The main results are error estimates bounded by $O(N^{-2}\ln^2 N) + \tau$ on Shishkin mesh and $O(N^{-2}) + \tau$ on Bakhvalov mesh, where τ is the mesh step in time, and $\varepsilon-$uniform convergence.

Linear time-dependent singularly perturbed reaction-diffusion Dirichlet boundary value problems are treated in [32] for two space variables on an infinite strip and on a rectangle. The main difficulty is that the differential operator contains two positive parameters that multiply the time derivative and the second-order space derivatives. The main method is layer-adapted meshes for each of the two domains and the result is uniform convergence. Large domains with respect to the space and time variables are considered by Shishkin in [33].

Time-dependent singularly perturbed reaction-diffusion is also the subject of [22]. In [15] the following linear problem is considered on $G = (0,1) \times (0,T]$:

$$\varepsilon^2\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}(x,t)\right) - c(x,t)u(x,t) - p(x,t)\frac{\partial u}{\partial t}(x,t) = f(x,t), \quad (x,t) \in G,$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S = \bar{G} \setminus G.$$

The functions a, c, p, f, φ are sufficiently smooth and bounded,

$$0 < a_0 \leq a(x,t), \quad 0 < p_0 \leq p(x,t), \quad c(x,t) \geq 0, \quad (x,t) \in \bar{G}$$

and $\varepsilon \in (0,1]$. When $\varepsilon \to 0$ the solution exhibits a boundary layer of width $O(\varepsilon)$. It is assumed that at the corner points $(0,0)$ and $(0,1)$ the following conditions hold true

$$\frac{\partial^k}{\partial x^k}\varphi(x,t) = \frac{\partial^{k_0}}{\partial t^{k_0}}\varphi(x,t) = 0, \quad k + 2k_0 \leq [\alpha] + 2n,$$

$$\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}f(x,t) = 0, \quad k + 2k_0 \leq [\alpha] + 2n - 2,$$

where $[\alpha]$ is the integer part of an arbitrary positive number $\alpha, n \geq 0$ is an integer and $[\alpha] + 2n \geq 2$. When a classical difference method is considered on the tensor product of a Shishkin mesh with N intervals on $[0, 1]$ and a uniform mesh with K intervals on $[0, T]$ (second-order differences in space and backward differences in time), a maximum principle is used. A discrete method based on defect correction is shown to yield $\varepsilon-$uniform convergence with accuracy $O(N^{-2} \ln N^2 + K^{-2})$ and $O(N^{-2} \ln N^2 + K^{-3})$, result which is consistent with [9]. Defect correction is also applied to singularly perturbed time-dependent convection-diffusion problems in [14]. A system of time-dependent reaction-diffusion problems in one space variable is studied in [12] and a time-dependent singularly perturbed equation in two space dimensions is discussed in [7]. In [4] a domain decomposition algorithm is considered for the singularly perturbed semilinear reaction-diffusion problem

$$\mu^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\partial u}{\partial t} = f(P, t, u), \quad P = (x, y),$$

$$(P, t) \in \Omega \times (0, t_F], \quad \Omega = \{P : 0 < x < 1, 0 < y < 1\},$$

where μ is a positive parameter. The initial-boundary conditions are

$$u(P, t) = g(P, t), \quad (P, t) \in \partial\Omega \times (0, t_F], \qquad u(P, 0) = u^0(P), \quad P \in \bar{\Omega},$$

where $\partial\Omega$ is the boundary of Ω. The functions $f(P, t, u)$, $g(P, t)$ and $u^0(P)$ are sufficiently smooth. For $\mu \ll 1$ the problem exhibits boundary layers near $\partial\Omega \times (0, t_F]$. The comparison/maximum principle and classical difference schemes on Shishkin meshes are used. Numerical experiments are performed on Bakhvalov and Shishkin meshes and uniform convergence of order one is achieved.

In [5] a $\theta-$time discretization, $0 \leq \theta \leq 1$, is performed for the nonlinear singularly perturbed reaction-diffusion problem

$$-\mu^2 (u_{xx} + u_{yy}) + u_t + f(x, y, t, u) = 0,$$

$$(x, y, t) \in Q = \omega \times (0, t_F], \quad \omega = \{0 < x < 1, 0 < y < 1\}$$

with initial and Dirichlet boundary conditions. Assumption $0 \leq f_u \leq c_*$, with c_* constant, is taken over the whole domain. For $\bar{v} = \max_{1 \leq i \leq N_x - 1} (v_{x,\, i-1} + v_{xi})$, $\quad \bar{w} = \max_{1 \leq j \leq N_y - 1} (w_{y,\, j-1} + w_{yj})$, the mesh size in time τ is required to satisfy $\tau(1 - \theta) \leq \frac{1}{\mu^2(\bar{v} + \bar{w}) + c_*}$, which yields existence of the discrete maximum principle. For $\theta = 1$, there is no restriction on τ. For uniform meshes of step sizes h_x and h_y, the condition is replaced by $(v_x + v_y)(1 - \theta) \leq 0.5$, $v_x = \frac{\mu^2 \tau}{h_x^2}$, $v_y = \frac{\mu^2 \tau}{h_y^2}$. The problem exhibits layers of widths $O(\mu)$ at the boundary $\partial\omega$. Uniform convergence of the weighted average scheme is achieved on piecewise uniform and log-meshes. On the piecewise uniform mesh, the implicit scheme ($\theta = 1$) is first order convergent with respect to the time step, while the Crank$-$Nicolson scheme ($\theta = 0.5$) is second order convergent.

Most analyses of singularly perturbed linear equations use the maximum principle, which cannot be applied to any nonlinear equation. There are very few numerical analysis results where the general condition $f_u > 0$ is not assumed in order to have maximum principle [34, 21, 19]. We are not aware of such

results for nonlinear time-dependent problems apart from the singularly perturbed reaction-diffusion initial-boundary value problem from [20].
In this paper we are concerned with the problem in [20] from the perspective of upper and lower solutions acting as substitutes for the maximum principle. We adopt and further develop the theory of upper and lower (or super- and sub-) solutions from the cited papers as a first step of a comparative study on how to deal with the lack of the maximum principle for nonlinear singularly perturbed problems. Generalizing the steady-state analysis from [21], we provide additional details on some results from [20] related to the exponential decay with a precise rate of the layer functions. Upon formulating the problem and the assumptions, in Section 2 we derive problems for the boundary, initial and corner layer functions depending on local, stretched variables. In Section 3 we perturb the asymptotic expansion of order one of the solution, from which we derive the upper and lower solutions. The main result, formulated by Theorem 3.1, is that these solutions provide existence and tight control of the solution.

1 Problem and assumptions

We are concerned with the singularly perturbed semilinear time-dependent reaction-diffusion equation

$$\mathcal{F}u \equiv \varepsilon^2[u_t - u_{xx}] + f(x,t,u) = 0 \qquad \text{for } (x,t) \in (0,1) \times (0,T], \tag{1}$$

with the singular perturbation positive parameter $\varepsilon \ll 1$, subject to the initial condition

$$u(x,0) = \varphi(x), \qquad x \in [0,1] \tag{2}$$

and the Dirichlet boundary conditions

$$\begin{cases} u(0,t) = g_0(t) \\ u(1,t) = g_1(t) \end{cases} \quad t \in [0,T], \tag{3}$$

where f, φ, g_0, g_1 are sufficiently smooth functions. We shall examine solutions of this problem that exhibit boundary, initial and corner layers induced by the small parameter ε. Solutions of (1), (2) may also have interior transition layers, as in the case of different values of the initial condition for different stable reduced solutions. As mentioned, we drop the usual restrictive assumption [23, 31] $f_u(x,t,u) > 0$ for all $(x,t,u) \in [0,1] \times [0,T] \times \mathbb{R}$ and consider the problem (1), (2), (3) under weaker assumptions:
A0. The reduced equation obtained by setting $\varepsilon = 0$ in (1):

$$f(x,t,u_0(x,t)) = 0 \quad \text{for } (x,t) \in (0,1) \times (0,T) \tag{4}$$

has a sufficiently smooth solution $u_0(x,t)$.
As an algebraic equation with f_u not necessarily positive, (4) may have multiple solutions, each of whom does not necessarily satisfy the initial condition (2) or the boundary condition (3).
In the steady-state case, equation (1) may have multiple solutions. For instance, the problem from [16]

$$-\varepsilon^2 u'' + (u^2 + u - 0.75)(u^2 + u - 3.75) = 0, \quad x \in (0,1), \quad u(0) = u(1) = 0$$

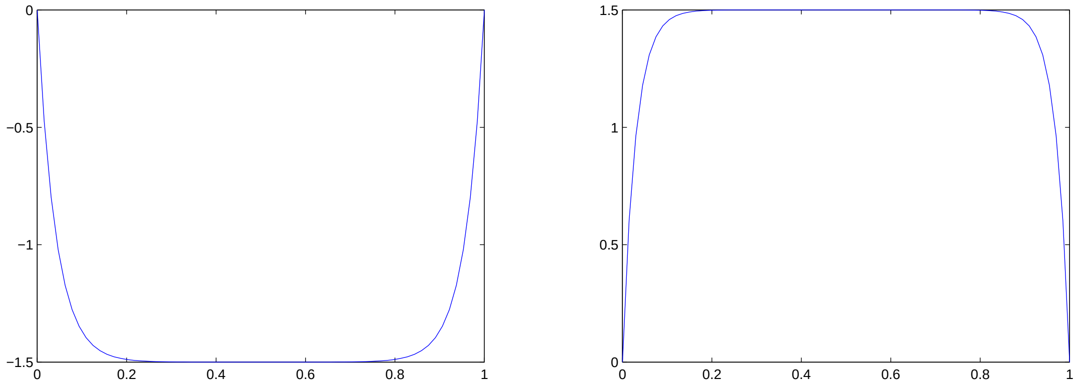

Figure 1: Example showing that the solution of a steady-state singularly perturbed reaction-diffusion problem is not unique

has a solution near the stable reduced solution -1.5 and another one near the stable reduced solution 1.5, as shown in Figure 1. In contrast, provided f_u is continuous for all $u \in \mathbb{R}$, problem (1), (2), (3) has at most one solution [29]. In general, uniqueness is not the case for equations of type (1). Take, for instance, $f(x, u) = u + (3 \sin^{2/3} x/2)u^{1/3}$, $0 < x < \pi$, and the Dirichlet boundary conditions $u(t, 0) = u(t, \pi) = 0$, $t > 0$ [29, §1.6]. Each of the functions $u_1 = -t^{3/2} \sin x$, $u_2 = 0$, $u_3 = t^{3/2} \sin x$ is a solution; for any $t_0 \geq 0$, the function

$$u(x,t) = \begin{cases} 0 & \text{for } 0 \leq t \leq t_0 \\ (t - t_0)^{3/2} \sin x & \text{for } t \geq t_0 \end{cases}$$

is also a solution. In accord with [20], we further assume:

A1. The reduced solution u_0 (whose existence is provided be assumption A0) is stable, i.e., there exists a constant γ such that

$$f_u(x, t, u_0(x, t)) > \gamma^2 > 0 \qquad \forall (x, t) \in [0, 1] \times [0, T];$$

A2. Let $(a_1, a_2]'$ denote $(a_1, a_2]$ when $a_1 < a_2$, $(a_2, a_1]$ when $a_1 > a_2$ and $\emptyset$ when $a_1 = a_2$. Then

$$\int_{u_0(0,t)}^{v} f(0, t, s)ds > 0 \quad \forall\, v \in (u_0(0, t), g_0(t)]', \qquad t \in [0, T]$$

and

$$\int_{u_0(1,t)}^{v} f(1, t, s)ds > 0 \quad \forall\, v \in (u_0(1, t), g_1(t)]', \qquad t \in [0, T].$$

A3. The initial condition belongs to the domain of attraction of the reduced solution, i.e.,

$$\frac{f(x, 0, u_0(x, 0) + s)}{s} > 0 \qquad \forall\, s \in (0, \varphi(x) - u_0(x, 0)]', \;\; x \in [0, 1]. \tag{5}$$

As shown in Figure 2, a solution of (1), (2), (3) exhibits boundary layers of widths $O(\varepsilon)$ near $x = 0$ and $x = 1$, an initial layer of width $O(\varepsilon^2)$ at $t = 0$ and corner layers around (0,0) and (1,0). Thus we employ the stretched variables

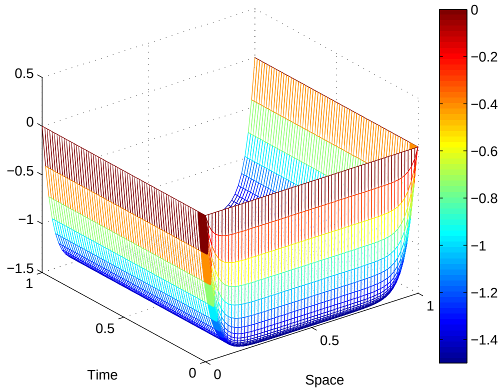

Figure 2: Solution of the initial-boundary value problem (1), (2), (3) with $\varepsilon = 0.01, f = (u+2.5)(u+1.5)(u-0.5)(u-1.5), \varphi(x) = 0.1,\ g_0 = 0,\ g_1 = 0.$

$$\xi = x/\varepsilon, \qquad \tau = t/\varepsilon^2$$

to define functions which describe these layers. For simplicity we only consider the boundary and corner layers in $x = 0$ and applying the boundary function method from [35], we approximate the solution of (1), (2), (3) by its asymptotic expansion of order one:

$$u_{as} := u_0(x,t) + [v_0(\xi,t) + \varepsilon v_1(\xi,t)] + w_0(x,\tau) + [q_0(\xi,\tau) + \varepsilon q_1(\xi,\tau)].$$

In this asymptotic solution we have identified the smooth component by the solution of the reduced problem u_0; the term $v_0(\xi,t) + \varepsilon v_1(\xi,t)$ characterizes the boundary-layer of width $O(\varepsilon)$ at $x = 0$; $w_0(x,\tau)$ is the initial-layer function and $q_0(\xi,\tau) + \varepsilon q_1(\xi,\tau)$ stands for the corner layer. We determine these functions by formally introducing them in the original problem and matching the coefficients multiplying the same powers of ε.

Compatibility conditions

We impose the zero-order compatibility conditions at the corner

$$\varphi(0) = g_0(0), \qquad \varphi(1) = g_1(0) \tag{6}$$

and to ensure that problem (1), (2), (3) has a sufficiently smooth solution, we impose compatibility condition of order one at the corners $(0,0)$ and $(1,0)$ by formally setting $x = 0$, $t = 0$ and $x = 1$, $t = 0$ in (1):

$$\varepsilon^2[g_l'(0) - \varphi''(l)] + f(l,0,\varphi(l)) = 0 \quad \text{for } l = 0,1. \tag{7}$$

Neglecting the $O(\varepsilon^2)$ terms in (7), we obtain

$$f(l,0,\varphi(l)) = 0 \ \text{ for } l = 0,1. \tag{8}$$

Note that choosing $s > 0$ in $A3$, for $x = 0$ we obtain

$$f(0,0,u_0(0,0)+s) > 0, \quad \forall\, s \in (0, \varphi(0) - u_0(0,0)]. \tag{9}$$

Presuming that $\varphi(0) - u_0(0,0) > 0$ and setting $s := \varphi(0) - u_0(0,0)$ in (9), we have

$$f(0, 0, \varphi(0)) > 0,$$

which contradicts (8). Therefore $\varphi(0) - u_0(0,0) \leq 0$. By (20), this yields $\varphi(0) = u_0(0,0)$. Applying a similar argument for $l = 1$, we see that

$$\varphi(l) = u_0(l, 0), \quad l = 0, 1. \tag{10}$$

Remark 1.1. Rigorously, the terms $\varepsilon^2[g_l'(0) - \varphi''(l)] = O(\varepsilon^2)$ should remain in (7). This would imply

$$u_0(l, 0) - \varphi(l) = O(\varepsilon^2), \quad l = 0, 1. \tag{11}$$

Therefore, neglecting these terms will not change the first-order asymptotic expansion.

Proof. We prove (11) for $l = 0$. Assume that $\varphi(0) - u_0(0,0) > 0$. Let

$$\tilde{f}(s) := f(0, 0, u_0(0,0) + s), \quad s \in (0, \varphi(0) - u_0(0,0)]$$

In a small vicinity of 0

$$\tilde{f}'(0) = \lim_{s \to 0} \frac{\tilde{f}(s) - \tilde{f}(0)}{s}$$

From (4) $\tilde{f}(0) = 0$, so

$$\lim_{s \to 0} \frac{\tilde{f}(s)}{s} = \tilde{f}'(0),$$

i.e. $\exists\, s_0 \in (0, \varphi(0) - u_0(0,0))$ such that

$$\left| \frac{\tilde{f}(s)}{s} - \tilde{f}'(0) \right| \leq \frac{f'(0)}{2} \qquad \forall s \in [0, s_0].$$

Hence

$$\tilde{f}(s) \geq Cs, \quad \forall s \in [0, s_0],$$

where

$$C = \tilde{f}'(0) - \frac{f'(0)}{2} \geq \frac{\gamma^2}{2}$$

and we used $A1$.
Outside $[0, s_0]$, let

$$m = \min_{s \in (s_0, \varphi(0) - u_0(0,0)]} \tilde{f}(s)$$

By $A3$, we have $m > 0$. Then

$$\tilde{f}(s) \geq \frac{m}{\varphi(0) - u_0(0,0)} s.$$

Therefore, choosing

$$\bar{C} = \min \left[\frac{\gamma^2}{2}, \min_{s \in (s_0, \varphi(0) - u_0(0,0)]} \frac{\tilde{f}(s)}{s} \right]$$

yields

$$f(0,0,u_0(0,0)+s) > \bar{C}s, \quad \forall\, s \in (0, \varphi(x) - u_0(x,0)]. \tag{12}$$

From (7) we obtain

$$f(0,0,\varphi(0)) = -\varepsilon^2[g_0'(0) - \varphi''(0)] = O(\varepsilon^2). \tag{13}$$

Relations (12) and (13) yield

$$O(\varepsilon^2) = f(0,0,\varphi(0)) = f(0,0,u_0(0,0)+s)|_{s=\varphi(0)-u_0(0,0)} \geq \bar{C}[\varphi(0)-u_0(0,0)] > 0,$$

which implies (11) for $l = 0$ and similarly for $l = 1$.
Hence (10) should be replaced by a more general relation

$$\varphi(l) = g_l(0) = u_0(l,0) + O(\varepsilon^2).$$

□

Yet, for simplicity, we keep the assumption (10).

2 Layer functions

Boundary-layer function

The solution of (1), (2), (3) near the boundary $\{(x,t)|\ x = 0\}$ is described by a boundary-layer function depending on the stretched variable $\xi = x/\varepsilon$. The spatial boundary region satisfies $u_t(x,t) = O(\varepsilon^2)$, which, upon formally introducing $u := u_0(x,t) + v_0(\xi,t) + \varepsilon v_1(\xi,t)$ in (1), yields

$$\varepsilon^2[u_0 + v_0 + \varepsilon v_1]_t = O(\varepsilon^2).$$

Due to u_0 being sufficiently smooth, $\varepsilon^2 u_{0,xx} = O(\varepsilon^2)$, giving

$$-(v_{0,\xi\xi} + \varepsilon v_{1,\xi\xi}) + f(x,t,u_0(x,t) + v_0(\xi,t) + \varepsilon v_1(\xi,t)) + O(\varepsilon^2) = 0.$$

Defining the functionals

$$F(x,t,s) := f(x,t,u_0(x,t)+s),$$
$$G(\varepsilon) := F(\varepsilon\xi, t, v_0 + \varepsilon v_1), \tag{14}$$

we have

$$f(x,t,u_0(x,t) + v_0(\xi,t) + \varepsilon v_1(\xi,t)) = G(\varepsilon).$$

Now expand $G(\varepsilon)$ as

$$G(\varepsilon) = G(0) + \varepsilon G'(0) + \frac{\varepsilon^2}{2} G''(\varepsilon^*), \tag{15}$$

with some ε^* between 0 and ε. Formally assuming $\varepsilon^2|G''(\varepsilon^*)| = O(\varepsilon^2)$ and setting $\varepsilon = 0$, in (14) we obtain $G(0) = F(0,t,v_0)$ and in (15)

$$G'(0) = v_1 F_s(0,t,v_0) + \xi F_x(0,t,v_0). \tag{16}$$

By (2), we have

$$-(v_{0,\xi\xi}+\varepsilon v_{1,\xi\xi})+f(x,t,u_0+v_0+\varepsilon v_1)+O(\varepsilon^2)=$$

$$=-(v_{0,\xi\xi}+\varepsilon v_{1,\xi\xi})+G(0)+\varepsilon G'(0)+O(\varepsilon^2)=0.$$

Combining this equality with (16), we obtain that the coefficient of the term in the right-hand side of (15) containing power zero of ε is $-v_0''+F(0,t,v_0)$. Thus the zero-order term of the boundary-layer function $v_0(\xi,t)$ must be a solution of

$$-\frac{\partial^2 v_0}{\partial \xi^2}+F(0,t,v_0)=0, \tag{17}$$

with $\xi>0$, subject to the boundary conditions

$$v_0(\infty,t)=0, \tag{18}$$

$$v_0(0,t)=g_0(t)-u_0(0,t), \tag{19}$$

where we assume

$$g_0(t)-u_0(0,t)\geq 0, \quad t\in[0,T]. \tag{20}$$

The coefficient of the term with power one of ε in the right-hand side of (15) is $-\frac{\partial^2 v_1}{\partial\xi^2}+F_s(0,t,v_0)v_1+\xi F_x(0,t,v_0)$, yielding the problem for v_1 :

$$\begin{cases} -\dfrac{\partial^2 v_1}{\partial \xi^2}+v_1F_s(0,t,v_0) & =-\xi F_x(0,t,v_0)\\ v_1(0,t) & =0\\ v_1(\infty,t) & =0. \end{cases} \tag{21}$$

In the sequel we provide a more detailed proof for a result from [20] concerning the existence and upper bounds of the zero-order term of the boundary-layer function and its derivatives.

Lemma 1. *Set*

$$\gamma_L^2:=\min_{t\geq 0} f_u(0,t,u_0(0,t))>\gamma^2, \tag{22}$$

where γ is the constant from assumption A1.
i) There exists a solution v_0 of the problem (17), (18), (19).
ii) For any arbitrarily small, but fixed $\delta\in(0,\gamma_L)$, there exists a positive constant C_δ depending on δ such that for $k=\overline{0,3}$ and in the maximum norm

$$\left|\frac{\partial^k v_0}{\partial \xi^k}\right|\leq C_\delta e^{-(\gamma_L-\delta)\xi}. \tag{23}$$

Proof. To prove existence and properties of the zero order term of the boundary-layer function, we generalize the steady-state analysis from [21, Lemma 2.1]. *i)* For any fixed t, writing (17) as the equivalent system

$$\frac{\partial v_0}{\partial \xi} \;=\; V(\xi,t) \tag{24}$$

$$\frac{\partial V}{\partial \xi} \;=\; F(0,t,v_0) \tag{25}$$

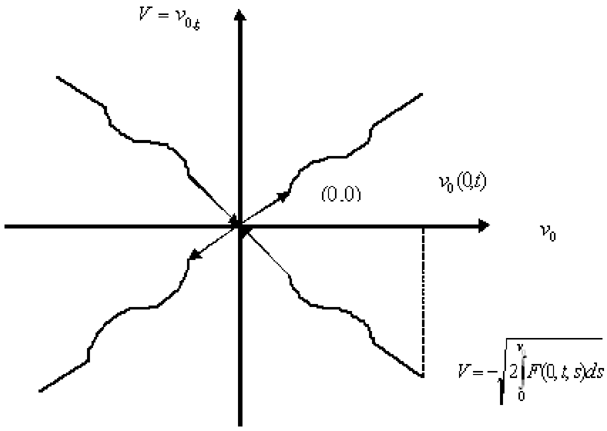

Figure 3: Phase plane which illustrates existence of the zero order term of the boundary-layer function

with the boundary conditions (19), (18) and

$$V(\infty, t) = 0, \tag{26}$$

we analyze the phase plane for (24) and (25) depicted in Figure 3. Assumption $A2$ implies that (24) and (25) have a stationary point at $(0, 0)$. A solution v_0 exists provided a phase plane trajectory leaves the point $(v_0(0, t), V(0, t))$ and enters $(0, 0)$ as $\xi \to \infty$. The Jacobian matrix associated with the right-hand side of (24), (25) is

$$\begin{pmatrix} 0 & 1 \\ F_s(0, t, v_0) & 0 \end{pmatrix}$$

with the eigenvalues $\pm\sqrt{F_s(0, t, v_0)}$. Assumption $A1$ implies that these eigenvalues are real and of opposite sign at the stationary point $(v_0, V) = (0, 0)$, hence $(0, 0)$ is a saddle for (24), (25), (26), i.e., a critical point where the eigenvalues of the associated matrix are real and of opposite sign [1]. By (24) and (25),

$$V = \pm\sqrt{2\int_0^{v_0} F(0, t, s)ds + C}, \tag{27}$$

with C a constant of integration. At $v_0 = 0$, $V = \frac{\partial v_0}{\partial \xi} = 0$, giving $C = 0$. By (18), the trajectory that leaves $(v_0(0, t), V(0, t))$ and decays to (0,0) as $\xi \to \infty$ is to be chosen in (27), while by (20), the boundary condition (19) is nonnegative, prompting towards the negative choice in (27).

Making a variable change $r = s - u_0(0, t)$ and setting $\hat{v} = v - u_0(0, t)$, we obtain an equivalent formulation of assumption $A2$:

$$\int_0^{\hat{v}} f(0, t, u_0(0, t)+r)\, dr = \int_0^{\hat{v}} F(0, t, s)\, ds > 0, \; \forall\, \hat{v} \in (0, g_0(t)-u_0(0, t)],\; t \in [0, T],$$

thus $\int_0^{g_0(t)-u_0(0,t)} F(0, t, s)ds > 0$ while v_0 decays from $v_0(0, t) = g_0(t) - u_0(0, t)$ to $v_0(\infty, t) = 0$. Therefore, the separatrix $V = -\sqrt{2\int_0^{v_0} F(0, t, s)ds}$ intersects

the line (19), providing a solution of (17), (18), (19).
ii) We have assumed in (4) that $g_0(t) - u_0(0,t) \geq 0$, which with (19) yields $v_0(0,t) > 0$. While $v_0(\xi,t) > 0$, $V(\xi,t) < 0$, thus $v_0(\xi,t)$ is decreasing. The inequality $F_s(0,t,0) \geq \gamma_L^2$ implies existence of a $s_0(t) > 0$ such that

$$(\gamma_L - \delta)^2 s \leq F(0,t,s) \qquad \text{for } 0 \leq s \leq \max_{t\geq 0} s_0(t).$$

If we had $v_0(\xi,t) > s_0(t)$ for all $\xi > 0$, there would exist a positive constant C such that $v_{0,\xi} \leq -C$, i.e., $V \leq -C$, yielding that $\lim_{\xi\to\infty} v_0(\xi,t) = -\infty$, which contradicts (18). Thus there exists ξ_0 such that $v_0(\xi_0,t) \leq s_0(t)$, implying that for all $\xi \geq \xi_0$, $v_0(\xi,t) \leq s_0(t)$. Then for $0 < v_0(\xi_0,t)$, (27) yields

$$V(\xi,t) \leq -\sqrt{2\int_0^{v_0(\xi,t)} (\gamma_L - \delta)^2 s\, ds},$$

giving

$$V(\xi,t) \leq -(\gamma_L - \delta) v_0(\xi,t) \quad \forall \xi > \xi_0, \tag{28}$$

i.e.,

$$\frac{\frac{\partial v_0(\xi,t)}{\partial \xi}}{v_0(\xi,t)} \leq -(\gamma_L - \delta) \qquad \forall\, \xi > \xi_0. \tag{29}$$

Integrating in (29) from ξ_0 to ξ, with $\xi_0 \leq \xi < \infty$, we obtain

$$v_0(\xi,t) \leq v_0(\xi_0,t) e^{(\gamma_L-\delta)\xi_0} e^{-(\gamma_L-\delta)\xi}, \tag{30}$$

showing that

$$\lim_{\xi\to\infty} v_0(\xi,t) = 0.$$

Taking $C_\delta \geq v_0(\xi_0,t)$ in (30), for $\xi \geq \xi_0$ we have

$$v_0(\xi,t) \leq C_\delta e^{-(\gamma_L-\delta)\xi}. \tag{31}$$

In case $0 \leq \xi \leq \xi_0$, by $s_0(t) \leq v_0(\xi,t) \leq v_0(0,t)$, taking $C_\delta = v_0(0,t) e^{(\gamma_L^2-\delta)\xi_0}$, we, again, obtain

$$v_0 \leq C_\delta e^{-(\gamma_L^2-\delta)\xi_0} \leq C_\delta e^{-(\gamma_L-\delta)\xi}.$$

Therefore, for a fixed ξ_0, (31) holds true for all $\xi \geq 0$, with

$$C_\delta = \max_{t\geq 0}\{\max[v_0(\xi_0,t), v_0(0,t)e^{(\gamma_L^2-\delta)\xi_0}]\}. \tag{32}$$

From (30) it follows that $|V(\xi,t)|$ has the same upper bound as in (31), i.e., (23) is satisfied for $k=1$. Writing (17) as

$$v_{0,\xi\xi} = F(0,t,v_0), \tag{33}$$

for $\hat{v}_0$ between 0 and v_0, (4) yields

$$v_{0,\xi\xi} = v_0 F_s(0,t,u_0(0,t) + \hat{v}_0)$$

with $\hat{v}_0$ bounded, thus

$$|v_{0,\xi\xi}| \leq C|v_0| \leq Ce^{-(\gamma_L-\delta)\xi}.$$

Differentiating (33) with respect to ξ and using the previous upper bounds for $|v_0|$ and its derivatives, we obtain (23) for $k=3$. $\square$

Lemma 2. *Set the non-negative functional*

$$\chi(\xi,t)=\begin{cases}\dfrac{v_{0,\xi}(\xi,t)}{v_{0,\xi}(0,t)} & \text{if } v_0(0,t)>0\\ e^{-\xi\sqrt{F_s(0,t,0)}} & \text{if } v_0(0,t)=0.\end{cases}\tag{34}$$

i) Then χ defined by (34) is a solution of the homogeneous problem

$$\begin{cases}-\dfrac{\partial^2\chi}{\partial\xi^2}+F_s(0,t,v_0)\chi(\xi,t) &=0\\ \lim_{\xi\to\infty}\chi(\xi,t) &=0\\ \chi(0,t) &=1\end{cases}\tag{35}$$

and there exist the positive constants C_1 and C_2 such that

$$C_1 v_0\le[g_0(t)-u_0(0,t)]\chi\le C_2 v_0.\tag{36}$$

ii) The solution of the problem

$$\begin{cases}-\dfrac{\partial^2\Phi}{\partial\xi^2}+F_s(0,t,v_0)\Phi(\xi,t) &=\Psi(\xi,t)\\ \Phi(0,t) &=A\\ \Phi(\infty,t) &=0\end{cases}\tag{37}$$

is given by

$$\Phi(\xi,t)=A\chi(\xi,t)+\chi(\xi,t)\int_0^\xi\frac{d\xi_0}{\chi^2(\xi_0,t)}\int_{\xi_0}^\infty\Psi(\eta,t)\chi(\eta,t)d\eta.\tag{38}$$

iii) If, for some $k\ge 0$,

$$|\Psi|\le C\chi(\xi^k+1),\tag{39}$$

then

$$|\Phi|\le C(A+1+\xi^{k+1})\chi.\tag{40}$$

Proof. *i)* The functional χ is well defined in (34), as $v_{0,\xi}(0,t)=0\Leftrightarrow v_0(0,t)=0$ and substituting (34) in (35), if $v_0(0,t)>0$, we obtain

$$\begin{cases}-\dfrac{v_{0,\xi\xi\xi}(\xi,t)}{v_{0,\xi}(0,t)}+F_s(0,t,v_0)\dfrac{v_{0,\xi}(\xi,t)}{v_{0,\xi}(0,t)} &=0\\ \lim_{\xi\to\infty}\dfrac{v_{0,\xi}(\xi,t)}{v_{0,\xi}(0,t)} &=0\\ \dfrac{v_{0,\xi}(0,t)}{v_{0,\xi}(0,t)} &=1.\end{cases}$$

The first equation above coincides with (17), or (33) differentiated by ξ. If, for some t, $g_0(t)-u_0(0,t)=0$, then $v_0(0,t)=0$ for all ξ, thus $F_s(0,t,v_0)=F_s(0,t,0)>0$ and so $\chi=e^{-\sqrt{F_s(0,t,0)}\xi}$ is well defined. Furthermore,

$$\begin{cases}-e^{\xi\sqrt{F_s(0,t,0)}}F_s(0,t,0)+e^{\xi\sqrt{F_s(0,t,0)}}F_s(0,t,0) &=0\\ \lim_{\xi\to\infty}e^{-\xi\sqrt{F_s(0,t,0)}} &=0\\ e^0 &=1.\end{cases}$$

The inequality (28) implies

$$|v_{0,\xi}| \geq \tilde{C}_2 v_0. \tag{41}$$

Similarly,

$$|v_{0,\xi}| \leq \tilde{C}_1 v_0.$$

Taking $C_1 := \tilde{C}_1^{-1}\tilde{C}_2$ and $C_2 := \tilde{C}_2^{-1}\tilde{C}_1$ yields (36).
ii) Setting

$$\Phi(\xi) = C_1(\xi)\chi(\xi)$$

and introducing it in (37), from standard variation of parameters for second order ordinary differential equations we have that the solution of a non-homogeneous equation is

$$\Phi = C_1(\xi)y_1(\xi) + C_2(\xi)y_2(\xi),$$

where y_1, y_2 are two linear independent solutions. We take $y_1 = \chi$, while y_2 is generally an increasing function, because, by [6] the Wronskian is

$$W = \begin{vmatrix} y_1' & y_2' \\ y_1 & y_2 \end{vmatrix} = \chi' y_2 - \chi y_2' = C,$$

so

$$\frac{y_2'\chi - y_2\chi'}{\chi^2} = \frac{C}{\chi^2}$$

$$\left(\frac{y_2}{\chi}\right)' = \frac{C}{\chi^2}$$

$$y_2 = \chi\left(A + \int \frac{C}{\chi^2}\right).$$

From (36) we see that while χ is exponentially decreasing, $\frac{C}{\chi^2}$ is exponentially increasing faster than χ. Therefore the term $C_2(\xi)y_2$ in the solution Φ of (37) is increasing, while satisfying a zero boundary condition at infinity, which prompts us to take $C_2(\xi) = 0$. Thus we look for a solution in the form $C_1(\xi)\chi(\xi)$. Introduced in (37), it gives

$$-(C_1\chi)_{\xi\xi} + F_s(0, t, v_0)C_1\chi = \Psi, \tag{42}$$

where

$$-C_1\chi_{\xi\xi} + F_s(0, t, v_0)C_1\chi = 0.$$

Setting $\hat{C}_1 = C_1'$ in (42), we obtain

$$\hat{C}_1'\chi + 2\hat{C}_1\chi' = \Psi.$$

The integrating factor here is $\rho := e^{\int 2\frac{\chi'}{\chi}} = \chi^2$ and $\hat{C}_1 = \frac{\int \frac{\Psi}{\chi}\rho + C}{\rho}$ yields

$$C_1 = \int_0^\xi \hat{C}_1 + A,$$

leading to (38).
Now we shall prove that (39) implies (40). Firstly, we estimate

$$I = I(\xi_0) := \int_{\xi_0}^\infty \Psi(\eta)\chi(\eta)d\eta \leq C\int_{\xi_0}^\infty \chi^2(\eta)(\eta^k + 1)d\eta,$$

where by (36)

$$\chi^2(\eta) = \chi(\eta)\chi(\eta) \leq$$

$$\leq \frac{C}{(g_0(t) - u_0(0,t))^2} v_{0,\eta}(\eta,t) v_0(\eta,t) = \frac{C}{(g_0(t) - u_0(0,t))^2} \left[\frac{v_0^2}{2}\right]_\eta .$$

Using integration by parts, we obtain

$$I \leq \frac{C}{(g_0(t) - u_0(0,t))^2} \int_{\xi_0}^{\infty} [v_0^2(\eta,t)]_\eta (\eta^k + 1) d\eta =$$

$$= \frac{C}{(g_0(t) - u_0(0,t))^2} \{[v_0^2(\eta)(\eta^k + 1)]|_{\xi_0}^{\infty} - \int_{\xi_0}^{\infty} v_0^2(\eta) k \eta^{k-1} d\eta\} \leq$$

$$\leq \frac{C}{(g_0(t) - u_0(0,t))^2} \left\{ |v_0^2(\xi_0)(\xi_0^k + 1)| + \left| \int_{\xi_0}^{\infty} [v_0(\eta)^2]_\eta \eta^{k-1} d\eta \right| \right\},$$

as $v_0 = 0$ at infinity. To estimate the second term above containing the integral, by (41), we have

$$v_0^2(\eta) \leq C v_0 v_{0,\eta} = C \left[\frac{v_0^2}{2}\right]_\eta$$

and so

$$I_1 =$$

$$= \int_{\xi_0}^{\infty} \eta^{k-1} v_0^2(\eta) d\eta \leq C \int_{\xi_0}^{\infty} \eta^{k-1} [v_0^2]_\eta d\eta = C \left[[v_0^2 \eta^{k-1}]|_{\xi_0}^{\infty} - \int_{\xi_0}^{\infty} v_0^2 (k-1) \eta^{k-2} d\eta \right].$$

Continuing in this manner and substituting I_{k-1} in I_k, we obtain

$$|I| \leq \frac{C}{(g_0(t) - u_0(0,t))^2} v_0^2(\xi_0)(\xi_0^k + \xi_0^{k-1} + ... + 1) \leq C \frac{(\xi_0^k + 1) v_0^2(\xi_0)}{(g_0(t) - u_0(0,t))^2}. \quad (43)$$

Using (43) in (38) yields

$$\chi(\xi,t) \frac{\int_0^{\xi} \frac{v_0^2(\xi_0)}{\chi^2(\xi_0,t)} d\xi_0}{(g_0(t) - u_0(0,t))^2} (\xi_0^k + 1) \leq C_2^2 \chi(\xi,t) \int_0^{\xi} (\xi_0^k + 1) d\xi_0 =$$

$$= C_2^2 \chi(\xi,t) \left(\frac{\xi^{k+1}}{k+1} + \xi \right),$$

which leads to (40). □

We now possess necessary information to derive upper estimates in the maximum norm of $\left|\frac{\partial^k v_0}{\partial t^k}\right|$ and existence and properties of v_1 and its derivatives, as they satisfy linear differential equations containing the differential operator featuring in (21).

Proposition 2.1. *(i) Problem (21) has a solution v_1.*
(ii) For the solution v_0 of the problem (17), (18), (19), $k = \overline{0,3}$ and $l = \overline{0,2}$, each of the maximum norm quantities $\left|\frac{\partial^l v_0}{\partial t^l}\right|$, $\left|\frac{\partial^k v_1}{\partial \xi^k}\right|$ and $\left|\frac{\partial^l v_1}{\partial t^l}\right|$ has the upper bound $C_\delta e^{-(\gamma_L - \delta)\xi}$, with γ_L defined by (22).

Proof. *(i)*The existence of v_1 is provided by Lemma 2.
(ii) Differentiating (17), (18), (19) with respect to t, with

$$\frac{\partial}{\partial t}F(0,t,v_0) = F_t(0,t,v_0) + F_s(0,t,v_0)v_{0,t}, \tag{44}$$

$v_{0,t}$ is given by the problem

$$\begin{cases} \left[-\frac{\partial^2}{\partial \xi^2} + F_s(0,t,v_0)\right] v_{0,t} & = -F_t(0,t,v_0) \\ v_{0,t}(0,t) & = g_{0,t}(t) - u_{0,t}(0,t) \\ v_{0,t}(\infty,t) & = 0 \end{cases} \tag{45}$$

We apply Lemma 2 to (45), for $\Phi = v_{0,t}$, $\Psi = -F_t(0,t,v_0)$ with $|-F_t(0,t,v_0)| \leq Cv_0$ and $k = 0$:

$$|v_{0,t}| \leq C(|g_{0,t}(t) - u_0(0,t)| + 1 + \xi)\chi \leq Ce^{-(\gamma_L - \delta)\xi}.$$

Differentiating (44) with respect to t, we obtain

$$\frac{\partial^2}{\partial t^2}F(0,t,v_0) = \frac{\partial}{\partial t}[F_t(0,t,v_0) + F_s(0,t,v_0)v_{0,t}] =$$

$$= F_{tt}(0,t,v_0) + 2F_{ts}(0,t,v_0)v_{0,t} + F_{ss}(0,t,v_0)v_{0,t}^2 + F_s(0,t,v_0)v_{0,tt}.$$

The problem for $v_{0,tt}$ is

$$\begin{cases} \left[-\frac{\partial^2}{\partial \xi^2} + F_s(0,t,v_0)\right] v_{0,tt} & = -F_{tt}(0,t,v_0) - 2F_{ts}(0,t,v_0)v_{0,t} - F_{ss}(0,t,v_0)v_{0,t}^2 \\ v_{0,tt}(0,t) & = g_{0,tt}(t) - u_{0,tt}(0,t) \\ v_{0,tt}(\infty,t) & = 0, \end{cases}$$

so in Lemma 2 we take $\Phi = v_{0,tt},\ A = g_{0,tt} - u_{0,tt}$ and

$$\Psi = -F_{tt}(0,\cdot,v_0) - 2F_{ts}(0,\cdot,v_0)v_{0,t} - F_{ss}(0,\cdot,v_0)v_{0,t}^2,$$

noting that

$$|-F_{tt}(0,t,v_0)| \leq Cv_0 \leq C\chi.$$

From (38) and

$$|\Psi| = |-2F_{ts}(0,t,v_0)v_{0,t} - F_{ss}(0,t,v_0)v_{0,t}^2| \leq C(1+\xi)\chi,$$

Lemma 2 with $k = 1$ yields

$$|v_{0,tt}| \leq C[(g_{0,tt}(t) - u_{0,tt}(0,t)) + 1 + \xi^2]\chi \leq Ce^{-(\gamma_L - \delta)\xi}.$$

Now take $v_1 = \Phi$ in (37), where $\Psi = -\xi F_x(0,t,v_0)$. If $v_0(0,t) > 0$, (36) yields

$$|\Psi| = |-\xi F_x(0,t,v_0)| \leq C\xi v_0 \leq C|g_0(t) - u_0(0,t)|\chi(\xi + 1),$$

If $v_0(0,t) = 0$, we also have $|\Psi| \leq C\chi(\xi + 1)$. We apply Lemma 2 with $k = 1$:

$$|v_1| \leq C(1+\xi^2)|g_0(t) - u_0(0,t)|\chi \leq C|g_0(t) - u_0(0,t)|e^{-(\gamma_L - \delta)\xi}. \tag{46}$$

The problem for $v_{1,t}$ is

$$\begin{cases} -\frac{\partial^2}{\partial\xi^2}v_{1,t} + v_{1,t}F_s(0,t,v_0) = \\ \qquad\quad = -v_1[F_{st}(0,t,v_0) + F_{ss}(0,t,v_0)v_{0,t}] - \xi[F_{xt}(0,t,v_0) + F_{xs}(0,t,v_0)v_{0,t}] \\ v_{1,t}(0,t) = 0 \\ v_{1,t}(\infty,t) = 0, \end{cases} \tag{47}$$

where we use the previous estimates for $|v_1|$, $|v_{0,t}|$ and that for positive arbitrary constants C,

$$|F_x(x,t,s)| \le C|s|, \quad |F_s(x,t,s)| \le C|s|, \quad |F_t(x,t,s)| \le C|s| \tag{48}$$

The right-hand side of the equation in (47) satisfies

$$|\Psi| \le |v_1| + C\xi(|v_0| + |v_0||v_{0,t}|) \le C(1+\xi)\chi,$$

where we use (46) and the estimates for $|v_0|$ and $|v_{0,t}|$. Applying Lemma 2 with $k = 1$ yields

$$|v_{1,t}| \le C(1+\xi^2)\chi \le Ce^{-(\gamma_L-\delta')\xi}, \qquad \delta' > 0,\ C = C(\delta').$$

Differentiating (47) with respect to t, we obtain an equation for $v_{1,tt}$ and the boundary conditions

$$v_{1,tt}(0,t) = 0,$$
$$v_{1,tt}(\infty,t) = 0.$$

We use (48), the estimates for $|v_1|$, $|v_{1,t}|, |v_0|, |v_{0,t}|, |v_{0,tt}|$ and apply Lemma 2 with $k = 2$.
Next, we derive upper bounds for the derivatives of v_1 with respect to ξ. For $k = 0$ we have (46). Differentiating (21) with respect to ξ, we obtain the problem

$$\begin{cases} -\frac{\partial^2}{\partial\xi^2}v_{1,\xi} + F_s(0,t,v_0)v_{1,\xi} = \\ \qquad\qquad = -v_1F_{ss}(0,t,v_0)v_{0,\xi} - F_x(0,t,v_0) - \xi F_{xs}(0,t,v_0)v_{0,\xi} \\ v_{1,\xi}(0,t) \quad = 0 \\ v_{1,\xi}(\infty,t) \quad = 0. \end{cases} \tag{49}$$

Using (48) and the estimates obtained for $|v_1|, |v_0|, |v_{0,\xi}|$ for

$$|\Psi| = |-v_1F_{ss}(0,t,v_0)v_{0,\xi} - F_x(0,t,v_0) - \xi F_{xs}(0,t,v_0)v_{0,\xi}| \le C\chi(1+\xi^1),$$

we apply Lemma 2 with $k = 1$. Differentiating (49) with respect to ξ we obtain the problem for $v_{1,\xi\xi}$, with the boundary conditions $v_{1,\xi\xi}(0,t) = 0, v_{1,\xi\xi}(\infty,t) = 0$, and apply Lemma 2 with $k = 2$. Differentiating again with respect to ξ, we obtain the problem for $v_{1,\xi\xi\xi}$ and apply Lemma 2 with $k = 3$. □

Noting that

$$\varepsilon^2\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right](v_0 + \varepsilon v_1) = -\frac{\partial^2}{\partial\xi^2}(v_0 + \varepsilon v_1) + O(\varepsilon^2),$$

we now prove the estimate from [20]:

$$\varepsilon^2\left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right](v_0 + \varepsilon v_1) + F(x,t,v_0 + \varepsilon v_1) = O(\varepsilon^2). \tag{50}$$

Using the functional defined in (14), $F(\varepsilon\xi, t, v_0+\varepsilon v_1) = G(\varepsilon)$, where $G(0) = \frac{\partial^2 v_0}{\partial \xi^2}$ and $G'(0) = \frac{\partial^2 v_1}{\partial \xi^2}$, (50) is

$$-G(0) - \varepsilon G'(0) + G(\varepsilon) + O(\varepsilon^2) = 0.$$

As by (15), $G(\varepsilon) = G(0) + \varepsilon G'(0) + \frac{\varepsilon^2}{2} G''(\varepsilon_*)$ for some $\varepsilon_* \in (0, \varepsilon)$, it remains to show that

$$G''(\varepsilon_*) \leq C, \quad \forall\, \varepsilon_* \in (0, \varepsilon). \tag{51}$$

By (14),

$$G'(\varepsilon_*) = \xi F_x(x, t, v_0 + \varepsilon_* v_1) + v_1 F_s(x, t, v_0 + \varepsilon_* v_1).$$

Hence

$$G''(\varepsilon_*) = \xi^2 F_{xx}(x, t, v_0 + \varepsilon_* v_1) + 2\xi v_1 F_{xs}(x, t, v_0 + \varepsilon_* v_1) + v_1^2 F_{ss}(x, t, v_0 + \varepsilon_* v_1).$$

The third term here is bounded, as $|v_1^2| \leq |v_1|$. By (48),

$$|F_{xx}(x, t, v_0 + \varepsilon_* v_1)| \leq C|v_0 + \varepsilon_* v_1|,$$

which yields $|\xi^2 F_{xx}(x, t, v_0 + \varepsilon_* v_1)| \leq C\xi^2 |v_0 + \varepsilon_* v_1| \leq C$, while

$$|2\xi v_1 F_{xs}(x, t, v_0 + \varepsilon_* v_1)| \leq C\xi |v_1| \leq C.$$

This proves (51), therefore, (50), which gives that

$$\mathcal{F}(u_0 + v_0 + \varepsilon v_1) = O(\varepsilon^2). \tag{52}$$

Initial-layer function

The behaviour of the solution of (1), (2), (3) in the layer $\{(x, t)|\ t = 0\}$ is described by a function $w_0(x, \tau)$ depending on the stretched variable $\tau = t/\varepsilon^2$ and given by the initial-value problem

$$\begin{cases} \frac{\partial w_0}{\partial \tau} & = -F(x, 0, w_0) \\ w_0(x, 0) & = \varphi(x) - u_0(x, 0) \\ \lim_{\tau \to \infty} w_0(x, \tau) & = 0. \end{cases} \tag{53}$$

Hence, the initial value belongs to the domain of attraction of the rest point 0 and the solution is local. To illustrate the term 'domain of attraction' from [35], consider the problem

$$\begin{cases} -\varepsilon^2 u_t + f(u) = 0, \quad t \in [0, 1] \\ u(x, 0) = \varphi(x) \end{cases}$$

where $f(u) = (u+1)u(u-1)(u-2)$ is depicted in Figure 4. The reduced equation has the roots $u_0 \in \{-1, 0, 1, 2\}$. We evaluate the derivative $f_u = 4u^3 - 6u^2 - 2u + 2$ in each root, to find out which ones are stable - if $f_u > 0$ - and which ones are unstable - if $f_u < 0$: $f_u(-1) = -6, \quad f_u(0) = 2, \quad f_u(1) = -2, \quad f_u(2) = 6.$ Thus -1 is unstable, 0 is stable, 1 is unstable and 2 is stable. If the initial condition φ is in the interval between -1 and 0, where $f < 0$, the solution increases towards 0. If it is between 0 and 1, where $f > 0$, the solution decreases

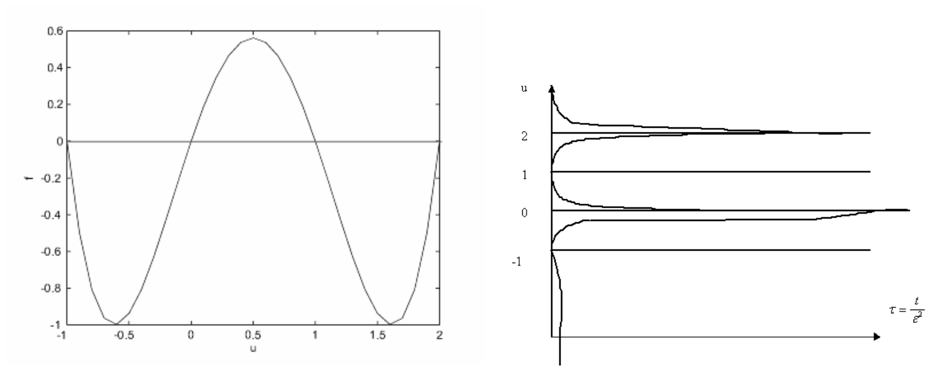

Figure 4: Example of a function f and its domains of attraction

towards the stable reduced solution 0. Thus, the domain of attraction of $u_0 = 0$ is $(-1, 1)$. If the initial condition φ is between $-\infty$ and -1, the solution goes to $-\infty$. Similarly, the domain of attraction of $u_0 = 2$ is $(1, \infty)$.

Corner layer function

Zero-order term of the corner layer function

The compatibility conditions (10) yield

$$u_0(0,0) = \varphi(0) = g_0(0)$$

and when ξ and τ tend to infinity the corner layer functions q_0 and q_1 should approach zero:

$$\lim_{\xi\to\infty} q_0(\xi,\tau) = 0, \qquad \lim_{\tau\to\infty} q_0(\xi,\tau) = 0, \tag{54}$$

$$\lim_{\xi\to\infty} q_1(\xi,\tau) = 0, \qquad \lim_{\tau\to\infty} q_1(\xi,\tau) = 0. \tag{55}$$

The corner layer along the time axis where ξ is close to is characterized by $u_{as}(x,t) = g_0(t)$, giving

$$u_0(0,t) + v_0(0,t) + \varepsilon v_1(0,t) + w_0(0,\tau) + q_0(0,\tau) + \varepsilon q_1(0,\tau) = g_0(t),$$

where $v_0(0,t) = g_0(t) - u_0(0,t)$ and $v_1(0,t) = 0$. We obtain

$$w_0(0,\tau) + q_0(0,\tau) + \varepsilon q_1(0,\tau) = 0,$$

yielding a boundary condition for q_0

$$q_0(0,\tau) = -w_0(0,\tau) \tag{56}$$

and a boundary condition for q_1

$$q_1(0,\tau) = 0. \tag{57}$$

In the horizontal part of the corner layer along ξ, where $\tau = 0$, we have

$$u_0(x,0) + v_0(\xi,0) + \varepsilon v_1(\xi,0) + w_0(\xi,0) + q_0(\xi,0) + \varepsilon q_1(\xi,0) = \varphi(x), \tag{58}$$

which yield the initial condition for q_1

$$q_1(\xi, 0) = -v_1(\xi, 0). \tag{59}$$

Due to the initial value $w_0(x, 0) = \varphi(x) - u_0(x, 0)$, (58) yields the initial condition for q_0

$$q_0(\xi, 0) = -v_0(\xi, 0). \tag{60}$$

Setting $x = t = 0$ in f and introducing $u_0(0, 0) + v_0(\xi, 0) + w_0(0, \tau) + q_0(\xi, \tau)$ in (1), we obtain the equation for $q_0(\xi, \tau)$:

$$-\frac{\partial^2}{\partial \xi^2} v_0(\xi, 0) + \frac{\partial}{\partial \tau} w_0(0, \tau) + \left[\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \xi^2}\right] q_0 +$$

$$+ f(0, 0, u_0(0, 0) + v_0(\xi, 0) + w_0(0, \tau) + q_0) = 0. \tag{61}$$

Using now the definition (76), (17) and (53), (61) becomes

$$\left[\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \xi^2}\right] q_0 =$$

$$-F(0, 0, v_0(\xi, 0) + w_0(0, \tau) + q_0) + F(0, 0, w_0(0, \tau)) - F(0, 0, v_0(\xi, 0)). \tag{62}$$

By $0 \leq v_0(\xi, t) \leq Ct$, we have

$$v_0(\xi, 0) = 0 \tag{63}$$

and $0 \leq w_0(x, \tau) \leq Cx$, yields

$$w_0(0, \tau) = 0 \tag{64}$$

From (63), (64) and $F(x, t, 0) = 0$ for all x, t, equation (62) becomes

$$\left[\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \xi^2}\right] q_0 = -F(0, 0, q_0). \tag{65}$$

Therefore, the zero-order term of the corner layer function q_0 is defined by (65), with the initial condition (60) and the boundary conditions (56), (54) becoming homogeneous. A solution of (62), (60), (56), (54) is $\underline{q_0(\xi, \tau) = 0}$.

Term of order one of the corner layer function

We obtain $q_1(\xi, \tau)$ by formally introducing

$$u_{as} = u_0(\varepsilon\xi, \varepsilon^2\tau) + v_0(\xi, \varepsilon^2\tau) + \varepsilon v_1(\xi, \varepsilon^2\tau) + w_0(\varepsilon\xi, \tau) + \varepsilon q_1(\xi, \tau)$$

in the original equation (1):

$$\varepsilon^2 \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) [u_0(\varepsilon\xi, \varepsilon^2\tau) + v_0(\xi, \varepsilon^2\tau) + \varepsilon v_1(\xi, \varepsilon^2\tau) + w_0(\varepsilon\xi, \tau) + \varepsilon q_1(\xi, \tau)] +$$

$$+ F(\varepsilon\xi, \varepsilon^2\tau, v_0(\xi, \varepsilon^2\tau) + \varepsilon v_1(\xi, \varepsilon^2\tau) + w_0(\varepsilon\xi, \tau) + \varepsilon q_1(\xi, \tau)) = 0.$$

Here we have

$$\varepsilon^2 \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u_0 = O(\varepsilon^2).$$

We use the estimates (50) and

$$\varepsilon^2 \left[\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right] w_0 + F(x,t,w_0) = O(\varepsilon^2), \qquad (66)$$

yielding

$$\varepsilon(q_{1,\tau} - q_{1,\xi\xi}) - F(\varepsilon\xi, \varepsilon^2\tau, w_0(\varepsilon\xi,\tau)) - F(\varepsilon\xi, \varepsilon^2\tau, v_0(\xi,\varepsilon^2\tau) + \varepsilon v_1(\xi,\varepsilon^2\tau)) +$$
$$+F(\varepsilon\xi, \varepsilon^2\tau, v_0(\xi,\varepsilon^2\tau) + w_0(\varepsilon\xi,\tau) + \varepsilon v_1(\xi,\varepsilon^2\tau) + \varepsilon q_1(\xi,\tau)) + O(\varepsilon^2) = 0.$$

Using the notation

$$\varsigma\big|_a^b := \varsigma(b) - \varsigma(a), \quad \varsigma\big|_{a;b}^c := \varsigma(c) - \varsigma(b) - \varsigma(a), \qquad (67)$$

this can be written

$$\varepsilon(q_{1,\tau} - q_{1,\xi\xi}) + F(\varepsilon\xi, \varepsilon^2\tau, \cdot)\big|_{[v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)]\,;\,w_0(\varepsilon\xi,\tau)}^{[v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)]+w_0(\varepsilon\xi,\tau)} +$$
$$+F(\varepsilon\xi, \varepsilon^2\tau, \cdot)\big|_{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)}^{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)} = O(\varepsilon^2). \qquad (68)$$

By Lemma 5, we have

$$|v_0 + \varepsilon v_1||w_0| \le C\varepsilon^2\tau e^{-(\gamma_0^2-\delta)\tau}. \qquad (69)$$

We obtain

$$F(\varepsilon\xi, \varepsilon^2\tau, \cdot)\big|_{[v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)]\,;\,w_0(\varepsilon\xi,\tau)}^{[v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)]+w_0(\varepsilon\xi,\tau)} =$$
$$= O(|[v_0(\xi,\varepsilon^2\tau) + \varepsilon v_1(\xi,\varepsilon^2\tau)]w_0(\varepsilon\xi,\tau)|) = O(\varepsilon^2).$$

Let

$$F(\varepsilon\xi, \varepsilon^2\tau, \cdot)\big|_{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)}^{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)} := G(\varepsilon). \qquad (70)$$

Then

$$G'(\varepsilon) = \xi F_x(\varepsilon\xi, \varepsilon^2\tau, \cdot)\big|_{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)}^{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)} +$$
$$+2\varepsilon\tau F_t(\varepsilon\xi, \varepsilon^2\tau, \cdot)\big|_{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)}^{v_0(\xi,\varepsilon^2\tau)+\varepsilon v_1(\xi,\varepsilon^2\tau)+w_0(\varepsilon\xi,\tau)+\varepsilon q_1(\xi,\tau)} +$$
$$+[2\varepsilon\tau v_{0,t}(\xi,\varepsilon^2\tau) + 2\varepsilon^2\tau v_{1,t}(\xi,\varepsilon^2\tau) + v_1(\xi,\varepsilon^2\tau) + \xi w_{0,\xi}(\varepsilon\xi,\tau) + q_1(\xi,\tau)]\cdot$$
$$\cdot F_s(\varepsilon\xi, \varepsilon^2\tau, v_0(\xi,\varepsilon^2\tau) + \varepsilon v_1(\xi,\varepsilon^2\tau) + w_0(\varepsilon\xi,\tau) + \varepsilon q_1(\xi,\tau)) -$$
$$-[2\varepsilon\tau v_{0,t}(\xi,\varepsilon^2\tau) + 2\varepsilon^2\tau v_{1,t}(\xi,\varepsilon^2\tau) + v_1(\xi,\varepsilon^2\tau) + \xi w_{0,\xi}(\varepsilon\xi,\tau)]\cdot$$
$$\cdot F_s(\varepsilon\xi, \varepsilon^2\tau, v_0(\xi,\varepsilon^2\tau) + \varepsilon v_1(\xi,\varepsilon^2\tau) + w_0(\varepsilon\xi,\tau)). \qquad (71)$$

A Taylor expansion gives

$$G(\varepsilon) = G(0) + \varepsilon G'(0) + \frac{\varepsilon^2}{2} G''(\tilde\varepsilon),$$

where $\tilde\varepsilon$ is between 0 and ε and

$$G(0) = F(0, 0, v_0(\xi, 0) + w_0(0,\tau)) = 0.$$

Formally assuming that

$$\varepsilon^2 |G''(\tilde{\varepsilon})| = O(\varepsilon^2),$$

we obtain

$$G(\varepsilon) = G(0) + \varepsilon G'(0) + O(\varepsilon^2),$$

where from (71)

$$G'(0) = q_1(\xi,\tau) F_s(0, 0, v_0(\xi, 0) + w_0(0,\tau)). \tag{72}$$

From $\varepsilon|v_1| \le Ct$, yielding

$$v_1(\xi, 0) = 0$$

and (63) and (64), we obtain

$$G'(0) = q_1(\xi,\tau) F_s(0, 0, 0). \tag{73}$$

Using (70) and (73), we match the terms in (68) containing power one of ε :

$$q_{1,\tau}(\xi,\tau) - q_{1,\xi\xi}(\xi,\tau) + q_1(\xi,\tau) F_s(0, 0, 0) = 0. \tag{74}$$

Setting $\xi = 0$ here and knowing that $v_0(\xi, 0) = v_1(\xi, 0) = w_0(0,\tau) = 0$, the equation for q_1 becomes

$$q_{1,\tau} - q_{1,\xi\xi} + F_s(0, 0, 0) q_1 = 0. \tag{75}$$

A solution of the problem (75), with the initial condition (59) and the boundary conditions (57) and (55) becoming homogeneous is $\underline{q_1 = 0}$.

3 Super-solutions and sub-solutions

3.1 Perturbation of the asymptotic expansion

We introduce a perturbation of the functional F by a sufficiently small parameter p :

$$\tilde{F}(x, t, s) = f(x, t, u_0(x, t) + s) - ps. \tag{76}$$

Genererally, for a sufficiently smooth function ς, under the notations (67), for any a and b, there is a θ with $|\theta| < |a| + |b|$ such that

$$\varsigma(a + b) - \varsigma(a) - \varsigma(b) + \varsigma(0) = a\, b\, \varsigma''(\theta)$$

and if $\varsigma(0) = 0$, $\varsigma\big|_{a;b}^{a+b} = O(|ab|)$, so by $\tilde{F}(x, t, 0) = 0$, we have

$$\tilde{F}(x, t, \cdot)\big|_{a;b}^{a+b} = O(|ab|). \tag{77}$$

In order to set a super-solution and a sub-solution of the problem (1), (2), (3), we define the perturbation $\tilde{v}_0(\xi, t; p)$ of $v_0(\xi, t)$ with $v_0(\xi, t) = \tilde{v}_0(\xi, t; 0)$ as solution of

$$-\frac{\partial^2 \tilde{v}_0}{\partial \xi^2} + \tilde{F}(0, t, \tilde{v}_0) = 0, \quad \xi > 0, \tag{78}$$

subject to the boundary conditions

$$\tilde{v}_0(0, t; p) = g_0(t) - u_0(0, t), \tag{79}$$

$$\tilde{v}_0(\infty, t; p) = 0. \tag{80}$$

Regarding the existence and properties of the perturbed boundary-layer function $\tilde{v}_0 = \tilde{v}_0(\xi, t; p)$, given by the problem (78), (79), (80) we have the following result from [21, 20].

Lemma 3. *i) Let γ_L be defined by (22). There exists $p_0 \in (0, \gamma_L)$ such that $\forall\, |p| \le p_0$, an analogue of assumption A1 in the form*

$$\tilde{F}_s(x, t, 0) > \tilde{\gamma}^2 > 0, \qquad \forall\, (x, t) \in [0, 1] \times [0, T]$$

is satisfied.
ii) Problem (78),(79),(80) has a solution $\tilde{v}_0$ whose initial value belongs to the domain of attraction of its rest point.
iii) There exists a positive constant C_δ depending on an arbitrary, but fixed $\delta \in (0, \sqrt{f_u(0, t, u_0(0, t)) - p})$, such that

$$0 < \frac{\partial \tilde{v}_0}{\partial p} \le C_\delta \exp(-(\gamma_* - \delta)\xi), \;\; 0 \le \xi \le \infty. \tag{81}$$

For any arbitrarily small but fixed $\delta \in (0, \gamma_L - \sqrt{p_0})$, there exists a positive constant C_δ such that for $\xi, t \ge 0$, $k = \overline{0, 3}$ and $l = \overline{0, 2}$, $\frac{\partial \tilde{v}_0}{\partial p} \ge 0$ and each of the quantities $\left|\frac{\partial \tilde{v}_0}{\partial \xi^k}\right|$ and $\left|\frac{\partial \tilde{v}_0}{\partial t^l}\right|$ is upper bounded in the maximum norm by $C_\delta e^{-(\gamma_L - \sqrt{p_0} - \delta)\xi}$.

For any fixed x, let the function $\tilde{w}_0(x, \tau; p)$ with $\tilde{w}_0(x, \tau; 0) := w_0(x, \tau)$ be a solution of the initial-value problem perturbed by the small parameter p:

$$\frac{\partial \tilde{w}_0}{\partial \tau} = -\tilde{F}(x, 0, \tilde{w}_0; p) \quad \tau > 0, \quad \tilde{w}_0(x, 0; p) = \varphi(x) - u_0(x, 0), \tag{82}$$

where $\tilde{F}$ is defined by (76). Since $\tilde{w}_0(x, \tau; p)$ describes a correction to $u_0(x, t)$ for small values of t, we search for a solution of (82) that satisfies the condition

$$\lim_{\tau \to \infty} \tilde{w}_0(x, \tau; p) = 0. \tag{83}$$

Setting

$$\gamma_0^2 := \min_{x \in [0,1]} f_u(x, 0, u_0(x, 0)) > \gamma^2, \tag{84}$$

where γ is from assumption $A1$, we prove in the following an analogue of assumption $A2$ for the perturbed function.

Lemma 4. *For $\tilde{F}$ defined by (76) and $|p| \le p_0$ with p_0 sufficiently small,*

$$\frac{\tilde{F}(x, 0, s)}{s} > 0 \quad \forall\, s \in (0, \varphi(x) - u_0(x, 0)]' \tag{85}$$

holds true provided (5) is satisfied.

Proof. By A1, $f(x, 0, u_0(x, 0)) = 0$. A Taylor expansion up to the power two gives

$$f(x, 0, u_0(x, 0) + s) =$$
$$= f(x, 0, u_0(x, 0)) + f_u(x, 0, u_0(x, 0))s + c_0 s^2 = f_u(x, 0, u_0(x, 0))s + c_0 s^2$$

for some constant $c_0 = c_0(x)$ which bounds the second order derivative of f with respect to u, $\forall\, s \in (0, \varphi(x) - u_0(x,0)]'$. From $f_u(x,0,u_0(x,0)) > \gamma_0^2$ and $p \leq p_0$ we have

$$f(x,0,u_0(x,0)+s) - ps \geq (\gamma_0^2 - p_0)s - c_0 s^2 \quad \forall\, s \in (0, \varphi(x) - u_0(x,0)]'. \tag{86}$$

If

$$0 < s < \frac{\gamma_0^2}{2c_0},$$

then

$$-c_0 s > -\frac{\gamma_0^2}{2},$$

yielding

$$\gamma_0^2 - c_0 s > \frac{\gamma_0^2}{2}. \tag{87}$$

Choosing

$$p_0 \leq \frac{\gamma_0^2}{2}, \tag{88}$$

(87) becomes

$$\gamma_0^2 - c_0 > p_0,$$

yielding

$$\gamma_0^2 - p_0 - c_0 s > 0,$$

so the right-hand side of (86) is positive and (85) holds true.
In case $s \geq \frac{\gamma_0^2}{2c_0}$, we take $s_{\max} = \varphi(x) - u_0(x,0) > \frac{\gamma_0^2}{2c_0}$ and

$$p_0 := \min\left\{ \min_{\substack{\frac{\gamma_0^2}{2c_0} \leq s \leq \varphi(x) - u_0(x,0) \\ x \in [0,1]}} \frac{f(x,0,u_0(x,0)+s)}{s}, \frac{\gamma_0^2}{2c_0} \right\},$$

which gives

$$ps \leq p_0(\varphi(x) - u_0(x,0)) \leq \min_{x\in[0,1]} f(x,0,u_0(x,0)+s) < f(x,0,u_0(x,0)+s),$$

yielding positivity of the right-hand side of (86), hence (85). □

Existence and properties of the perturbed initial-layer function are derived in [20]:

Lemma 5. *1. There exists $p_0 \in (0, \gamma_0^2)$ such that for all p with $|p| \leq p_0$, problem (82) has a solution $\tilde{w}_0(x,\tau;p)$.*

2. For w_0 and $\tilde{w}_0$ we have

$$0 \leq w_0(x,\tau) \leq Cx, \quad \frac{\partial \tilde{w}_0}{\partial p} \geq 0, \qquad \forall\, x \in [0,1],\ \tau \geq 0 \tag{89}$$

and for an arbitrarily small but fixed $\delta \in (0, \gamma_0^2 - p_0)$, there exists a constant C_δ such that for all $x \in [0,1]$, $\tau \geq 0$, the maximum norm quantities $\left|\frac{\partial^l \tilde{w}_0}{\partial \tau^l}\right|$, $\left|\frac{\partial^k \tilde{w}_0}{\partial x^k}\right|$ and $\left|\frac{\partial \tilde{w}_0}{\partial p}\right|$ with $k = \overline{0,3}$, $l = \overline{0,2}$ have the upper bound $C_\delta e^{-(\gamma_0^2 - |p| - \delta)\tau}$, where γ_0 is given by (84).

We construct sub-solutions and super-solutions by adding the term C_0p with C_0 a positive constant and p the perturbation parameter with $|p|$ sufficiently small to the perturbed the asymptotic expansion containing perturbed boundary and initial layer functions $\tilde{v}_0$ and $\tilde{w}_0$:

$$\beta(x,\tau;p) := u_0(x,t) + [\tilde{v}_0(\xi,t;p) + \varepsilon v_1(\xi,t)] + \tilde{w}_0(x,\tau;p) + C_0p,$$

or

$$\beta = u_{as} + V + W + C_0p, \tag{90}$$

where

$$V(\xi,t;p) := v - v_0 = p\frac{\partial v_0(\xi,t;\tilde{p})}{\partial p}$$

(not to be confused with V from (24)) and

$$W := W(x,\tau;p) = w_0(x,\tau;p) - w_0(x,\tau) = p\frac{\partial w_0(x,\tau;\tilde{p})}{\partial p},$$

with $\tilde{p} \in (0,p)$ and C, C_0 positive constants. By the estimates for $\frac{\partial}{\partial p}\tilde{v}_0$ and $\frac{\partial}{\partial p}\tilde{w}_0$,

$$(1+\xi)|V| \leq Cp, \quad (1+\tau)|W| \leq Cp. \tag{91}$$

Lemma 6. *If $p_1 \leq p_2$, then the super-solution satisfies $\beta(x,t,p_1) \leq \beta(x,t,p_2)$, which is*

$$\frac{\partial \beta}{\partial p} \geq 0.$$

Proof. From (81) and $\dfrac{\partial \tilde{v}_0}{\partial p} > 0$ we have $V(\xi,t;p_1) \leq V(\xi,t;p_2)$ and from (89) we have $W(\xi,t;p_1) \leq W(\xi,t;p_2)$. □

Lemma 7. *For the function β given by (90), we have*

$$\beta = u_{as} + O(p). \tag{92}$$

For all $(x,t) \in [0,1] \times [0,T]$

$$\beta(x,t;-|p|) \leq u_{as} - C_0p \leq u_{as} + C_0p \leq \beta(x,t;|p|). \tag{93}$$

Proof. Assertion (92) follows from (90) and (91). Noting that $u_{as}(x,t) = \beta(x,t;0)$ and using the bounds $\frac{\partial \tilde{v}_0}{\partial p} \geq 0$ and $\frac{\partial \tilde{w}_0}{\partial p} \geq 0$ yields the second assertion (93). □

Therefore, we set the upper solution $\beta(x,t;|p|)$ and the lower solution as $\beta(x,t;-|p|)$, with $|p|$ small.

3.2 Existence of solution between sub-solution and super-solution

The role of the sub-solutions and the super-solutions (or lower and upper solutions) is to provide tight control of the solution of problem (1), (2), (3).

Definition 3.1. Let $\alpha(x,t)$, $\beta(x,t)$ be functions continuously mapping $[0,1] \times [0,T]$ into $\mathbb{R}$, twice continuously differentiable in x and continuously differentiable in t for $(x,t) \in D$. Then $\alpha(x,t)$, $\beta(x,t)$ are called sub- and super- solutions of (1), (2), (3) if:

$$\alpha(x,t) \le \beta(x,t), \quad (x,t) \in [0,1] \times [0,T]; \tag{94}$$

$$\mathcal{F}\alpha \le 0 \le \mathcal{F}\beta, \qquad (x,t) \in [0,1] \times [0,T]; \tag{95}$$

$$\alpha(0,t) \le g_0(t) \le \beta(0,t), \quad \alpha(1,t) \le g_1(t) \le \beta(1,t), \quad t \in [0,T]; \tag{96}$$

$$\alpha(x,0) \le \varphi(x) \le \beta(x,0), \qquad x \in [0,1]. \tag{97}$$

The function $f_u(x,t,u)$ is continuous, so when u is in the sector $[\alpha,\beta]$, this function is bounded by a function

$$K(x,t) = \max_{u \in [\alpha(x,t),\beta(x,t)]} |f_u(x,t,u)|$$

and

$$-(u_1 - u_2)K(x,t) \le f(x,t,u_1) - f(x,t,u_2), \quad \alpha \le u_2 \le u_1 \le \beta. \tag{98}$$

The function $K(x,t)$ being continuous, the function

$$B(x,t,u) \equiv K(x,t)u - f(x,t,u) \tag{99}$$

is also continuous in $[0,1] \times [0,T] \times [\alpha,\beta]$ and monotone nondecreasing in $u \in [\alpha,\beta]$:

$$B(x,t,u_1) - B(x,t,u_2) \ge 0, \qquad \beta \ge u_1 \ge u_2 \ge \alpha. \tag{100}$$

Theorem 3.1. *Existence of lower and upper solutions provides existence of a solution $u(x,t)$ of (1), (2), (3), with*

$$\alpha(x,t) \le u(x,t) \le \beta(x,t). \tag{101}$$

Proof. We adapt the proof from [29, p.59] by defining an operator

$$\mathbb{L}[u] \equiv \varepsilon^2[u_t - u_{xx}] + K(x,t)u \tag{102}$$

and consider the differential equation that is equivalent to (1)

$$\mathbb{L}[u] = B(x,t,u) \tag{103}$$

with the initial and boundary conditions

$$u(l,t) = g_l(t), \quad l = 0,1; \qquad u(0,x) = \varphi(x), \tag{104}$$

where B is given by (99). Then Definition 3.1 for (1), (2), (3) is equivalent to the Definition for the problem (103), (104), with $\mathcal{F}$ replaced by $\mathbb{L}$ in (95) .
We construct the sequences $\{\alpha^{(k)}\}$ defined by

$$\mathbb{L}[\alpha^{(k)}] = B(x,t,\alpha^{(k-1)}), \quad \alpha^{(0)} = \alpha \tag{105}$$

and $\{\beta^{(k)}\}$ by

$$\mathbb{L}[\beta^{(k)}] = B(x,t,\beta^{(k-1)}), \quad \beta^{(0)} = \beta, \tag{106}$$

with

$$\alpha^{(k)}(0,t) = g_0(t), \quad \alpha^{(k)}(1,t) = g_1(t), \qquad k = 1, 2, ...$$

$$\alpha^{(k)}(0,x) = \varphi(x), \tag{107}$$

$$\beta^{(k)}(0,t) = g_0(t), \quad \beta^{(k)}(1,t) = g_1(t),$$

$$\beta^{(k)}(0,x) = \varphi(x), \tag{108}$$

and refer to them as lower and upper sequences. We start by proving the following statement.
(i) Each $\beta^{(k)}$ is a super-solution and each $\alpha^{(k)}$ is a sub-solution; the lower and upper sequences possess the monotone property in $[0,1] \times [0,T]$:

$$\alpha \leq \alpha^{(k)} \leq \alpha^{(k+1)} \leq \beta^{(k+1)} \leq \beta^{(k)} \leq \beta. \tag{109}$$

Let $w = \beta^{(0)} - \beta^{(1)} = \beta - \beta^{(1)}$. Setting the operator $\mathcal{F}$ in (95) of Definition 3.1 as $\mathbb{L}$, gives

$$\mathbb{L}[w] = \mathbb{L}[\beta^{(0)}] - B(x,t,\beta^{(0)}) \geq 0$$

From (106) and (108) we have

$$w(0,t) = \beta(0,t) - \beta^{(1)}(0,t) \geq 0$$

$$w(x,0) = \beta(x,0) - \varphi(x) \geq 0.$$

By the maximum principle [30, Chapter 3] , $w(x,t) \geq 0$, so $\beta^{(1)} \leq \beta^{(0)}$. Similarly, $\alpha^{(1)} \geq \alpha^{(0)}$. Let $w^{(1)} = \beta^{(1)} - \alpha^{(1)}$. Then by (105) to (108) and by the monotone property of B in (100),

$$\mathbb{L}[w^{(1)}] = B(x,t,\beta^{(0)}) - B(x,t,\alpha^{(0)}) \geq 0$$

$$w^{(1)}(0,t) = 0$$

$$w^{(1)}(x,0) = 0.$$

From the maximum principle it follows that $w^{(1)} \geq 0$ in $[0,1] \times [0,T]$. Hence

$$\alpha^{(0)} \leq \alpha^{(1)} \leq \beta^{(1)} \leq \beta^{(0)} \quad \text{in } [0,1] \times [0,T].$$

Next, assume by induction that

$$\alpha^{(k-1)}(x,t) \leq \alpha^{(k)}(x,t) \leq \beta^{(k)}(x,t) \leq \beta^{(k-1)}(x,t) \quad \text{in } [0,1] \times [0,T].$$

Then (105) to (108) and the monotone property of B in (100) imply that $w^{(k)} = \beta^{(k)} - \beta^{(k+1)}$ satisfies

$$\mathbb{L}w^{(k)} = B(x,t,\beta^{(k-1)}) - B(x,t,\beta^{(k)}) \geq 0$$

and the boundary and initial conditions as for $w^{(1)}$. Hence $w^{(k)} \geq 0$ and $\beta^{(k+1)} \leq \beta^{(k)}$. Similarly,

$$\alpha^{(k+1)} \geq \alpha^{(k)} \qquad \text{and} \quad \beta^{(k+1)} \geq \alpha^{(k+1)}.$$

By the principle of induction, the first part of assertion (i) is established for all k.

To prove that each element of the lower sequence is a sub-solution, using (105) and (107), we write

$$\varepsilon^2[\alpha_t^{(k)} - \alpha_{xx}^{(k)}] = K(x,t)(\alpha^{(k-1)} - \alpha^{(k)}) - f(x,t,\alpha^{(k-1)})$$

$$= K(x,t)(\alpha^{(k-1)} - \alpha^{(k)}) + [f(x,t,\alpha^{(k)}) - f(x,t,\alpha^{(k-1)})] - f(x,t,\alpha^{(k)}),$$

which, by (98) and (109), yields

$$\mathcal{F}\alpha^{(k)} = \varepsilon^2[\alpha_t^{(k)} - \alpha_{xx}^{(k)}] + f(x,t,\alpha^{(k)}) \leq 0.$$

Hence (95) is satisfied. From the boundary and initial conditions (107) we have that (96) and (97) are also satisfied by $\alpha^{(k)}$.
From (106) and (108)

$$\varepsilon^2[\beta_t^{(k)} - \beta_{xx}^{(k)}] = K(x,t)(\beta^{(k-1)} - \beta^{(k)}) - f(x,t,\beta^{(k-1)})$$

$$= K(x,t)(\beta^{(k-1)} - \beta^{(k)}) + [f(x,t,\beta^{(k)}) - f(x,t,\beta^{(k-1)})] - f(x,t,\beta^{(k)}),$$

so by (98) and (109)

$$\mathcal{F}\beta^{(k)} = \varepsilon^2[\beta_t^{(k)} - \beta_{xx}^{(k)}] + f(x,t,\beta^{(k)}) \geq 0.$$

From the boundary and initial conditions (108), we have that (96) and (97) are also satisfied by $\beta^{(k)}$ and from (109) we have $\alpha^{(k)} \leq \beta^{(k)}$. Therefore, according to Definition 3.1, $\alpha^{(k)}$ is a sub-solution and $\beta^{(k)}$ is a super-solution of (1), (2), (3).
We shall prove two further statements:
(ii) The pointwise limits

$$\lim_{k\to\infty} \beta^{(k)}(x,t) \qquad \text{and} \qquad \lim_{k\to\infty} \alpha^{(k)}(x,t) \tag{110}$$

exist and in $[0,1]\times[0,T]$ satisfy

$$\alpha(x,t) \leq \alpha^{(k)}(x,t) \leq \alpha^{(k+1)}(x,t) \leq \lim_{k\to\infty} \alpha^{(k)}(x,t) \leq$$

$$\leq \lim_{k\to\infty} \beta^{(k)}(x,t) \leq \beta^{(k+1)}(x,t) \leq \beta^{(k)}(x,t) \leq \beta(x,t). \tag{111}$$

Indeed, since by (i), the sequence $\{\beta^{(k)}\}$ is monotone nonincreasing and bounded from below and the sequence $\{\alpha^{(k)}\}$ is monotone nondecreasing and bounded from above, the pointwise limits of these sequences exist and by (109), satisfy (111).
(iii) If the limits (110) are solutions of (1), (2), (3), then

$$\lim_{k\to\infty} \beta^{(k)}(x,t) = \lim_{k\to\infty} \alpha^{(k)}(x,t)$$

and this common limit in $[\alpha,\beta]$ is the solution of (1), (2), (3).
Indeed, let $d = \lim_{k\to\infty} \alpha^{(k)}(x,t) - \lim_{k\to\infty} \beta^{(k)}(x,t) \leq 0$. Then d satisfies

$$\varepsilon^2[d_t - d_{xx}] = -f(x,t,\alpha) + f(x,t,\beta) \geq$$

$$\geq -K(x,t)[\lim_{k\to\infty} \beta^{(k)}(x,t) - \lim_{k\to\infty} \alpha^{(k)}(x,t)] = K(x,t)d$$

and the boundary and initial conditions

$$d(0,t) = 0, \quad t \in [0,T]; \qquad d(x,0) = 0, \quad x \in [0,1].$$

By the maximum principle, $d \geq 0$ in $[0,1] \times [0,T]$, which provides

$$\lim_{k\to\infty} \beta^{(k)}(x,t) = \lim_{k\to\infty} \alpha^{(k)}(x,t).$$

□

According to [20], we have

Lemma 8. *For all* $(x,t) \in (0,1) \times (0,T]$

$$\mathcal{F}\beta = C_0 p f_u(x,t,u_0) + p(1 + C_0\gamma)(v_0 + w_0) + O(\varepsilon^2 + p^2), \tag{112}$$

where $\lambda = \lambda(x,t) = f_{uu}(x,t,u_0 + \vartheta(v_0 + w_0))$ *and* $\vartheta = \vartheta(x,t) \in (0,1)$.

Corollary 3.2. *There exist the constants* $C_0 > 0$ *and* $C_1 > 0$ *such that for all* $|p| \leq p_0$

$$\mathcal{F}\beta \geq C_0 p\gamma^2 - C_1(\varepsilon^2 + p^2), \quad \textit{if } p > 0,$$
$$\mathcal{F}\beta \leq -C_0|p|\gamma^2 + C_1(\varepsilon^2 + p^2), \quad \textit{if } p < 0.$$

Proof. Recalling $A1$, $v_0 \geq 0$, $w_0 \geq 0$ and (89), choose $0 < C_0 \leq |\lambda(x,t)|^{-1}$ for all x and t, such that $1 + C_0\lambda \geq 0$. □

At this point, we have the necessary information to establish existence and uniqueness of solution for (1), (2), (3) in a neighbourhood of the asymptotic expansion u_{as}.

Theorem 3.3. *There exists a sufficiently small* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \leq \varepsilon_0$, *a solution of the problem (1), (2), (3) exists and is unique. Furthermore, this solution satisfies*

$$|u(x,t) - u_{as}(x,t)| \leq C\varepsilon^2 \quad \textit{for all } (x,t) \in [0,1] \times [0,T].$$

Proof. From

$$u_{as}(x,0) = \varphi(x)$$

we have

$$u_{as}(x,0) = u(x,0).$$

Also,

$$u_{as}(0,t) = g_0(t)$$

and

$$u_{as}(1,t) = g_1(t) + O(\varepsilon^2). \tag{113}$$

Set $\bar{p} = C_2\varepsilon^2$, where $C_2 \geq 2C_1/(C_0\lambda^2)$ so that $C_0\bar{p}\gamma^2 \geq 2C_1\varepsilon^2$. By Corollary 3.2 with $\varepsilon \leq 1/C_2$, we obtain $\bar{p} \leq \varepsilon$ so $C_1(\varepsilon^2 + p^2) \leq 2C_1\varepsilon^2$. Hence

$$\mathcal{F}\beta(x,t;-\bar{p}) \leq 0 \leq \mathcal{F}\beta(x,t;\bar{p}). \tag{114}$$

In view of (93),

$$\beta(x,0;-\bar{p}) \leq \varphi(x) \leq \beta(x,0;\bar{p}) \tag{115}$$

and

$$\beta(0,t;-\bar{p}) \leq g_0(t) \leq \beta(0,t;\bar{p}). \tag{116}$$

Choosing C_2 sufficiently large, so that $C_0\bar{p} = C_0C_2\varepsilon^2$ dominates the term $O(\varepsilon^2)$ in (113), (93) yields

$$\beta(1,t;-\bar{p}) \leq g_1(t) \leq \beta(1,t;\bar{p}). \tag{117}$$

By (93), we also have

$$\beta(x,t;-\bar{p}) \leq \beta(x,t;\bar{p}). \tag{118}$$

Comparing (114), (115), (116), (117), (118) with (1), we see that $\beta(x,t;-\bar{p})$ and $\beta(x,t;\bar{p})$ are ordered lower and upper solutions, respectively, for problem (1), (2), (3). Applying [29, Theorem 5.1] yields existence of a solution u between $\beta(x,t;-\bar{p})$ and $\beta(x,t;\bar{p})$:

$$\beta(x,t;-\bar{p}) \leq u(x,t) \leq \beta(x,t;\bar{p}).$$

By Theorem 5.1 of Pao in [29, p.23], if f_u is continuous, the problem (1), (2), (3) has at most one solution. Since by Lemma 7,

$$\beta(x,t;\pm\bar{p}) = u_{as} + O(\bar{p}) = u_{as} + O(\varepsilon^2),$$

we have $|u - u_{as}| \leq C\varepsilon^2$. □

4 Conclusion

The maximum principle is a necessary and sought feature for the existence of exact and computed solutions of partial differential equations. An assumption often used for singularly perturbed problems in order to have maximum principle is that $f_u > 0$ for all x, t and u. Instead, we assumed that the reduced equation obtained by taking $\varepsilon = 0$ in (1), $f(x,t,u) = 0$, has a sufficiently smooth solution $u_0 = u_0(x,t)$ in which this assumption is satisfied: $f_u(x,t,u_0) > \gamma^2 > 0$ for all x and t. The latter assumption, providing a stable reduced solution, is justified by dynamical systems theory and it has the advantage of localness over the former. We constructed the upper and lower solutions as sums of a perturbed asymptotic expansion of order one of the solution and an additional term related to the nature of the boundary conditions. Therefore, the theory of sub-solutions and super-solutions (or lower and upper solutions) works satisfactory for singularly perturbed time-dependent semilinear reaction-diffusion problems as a substitute for the maximum principle, by providing existence and tight control of solutions for semilinear singularly perturbed time-dependent reaction-diffusion problems.

References

[1] D. Arrowsmith and C. Place, *An introduction to dynamical systems*, Cambridge University Press, 1990.

[2] N. Bakhvalov, *On the optimization of methods for solving boundary value problems with boundary layers*, Zh. Vychisl. Mat. mal. Fiz., 9 (1969), pp. 841–859.

[3] ——, *The optimization of methods of solving boundary value problems with a boundary layer*, USSR Comp. Math. Math. Phys., 9 (1969), pp. 139–166.

[4] I. Boglaev, *On a domain decomposition algorithm for a singularly perturbed reaction-diffusion problem*, J. Comput. and Appl. Math., 98 (1998), pp. 213–232.

[5] I. Boglaev and M. Hardy, *Uniform convergence of a weighted average scheme for a nonlinear reaction-diffusion problem*, J. Comput. and Appl. Math., 200 (2007), pp. 705–721.

[6] W. Boyce and R. DiPrima, *Calculus*, New York ; Chichester : Wiley, 1988.

[7] G. Clavero, J. Jorge, F. Lisbona, and G. Shishkin, *An alternating direction scheme on a nonuniform mesh for reaction-diffusion parabolic problems*, IMA J. Numer. Anal., 20 (2000), pp. 263–280.

[8] P. A. Farrell, A. Hegarty, J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, *Robust Computational Techniques for Boundary Layers*, Appl. Math. and Mathematical Computation Series, Chapman and Hall/Crc, 2000.

[9] P. A. Farrell, P. W. Hemker, and G. I. Shishkin, *Discrete approximations for singularly perturbed boundary value problems with parabolic layers I*, J. of Comput. Math., 14 (1995), pp. 71 – 97.

[10] ——, *Discrete approximations for singularly perturbed boundary value problems with parabolic layers II*, J. of Comput. Math., 14 (1996), pp. 183 – 194.

[11] ——, *Discrete approximations for singularly perturbed boundary value problems with parabolic layers III*, J. of Comput. Math., 14 (1996), pp. 273 – 290.

[12] J. Gracia and F. Lisbona, *A uniformly convergent scheme for a system of reaction-diffusion equations*, J. Comput. Appl. Math., 206 (2007), pp. 1–16.

[13] P. Grindrod, *Patterns and Waves: the Theory and Applications of Reaction-Diffusion Equations*, Clarendon Press, 1991.

[14] P. W. Hemker, G. I. Shishkin, and L. P. Shishkina, *The use of defect correction for the solution of parabolic singular perturbation problems*, Z. Angew. Math. Mech., 77 (1997), pp. 59 – 74.

[15] ——, *$\varepsilon-$uniform schemes with high-order time-accuracy for parabolic singular perturbation problems*, IMA J. Numer. Anal., 20 (2000), pp. 99–121.

[16] D. Herceg, *Uniform fourth order difference scheme for a singular perturbation problem*, Numer. Math., 56 (1990), pp. 675–693.

[17] Y. Kanon and M. Mimura, *Singular perturbation approach to a 3-component reaction-diffusion system arising in population dynamics*, SIAM J. Math. Anal., 29 (1998), pp. 1519–1536.

[18] N. Kopteva, *Uniform pointwise convergence of difference schemes for convection-diffusion problems on layer-adapted meshes*, Computing, 66 (2001), pp. 179–197.

[19] ——, *Maximum norm error analysis of a 2D singularly perturbed semilinear reaction-diffusion problem*, Math. Comp., 76 (2007), pp. 631–646.

[20] N. Kopteva and S.-B. Savescu, *Pointwise error estimates for a singularly perturbed time-dependent semilinear reaction-diffusion problem*, IMA J. Numer. Anal., (2010). doi: 10.1093/imanum/drp032.

[21] N. Kopteva and M. Stynes, *Numerical analysis of a singularly perturbed nonlinear reaction-diffusion problem with multiple solutions*, Appl. Numer. Math., 51 (2004), pp. 273–288.

[22] N. Madden and T. Linss, *Layer-adapted meshes for time-dependent reaction-diffusion*, PAMM, 6 (2007), pp. 749–750. GAMMM Annual Meeting - Berlin 2006.

[23] J. Miller, E. O'Riordan, and G. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems. Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions*, World Scientific Singapore, 1996.

[24] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, and L. P. Shishkina, *Fitted mesh methods for problems with parabolic boundary layers*, Math. Proc. Roy. Irish Acad., 98A (1998), pp. 173 – 190.

[25] J. Murray, *Mathematical Biology*, Springer Berlin, 2 ed., 1993. §14.7.

[26] N. Nefedov, *The method of differential inequalities for some classes of nonlinear singularly perturbed problems with internal layers*, Differ.Uravn., 31 (1995), pp. 1142–1149. translation in Differ.Equ., 1077-1085.

[27] N.S. Bakhvalov and G. Panasenko, *Homogenization: Averaging Processes in Periodic Media. Mathematical Problems in the Mechanics of Composite Materials*, vol. 36 of Mathematics and Its Applications (Soviet Series), Kluwer Academic Publishers, 1989, p. 8.

[28] R. O'Malley, *Singular Perturbation Methods for Ordinary Differential Equations*, vol. 89 of Appl. Math. Sci., Springer, New York, 1991.

[29] C. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York and London, 1992, p. 59.

[30] M. Protter and H. Weinberger, *Maximum Principles in Differential Equations*, Springer, 1984.

[31] H.-G. Roos, M. Stynes, and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations: Convection-Diffusion and Flow Problems*, vol. 24, Springer Series in Comput. Math., Berlin, 1996.

[32] G. I. Shishkin, *Grid approximation of parabolic equations with small parameters multiplying the space and time derivatives. reaction-diffusion equations*, Math. Balkanica (N.S.), 12 (1998), pp. 179 – 214.

[33] ——, Comput. Math. and Mathematical Physics, 46 (2006), pp. 1953 – 1971.

[34] G. Sun and M. Stynes, *A uniformly convergent method for a singularly perturbed semilinear reaction-diffusion problem with multiple solutions*, Math. Comp., 65 (1996), pp. 1085–1109.

[35] A. Vasil'eva, V. Butuzov, and L. Kalachev, *The Boundary Function Method for Singularly Perturbed Problems*, vol. 14, SIAM Studies in Applied Mathematics, 1995, p. 98.

[36] L.-C. Yeh, *A singularly perturbed reaction−diffusion problem with inhomogeneous environment*, J. Comput. and Appl. Math., 166 (2004), pp. 321–341.