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Maximal parabolic regularity for divergence operators on distribution spaces

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Abstract. We show that elliptic second order operators A of divergence type fulfill maximal parabolic regularity on distribution spaces, even if the underlying domain is highly non-smooth, the coefficients of A are discontinuous and A is complemented with mixed boundary conditions. Applications to quasilinear parabolic equations with non-smooth data are presented.

1. Introduction

It is the aim of this paper to give an abridged version of our work [39]. The goal is to provide a text with only very few proofs and with a considerable reduction of the sophisticated technicalities. In particular, we present a direct way to carry over maximal parabolic regularity from L^p spaces to the distribution spaces, avoiding the Dore-Venni argument. So our hope is to produce a more readable paper for colleagues who are only interested in the principal ideas and results of [39].

Our motivation was to find a concept which allows to treat nonlinear parabolic equations of the formal type

$$\begin{cases} u' - \nabla \cdot \mathcal{G}(u)\mu\nabla u = \mathcal{R}(t,u), \\ u(T_0) = u_0, \end{cases} \tag{1.1}$$

combined with mixed, nonlinear boundary conditions:

$$\nu \cdot \mathcal{G}(u)\mu\nabla u + b(u) = g \text{ on } \Gamma \qquad \text{and} \qquad u = 0 \text{ on } \partial\Omega \setminus \Gamma, \tag{1.2}$$

where Γ is a suitable open subset of $\partial\Omega$.

The main feature is here – in contrast to [42] – that inhomogeneous Neumann conditions and the appearance of distributional right-hand sides (e.g. surface densities) should be admissible. Thus, one has to consider the equations in suitably chosen distribution spaces. The concept to solve (1.1) is to apply a theorem of Prüss (see [50], see also [15]) which bases on maximal parabolic regularity. This has the advantage that right-hand sides are admissable which depend discontinuously on time, what is desirable in many applications. Pursuing this idea, one has, of course, to prove that the occurring elliptic operators satisfy maximal parabolic regularity on the chosen distribution spaces.

In fact, we show that, under very mild conditions on the domain Ω, the Dirichlet boundary part $\partial\Omega \setminus \Gamma$ and the coeffcient function, elliptic divergence operators with real L^∞-coefficients satisfy maximal parabolic regularity on a huge variety of spaces, among which are Sobolev, Besov and Lizorkin-Triebel spaces, provided that the differentiability index is between 0 and -1 (cf. Theorem 5.19). We consider this as the first main result of this work, also interesting in itself. Up to now, the only existing results for mixed boundary conditions in distribution spaces (apart from the Hilbert space situation) are, to our knowledge, that of Gröger [35] and the recent one of Griepentrog [29]. Concerning the Dirichlet case, compare [10] and references therein.

Let us point out some ideas, which will give a certain guideline for the paper:

In principle, our strategy for proving maximal parabolic regularity for divergence operators on $H_\Gamma^{-1,q}$ was to show an analog of the central result of [9], this time in case of mixed boundary conditions, namely that

$$(1.3) \qquad \big(-\nabla \cdot \mu\nabla + 1\big)^{-1/2} : L^q \to H_\Gamma^{1,q}$$

provides a topological isomorphism for suitable q. This would give the possibility of carrying over the maximal parabolic regularity, known for L^q, to the dual of $H_\Gamma^{1,q'}$, because, roughly spoken, $(-\nabla\cdot\mu\nabla+1)^{-1/2}$ commutes with the corresponding parabolic solution operator. Unfortunately, we were only able to prove the continuity of (1.3) within the range $q \in \,]1, 2]$, due to a result of Duong and M$^{\rm c}$Intosh [21], but did not succeed in proving the continuity of the inverse in general.

It turns out, however, that (1.3) provides a topological isomorphism, if $\Omega \cup \Gamma$ is the image under a volume-preserving and bi-Lipschitz mapping of one of Gröger's model sets [34], describing the geometric configuration in neighborhoods of boundary points of Ω. Thus, in these cases one may carry over the maximal parabolic regularity from L^q to $H_\Gamma^{-1,q}$. Knowing this, we localize the linear parabolic problem, use the 'local' maximal parabolic information and interpret this again in the global context at the end. Interpolation with the L^p result then yields maximal parabolic regularity on the corresponding interpolation spaces.

Let us explicitely mention that the concept of Gröger's regular sets, where the domain itself is a Lipschitz domain, seems adequate to us, because it covers many realistic geometries that fail to be domains with Lipschitz boundary. One striking example are the two crossing beams, cf. [39, Subsection 7.3].

The strategy for proving that (1.1), (1.2) admit a unique local solution is as follows. We reformulate (1.1) by adding the distributional terms, corresponding to the boundary condition (1.2) to the right hand side of (1.1). Assuming additionally that the elliptic operator $-\nabla\cdot\mu\nabla+1 : H_\Gamma^{1,q} \to H_\Gamma^{-1,q}$ provides a topological isomorphism for a q larger than the space dimension d, the above mentioned result of Prüss for abstract quasilinear equations applies to the resulting quasilinear parabolic equation. The detailed discussion how to assure all requirements of [50], including the adequate choice of the Banach space, is presented in Section 6. Let us further emphasize that the presented setting allows for coefficient functions that really jump at hetero interfaces of the material and permits mixed boundary conditions, as well as domains which do not possess a Lipschitz boundary. It is well known that this is highly desirable when modelling real world problems. One further advantage is that nonlinear, nonlocal boundary conditions are admissible in our concept, despite the fact that the data is highly non-smooth, compare [2]. It is remarkable that, irrespective of the discontinuous right-hand sides, the solution is Hölder continuous simultaneously in space and time, see Corollary 6.16 below.

In Section 7 we give examples for geometries, Dirichlet boundary parts and coefficients in three dimensions for which our additional supposition, the isomorphy $-\nabla \cdot \mu\nabla + 1 : H_\Gamma^{1,q} \to H_\Gamma^{-1,q}$ really holds for a $q > d$.

Finally, some concluding remarks are given in Section 8.

2. Notation and general assumptions

Throughout this article the following assumptions are valid.

- $\Omega \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain (cf. Assumption 3.1 a)) and Γ is an open subset of $\partial\Omega$.
- The coefficient function μ is a Lebesgue measurable, bounded function on Ω taking its values in the set of real, symmetric, positive definite $d \times d$ matrices, satisfying the usual ellipticity condition.

Remark 2.1. Concerning the notions 'Lipschitz domain' and 'domain with Lipschitz boundary' (synonymous: strongly Lipschitz domain) we follow the terminology of Grisvard [33].

For $\varsigma \in]0,1]$ and $1 < q < \infty$ we define $H^{\varsigma,q}_\Gamma(\Omega)$ as the closure of

$$C^\infty_\Gamma(\Omega) := \{\psi|_\Omega : \psi \in C^\infty(\mathbb{R}^d),\ \mathrm{supp}(\psi) \cap (\partial\Omega \setminus \Gamma) = \emptyset\}$$

in the Sobolev space $H^{\varsigma,q}(\Omega)$. Concerning the dual of $H^{\varsigma,q}_\Gamma(\Omega)$, we have to distinguish between the space of linear and the space of anti-linear forms on this space. We define $H^{-\varsigma,q}_\Gamma(\Omega)$ as the space of continuous, linear forms on $H^{\varsigma,q'}_\Gamma(\Omega)$ and $\breve{H}^{-\varsigma,q}_\Gamma(\Omega)$ as the space of anti-linear forms on $H^{\varsigma,q'}_\Gamma(\Omega)$ if $1/q + 1/q' = 1$. Note that L^p spaces may be viewed as part of $\breve{H}^{-\varsigma,q}_\Gamma$ for suitable ς, q via the identification of an element $f \in L^p$ with the anti-linear form $H^{\varsigma,q'}_\Gamma \ni \psi \mapsto \int_\Omega f\overline{\psi}\,\mathrm{dx}$.

If misunderstandings are not to be expected, we drop the Ω in the notation of spaces, i.e. function spaces without an explicitely given domain are to be understood as function spaces on Ω.

By K we denote the open unit cube $]-1,1[^d$ in $\mathbb{R}^d$, by K_- the lower half cube $K \cap \{\mathrm{x} : x_d < 0\}$, by $\Sigma = K \cap \{\mathrm{x} : x_d = 0\}$ the upper plate of K_- and by Σ_0 the left half of Σ, i.e. $\Sigma_0 = \Sigma \cap \{\mathrm{x} : x_{d-1} < 0\}$.

Throughout the paper we will use $\mathrm{x}, \mathrm{y}, \ldots$ for vectors in $\mathbb{R}^d$.

If B is a closed operator on a Banach space X, then we denote by $dom_X(B)$ the domain of this operator. $\mathcal{L}(X,Y)$ denotes the space of linear, continuous operators from X into Y; if $X = Y$, then we abbreviate $\mathcal{L}(X)$. Furthermore, we will write $\langle\cdot,\cdot\rangle_{X'}$ for the dual pairing of elements of X and the space X' of anti-linear forms on X.

Finally, the letter c denotes a generic constant, not always of the same value.

3. Preliminaries

In this section we will properly define the elliptic divergence operator and afterwards collect properties of the L^p realizations of this operator which will be needed in the subsequent sections. Let us introduce an assumption on Ω and Γ which will define the geometric framework relevant for us in the sequel.

Assumption 3.1. a) *For any point* $\mathrm{x} \in \partial\Omega$ *there is an open neighborhood* Υ_x *of* x *and a bi-Lipschitz mapping* ϕ_x *from* Υ_x *into* $\mathbb{R}^d$*, such that* $\phi_\mathrm{x}(\mathrm{x}) = 0$

and $\phi_{\mathrm{x}}\big((\Omega\cup\Gamma)\cap\Upsilon_{\mathrm{x}}\big)=\alpha K_-$ *or* $\alpha(K_-\cup\Sigma)$ *or* $\alpha(K_-\cup\Sigma_0)$ *for some positive* $\alpha=\alpha(\mathrm{x})$.

b) *Each mapping* ϕ_{x} *is, in addition, volume-preserving.*

Remark 3.2. Assumption 3.1 a) exactly characterizes Gröger's regular sets, introduced in his pioneering paper [34]. Note that the additional property 'volume-preserving' also has been required in several contexts (see [30] and [35]).

It is not hard to see that every Lipschitz domain and also its closure is regular in the sense of Gröger, the corresponding model sets are then K_- or $K_-\cup\Sigma$, respectively, see [33, Ch 1.2]. A simplifying topological characterization of Gröger's regular sets for $d=2$ and $d=3$ will be given in Remark 8.5.

In particular, all domains with Lipschitz boundary (strongly Lipschitz domains) satisfy Assumption 3.1: if, after a shift and an orthogonal transformation, the domain lies locally beyond a graph of a Lipschitz function ψ, then one defines $\phi(x_1,\dots,x_d)=(x_1-\psi(x_2,\dots,x_d),x_2,\dots,x_d)$. Obviously, ϕ is then bi-Lipschitz and the determinant of its Jacobian is identically 1.

Following [24, Ch. 3.3.4 C], for every Lipschitz hypersurface $\mathcal{H}$ one can introduce a surface measure σ on $\mathcal{H}$. This is in particular true for $\mathcal{H}=\partial\Omega$, see also [40]. Having this at hand, one can prove the following trace theorem.

Proposition 3.3. *Assume* $q\in\,]1,\infty[$ *and* $\theta\in\,]\frac{1}{q},1[$. *Let* Π *be a Lipschitz hypersurface in* $\overline{\Omega}$ *and let* ϖ *be any measure on* Π *which is absolutely continuous with respect to the surface measure* σ. *If the corresponding Radon-Nikodym derivative is essentially bounded (with respect to* σ*), then the trace operator* Tr *is continuous from* $H^{\theta,q}(\Omega)$ *to* $L^q(\Pi,\varpi)$.

Later we will repeatedly need the following interpolation result.

Proposition 3.4. *Let* Ω *and* Γ *satisfy Assumption 3.1 a) and let* $\theta\in\,]0,1[$.

Then for $q_0,q_1\in\,]1,\infty[$ *and* $\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$ *one has*

$$H^{1,q}_\Gamma=\big[H^{1,q_0}_\Gamma,H^{1,q_1}_\Gamma\big]_\theta \quad \textit{and} \quad \breve{H}^{-1,q}_\Gamma=\big[\breve{H}^{-1,q_0}_\Gamma,\breve{H}^{-1,q_1}_\Gamma\big]_\theta. \tag{3.1}$$

We define the operator $A: H^{1,2}_\Gamma\to\breve{H}^{-1,2}_\Gamma$ by

$$\langle A\psi,\varphi\rangle_{\breve{H}^{-1,2}_\Gamma}:=\int_\Omega \mu\nabla\psi\cdot\nabla\overline{\varphi}\,\mathrm{dx}+\int_\Gamma \varkappa\,\psi\,\overline{\varphi}\,\mathrm{d}\sigma,\quad \psi,\varphi\in H^{1,2}_\Gamma, \tag{3.2}$$

where $\varkappa\in L^\infty(\Gamma,\mathrm{d}\sigma)$. Note that in view of Proposition 3.3 the form in (3.2) is well defined.

In the special case $\varkappa=0$, we write more suggestively $-\nabla\cdot\mu\nabla$ instead of A.

The L^2 realization of A, i.e. the maximal restriction of A to the space L^2, we denote by the same symbol A; clearly this is identical with the operator which is induced by the form on the right-hand side of (3.2). If B is a self-adjoint operator on L^2, then by the L^p realization of B we mean its restriction to L^p if $p>2$ and the L^p closure of B if $p\in[1,2[$.

First, we collect some basic facts on A.

Proposition 3.5. i) *$\nabla \cdot \mu\nabla$ generates an analytic semigroup on $\breve{H}_\Gamma^{-1,2}$.*

ii) *$-\nabla \cdot \mu\nabla$ is self-adjoint on L^2 and bounded by 0 from below. The restriction of $-A$ to L^2 is densely defined and generates an analytic semigroup there.*

iii) *If $\lambda > 0$ then the operator $(-\nabla\cdot\mu\nabla+\lambda)^{1/2} : H_\Gamma^{1,2} \to L^2$ provides a topological isomorphism; in other words: the domain of $(-\nabla \cdot \mu\nabla + \lambda)^{1/2}$ on L^2 is the form domain $H_\Gamma^{1,2}$.*

iv) *The form domain $H_\Gamma^{1,2}$ is invariant under multiplication with functions from $H^{1,q}$, if $q > d$.*

One essential instrument for our subsequent considerations are (upper) Gaussian estimates.

Theorem 3.6. *The semigroup generated by $\nabla \cdot \mu\nabla$ in L^2 satisfies upper Gaussian estimates, precisely:*

$$(\mathrm{e}^{t\nabla\cdot\mu\nabla} f)(\mathrm{x}) = \int_\Omega K_t(\mathrm{x},\mathrm{y})f(\mathrm{y})\,\mathrm{dy}, \quad \mathrm{x} \in \Omega,\ f \in L^2,$$

for some measurable function $K_t : \Omega\times\Omega \to \mathbb{R}_+$ and for all $\varepsilon > 0$ there exist constants $c, b > 0$, such that

$$0 \le K_t(\mathrm{x},\mathrm{y}) \le \frac{c}{t^{d/2}}\ \mathrm{e}^{-b\frac{|\mathrm{x}-\mathrm{y}|^2}{t}}\ \mathrm{e}^{\varepsilon t}, \quad t > 0,\ a.a.\ \mathrm{x},\mathrm{y} \in \Omega. \tag{3.3}$$

This follows from Theorem 6.10 in [48], see also [7].

4. Mapping properties for $(-\nabla \cdot \mu\nabla)^{1/2}$

In this chapter we prove that, under certain topological conditions on Ω and Γ, the mapping

$$(-\nabla \cdot \mu\nabla + 1)^{1/2} : H_\Gamma^{1,q} \to L^q$$

is a topological isomorphism for $q \in\,]1, 2[$. We abbreviate $-\nabla\cdot\mu\nabla$ by A_0 throughout this chapter. Let us introduce the following

Assumption 4.1. *There is a bi-Lipschitz, volume-preserving mapping ϕ from a neighborhood of $\overline{\Omega}$ into $\mathbb{R}^d$ such that $\phi(\Omega \cup \Gamma) = \alpha K_-$ or $\alpha(K_- \cup \Sigma)$ or $\alpha(K_- \cup \Sigma_0)$ for some $\alpha > 0$.*

Remark 4.2. It is known that a bi-Lipschitz mapping is volume-preserving, iff the absolute value of the determinant of its Jacobian is one almost everywhere (see [24, Ch. 3]).

The main results of this section are the following three theorems.

Theorem 4.3. *Under the general assumptions made in Section 2 the following holds true: For every $q \in\,]1, 2]$, the operator $(A_0 + 1)^{-1/2}$ is a continuous operator from L^q into $H_\Gamma^{1,q}$. Hence, it continuously maps $\breve{H}_\Gamma^{-1,q}$ into L^q for any $q \in [2, \infty[$.*

Theorem 4.4. *Let in addition Assumption 4.1 be fulfilled. Then*

i) *For $q \in\,]1, 2]$, the operator $A_0^{1/2}$ maps $H_\Gamma^{1,q}$ continuously into L^q. Hence, it continuously maps L^q into $\breve{H}_\Gamma^{-1,q}$ for any $q \in [2, \infty[$.*

ii) *The same is true for* A_0+1 *instead of* A_0.

Putting these two results together, one immediately gets the following isomorphism property of the square root of A_0+1.

Theorem 4.5. *Under Assumption 4.1,* $(A_0+1)^{1/2}$ *provides a topological isomorphism between* $H^{1,q}_\Gamma$ *and* L^q *for* $q \in]1,2]$ *and a topological isomorphism between* L^q *and* $\breve{H}^{-1,q}_\Gamma$ *for any* $q \in [2,\infty[$.

Remark 4.6. In all three theorems the second assertion follows from the first by the self-adjointness of A_0 on L^2 and duality; thus one may focus on the proof of the first assertions.

Let us first prove the continuity of the operator $(A_0+1)^{-1/2} : L^q \to H^{1,q}_\Gamma$. In order to do so, we observe that this follows, whenever

1. The Riesz transform $\nabla(A_0+1)^{-1/2}$ is a bounded operator on L^q, and, additionally,
2. $(A_0+1)^{-1/2}$ maps L^q into $H^{1,q}_\Gamma$.

The first item can be deduced from the following result of Duong and M^c^Intosh (see [21, Thm. 2]) that is even true in a much more general setting.

Proposition 4.7. *Let* B *be a positive, self-adjoint operator on* L^2*, having the space* W *as its form domain and admitting the estimate* $\|\nabla\psi\|_{L^2} \le c\|B^{1/2}\psi\|_{L^2}$ *for all* $\psi \in W$. *Assume that* W *is invariant under multiplication by bounded functions with bounded, continuous first derivatives and that the kernel* K_t *of the semigroup* e^{-tB} *satisfies bounds*

$$|K_t(\mathrm{x},\mathrm{y})| \le \frac{C}{t^{d/2}}\left(1+\frac{|\mathrm{x}-\mathrm{y}|^2}{t}\right)^{-\beta} \tag{4.1}$$

for some $\beta > d/2$. *Then the operator* $\nabla B^{-1/2}$ *is of weak type (1,1), and, thus can be extended from* L^2 *to a bounded operator on* L^q *for all* $q \in]1,2[$.

Proof of Theorem 4.3. According to Theorem 3.6 the semigroup kernels corresponding to the operator A_0 satisfy the estimate (3.3). Thus, the corresponding kernels for the operator A_0+1 satisfy again (3.3), but without the factor $\mathrm{e}^{\varepsilon t}$ now. Next, we verify that $B := A_0+1$ and $W := H^{1,2}_\Gamma$ satisfy the assumptions of Proposition 4.7. By Proposition 3.5, $W = H^{1,2}_\Gamma$ is the domain for $(A_0+1)^{1/2}$, thus $\|\nabla\psi\|_{L^2} \le c\|(A_0+1)^{1/2}\psi\|_{L^2}$ holds for all $\psi \in W$. The invariance property of W under multiplication is ensured by Proposition 3.5 iv). Concerning the bound (4.1), it is easy to see that the resulting Gaussian bounds from Theorem 3.6 are even much stronger, since the function $r \mapsto (1+r)^\beta\,\mathrm{e}^{-br}$, $r \ge 0$, is bounded for every $\beta > 0$. All this shows that $(A_0+1)^{-1/2} : L^q \to H^{1,q}$ is continuous for $q \in]1,2]$.

It remains to show 2. The first point makes clear that $(A_0+1)^{-1/2}$ maps L^q continuously into $H^{1,q}$, thus one has only to verify the correct boundary behavior of the images. If $f \in L^2 \hookrightarrow L^q$, then one has $(A_0+1)^{-1/2}f \in H^{1,2}_\Gamma \hookrightarrow H^{1,q}_\Gamma$. Thus, the assertion follows from 1. and the density of L^2 in L^q. $\square$

Remark 4.8. Theorem 4.3 is not true for other values of q in general, see [8, Ch. 4] for a further discussion.

It follows the proof of Theorem 4.4 i). It will be deduced from the subsequent deep result on divergence operators with Dirichlet boundary conditions and some permanence principles.

Proposition 4.9 (Auscher/Tchamitchian, [9])**.** *Let $q \in]1,\infty[$ and Ω be a strongly Lipschitz domain. Then the root of the operator A_0, combined with a homogeneous Dirichlet boundary condition, maps $H_0^{1,q}(\Omega)$ continuously into $L^q(\Omega)$.*

For further reference we mention the following immediate consequence of Theorem 4.3 and Proposition 4.9.

Corollary 4.10. *Under the hypotheses of Proposition 4.9 the operator $A_0^{-1/2}$ provides a topological isomorphism between L^q and $H_0^{1,q}$, if $q \in]1,2]$.*

In view of Assumption 4.1 it is a natural idea to reduce our considerations to the three model constellations mentioned there. In order to do so, we have to show that the assertion of Theorem 4.4 is invariant under volume-preserving bi-Lipschitz transformations of the domain. The proof will stem on the following lemma.

Lemma 4.11. *Assume that ϕ is a mapping from a neighborhood of $\overline{\Omega}$ into $\mathbb{R}^d$ that is additionally bi-Lipschitz. Let us denote $\phi(\Omega) = \Omega_\triangle$ and $\phi(\Gamma) = \Gamma_\triangle$. Define for any function $f \in L^1(\Omega_\triangle)$*

$$(\Phi f)(\mathrm{x}) = f(\phi(\mathrm{x})) = (f \circ \phi)(\mathrm{x}), \quad \mathrm{x} \in \Omega.$$

Then

i) *The restriction of Φ to any $L^p(\Omega_\triangle)$, $1 \le p < \infty$, provides a linear, topological isomorphism between this space and $L^p(\Omega)$.*

ii) *For any $p \in]1,\infty[$, the mapping Φ induces a linear, topological isomorphism*

$$\Phi_p : H^{1,p}_{\Gamma_\triangle}(\Omega_\triangle) \to H^{1,p}_{\Gamma}(\Omega).$$

iii) *$\Phi^*_{p'}$ is a linear, topological isomorphism between $\breve{H}^{-1,p}_{\Gamma}(\Omega)$ and $\breve{H}^{-1,p}_{\Gamma_\triangle}(\Omega_\triangle)$ for any $p \in]1,\infty[$.*

iv) *One has*

$$\Phi^*_{p'} A_0 \Phi_p = -\nabla \cdot \mu_\triangle \nabla$$

with

$$\mu_\triangle(\mathrm{y}) = \frac{1}{\left|\det(D\phi)(\phi^{-1}(\mathrm{y}))\right|}(D\phi)(\phi^{-1}(\mathrm{y}))\, \mu(\phi^{-1}(\mathrm{y}))\, \big(D\phi\big)^T(\phi^{-1}(\mathrm{y}))$$

for almost all $\mathrm{y} \in \Omega_\triangle$. Here, $D\phi$ denotes the Jacobian of ϕ and $\det(D\phi)$ the corresponding determinant.

v) *$\mu_\triangle$ also is bounded, Lebesgue measurable, elliptic and takes real, symmetric matrices as values.*

vi) *The restriction of $\Phi_2^*\Phi$ to $L^2(\Omega_\triangle)$ equals the multiplication operator which is given by the function $\left|\det(D\phi)(\phi^{-1}(\cdot))\right|^{-1}$. Consequently, if $|\det(D\phi)| = 1$ a.e., then the restriction of $\Phi_2^*\Phi$ to $L^2(\Omega_\triangle)$ is the identity operator on $L^2(\Omega_\triangle)$, or, equivalently, $(\Phi_2^*)^{-1}|_{L^2(\Omega_\triangle)} = \Phi|_{L^2(\Omega_\triangle)}$.*

Lemma 4.12. *Let $p \in]1,\infty[$. Suppose further that $\partial\Omega \setminus \Gamma$ does not have boundary measure zero and that $|\det(D\phi)| = 1$ almost everywhere in Ω. Then, in the notation of the preceding lemma, the operator $\big(-\nabla\cdot\mu_\vartriangle\nabla\big)^{1/2}$ maps $H^{1,p}_{\Gamma_\vartriangle}(\Omega_\vartriangle)$ continuously into $L^p(\Omega_\vartriangle)$ if and only if $A_0^{1/2}$ maps $H^{1,p}_\Gamma(\Omega)$ continuously into $L^p(\Omega)$.*

Let us give the main ideas of the proof: making use of vi) in Lemma 4.11, we get the following operator equation on $L^2(\Omega_\vartriangle)$:

$$\Phi^{-1}\big(A_0 + t\big)^{-1}\Phi|_{L^2(\Omega_\vartriangle)} = \big(-\nabla\cdot\mu_\vartriangle\nabla + t\big)^{-1}. \tag{4.2}$$

Employing then the formula

$$B^{-1/2} = \frac{1}{\pi}\int_0^\infty t^{-1/2}(B+t)^{-1}\,\mathrm{d}t, \tag{4.3}$$

one ends up with

$$\Phi^{-1}A_0^{1/2}\Phi_2 = \big(-\nabla\cdot\mu_\vartriangle\nabla\big)^{1/2}. \tag{4.4}$$

From this, the assertion may be deduced by straightforward arguments.

Remark 4.13. It is the property of 'volume-preserving' which leads, due to vi) of Lemma 4.11, to (4.2) and then to (4.4) and thus allows to hide the complicated geometry of the boundary in Φ and $\mu_\vartriangle$.

It turns out that 'bi-Lipschitz' together with 'volume-preserving' is not a too restrictive condition. In particular, there are such mappings – although not easy to construct – which map the ball onto the cylinder, the ball onto the cube and the ball onto the half ball, see [31], see also [25]. The general message is that this class has enough flexibility to map 'non-smooth objects' onto smooth ones.

Lemma 4.12 allows to reduce the proof of Theorem 4.4 i) to $\Omega = \alpha K_-$ and the three cases $\Gamma = \emptyset$, $\Gamma = \alpha\Sigma$ or $\Gamma = \alpha\Sigma_0$. The first case, $\Gamma = \emptyset$, is already contained in Proposition 4.9. In order to treat the second one, we use a reflection argument. Let us point out the main ideas for this: First, one defines the operator $\mathfrak{E} : L^1(K_-) \to L^1(K)$ which assigns to every function from $L^1(K_-)$ its symmetric extension. Let us further denote by $\mathfrak{R} : L^1(K) \to L^1(K_-)$ the restriction operator. Finally, one defines $-\nabla\cdot\hat{\mu}\nabla : H^{1,2}_0(K) \to \breve{H}^{-1,2}(K)$ as the symmetric extension of $-\nabla\cdot\mu\nabla$ to K. Note that this latter operator is then combined with homogeneous Dirichlet conditions. The definition of $-\nabla\cdot\hat{\mu}\nabla$ in particular implies

$$\big(A_0 + t\big)^{-1} f = \mathfrak{R}\big(-\nabla\cdot\hat{\mu}\nabla + t\big)^{-1}\mathfrak{E} f \quad \text{for all} \quad f \in L^2(K_-).$$

Multiplying this equation by $\frac{t^{-1/2}}{\pi}$ and integrating over t, one obtains in accordance with (4.3)

$$A_0^{-1/2} f = \mathfrak{R}\big(-\nabla\cdot\hat{\mu}\nabla\big)^{-1/2}\mathfrak{E} f, \quad f \in L^2(K_-).$$

This equation extends to all $f \in L^p(K_-)$ with $p \in]1,2[$. Now one exploits the fact that $\big(-\nabla\cdot\hat{\mu}\nabla\big)^{-1/2}$ is a surjection onto the whole $H^{1,p}_0(K)$ by Corollary 4.10. Then some straightforward arguments show that $A_0^{-1/2} : L^p(K_-) \to H^{1,p}_\Sigma(K_-)$ also is a surjection. Since, by Theorem 4.3 $A_0^{-1/2} : L^p(K_-) \to H^{1,p}_\Sigma(K_-)$ is continuous, the continuity of the inverse finally is implied by the open mapping theorem.

In order to prove the same for the third model constellation, i.e. $\Gamma = \Sigma_0$, one shows

Lemma 4.14. *For every $\alpha > 0$ there is a volume-preserving, bi-Lipschitz mapping $\phi : \mathbb{R}^d \to \mathbb{R}^d$ that maps $\alpha(K_- \cup \Sigma_0)$ onto $\alpha(K_- \cup \Sigma)$.*

Thus, the proof of Theorem 4.4 i) in the case $\Gamma = \alpha\Sigma_0$ results from the case $\Gamma = \alpha\Sigma$ and Lemmas 4.12 and 4.14.

It remains to prove part ii) of Theorem 4.4. Writing $(A_0+1)^{1/2} = (A_0+1)^{1/2}A_0^{-1/2}A_0^{1/2}$, the assertion follows, if one shows that $(A_0+1)^{1/2}A_0^{-1/2} = (1+A_0^{-1})^{1/2} : L^2 \to L^2$ extends to a continous mapping from L^q into itself. Since Assumption 4.1 implies that the boundary measure of $\partial\Gamma \setminus \Gamma$ is nonzero, the L^2 spectrum of A_0 is contained in an interval $[\varepsilon, \infty[$, $\varepsilon > 0$. But the spectrum of A_0, considered on L^q, is independent from q (see [48, Thm. 7.10]). Hence, A_0^{-1} is well defined and continuous on every L^q. Moreover, the spectrum of $1 + A_0^{-1}$ (considered as an operator on L^q) is thus contained in a bounded interval $[1, \delta]$ by the spectral mapping theorem, see [44, Ch. III.6.3]. Consequently, $(1 + A_0^{-1})^{1/2} : L^q \to L^q$ is also a continuous operator by classical functional calculus (see [20, Ch. VII.3]).

Remark 4.15. Let us mention that Lemma 4.12, only applied to $\Omega = K$ and $\Gamma = \emptyset$ (the pure Dirichlet case) already provides a zoo of geometries which is not covered by [9]. Notice in this context that the image of a strongly Lipschitz domain under a bi-Lipschitz transformation needs not to be a strongly Lipschitz domain at all, cf. [33, Ch. 1.2].

5. Maximal parabolic regularity for A

In this section we intend to prove the first main result of this work announced in the introduction, i.e. maximal parabolic regularity of A in spaces with negative differentiability index. Let us first recall the notion of maximal parabolic L^s regularity.

Definition 5.1. Let $1 < s < \infty$, let X be a Banach space and let $J :=]T_0, T[\subseteq \mathbb{R}$ be a bounded interval. Assume that B is a closed operator in X with dense domain D (in the sequel always equipped with the graph norm). We say that B satisfies *maximal parabolic $L^s(J;X)$ regularity*, if for any $f \in L^s(J;X)$ there exists a unique function $u \in W^{1,s}(J;X) \cap L^s(J;D)$ satisfying

$$u' + Bu = f, \qquad u(T_0) = 0,$$

where the time derivative is taken in the sense of X-valued distributions on J (see [4, Ch III.1]).

Remark 5.2. i) It is well known that the property of maximal parabolic regularity of an operator B is independent of $s \in]1, \infty[$ and the specific choice of the interval J (cf. [19]). Thus, in the following we will say for short that B admits maximal parabolic regularity on X.

ii) If an operator satisfies maximal parabolic regularity on a Banach space X, then its negative generates an analytic semigroup on X (cf. [19]). In particular, a suitable left half plane belongs to its resolvent set.

iii) If X is a Hilbert space, the converse is also true: The negative of every generator of an analytic semigroup on X satisfies maximal parabolic regularity, cf. [18] or [19].

iv) If $-B$ is a generator of an analytic semigroup on a Banach space X, and S_X indicates the space of X-valued step functions on J, then we define

$$B\big(\frac{\partial}{\partial t} + B\big)^{-1} : S_X \to C(\overline{J}; X) \hookrightarrow L^s(J; X)$$

by

$$\Big(B\big(\frac{\partial}{\partial t} + B\big)^{-1} f\Big)(t) := B \int_{T_0}^{t} \mathrm{e}^{-(t-s)B} f(s)\, \mathrm{d}s,$$

compare (5.4) below. It is known that S_X is a dense subspace of $L^s(J; X)$, if $s \in [1, \infty[$, see [27, Lemma IV.1.3]. Using this, it is easy to see that B has maximal parabolic regularity on X if and only if the operator $B\big(\frac{\partial}{\partial t} + B\big)^{-1}$ continuously extends to an operator from $L^s(J; X)$ into itself.

v) Observe that

$$W^{1,s}(J; X) \cap L^s(J; D) \hookrightarrow C(\overline{J}; (X, D)_{1-\frac{1}{s},s}). \tag{5.1}$$

The next theorem will be the cornerstone on maximal parabolic regularity of this work:

Theorem 5.3. *Let Ω, Γ fulfill Assumption 3.1 and set $q_{\mathrm{iso}} := \sup M_{\mathrm{iso}}$, where*

$$M_{\mathrm{iso}} := \{q \in [2, \infty[\ :\ -\nabla \cdot \mu \nabla + 1 : H^{1,q}_{\Gamma} \to \breve{H}^{-1,q}_{\Gamma} \text{ is a topological isomorphism}\}.$$

Then $-\nabla \cdot \mu \nabla$ satisfies maximal parabolic regularity on $\breve{H}^{-1,q}_{\Gamma}$ for all $q \in [2, q^*_{\mathrm{iso}}[$, where by r^* we denote the Sobolev conjugated index of r, i.e.

$$r^* = \begin{cases} \infty, & \text{if } r \geq d, \\ \big(\frac{1}{r} - \frac{1}{d}\big)^{-1}, & \text{if } r \in [1, d[\,. \end{cases}$$

Remark 5.4. i) If Ω, Γ fulfill Assumption 3.1 a), then $q_{\mathrm{iso}} > 2$, see [36] and also [34].

ii) It is clear by Lax-Milgram and interpolation (see Proposition 3.4) that M_{iso} is the interval $[2, q_{\mathrm{iso}}[$ or $[2, q_{\mathrm{iso}}]$. Moreover, it can be concluded from a deep theorem of Sneiberg [53] (see also [8, Lemma 4.16]) that the second case cannot occur.

Proposition 5.5. *If Ω is a bounded Lipschitz domain and Γ is any closed subset of $\partial\Omega$. Then $-\nabla \cdot \mu \nabla$ satisfies maximal parabolic regularity on L^p for all $p \in \,]1, \infty[$.*

Proof. It is known that the Gaussian estimates, stated in Theorem 3.6, imply maximal parabolic regularity on L^p, if $p \in \,]1, \infty[$, see [41] or [16]. □

Remark 5.6. Alternatively, the assertion of Proposition 5.5 may be deduced as follows: First one observes that the induced semigroup on any L^p is contractive, see [48, Thm. 4.28/Prop. 4.11]. Then one applies [46, Cor. 1.1].

Lemma 5.7. *Let Ω, Γ fulfill Assumption 4.1. Then $\nabla \cdot \mu \nabla$ generates an analytic semigroup on $\breve{H}^{-1,q}_{\Gamma}$ for all $q \in [2, \infty[$.*

Proof. One has the following operator identity

(5.2)
$$\big(-\nabla\cdot\mu\nabla+\lambda\big)^{-1}=\big(-\nabla\cdot\mu\nabla+1\big)^{1/2}\big(-\nabla\cdot\mu\nabla+\lambda\big)^{-1}\big(-\nabla\cdot\mu\nabla+1\big)^{-1/2},\quad \operatorname{Re}\lambda\geq 0,$$

on L^q. Under Assumption 4.1 $(-\nabla\cdot\mu\nabla+1)^{1/2}$ is a topological isomorphism between L^q and $\breve{H}_\Gamma^{-1,q}$ for every $q\in[2,\infty[$, thanks to Theorem 4.5. Thus, via (5.2), the corresponding resolvent estimate carries over from L^q to $\breve{H}_\Gamma^{-1,q}$ by the density of L^q in $\breve{H}_\Gamma^{-1,q}$. □

In the next step we show

Theorem 5.8. *Let Ω,Γ fulfill Assumption 4.1. Then $-\nabla\cdot\mu\nabla$ satisfies maximal parabolic regularity on $\breve{H}_\Gamma^{-1,q}$ for all $q\in[2,\infty[$.*

This will be a consequence of Theorem 4.5 and the following two lemmata.

Lemma 5.9. *Assume that the operator B fullfils maximal parabolic regularity on a Banach space X and has no spectrum in $]-\infty,0[$. If S_X again denotes the space of X-valued step functions on J, then one has for every $\alpha\in]0,1[$*

$$B\big(\frac{\partial}{\partial t}+B\big)^{-1}\psi=(B+1)^{\alpha}B\big(\frac{\partial}{\partial t}+B\big)^{-1}(B+1)^{-\alpha}\psi\quad\textit{for all}\quad\psi\in S_X. \tag{5.3}$$

Proof. First, B satisfies a resolvent estimate $\|(B+\lambda)^{-1}\|_{\mathcal{L}(X)}\leq\frac{c}{|\lambda|}$ for all λ from a suitable right half space. Since, additionally, B has no spectrum in $]-\infty,0[$, the operators $(B+1)^{-\alpha}$ and $(B+1)^{\alpha}$ are well defined on X.

If $x\in X$ and χ_I denotes the indicator function of an interval $I=]a,b[\subseteq J$, then one calculates by the definition of $B(\frac{\partial}{\partial t}+B)^{-1}$

(5.4)
$$\Big[B\big(\frac{\partial}{\partial t}+B\big)^{-1}\chi_I x\Big](t)=B\int_{T_0}^{t}e^{-(t-s)B}x\chi_I(s)\,ds=\begin{cases}0, & \text{if } t<a\\ \big(1-e^{(a-t)B}\big)x, & \text{if } t\in[a,b],\\ \big(e^{(b-t)B}-e^{(a-t)B}\big)x, & \text{if } t>b,\end{cases}$$

compare [49, Ch. 1.2]. This gives for every step function $\sum_{l=1}^{N}\chi_{I_l}x_l\in S_X$

$$\begin{aligned}B\big(\frac{\partial}{\partial t}+B\big)^{-1}(B+1)^{-\alpha}\sum_{l=1}^{N}\chi_{I_l}x_l&=\sum_{l=1}^{N}B\big(\frac{\partial}{\partial t}+B\big)^{-1}\chi_{I_l}(B+1)^{-\alpha}x_l\\&=(B+1)^{-\alpha}B\big(\frac{\partial}{\partial t}+B\big)^{-1}\sum_{l=1}^{N}\chi_{I_l}x_l,\end{aligned}$$

since $(B+1)^{-\alpha}$ commutes with the semigroup e^{-tB}. □

Remark 5.10. By the density of S_X in $L^s(J;X)$ for $s\in[1,\infty[$, equation (5.3) extends to the whole of $L^s(J;X)$ since the left-hand side is a continuous operator on $L^s(J;X)$ by maximal regularity of B.

Lemma 5.11. *Assume that X,Y are Banach spaces, where X continuously and densely injects into Y. Suppose B to be an operator on Y, whose maximal restriction $B|_X$ to X satisfies maximal parabolic regularity there. If $B|_X$ has no spectrum in*

$]-\infty,0[$ and $(B+1)^\alpha$ provides a topological isomorphism from X onto Y, then B also satisfies maximal parabolic regularity on Y.

Proof. Let S_X be the set of step functions on J, taking their values in X. By the density of X in Y, S_X is also dense in $L^s(J;Y)$. Due to Lemma 5.9, we may estimate for any $\psi \in S_X$:

$$\|B\big(\frac{\partial}{\partial t}+B\big)^{-1}\psi\|_{\mathcal{L}(L^s(J;Y))} = \|(B+1)^\alpha B\big(\frac{\partial}{\partial t}+B\big)^{-1}(B+1)^{-\alpha}\psi\|_{\mathcal{L}(L^s(J;Y))}$$
$$\le c\|(B+1)^\alpha\|_{\mathcal{L}(L^s(J;X);L^s(J;Y))}\|B\big(\frac{\partial}{\partial t}+B\big)^{-1}\|_{\mathcal{L}(L^s(J;X))}\|(B+1)^{-\alpha}\psi\|_{L^s(J;X)}$$
$$\le c\|(B+1)^\alpha\|_{\mathcal{L}(X;Y)}\|B\big(\frac{\partial}{\partial t}+B\big)^{-1}\|_{\mathcal{L}(L^s(J;X))}\|(B+1)^{-\alpha}\|_{\mathcal{L}(Y;X)}\|\psi\|_{L^s(J;Y)}.$$

By density, $B\big(\frac{\partial}{\partial t}+B\big)^{-1}$ extends to a continuous operator on the whole of $L^s(J;Y)$ and the assertion follows by Remark 5.2 iv). □

The proof of Theorem 5.8 is now obtained by the isomorphism property $\big(-\nabla\cdot\mu\nabla+1\big)^{1/2} : L^q \to \breve{H}^{-1,q}_\Gamma$, assured by Theorem 4.5, and afterwards applying Proposition 5.5 and Lemma 5.11, putting there $X := L^q$, $Y := \breve{H}^{-1,q}_\Gamma$, $B := -\nabla\cdot\mu\nabla$ and $\alpha := 1/2$.

Now we intend to 'globalize' Theorem 5.8, in other words: We prove that $-\nabla\cdot\mu\nabla$ satisfies maximal parabolic regularity on $\breve{H}^{-1,q}_\Gamma$ for suitable q if Ω, Γ satisfy only Assumption 3.1, i.e. if αK_-, $\alpha(K_-\cup\Sigma)$ and $\alpha(K_-\cup\Sigma_0)$ need only to be model sets for the constellation around boundary points. Obviously, then the variety of admissible Ω's and Γ's increases considerably; in particular, Γ may have more than one connected component.

5.1. **Auxiliaries.** We continue with a result that allows to 'localize' the elliptic operator.

Lemma 5.12. *Let Ω,Γ satisfy Assumption 3.1 and let $\Upsilon \subseteq \mathbb{R}^d$ be open, such that $\Omega_\bullet := \Omega\cap\Upsilon$ is also a Lipschitz domain. Furthermore, we put $\Gamma_\bullet := \Gamma\cap\Upsilon$ and fix an arbitrary, real valued function $\eta \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\eta)\subseteq\Upsilon$. Denote by $\mu_\bullet$ the restriction of the coefficient function μ to $\Omega_\bullet$ and assume $v \in H^{1,2}_\Gamma(\Omega)$ to be the solution of*

$$-\nabla\cdot\mu\nabla v = f \in \breve{H}^{-1,2}_\Gamma(\Omega).$$

Then the following holds true:

i) *For all $q \in]1,\infty[$ the anti-linear form*

$$f_\bullet : w \mapsto \langle f, \widetilde{\eta w}\rangle_{\breve{H}^{-1,2}_\Gamma}$$

(where $\widetilde{\eta w}$ again means the extension of ηw by zero to the whole Ω) is well defined and continuous on $H^{1,q'}_{\Gamma_\bullet}(\Omega_\bullet)$, whenever f is an anti-linear form from $\breve{H}^{-1,q}_\Gamma(\Omega)$. The mapping $\breve{H}^{-1,q}_\Gamma(\Omega)\ni f\mapsto f_\bullet \in \breve{H}^{-1,q}_{\Gamma_\bullet}(\Omega_\bullet)$ is continuous.

ii) *If we denote the anti-linear form*

$$H^{1,2}_{\Gamma_\bullet}(\Omega_\bullet) \ni w \mapsto \int_{\Omega_\bullet} v\mu_\bullet \nabla\eta \cdot \nabla\overline{w}\,\mathrm{dx}$$

by I_v, *then* $u := \eta v|_{\Omega_\bullet}$ *satisfies*

$$-\nabla\cdot\mu_\bullet\nabla u = -\mu_\bullet\nabla v|_{\Omega_\bullet}\cdot\nabla\eta|_{\Omega_\bullet} + I_v + f_\bullet.$$

iii) *For every* $q \geq 2$ *and all* $r \in [2, q^*[$ *(*q^* *denoting again the Sobolev conjugated index of* q*) the mapping*

$$H^{1,q}_\Gamma(\Omega) \ni v \mapsto -\mu_\bullet\nabla v|_{\Omega_\bullet}\cdot\nabla\eta|_{\Omega_\bullet} + I_v \in \breve{H}^{-1,r}_{\Gamma_\bullet}(\Omega_\bullet)$$

is well defined and continuous.

Remark 5.13. It is the lack of integrability for the gradient of v (see the counterexample in [23, Ch. 4]) together with the quality of the needed Sobolev embeddings which limits the quality of the correction terms. In the end it is this effect which prevents the applicability of the localization procedure in Subsection 5.2 in higher dimensions – at least when one aims at a $q > d$.

Remark 5.14. If $v \in L^2(\Omega)$ is a regular distribution, then $v_\bullet$ is the regular distribution $(\eta v)|_{\Omega_\bullet}$.

5.2. **Core of the proof of Theorem 5.3.** We are now in the position to start the proof of Theorem 5.3. We first note that in any case the operator $-\nabla\cdot\mu\nabla$ admits maximal parabolic regularity on the Hilbert space $\breve{H}^{-1,2}_\Gamma$, since its negative generates an analytic semigroup on this space by Proposition 3.5, cf. Remark 5.2 iii). Thus, defining

$$M_{\mathrm{MR}} := \{q \geq 2 \,:\, -\nabla\cdot\mu\nabla \text{ admits maximal regularity on } \breve{H}^{-1,q}_\Gamma\}$$

and exploiting (3.1) and Lemma 5.18 we see by interpolation that M_{MR} is $\{2\}$ or an interval with left endpoint 2.

The main step of the proof for Theorem 5.3 is contained in the following lemma.

Lemma 5.15. *Let* Ω, Γ, Υ, η, $\Omega_\bullet$, $\Gamma_\bullet$, $\mu_\bullet$ *be as before. Assume that* $-\nabla\cdot\mu_\bullet\nabla$ *satisfies maximal parabolic regularity on* $\breve{H}^{-1,q}_{\Gamma_\bullet}(\Omega_\bullet)$ *for all* $q \in [2,\infty[$ *and that* $-\nabla\cdot\mu\nabla$ *satisfies maximal parabolic regularity on* $\breve{H}^{-1,q_0}_\Gamma(\Omega)$ *for some* $q_0 \in [2, q_{\mathrm{iso}}[$. *If* $r \in [q_0, q_0^*[$ *and* $G \in L^s(J;\breve{H}^{-1,r}_\Gamma(\Omega)) \hookrightarrow L^s(J;\breve{H}^{-1,q_0}_\Gamma(\Omega))$, *then the unique solution* $V \in W^{1,s}(J;\breve{H}^{-1,q_0}_\Gamma(\Omega)) \cap L^s(J; dom_{\breve{H}^{-1,q_0}_\Gamma(\Omega)}(-\nabla\cdot\mu\nabla))$ *of*

$$V' - \nabla\cdot\mu\nabla V = G, \qquad V(T_0) = 0,$$

even satisfies

$$\eta V \in W^{1,s}(J;\breve{H}^{-1,r}_\Gamma(\Omega)) \cap L^s(J; dom_{\breve{H}^{-1,r}_\Gamma(\Omega)}(-\nabla\cdot\mu\nabla)).$$

Proof of Theorem 5.3. For every $\mathrm{x} \in \Omega$ let $\Xi_{\mathrm{x}} \subseteq \Omega$ be an open cube, containing x. Furthermore, let for any point $\mathrm{x} \in \partial\Omega$ an open neighborhood be given according to the supposition of the theorem (see Assumption 3.1). Possibly shrinking this neighborhood to a smaller one, one obtains a new neighborhood Υ_{x}, and a bi-Lipschitz, volume-preserving mapping ϕ_{x} from a neighborhood of $\overline{\Upsilon_{\mathrm{x}}}$ into $\mathbb{R}^d$ such that $\phi_{\mathrm{x}}(\Upsilon_{\mathrm{x}} \cap (\Omega \cup \Gamma)) = \beta K_-$, $\beta(K_- \cup \Sigma)$ or $\beta(K_- \cup \Sigma_0)$ for some $\beta = \beta(\mathrm{x}) > 0$.

Obviously, the Ξ_{x} and Υ_{x} together form an open covering of $\overline{\Omega}$. Let $\Xi_{\mathrm{x}_1}, \dots, \Xi_{\mathrm{x}_k}$, $\Upsilon_{\mathrm{x}_{k+1}}, \dots, \Upsilon_{\mathrm{x}_l}$ be a finite subcovering and $\eta_1, \dots, \eta_l$ a C^∞ partition of unity, subordinate to this subcovering. Set $\Omega_j := \Xi_{\mathrm{x}_j} = \Xi_{\mathrm{x}_j} \cap \Omega$ for $j \in \{1, \dots, k\}$ and $\Omega_j := \Upsilon_{\mathrm{x}_j} \cap \Omega$ for $j \in \{k+1, \dots, l\}$. Moreover, set $\Gamma_j := \emptyset$ for $j \in \{1, \dots, k\}$ and $\Gamma_j := \Upsilon_{\mathrm{x}_j} \cap \Gamma$ for $j \in \{k+1, \dots, l\}$.

Denoting the restriction of μ to Ω_j by μ_j, each operator $-\nabla \cdot \mu_j \nabla$ satisfies maximal parabolic regularity in $\breve{H}^{-1,q}_{\Gamma_j}(\Omega_j)$ for all $q \in [2, \infty[$ and all j, according to Theorem 5.8. Thus, we may apply Lemma 5.15, the first time taking $q_0 = 2$. Then $-\nabla \cdot \mu \nabla$ satisfies maximal parabolic regularity on $\breve{H}^{-1,r}_\Gamma(\Omega)$ for all $r \in [2, 2^*[$. Next taking q_0 as any number from the interval $[2, \min(2^*, q_{\mathrm{iso}})[$ and continuing this way, one improves the information on r step by step. Since the augmentation in r increases in every step, any number below q^*_{iso} is indeed achieved. □

Remark 5.16. Note that Theorem 5.3 already yields maximal regularity of $-\nabla \cdot \mu \nabla$ on $\breve{H}^{-1,q}_\Gamma$ for all $q \in [2, 2^*[$ without any additional information on $dom_{\breve{H}^{-1,q}_\Gamma}(-\nabla \cdot \mu \nabla)$ nor on $dom_{\breve{H}^{-1,q}_{\Gamma_j}(\Omega_j)}(-\nabla \cdot \mu_j \nabla)$.

In the 2-d case this already implies maximal regularity for every $q \in [2, \infty[$. Taking into account Remark 5.4 i), without further knowledge on the domains we get in the 3-d case every $q \in [2, 6 + \varepsilon[$ and in the 4-d case every $q \in [2, 4 + \varepsilon[$, where ε depends on Ω, Γ, μ.

5.3. **The operator** A. Next we carry over the maximal parabolic regularity result, up to now proved for $-\nabla \cdot \mu \nabla$ on the spaces $\breve{H}^{-1,q}_\Gamma$, to the operator A and to a much broader class of distribution spaces. For this we first need the following perturbation result on relative boundedness of the boundary part of the operator A.

Lemma 5.17. *Suppose $q \geq 2$, $\varsigma \in]1 - \frac{1}{q}, 1]$ and $\varkappa \in L^\infty(\Gamma, \mathrm{d}\sigma)$ and let Ω, Γ satisfy Assumption 3.1. If we define the mapping $Q : dom_{\breve{H}^{-\varsigma,q}_\Gamma}(-\nabla \cdot \mu \nabla) \to \breve{H}^{-\varsigma,q}_\Gamma$ by*

$$\langle Q\psi, \varphi \rangle_{\breve{H}^{-\varsigma,q}_\Gamma} := \int_\Gamma \varkappa \, \psi \, \overline{\varphi} \, \mathrm{d}\sigma, \quad \varphi \in H^{\varsigma,q'}_\Gamma,$$

then Q is well defined and continuous. Moreover, it is relatively bounded with respect to $-\nabla \cdot \mu \nabla$, when considered on the space $\breve{H}^{-\varsigma,q}_\Gamma$, and the relative bound may be taken arbitrarily small.

The instrument which will allow us to carry over maximal parabolic regularity from L^q and $\breve{H}^{-1,q}_\Gamma$ to various distribution spaces is the following interpolation result:

Lemma 5.18. *Suppose that X, Y are Banach spaces, which are contained in a third Banach space Z with continuous injections. Let B be a linear operator on Z whose restrictions to each of the spaces X, Y induce closed, densely defined operators there. Assume that the induced operators fulfill maximal parabolic regularity on X*

and Y, respectively. Then B satisfies maximal parabolic regularity on each of the interpolation spaces $[X,Y]_\theta$ and $(X,Y)_{\theta,s}$ with $\theta \in \,]0,1[$ and $s \in \,]1,\infty[$.

Having this property at hand, we can prove our main result for the operator A.

Theorem 5.19. *Suppose $q \geq 2$, $\varkappa \in L^\infty(\Gamma, \mathrm{d}\sigma)$ and let Ω, Γ satisfy Assumption 3.1.*

i) *If $\varsigma \in \,]1-\frac{1}{q}, 1]$, then $\mathrm{dom}_{\breve{H}_\Gamma^{-\varsigma,q}}(-\nabla \cdot \mu\nabla) = \mathrm{dom}_{\breve{H}_\Gamma^{-\varsigma,q}}(A)$.*

ii) *If $\varsigma \in \,]1-\frac{1}{q}, 1]$ and $-\nabla \cdot \mu\nabla$ satisfies maximal parabolic regularity on $\breve{H}_\Gamma^{-\varsigma,q}$, then A also does.*

iii) *The operator A satisfies maximal parabolic regularity on L^2. If $\varkappa \geq 0$, then A satisfies maximal parabolic regularity on L^p for all $p \in \,]1,\infty[$.*

iv) *Suppose that $-\nabla \cdot \mu\nabla$ satisfies maximal parabolic regularity on $\breve{H}_\Gamma^{-1,q}$. Then A satisfies maximal parabolic regularity on any of the interpolation spaces*

$$[L^2, \breve{H}_\Gamma^{-1,q}]_\theta, \quad \theta \in [0,1],$$

and

$$(L^2, \breve{H}_\Gamma^{-1,q})_{\theta,s}, \quad \theta \in [0,1], \; s \in \,]1,\infty[\,.$$

Let $\varkappa \geq 0$ and $p \in \,]1,\infty[$ in case of $d = 2$ or $p \in [\left(\frac{1}{2}+\frac{1}{d}\right)^{-1}, \infty[$ if $d \geq 3$. Then A also satisfies maximal parabolic regularity on any of the interpolation spaces

$$[L^p, \breve{H}_\Gamma^{-1,q}]_\theta, \quad \theta \in [0,1], \tag{5.5}$$

and

$$(L^p, \breve{H}_\Gamma^{-1,q})_{\theta,s}, \quad \theta \in [0,1], \; s \in \,]1,\infty[\,. \tag{5.6}$$

Proof. i) follows from Lemma 5.17 and classical perturbation theory.

ii) The assertion is implied by Lemma 5.17 and a perturbation theorem for maximal parabolic regularity, see [6, Prop. 1.3].

iii) The first assertion follows from Proposition 3.5 ii) and Remark 5.2 iii). The second is shown in [32, Thm. 7.4].

iv) Under the given conditions on p, we have the embedding $L^p \hookrightarrow \breve{H}_\Gamma^{-1,2}$. Thus, the assertion follows from the preceding points and Lemma 5.18. □

Remark 5.20. The interpolation spaces $[L^p, H_\Gamma^{-1,q}]_\theta$, $\theta \in [0,1]$, and $(L^p, H_\Gamma^{-1,q})_{\theta,s}$, $\theta \in [0,1]$, $s \in \,]1,\infty[$, are characterized in [30], see in particular Remark 3.6 therein. Identifying each $f \in L^q$ with the anti-linear form $L^{q'} \ni \psi \to \int_\Omega f\overline{\psi}\,\mathrm{d}x$ and using the retraction/coretraction theorem with the coretraction which assigns to $f \in \breve{H}_\Gamma^{-1,r}$ the linear form $H_\Gamma^{1,r'} \ni \psi \to \langle f, \overline{\psi}\rangle_{\breve{H}_\Gamma^{-1,r}}$, one easily identifies the interpolation spaces in (5.5) and (5.6). In particular, this yields $\left[L^{q_0}, \breve{H}_\Gamma^{-1,q_1}\right]_\theta = \breve{H}_\Gamma^{-\theta,q}$ if $\theta \neq 1-\frac{1}{q}$.

Corollary 5.21. *Let Ω and Γ satisfy Assumption 3.1. The operator $-A$ generates analytic semigroups on all spaces $\breve{H}_\Gamma^{-1,q}$ if $q \in [2, q_{\mathrm{iso}}^*[$ and on all the interpolation* *spaces occurring in Theorem 5.19, there q also taken from $[2, q^*_{\rm iso}[$. Moreover, if $\varkappa \geq 0$, the following resolvent estimates are valid:*

$$\|(A+1+\lambda)^{-1}\|_{\mathcal{L}(\breve{H}_\Gamma^{-1,q})} \leq \frac{c_q}{1+|\lambda|}, \quad \operatorname{Re}\lambda \geq 0.$$

6. Nonlinear parabolic equations

In this section we will apply maximal parabolic regularity for the treatment of quasilinear parabolic equations which are of the (formal) type (1.1). Concerning all the occurring operators we will formulate precise requirements in Assumption 6.11 below.

The outline of the section is as follows: First we give a motivation for the choice of the Banach space we will regard (1.1)/(1.2) in. Afterwards we show that maximal parabolic regularity, combined with regularity results for the elliptic operator, allows to solve this problem. Below we will consider (1.1)/(1.2) as a quasilinear problem

$$\left\{\begin{aligned} u'(t) + \mathcal{B}\big(u(t)\big)u(t) &= \mathcal{S}(t, u(t)), \quad t \in J, \\ u(T_0) &= u_0. \end{aligned}\right. \tag{6.1}$$

To give the reader already here an idea what properties of the operators $-\nabla \cdot \mathcal{G}(u)\mu\nabla$ and of the corresponding Banach space are required, we first quote the result on existence and uniqueness for abstract quasilinear parabolic equations (due to Clément/Li [15] and Prüss [50]) on which our subsequent considerations will base.

Proposition 6.1. *Suppose that B is a closed operator on some Banach space X with dense domain D, which satisfies maximal parabolic regularity on X. For some $s > 1$ suppose further $u_0 \in (X, D)_{1-\frac{1}{s},s}$ and $\mathcal{B} : J \times (X, D)_{1-\frac{1}{s},s} \to \mathcal{L}(D, X)$ to be continuous with $B = \mathcal{B}(T_0, u_0)$. Let, in addition, $\mathcal{S} : J \times (X, D)_{1-\frac{1}{s},s} \to X$ be a Carathéodory map and assume the following Lipschitz conditions on $\mathcal{B}$ and $\mathcal{S}$:*

(**B**) *For every $M > 0$ there exists a constant $C_M > 0$, such that for all $t \in J$*

$$\|\mathcal{B}(t, u) - \mathcal{B}(t, \tilde{u})\|_{\mathcal{L}(D,X)} \leq C_M \, \|u - \tilde{u}\|_{(X,D)_{1-\frac{1}{s},s}} \ \text{ if } \|u\|_{(X,D)_{1-\frac{1}{s},s}}, \|\tilde{u}\|_{(X,D)_{1-\frac{1}{s},s}} \leq M.$$

(**S**) *$\mathcal{S}(\cdot, 0) \in L^s(J; X)$ and for each $M > 0$ there is a function $h_M \in L^s(J)$, such that*

$$\|\mathcal{S}(t, u) - \mathcal{S}(t, \tilde{u})\|_X \leq h_M(t) \, \|u - \tilde{u}\|_{(X,D)_{1-\frac{1}{s},s}}$$

holds for a.a. $t \in J$, if $\|u\|_{(X,D)_{1-\frac{1}{s},s}}, \|\tilde{u}\|_{(X,D)_{1-\frac{1}{s},s}} \leq M$.

Then there exists $T^* \in J$, such that (6.1) admits a unique solution u on $]T_0, T^*[$ satisfying

$$u \in W^{1,s}(]T_0, T^*[; X) \cap L^s(]T_0, T^*[; D).$$

Remark 6.2. Up to now we were free to consider complex Banach spaces. But the context of equations like (1.1) requires real spaces, in particular in view of the quality of the operator $\mathcal{G}$ which often is a superposition operator. Therefore, from this moment on we use the real versions of the spaces. In particular, $H_\Gamma^{-\varsigma,q}$ is now understood as the dual of the real space $H_\Gamma^{\varsigma,q'}$ and clearly can be identified with the set of anti-linear forms on the complex space $H_\Gamma^{\varsigma,q'}$ that take real values when applied to real functions.

Fortunately, the property of maximal parabolic regularity is maintained for the restriction of the operator A to the real spaces in case of a real function $\varkappa$, as A then commutes with complex conjugation.

We will now give a motivation for the choice of the Banach space X we will use later. In view of the applicability of Proposition 6.1 and the non-smooth characteristic of (1.1)/(1.2) it is natural to require the following properties.

a) The operators A, or at least the operators $-\nabla \cdot \mu\nabla$, defined in (3.2), must satisfy maximal parabolic regularity on X.
b) As in the classical theory (see [45], [28], [54] and references therein) quadratic gradient terms of the solution should be admissible for the right-hand side.
c) The operators $-\nabla \cdot \mathcal{G}(u)\mu\nabla$ should behave well concerning their dependence on u, see condition (**B**) above.
d) X has to contain certain measures, supported on Lipschitz hypersurfaces in Ω or on $\partial\Omega$ in order to allow for surface densities on the right-hand side or/and for inhomogeneous Neumann conditions.

The condition in a) is assured by Theorems 5.3 and 5.19 for a great variety of Banach spaces, among them candidates for X. Requirement b) suggests that one should have $dom_X(-\nabla \cdot \mu\nabla) \hookrightarrow H_\Gamma^{1,q}$ and $L^{\frac{q}{2}} \hookrightarrow X$. Since $-\nabla \cdot \mu\nabla$ maps $H_\Gamma^{1,q}$ into $H_\Gamma^{-1,q}$, this altogether leads to the necessary condition

$$L^{\frac{q}{2}} \hookrightarrow X \hookrightarrow H_\Gamma^{-1,q}. \tag{6.2}$$

Sobolev embedding shows that q cannot be smaller than the space dimension d. Taking into account d), it is clear that X must be a space of distributions which (at least) contains surface densities. In order to recover the desired property $dom_X(-\nabla \cdot \mu\nabla) \hookrightarrow H_\Gamma^{1,q}$ from the necessary condition in (6.2), we make for all what follows this general

Assumption 6.3. *There is a $q > d$, such that $-\nabla \cdot \mu\nabla + 1 : H_\Gamma^{1,q} \to H_\Gamma^{-1,q}$ is a topological isomorphism.*

Remark 6.4. By Remark 5.4 i) Assumption 6.3 is always fulfilled for $d = 2$. On the other hand for $d \geq 4$ it is generically false in case of mixed boundary conditions, see [52] for the famous counterexample. Moreover, even in the Dirichlet case, when the domain Ω has only a Lipschitz boundary or the coefficient function μ is constant within layers, one cannot expect $q \geq 4$, see [43] and [22].

In Section 7 we will present examples for domains Ω, coefficient functions μ and Dirichlet boundary parts $\Omega \setminus \Gamma$, for which Assumption 6.3 is fulfilled.

From now on we fix some $q > d$, for which Assumption 6.3 holds.

As a first step, one shows that Assumption 6.3 carries over to a broad class of modified operators:

Lemma 6.5. *Assume that ξ is a real valued, uniformly continuous function on Ω that admits a lower bound $\underline{\xi} > 0$. Then the operator $-\nabla\cdot\xi\mu\nabla+1$ also is a topological isomorphism between $H^{1,q}_\Gamma$ and $H^{-1,q}_\Gamma$.*

In this spirit, one could now suggest $X := H^{-1,q}_\Gamma$ to be a good choice for the Banach space, but in view of condition (**S**) the right-hand side of (6.1) has to be a continuous mapping from an interpolation space $(dom_X(A), X)_{1-\frac{1}{s},s}$ into X. Chosen $X := H^{-1,q}_\Gamma$, for elements $\psi \in (dom_X(A), X)_{1-\frac{1}{s},s} = (H^{1,q}_\Gamma, H^{-1,q}_\Gamma)_{1-\frac{1}{s},s}$ the expression $|\nabla\psi|^2$ cannot be properly defined and, if so, will not lie in $H^{-1,q}_\Gamma$ in general. This shows that $X := H^{-1,q}_\Gamma$ is not an appropriate choice, but we will see that $X := H^{-\varsigma,q}_\Gamma$, with ς properly chosen, is.

Lemma 6.6. *Put $X := H^{-\varsigma,q}_\Gamma$ with $\varsigma \in [0,1[\setminus \{\frac{1}{q}, 1-\frac{1}{q}\}$. Then*

i) *For every $\tau \in]\frac{1+\varsigma}{2}, 1[$ there is a continuous embedding $(X, dom_X(-\nabla\cdot\mu\nabla))_{\tau,1} \hookrightarrow H^{1,q}_\Gamma$.*
ii) *If $\varsigma \in [\frac{d}{q}, 1]$, then X has a predual $X_* = H^{\varsigma,q'}_\Gamma$ which admits the continuous, dense injections $H^{1,q'}_\Gamma \hookrightarrow X_* \hookrightarrow L^{(\frac{q}{2})'}$ that by duality clearly imply (6.2). Furthermore, $H^{1,q}_\Gamma$ is a multiplier space for X_*.*

Next we will consider requirement c), see condition (**B**) in Proposition 6.1.

Lemma 6.7. *Let q be a number from Assumption 6.3 and let X be a Banach space with predual X_* that admits the continuous and dense injections*

$$H^{1,q'}_\Gamma \hookrightarrow X_* \hookrightarrow L^{(\frac{q}{2})'}.$$

i) *If $\xi \in H^{1,q}$ is a multiplier on X_*, then $dom_X(-\nabla\cdot\mu\nabla) \hookrightarrow dom_X(-\nabla\cdot\xi\mu\nabla)$.*
ii) *If $H^{1,q}$ is a multiplier space for X_*, then the (linear) mapping $H^{1,q} \ni \xi \mapsto -\nabla\cdot\xi\mu\nabla \in \mathcal{L}(dom_X(-\nabla\cdot\mu\nabla), X)$ is well defined and continuous.*

Corollary 6.8. *If ξ additionally to the hypotheses of Lemma 6.7 i) has a positive lower bound, then*

$$dom_X(-\nabla\cdot\xi\mu\nabla) = dom_X(-\nabla\cdot\mu\nabla).$$

Next we will show that functions on $\partial\Omega$ or on a Lipschitz hypersurface, which belong to a suitable summability class, can be understood as elements of the distribution space $H^{-\varsigma,q}_\Gamma$.

Theorem 6.9. *Assume $q \in]1,\infty[$, $\varsigma \in]1-\frac{1}{q}, 1[\setminus \{\frac{1}{q}\}$ and let Π, ϖ be as in Proposition 3.3. Then the adjoint trace operator $(\mathrm{Tr})^*$ maps $L^q(\Pi)$ continuously into $\big(H^{\varsigma,q'}(\Omega)\big)' \hookrightarrow H^{-\varsigma,q}_\Gamma$.*

Proof. The result is obtained from Proposition 3.3 by duality. □

Remark 6.10. Here we restricted the considerations to the case of Lipschitz hypersurfaces, since this is the most essential insofar as it gives the possibility of prescribing jumps in the normal component of the current $j := \mathcal{G}(u)\mu\nabla u$ along hypersurfaces where the coefficient function jumps. This case is of high relevance in view of applied problems and has attracted much attention also from the numerical point of view, see e.g. [1], [11] and references therein.

From now on we fix once and for all a number $\varsigma \in \,]\max\{1-\frac{1}{q},\frac{d}{q}\},1[$ and set for all what follows $X := H_\Gamma^{-\varsigma,q}$.

Next we introduce the requirements on the data of problem (1.1)/(1.2).

Assumption 6.11. **Op)** *For all what follows we fix a number* $s > \frac{2}{1-\varsigma}$.

Ga) *The mapping* $\mathcal{G} : H^{1,q} \to H^{1,q}$ *is locally Lipschitz continuous.*

Gb) *For any ball in* $H^{1,q}$ *there exists* $\delta > 0$*, such that* $\mathcal{G}(u) \ge \delta$ *for all* u *from this ball.*

Ra) *The function* $\mathcal{R} : J \times H^{1,q} \to X$ *is of Carathéodory type, i.e.* $\mathcal{R}(\cdot,u)$ *is measurable for all* $u \in H^{1,q}$ *and* $\mathcal{R}(t,\cdot)$ *is continuous for a.a.* $t \in J$.

Rb) $\mathcal{R}(\cdot,0) \in L^s(J;X)$ *and for* $M > 0$ *there exists* $h_M \in L^s(J)$*, such that*

$$\|\mathcal{R}(t,u) - \mathcal{R}(t,\tilde{u})\|_X \le h_M(t)\|u-\tilde{u}\|_{H^{1,q}}, \quad t \in J,$$

provided $\max(\|u\|_{H^{1,q}}, \|\tilde{u}\|_{H^{1,q}}) \le M$.

BC) b *is an operator of the form* $b(u) = Q(b_\circ(u))$*, where* $b_\circ$ *is a (possibly nonlinear), locally Lipschitzian operator from* $C(\overline{\Omega})$ *into itself (see Lemma 5.17).*

Gg) $g \in L^q(\Gamma)$.

IC) $u_0 \in (X, dom_X(-\nabla\cdot\mu\nabla))_{1-\frac{1}{s},s}$.

Remark 6.12. i) At the first glance the choice of s seems indiscriminate. The point is, however, that generically in applications the explicit time dependence of the reaction term $\mathcal{R}$ is essentially bounded. Thus, in view of condition **Rb)** it is justified to take s as any arbitrarily large number, whose magnitude needs not to be controlled explicitely.

ii) Note that the requirement on $\mathcal{G}$ allows for nonlocal operators. This is essential if the current depends on an additional potential governed by an auxiliary equation, what is usually the case in drift-diffusion models, see [3], [26] or [51].

iii) The conditions **Ra)** and **Rb)** are always satisfied if $\mathcal{R}$ is a mapping into $L^{q/2}$ with the analog boundedness and continuity properties, see Lemma 6.6 ii).

iv) It is not hard to see that Q in fact is well defined on $C(\overline{\Omega})$, therefore condition **BC)** makes sense. In particular, $b_\circ$ may be a superposition operator, induced by a $C^1(\mathbb{R})$ function. Let us emphasize that in this case the inducing function needs not to be positive. Thus, non-dissipative boundary conditions are included.

v) Finally, the condition **IC)** is an 'abstract' one and hardly to verify, because one has no explicit characterization of $(X, dom_X(-\nabla\cdot\mu\nabla))_{1-\frac{1}{s},s}$ at hand. Nevertheless, the condition is reproduced along the trajectory of the solution by means of the embedding (5.1).

In order to solve (1.1)/(1.2), we will consider (6.1) with

$$\mathcal{B}(u) := -\nabla\cdot\mathcal{G}(u)\mu\nabla \tag{6.3}$$

and the right-hand side $\mathcal{S}$

$$\mathcal{S}(t,u) := \mathcal{R}(t,u) - Q(b_\circ(u)) + (\mathrm{Tr})^* g, \tag{6.4}$$

seeking the solution in the space $W^{1,s}(J;X) \cap L^s(J; dom_X(-\nabla\cdot\mu\nabla))$.

Remark 6.13. Let us explain this reformulation: as it is well known in the theory of boundary value problems, the boundary condition (1.2) is incorporated by introducing the boundary terms $-\varkappa b_\circ(u)$ and g on the right-hand side. In order to understand both as elements from X, we write $Q(b_\circ(u))$ and $(\mathrm{Tr})^* g$, see Lemma 5.17 and Theorem 6.9.

Theorem 6.14. *Let Assumption 6.3 be satisfied and assume that the data of the problem satisfy Assumption 6.11. Then* (6.1) *has a local in time, unique solution in* $W^{1,s}(J;X) \cap L^s(J; dom_X(-\nabla\cdot\mu\nabla))$*, provided that* $\mathcal{B}$ *and* $\mathcal{S}$ *are given by* (6.3) *and* (6.4)*, respectively.*

Proof. First of all we note that, due to **Op)**, $1-\frac{1}{s} > \frac{1+\varsigma}{2}$. Thus, if $\tau \in \left]\frac{1+\varsigma}{2}, 1-\frac{1}{s}\right[$ by a well-known interpolation result (see [55, Ch. 1.3.3]) and Lemma 6.6 i) we have

$$(X, dom_X(-\nabla\cdot\mu\nabla))_{1-\frac{1}{s},s} \hookrightarrow (X, dom_X(-\nabla\cdot\mu\nabla))_{\tau,1} \hookrightarrow H^{1,q}. \tag{6.5}$$

Hence, by **IC)**, $u_0 \in H^{1,q}$. Consequently, due to the suppositions on $\mathcal{G}$, both the functions $\mathcal{G}(u_0)$ and $\frac{1}{\mathcal{G}(u_0)}$ belong to $H^{1,q}$ and are bounded from below by a positive constant. Denoting $-\nabla\cdot\mathcal{G}(u_0)\mu\nabla$ by B, Corollary 6.8 gives $dom_X(-\nabla\cdot\mu\nabla) = dom_X(B)$. This implies $u_0 \in (X, dom_X(B))_{1-\frac{1}{s},s}$. Furthermore, the so defined B has maximal parabolic regularity on X, thanks to (5.5) in Theorem 5.19 with $p=q$.

Condition (**B**) from Proposition 6.1 is implied by Lemma 6.7 ii) in cooperation with Lemma 6.6, the fact that the mapping $H^{1,q} \ni \phi \mapsto \mathcal{G}(\phi) \in H^{1,q}$ is boundedly Lipschitz and (6.5).

It remains to show that the 'new' right-hand side $\mathcal{S}$ satisfies condition (**S**) from Proposition 6.1. We do this for every term in (6.4) separately, beginning from the left: concerning the first, one again uses (6.5) together with the asserted conditions **Ra)** and **Rb)** on $\mathcal{R}$. The assertion for the last two terms results from (6.5), the assumptions **BC)/Gg)**, Lemma 5.17 and Theorem 6.9. ☐

Remark 6.15. Note that, if $\mathcal{R}$ takes its values only in the space $L^{q/2} \hookrightarrow X$, then – in the light of Lemma 5.17 – the elliptic operators incorporate the boundary conditions (1.2) in a generalized sense, see [27, Ch. II.2] or [14, Ch. 1.2].

Finally, it can be shown that the solution u is Hölder continuous simultaneously in space and time, even more:

Corollary 6.16. *There exist* $\alpha, \beta > 0$ *such that the solution* u *of* (1.1)/(1.2) *belongs to the space* $C^\beta(J; H^{1,q}_\Gamma(\Omega)) \hookrightarrow C^\beta(J; C^\alpha(\Omega))$.

7. Examples

In this section we describe geometric configurations for which our Assumption 6.3 holds true and we present concrete examples of mappings $\mathcal{G}$ and reaction terms $\mathcal{R}$ fitting into our framework.

7.1. **Geometric constellations.** While our results in Sections 4 and 5 on the square root of $-\nabla \cdot \mu\nabla$ and maximal parabolic regularity are valid in the general geometric framework of Assumption 3.1, we additionally had to impose Assumption 6.3 for the treatment of quasilinear equations in Section 6. Here we shortly describe geometric constellations, in which this additional condition is satisfied.

Let us start with the observation that the 2-d case is covered by Remark 5.4 i). Admissible three-dimensional settings may be described as follows.

Proposition 7.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exists a $q > 3$ such that $-\nabla \cdot \mu\nabla + 1$ is a topological isomorphism from $H^{1,q}_\Gamma$ onto $H^{-1,q}_\Gamma$, if one of the following conditions is satisfied:*

i) *Ω has a Lipschitz boundary. $\Gamma = \emptyset$ or $\Gamma = \partial\Omega$. $\Omega_\circ \subseteq \Omega$ is another domain which is C^1 and which does not touch the boundary of Ω. $\mu|_{\Omega_\circ} \in BUC(\Omega_\circ)$ and $\mu|_{\Omega\setminus\overline{\Omega_\circ}} \in BUC(\Omega \setminus \overline{\Omega_\circ})$.*
ii) *Ω has a Lipschitz boundary. $\Gamma = \emptyset$. $\Omega_\circ \subseteq \Omega$ is a Lipschitz domain, such that $\partial\Omega_\circ \cap \Omega$ is a C^1 surface and $\partial\Omega$ and $\partial\Omega_\circ$ meet suitably (see [23] for details). $\mu|_{\Omega_\circ} \in BUC(\Omega_\circ)$ and $\mu|_{\Omega\setminus\overline{\Omega_\circ}} \in BUC(\Omega \setminus \overline{\Omega_\circ})$.*
iii) *Ω is a three-dimensional Lipschitzian polyhedron. $\Gamma = \emptyset$. There are hyperplanes $\mathcal{H}_1, \dots, \mathcal{H}_n$ in $\mathbb{R}^3$ which meet at most in a vertex of the polyhedron such that the coefficient function μ is constantly a real, symmetric, positive definite 3×3 matrix on each of the connected components of $\Omega \setminus \cup_{l=1}^n \mathcal{H}_l$. Moreover, for every edge on the boundary, induced by a hetero interface $\mathcal{H}_l$, the angles between the outer boundary plane and the hetero interface do not exceed π and at most one of them may equal π.*
iv) *Ω is a convex polyhedron. $\overline{\Gamma} \cap (\partial\Omega \setminus \Gamma)$ is a finite union of line segments. $\mu \equiv 1$.*
v) *$\Omega \subseteq \mathbb{R}^3$ is a prismatic domain with a triangle as basis. Γ equals either one half of one of the rectangular sides or one rectangular side or two of the three rectangular sides. There is a plane which intersects Ω such that the coefficient function μ is constant above and below the plane.*
vi) *Ω is a bounded domain with Lipschitz boundary. Additionally, for each $\mathrm{x} \in \overline{\Gamma} \cap (\partial\Omega \setminus \Gamma)$ the mapping ϕ_x defined in Assumption 3.1 is a C^1-diffeomorphism from Υ_x onto its image. $\mu \in BUC(\Omega)$.*

The assertions i) and ii) are shown in [23], while iii) is proved in [22] and iv) is a result of Dauge [17]. Recently, v) was obtained in [37] and vi) will be published in a forthcoming paper. □

Remark 7.2. The assertion remains true, if there is a finite open covering $\Upsilon_1, \dots, \Upsilon_l$ of $\overline{\Omega}$, such that each of the pairs $\Omega_j := \Upsilon_j \cap \Omega$, $\Gamma_j := \Gamma \cap \Upsilon_j$ fulfills one of the points i) – vi) after a volume-preserving bi-Lipschitz transformation.

This provides a huge zoo of geometries and boundary constellations, for which $-\nabla \cdot \mu\nabla$ provides the required isomorphism. We intend to complete this in the future.

7.2. **Nonlinearities and reaction terms.** The most common case is that where $\mathcal{G}$ is the exponential or the Fermi-Dirac distribution function $\mathcal{F}_{1/2}$ given by

$$\mathcal{F}_{1/2}(t) := \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{s}}{1+\mathrm{e}^{s-t}} \, \mathrm{d}s.$$

As a second example we present a nonlocal operator arising in the diffusion of bacteria; see [12], [13] and references therein.

Example 7.3. Let ζ be a continuously differentiable function on $\mathbb{R}$ which is bounded from above and below by positive constants. Assume $\varphi \in L^2(\Omega)$ and define

$$\mathcal{G}(u) := \zeta\Big(\int_\Omega u\varphi \, \mathrm{dx}\Big), \quad u \in H^{1,q}.$$

Now we give an example for a mapping $\mathcal{R}$.

Example 7.4. Assume $\iota : \mathbb{R} \to]0, \infty[$ to be a continuously differentiable function. Furthermore, let $\mathcal{T} : H^{1,q} \to H^{1,q}$ be the mapping which assigns to $v \in H^{1,q}$ the solution φ of the elliptic problem (including boundary conditions)

$$-\nabla \cdot \iota(v)\nabla\varphi = 0. \tag{7.1}$$

If one defines

$$\mathcal{R}(v) = \iota(v)|\nabla(\mathcal{T}(v))|^2,$$

then, under reasonable suppositions on the data of (7.1), the mapping $\mathcal{R}$ satisfies Assumption **Ra)**.

The example comes from a model which describes electrical heat conduction; see [5] and the references therein.

8. Concluding Remarks

Remark 8.1. Under the additional Assumption 6.3, Theorem 5.3 implies maximal parabolic regularity for $-\nabla \cdot \mu\nabla$ on $H_\Gamma^{-1,q}$ for every $q \in [2, \infty[$, as in the 2-d case.

Besides, the question arises whether the limitation for the exponents, caused by the localization procedure, is principal in nature or may be overcome when applying alternative ideas and techniques (cf. Theorem 4.4). We suggest that the latter is the case.

Remark 8.2. Equations of type (1.1)/(1.2) may be treated in an analogous way, if under the time derivative a suitable superposition operator is present, see [39] for details.

Remark 8.3. In the semilinear case, it turns out that one can achieve satisfactory results without Assumption 6.3, at least when the nonlinear term on the right-hand side depends only on the function itself and not on its gradient.

Remark 8.4. Let us explicitely mention that Assumption 6.3 is not always fulfilled in the 3-d case. First, there is the classical counterexample of Meyers, see [47], a simpler (and somewhat more striking) one is constructed in [22], see also [23]. The point, however, is that not the mixed boundary conditions are the obstruction but a somewhat 'irregular' behavior of the coefficient function μ in the inner of the domain. If one is confronted with this, spaces with weight may be the way out.

Remark 8.5. In two and three space dimensions one can give the following simplifying characterization for a set $\Omega \cup \Gamma$ to be regular in the sense of Gröger, i.e. to satisfy Assumption 3.1 a), see [38]:

If $\Omega \subseteq \mathbb{R}^2$ is a bounded Lipschitz domain and $\Gamma \subseteq \partial\Omega$ is relatively open, then $\Omega \cup \Gamma$ is regular in the sense of Gröger, iff $\partial\Omega \setminus \Gamma$ is the finite union of (non-degenerate) closed arc pieces.

In $\mathbb{R}^3$ the following characterization can be proved:

If $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain and $\Gamma \subset \partial\Omega$ is relatively open, then $\Omega \cup \Gamma$ is regular in the sense of Gröger, iff the following two conditions are satisfied:

i) $\partial\Omega \setminus \Gamma$ is the closure of its interior (within $\partial\Omega$).
ii) for any $\mathrm{x} \in \overline{\Gamma} \cap (\partial\Omega \setminus \Gamma)$ there is an open neighborhood $\mathcal{U} \ni \mathrm{x}$ and a bi-Lipschitz mapping $\kappa : \mathcal{U} \cap \overline{\Gamma} \cap (\partial\Omega \setminus \Gamma) \to]-1, 1[$.

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