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A priori error analysis for state constrained boundary control problems. Part II: Full discretization

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Abstract. This is the second of two papers concerned with a state-constrained optimal control problems with boundary control, where the state constraints are only imposed in an interior subdomain. We apply the virtual control concept introduced in [26] to regularize the problem. The arising regularized optimal control problem is discretized by finite elements and linear and continuous ansatz functions for the boundary control. In the first part of the work, we investigate the errors induced by the regularization and the discretization of the boundary control. The second part deals with the error arising from discretization of the PDE. Since the state constraints only appear in an inner subdomain, the obtained order of convergence exceeds the known results in the field of a priori analysis for state-constrained problems. The theoretical results are illustrated by numerical computations.

1. Introduction. This is the second of two papers, where we consider the following linear-quadratic optimal control problem with Neumann boundary control and pointwise state and control constraints:

$$\left.\begin{aligned} \min \quad J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} \\ -\Delta y + y = 0 \quad \text{in } \Omega \\ \partial_n y = u \quad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. on } \Gamma \\ y(x) \ge y_c(x) \quad \text{a.e. in } \Omega', \end{aligned}\right\} \tag{P}$$

where Ω' is an inner subdomain that is strictly contained in Ω. The precise hypothesis on the given quantities in (P) are given in Assumption 1.1 below.

It is well known that solutions to state-constrained optimal control problems in general provide a substantial lack of regularity caused by the low regularity of the Lagrange multipliers, see e.g. Casas [8]. Therefore, several regularization techniques have been developed to overcome this problem. Here, we use the *virtual control concept* introduced in [26] and refer to the first part of this work, where a more comprehensive overview over several regularization approaches is given. The regularized control problem reads as follows:

$$\left.\begin{aligned} \min \quad J_\varepsilon(y,u,v) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|v\|^2_{L^2(\Omega)} \\ -\Delta y + y = \phi(\varepsilon) v \quad \text{in } \Omega \\ \partial_n y = u \quad \text{on } \Gamma \\ u_a \le u(x) \le u_b \quad \text{a.e. on } \Gamma \\ y(x) \ge y_c(x) + \xi(\varepsilon) v(x) \quad \text{a.e. in } \Omega', \end{aligned}\right\} \tag{P$^\varepsilon$}$$

where v is the virtual control and $\varepsilon > 0$ a regularization parameter.

In both papers, we consider the finite element discretization of (P$^\varepsilon$) and the associated a priori analysis. In the first paper [25], we established estimates for the regularization error and the error induced the discretization of the boundary control by linear and continuous ansatz functions. Based on these results, we incorporate the finite element discretization of the PDE into the a priori analysis in this part of the work. In this way, we finally obtain an error estimate for the *full discretization* of (P$^\varepsilon$) that also involves the regularization error and therefore suggests a coupling of mesh size and regularization parameter ε.

The lack of regularity, typical for state-constrained problems, essential impairs the behavior of finite element discretizations. In the context of control-constrained optimal control problems, the finite element error analysis is well investigated, see for instance [11, 12, 13, 34] for the case of boundary controls as in (P). In the recent past, several authors turned to the a priori analysis for state-constrained problems with distributed control. We refer to Casas [9], where the authors dealt with finitely many state constraints. In [17] and [18] Deckelnick and Hinze established error estimates for the so-called variational discrete approach introduced in [22]. Both papers are concerned with linear-quadratic elliptic problems with distributed control. Finite element error estimates for the fully discretized case are developed in [29]. In [15], the underlying analysis is transferred to the state-constrained optimal control of the Stokes system. Problems with pointwise constraints on the gradient of the state are analyzed in [16]. Moreover, semilinear problems with state constraints are considered in [10] and [23].

Furthermore, a priori error analysis for problems with regularized pointwise state constraints is carried out in several other contributions. We refer to [14] and [28], where finite element error estimates for Lavrentiev regularized problems were established. Moreover, in [24], Hinze and Schiela investigated the finite element discretization in case of interior point regularization, whereas the a priori analysis for a Moreau Yosida based regularization of pointwise state constraints is considered in [20].
Here, we apply a technique which contains contributions of the analysis presented in [14, 29] and in [28]. In contrast to all contributions, mentioned so far, we derive error estimates for state-constrained optimal control problems with boundary controls. The state constraints in (P) are only imposed in an inner subdomain of Ω. By exploiting this property of the state constraints the order of convergence can be increased substantially. We point out that, in many applications, pointwise state constraints only appear in the interior of the respective domain. An example is mentioned in the first part [25]. These two aspects –the consideration of boundary controls as well as the special structure of the state constraints– represent the genuine contributions of the two papers. The final result contains a coupling of regularization parameter and mesh size that leads to the following error estimate:

$$\|\bar{u} - \bar{u}_h^\varepsilon\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h^\varepsilon\|_{L^2(\Omega)} \leq ch|\log h|^{1/2}, \tag{1.1}$$

where $(\bar{y}, \bar{u})$ is the solution to (P), while $(\bar{y}_h^\varepsilon, \bar{u}_h^\varepsilon)$ solves the discrete version of (P^ε). We emphasize that the above estimate is independent of the spatial dimension and thus also holds in the three dimensional case. With a minor modification, the theory also includes the case without regularization, i.e., a pure discretization of (P). The order of convergence in this case is same as in (1.1). Nevertheless, a regularization of (P) is reasonable to improve the performance of standard optimization algorithms, see Remark 5.5 below.
The paper is organized as follows: In Section 2, the original problem (P) is analyzed concerning first-order optimality conditions and several regularity and boundedness results for the optimal solution. Section 3 is devoted to the finite element discretization of the regularized problem (P^ε). Moreover, we establish several boundedness results for the discrete optimal solution based on a modification of the results in [7]. In Section 4, we construct feasible controls for the original problem (P) and for the discretized counterpart of (P^ε), respectively. This two-way feasibility is the basis for the following a priori error analysis at the end of Section 4. The purely discretized counterpart to the original problem is considered in Section 5. Here it is shown how to apply the theory for the discretization of (P^ε) to this case. Finally, Section 6 presents a numerical example illustrating the theoretical results.

1.1. Assumptions and Notations. Let us briefly introduce the main notations used throughout the paper. If X is a Banach space, we denote its dual by X^*, and the associated dual pairing is $\langle \cdot\,, \cdot \rangle_{X,X^*}$. The space of regular Borel measures over a domain Ω is denoted by $\mathcal{M}(\Omega)$. If Ω_1 and Ω_2 are two open, bounded domains in $\mathbb{R}^d$, then we mean by $\Omega_1 \subset\subset \Omega_2$ that Ω_1 is strictly contained in Ω_2, i.e.,

$$\operatorname{dist}\{\overline{\Omega_1}, \partial\Omega_2\} := \inf_{x \in \overline{\Omega_1}, y \in \partial\Omega_2} \|x - y\|_{\mathbb{R}^d} > 0,$$

where $\|\,.\,\|_{\mathbb{R}^d}$ is the Euclidian norm. Next we state the basic assumptions, we require for the discussion of (P) and (P^ε), respectively:
Assumption 1.1. *The domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is open, bounded, and convex with a polygonal ($d = 2$) or polyhedral ($d = 3$) boundary Γ. Moreover, $\Omega' \subset\subset \Omega$ is an inner subdomain. Furthermore, $y_d \in L^2(\Omega)$, and $y_c \in C^{0,1}(\bar{\Omega})$ are given functions and $u_a \leq u_b$, $\nu > 0$ are real numbers.*
Assumption 1.2. *The functions ψ, ϕ and ξ are positive and real valued.*

2. The purely state constrained problem. This section is devoted to the analysis of the original problem (P) concerning regularity results of the state equation and first order optimality conditions. Furthermore, we recall several boundedness results, derived in the first part of this work, for the optimal solution of problem (P).

2.1. Regularity for the state equation. We start with the definition of a solution operator associated with the state equation of problem (P). Introducing the corresponding bilinear form a :

$H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, the weak formulation of the state equation for an arbitrary element $f \in H^1(\Omega)^*$ is given by

$$a(y,z) := \int_\Omega (\nabla y \cdot \nabla z + yz)dx = \langle f\,,\,z\rangle_{H^1(\Omega)^*,H^1(\Omega)}, \quad \forall z \in H^1(\Omega). \tag{2.1}$$

The Lax-Milgram Lemma gives the existence of a unique solution to (2.1) in $H^1(\Omega)$ for every element $f \in H^1(\Omega)^*$. The corresponding linear and continuous solution operator is denoted by $G : H^1(\Omega)^* \to H^1(\Omega)$. Due to

$$\langle \tau^* u\,,\,z\rangle_{H^1(\Omega)^*,H^1(\Omega)} := (u\,,\,\tau z)_{L^2(\Gamma)} = \int_\Gamma u\tau z\,ds, \tag{2.2}$$

where $\tau : H^1(\Omega) \to L^2(\Gamma)$ denotes the trace operator, the control $u \in L^2(\Gamma)$ defines an element in $H^1(\Omega)^*$. Since the space $H^1(\Omega)$ is continuously embedded in $L^2(\Omega)$, we consider the solution of (2.1) as an element in $L^2(\Omega)$. Introducing the embedding operator $E_H : H^1(\Omega) \to L^2(\Omega)$ and using (2.2), the control-to-state mapping $S : u \mapsto y$ is given by

$$y = S\tau^* u = E_H G \tau^* u \tag{2.3}$$

with the solution operator $S : H^1(\Omega)^* \to L^2(\Omega)$.

In the sequel we recall a regularity result for solutions of the state equation in (P). It improves the interior regularity of the solution which is essential for the upcoming analysis.

PROPOSITION 2.1. *Let $y = S\tau^* u \in H^1(\Omega)$ for a given $u \in L^2(\Gamma)$. Furthermore, let $\Omega' \subset\subset \Omega$ be an inner subdomain of Ω. Then y is an element of $W^{2,\infty}(\Omega')$, and there exist a constant $c > 0$, depending only on Ω and Ω', such that*

$$\|y\|_{W^{2,\infty}(\Omega')} \le c\|y\|_{L^2(\Omega)}. \tag{2.4}$$

The proof of Proposition 2.1 is based on classical interior regularity results and is described in [25, Theorem 2.1–Corollary 2.3] in more details. For the L^2-norm of the weak solution $y = S\tau^* u$ appearing in the previous estimate, we recall another result of the first part of this work:

LEMMA 2.2. *Let $y = S\tau^* u \in H^1(\Omega)$ for a given $u \in L^2(\Gamma)$. Then there is a constant $c > 0$, independent of u, such that*

$$\|y\|_{L^2(\Omega)} \le c\,\|u\|_{H^1(\Gamma)^*}.$$

Proof. The assertion follows by duality arguments and the continuity of the trace operator from $H^2(\Omega)$ to $H^1(\Gamma)$, see [30, Theorem II.4.11.]. For a detailed proof, we refer to [25, Lemma 2.4]. ∎

2.2. Karush-Kuhn-Tucker conditions and boundedness results. This section concerns the derivation of first order optimality conditions for problem (P) using a Lagrange multiplier approach for the pure state constraints. Furthermore, we recall smoothness and boundedness results for the optimal control of problem (P) that follow from the optimality conditions. We start with the following assumption concerning the existence of an inner point with respect to the state constraints.

ASSUMPTION 2.3. There exists a function $\hat{u} \in H^1(\Gamma)$ with $u_a \le \hat{u}(x) \le u_b$ a.e. on Γ and $\hat{y}(x) \ge y_c + \gamma$ a.e. in Ω' with $\gamma > 0$, where $\hat{y} = S\tau^*\hat{u}$.

Due to this assumption, the admissible set of problem (P) is nonempty. Moreover, the set is convex and closed. Since the cost functional is strictly convex and radially unbounded, the existence and uniqueness of the optimal solution is obtained by standard arguments. It is to be noted that the existence of an optimal solution for problem (P) requires only the existence of a feasible point. The stricter Assumption 2.3 is needed for the existence of Lagrange multipliers associated with the state constraints in (P).

It is well known that Lagrange multipliers associated with pointwise state constraints are in general only regular Borel measures. Thanks to Proposition 2.1, the solution of the state equation is continuous in $\overline{\Omega'}$. Hence, Assumption 2.3 gives the existence of a Slater point with respect to the $C(\overline{\Omega'})$-topology such that the generalized Karush-Kuhn-Tucker theory implies the existence of a Lagrange multiplier, for details see Casas [8]. The control constraints on the boundary are included via the following admissible set

$$U_{ad}^L := \{u \in L^2(\Gamma) : \, u_a \le u \le u_b \text{ a.e. on } \Gamma\}. \tag{2.5}$$

By adapting the theory of Casas in [8] to our problem (P), we obtain the following result:
THEOREM 2.4. *Suppose that Assumption 2.3 is fullfilled. Moreover, let* $(\bar y, \bar u)$ *be the optimal solution of problem* (P). *Then a regular Borel measure* $\mu \in \mathcal{M}(\overline{\Omega'})$ *and an adjoint state* $p \in W^{1,s}(\Omega)$, $s < d/(d-1)$ *exist such that the following optimality system is satisfied:*

$$\begin{aligned} -\Delta \bar y + \bar y &= 0 & \qquad -\Delta p + p &= \bar y - y_d - \chi^*_{\overline{\Omega'}}\mu \\ \partial_n \bar y &= \bar u & \partial_n p &= 0 \end{aligned} \tag{2.6}$$

$$(\tau p + \nu \bar u \,,\, u - \bar u)_{L^2(\Gamma)} \ge 0, \quad \forall u \in U_{ad}^L \tag{2.7}$$

$$\begin{aligned} &\int_{\overline{\Omega'}} (y_c - \bar y) d\mu = 0, \quad \bar y(x) \ge y_c(x) \quad \textit{a.e in } \Omega' \\ &\int_{\overline{\Omega'}} \varphi d\mu \ge 0 \quad \forall \varphi \in C(\overline{\Omega'}), \quad \varphi(x) \ge 0 \quad \forall x \in \overline{\Omega'}. \end{aligned} \tag{2.8}$$

Here, $\chi^*_{\overline{\Omega'}} : \mathcal{M}(\overline{\Omega'}) \to \mathcal{M}(\Omega)$ *denotes the adjoint of the restriction operator from* Ω *to* $\overline{\Omega'}$.
REMARK 2.5. *For convenience of the reader, we recall the notion of solutions to PDEs involving measures as in case of the adjoint equation in* (2.6). *We say that* $p \in W^{1,s}(\Omega)$, $s < d/(d-1)$, *solves*

$$\begin{aligned} -\Delta\, p + p &= \chi^*_{\overline{\Omega'}}\mu \quad && \textit{in } \Omega \\ \partial_n p &= 0 && \textit{on } \Gamma, \end{aligned} \tag{2.9}$$

if it satisfies

$$a(z,p) = \int_{\overline{\Omega'}} z \, d\mu \quad \forall\, z \in W^{1,s'}(\Omega), \tag{2.10}$$

where $a : W^{1,s}(\Omega) \times W^{1,s'}(\Omega) \to \mathbb{R}$ *is the bilinear form defined in* (2.1). *Note that the right hand side in* (2.10) *is well defined due to* $W^{1,s'}(\Omega) \hookrightarrow C(\overline{\Omega'})$ *since* $s' = s/(s-1) > d$. *Based on results of Gröger [19] and Zanger [35] for the associated dual problem, it can be shown by standard arguments that there exists a unique solution* $p \in W^{1,s}(\Omega)$ *for every* $\mu \in \mathcal{M}(\overline{\Omega'})$ *which depends continuously on the data.*
Let us now recall an additional regularity result for the optimal control $\bar u$ that were proven in the first part of this work, see [25, Section 2.2]. Since the measure valued Lagrange multiplier is only localized in the inner subdomain Ω', the adjoint state p provides higher regularity close to the boundary and on the boundary itself. To be more precise, one derives H^1-regularity of p on Γ. As the optimal control $\bar u$ is the L^2-projection of $(-1/\nu)\tau\, p$ by the variational inequality (2.7), and the L^2-projection operator is stable in $H^1(\Gamma)$, we finally obtain that $\bar u$ is an element of $H^1(\Gamma)$, which is stated in the Proposition below. For a detailed description of the underlying arguments, we refer to [25, Lemma 2.7–Lemma 2.9].
PROPOSITION 2.6. *The optimal solution of* (P), *denoted by* $\bar u$ *satisfies*

$$\|\bar u\|_{H^1(\Gamma)} \le C$$

with a constant $C > 0$.
REMARK 2.7. *We point out that the above Proposition 2.6 represents a substantial gain of regularity compared to the case where the state constraints are imposed in the whole domain* Ω. *We will benefit from the additional regularity in various ways throughout the overall error analysis presented in the following.*

3. Regularization and discretization. The regularization of state constrained optimal control problems was motivated in the introduction. Instead of problem (P), we investigate a family of regularized problems (P^ε). This section is devoted to the finite element approximation of the regularized optimal control problems (P^ε), i.e. the regularization and discretization of the original problem (P) is considered simultaneously.

3.1. Discretization of the state equation. We start with the introduction of a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\bar\Omega$. The mesh $\mathcal{T}_h$ consists of open and pairwise disjoint cells (triangles, tetrahedra) such that

$$\bar\Omega = \bigcup_{T\in\mathcal{T}_h} \bar T.$$

Note that the domain Ω is polygonally or polyhedrally bounded. The vertices of the elements of $\mathcal{T}_h$ are denoted by $x_1,\dots,x_n$. With each element $T\in\mathcal{T}_h$, we associate two parameters $\rho(T)$ and $R(T)$, where $\rho(T)$ denotes the diameter of the element T and $R(T)$ is the diameter of the largest ball contained in T. The mesh size of $\mathcal{T}_h$ is defined by

$$h = \max_{T\in\mathcal{T}_h} \rho(T).$$

We suppose the following regularity assumption for $\mathcal{T}_h$:
Assumption 3.1. There exist two positive constants ρ and R such that

$$\frac{\rho(T)}{R(T)} \le R, \quad \frac{h}{\rho(T)} \le \rho$$

hold for all $T\in\mathcal{T}_h$ and all $h>0$.
For a fixed mesh size $h>0$, we denote by $\{T_j\}_{j=1}^{n_\Gamma}$ the family of elements of $\mathcal{T}_h$ with at least one side lying on the boundary Γ. Furthermore, we set $e_j := \overline{T_j}\cap\Gamma$, $j=1,\dots,n_\Gamma$. The vertices of the boundary elements e_j, $j=1,\dots,n_\Gamma$, are denoted by x_i^Γ, $i=1,\dots,n_e$.
Based on this triangulation, we define the space of linear finite elements:

$$V_h = \{v\in C(\bar\Omega)\,|\, v|_T\in\mathcal{P}_1\,\forall T\in\mathcal{T}_h\},$$

where $\mathcal{P}_1$ is the space of polynomials of degree less than or equal one. Let us introduce the discrete solution operator. For each element $f\in H^1(\Omega)^*$, we denote by y_h the unique element of V_h that satisfies

$$a(y_h,z_h) = \langle f\,,\, z_h\rangle_{H^1(\Omega)^*,H^1(\Omega)} \quad \forall\, z_h\in V_h\subset H^1(\Omega), \tag{3.1}$$

where the bilinear form $a: V_h\times V_h\to\mathbb{R}$ is as defined in (2.1). The existence and uniqueness of $y_h\in V_h$ directly follows from the Lax-Milgram Lemma. Then, the discrete solution operator $S_h: H^1(\Omega)^*\to L^2(\Omega)$ is defined as follows:

$$y_h = S_h f \quad\Longleftrightarrow\quad a(y_h,z_h) = \langle f\,,\, z_h\rangle_{H^1(\Omega)^*,H^1(\Omega)} \quad \forall z_h\in V_h. \tag{3.2}$$

Hence, the discrete counterpart of the state equation in problem (P^ε) is given by:

$$y_h^\varepsilon = S_h(\tau^* u + \phi(\varepsilon)E_H^* v) \Leftrightarrow a(y_h,z_h) = \int\limits_\Gamma u\tau z_h\,ds + \int\limits_\Omega \phi(\varepsilon)vE_H z_h\,dx \;\; \forall z_h\in V_h \tag{3.3}$$

where, as above, $\tau: H^1(\Omega)\to L^2(\Gamma)$ denotes the trace operator, and $E_H: H^1(\Omega)\to L^2(\Omega)$ the continuous embedding operator from $H^1(\Omega)$ to $L^2(\Omega)$. We will make use of the following standard approximation properties of the discrete solution operator.
Theorem 3.2. *There exist a positive constant c, independent of h, such that*
(i) for all $v\in L^2(\Omega)$

$$\|(S-S_h)E_H^* v\|_{L^2(\Omega)} \le ch^2\|v\|_{L^2(\Omega)} \tag{3.4}$$

$$\|(S-S_h)E_H^* v\|_{L^2(\Gamma)} \le ch^{3/2}\|v\|_{L^2(\Omega)}. \tag{3.5}$$

(ii) for all $u \in H^s(\Gamma)$, $0 \le s \le 1/2$

$$\|(S - S_h)\tau^* u\|_{L^2(\Omega)} \le ch^{3/2+s}\|u\|_{H^s(\Gamma)} \tag{3.6}$$

For the corresponding proof, we refer for instance to [5]. In the sequel, we will frequently rely on interior maximum norm estimates for the finite element approximation of the state equation. The next Theorem provides a corresponding result. In [32] Schatz and Wahlbin developed L^∞-approximation error estimates for finite element solutions in an inner subdomain of Ω for a general class of elliptic PDEs. The authors estimated the finite element approximation error in the L^∞-norm by the best approximation plus the error in a weaker norm on slightly larger domain. Thanks to this work, we state the following result.

THEOREM 3.3. *Let Ω' be a subdomain of Ω satisfying $\Omega' \subset\subset \Omega$. There exist a positive constant c, depending only on Ω and Ω', and a sufficient small mesh size $0 < h_0 < 1$, such that*

(i) for every $u \in L^2(\Gamma)$

$$\|(S - S_h)\tau^* u\|_{L^\infty(\Omega')} \le ch^{3/2}\|u\|_{L^2(\Gamma)} \tag{3.7}$$

(ii) for every $u \in H^{1/2}(\Gamma)$

$$\|(S - S_h)\tau^* u\|_{L^\infty(\Omega')} \le ch^2|\log h|\|u\|_{H^{1/2}(\Gamma)} \tag{3.8}$$

is satisfied for $0 < h \le h_0$.

Proof. The proof is an immediate consequence of a result for a general class of elliptic PDEs derived by Schatz and Wahlbin in [32]. For convenience of the reader we present how to apply the results of [32] in our case. Throughout the proof, we use abbreviations $y = S\tau^* u$ and $y_h = S_h\tau^* u$. Since $\operatorname{dist}(\overline{\Omega'}, \Gamma) > 0$ by assumption, there is another subdomain, denoted by Ω'', that fulfills $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Then, due to [32, Theorem 5.1], there is a mesh size $0 < h_0 < 1$ and a constant c, independent of h, such that

$$\|y - y_h\|_{L^\infty(\Omega')} \le c(|\log h|\|y - I_h y\|_{L^\infty(\Omega'')} + \|y - y_h\|_{L^2(\Omega)}), \tag{3.9}$$

is satisfied for all $0 < h \le h_0$, where I_h denotes the standard nodal interpolation operator. Now, we define the set Ω_h by

$$\Omega_h := \bigcup_{T\in\mathcal{T}_h,\, \overline{T}\cap\overline{\Omega''}\neq\emptyset} T$$

such that Ω_h is a union of elements that contains Ω''. If $h_0 > 0$ is chosen sufficiently small, then, for each $h < h_0$, Ω_h clearly exists and is strictly contained in a subdomain Ω''' with $\Omega''' \subset\subset \Omega$. Due to Proposition 2.1, we thus have $y \in W^{2,\infty}(\Omega''')$, and one obtains by well known interpolation error estimates (cf. e.g. [5])

$$\begin{aligned}\|y - I_h y\|_{L^\infty(\Omega'')} &\le \|y - I_h y\|_{L^\infty(\Omega_h)} \\ &\le ch^2\|y\|_{W^{2,\infty}(\Omega_h)} \le ch^2\|y\|_{W^{2,\infty}(\Omega''')} \le ch^2\|u\|_{L^2(\Gamma)},\end{aligned}$$

with a constant $c > 0$ independent of h. Finally, by applying the standard approximation error estimates, given in Theorem 3.2, for the second addend in the right hand side of (3.9), the assertion is proven. ∎

3.2. Discretization of the boundary control. We proceed with the discretization of the boundary control. This was already discussed in the first part of this work, see [25, Section 3.1.]. The space of discrete controls on the boundary is defined by

$$U_h = \{u \in C(\Gamma)\,|\,u|_{e_j} \in \mathcal{P}_1 \text{ for } j = 1, \ldots, n_e\},$$

i.e., the space of edgewise or facewise linear finite elements, respectively. Notice that the restriction of the finite element space V_h to the boundary Γ coincides with the space U_h by construction of the mesh.

DEFINITION 3.4. *As basis function for the space U_h, we choose functions $\psi_i \in U_h$, $i = 1, \ldots, n_e$, satisfying the following conditions:*

$$\psi_i(x_j^\Gamma) = \delta_{ij}, \qquad \psi_i(x) \ge 0 \text{ a.e. on } \Gamma, \qquad \sum_{i=1}^{n_e} \psi_i(x) = 1 \tag{3.10}$$

for every $i, j = 1, \ldots, n_e$.

REMARK 3.5. *We define by*

$$\omega_i := supp\ \psi_i \qquad i = 1, \ldots, n_e,$$

the patch ω_i *that consists of the* M_i *adjacent elements of* $\{e_j\}_{j=1}^{n_\Gamma}$ *that share the vertex* x_i^Γ. *Assumption 3.1 implies the existence of a constant* $M \in \mathbb{N}$, *independent of* h, *such that* $M_i \leq M$ *for all* $i = 1, \ldots, n_e$. One can easily see that for two-dimensional convex polygonal domains every patch ω_i consists of two edges e_j, i.e. $M = 2$ in this case.

Now, we define a quasi-interpolation operator for arbitrary functions in $L^2(\Gamma)$ that preserves the feasibility w.r.t. the control constraints in (P) and (P$^\varepsilon$), respectively, i.e.:

$$u_a \leq u(x) \leq u_b \text{ a.e. on } \Gamma \quad \Rightarrow \quad u_a \leq (\Pi_h u)(x) \leq u_b \text{ a.e. on } \Gamma. \tag{3.11}$$

For an arbitrary $u \in L^1(\Gamma)$, the operator is constructed as follows:

$$\Pi_h u = \sum_{i=1}^{n_e} \pi_i(u)\varphi_i, \quad \pi_i(u) = \frac{\int_{\omega_i} u\varphi_i ds}{\int_{\omega_i} \varphi_i ds}. \tag{3.12}$$

This quasi-interpolation operator was introduced in [6]. One can easily see that Π_h, constructed in this way, satisfies (3.11). In the following lemmata, we recall approximation properties of the quasi-interpolation operator. For the corresponding proofs, we refer to [15, Lemma 4.4 and 4.5] and [25, Section 3.1].

LEMMA 3.6. *The quasi-interpolation operator is stable w.r.t. the* L^2*- and* H^1*-norm, i.e.*

$$\begin{aligned} \|\Pi_h u\|_{L^2(\Gamma)} &\leq c\,\|u\|_{L^2(\Gamma)} \quad \forall\, u \in L^2(\Gamma) \\ \|\Pi_h u\|_{H^1(\Gamma)} &\leq c\,\|u\|_{H^1(\Gamma)} \quad \forall\, u \in H^1(\Gamma) \end{aligned}$$

hold true with a constant $c > 0$, *independent of* h.

LEMMA 3.7. *There is a constant* c, *independent of* h, *such that*

$$\|u - \Pi_h u\|_{L^2(\Gamma)} \leq ch\|u\|_{H^1(\Gamma)} \tag{3.13}$$

$$\|u - \Pi_h u\|_{H^1(\Gamma)^*} \leq ch^2\|u\|_{H^1(\Gamma)} \tag{3.14}$$

for all $u \in H^1(\Gamma)$.

3.3. The fully discretized and regularized problem. First, we mention that the virtual control v_ε of problem (P$^\varepsilon$) is discretized by piecewise linear ansatz functions. Associated with the finite element spaces V_h and U_h, respectively, introduced in the previous sections, the regularized and discretized optimal control problem is given by

$$\left.\begin{aligned} &\min \quad J_\varepsilon(y_h^\varepsilon, u_h^\varepsilon, v_h^\varepsilon) = \frac{1}{2}\|y_h^\varepsilon - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h^\varepsilon\|^2_{L^2(\Gamma)} + \frac{\psi(\varepsilon)}{2}\|v_h^\varepsilon\|^2_{L^2(\Omega)} \\ &\text{s.t.} \qquad y_h^\varepsilon = S_h(\tau^* u_h^\varepsilon + \phi(\varepsilon)E_H^* v_h^\varepsilon) \quad \text{and} \quad (u_h^\varepsilon, v_h^\varepsilon) \in V_{ad}^{\varepsilon,h}, \end{aligned}\right\} \tag{P$_h^\varepsilon$}$$

where the discrete admissible set is defined by

$$\begin{aligned} V_{ad}^{\varepsilon,h} := \{(u_h, v_h) \in U_h \times V_h \,|\, &u_a \leq u_h(x) \leq u_b \text{ a.e. on } \Gamma, \\ &S_h(\tau^* u_h + \phi(\varepsilon)E_H^* v_h)(x) \geq y_c(x) - \xi(\varepsilon)v_h(x) \text{ a.e. in } \Omega'\}. \end{aligned}$$

The admissible set is convex and closed. Furthermore, the next lemma shows that the admissible set is nonempty for sufficiently small mesh sizes h. Thus, the problem (P$_h^\varepsilon$) admits a unique solution by standard arguments.

LEMMA 3.8. *There is a mesh size* $0 < h_0 < 1$ *such that, for all* $h \leq h_0$

$$\hat{y}_h(x) = (S_h \tau^* \Pi_h \hat{u})(x) \geq y_c(x) + \gamma_0, \qquad a.e.\ in\ \Omega'$$

is valid with a constant γ_0 independent of h.

Proof. Since $\hat{u}$ satisfies the control constraints and the quasi-interpolation operator Π_h by (3.12) preserves this property, we obtain $u_a \le \Pi_h \hat{u} \le u_b$. By means of Assumption 2.3, we continue with

$$\begin{aligned}(S_h\tau^*\Pi_h\hat{u})(x) &= (S\tau^*\hat{u})(x) + (S\tau^*(\Pi_h\hat{u}-\hat{u}))(x) + ((S_h-S)\tau^*\Pi_h\hat{u})(x)\\ &\ge y_c(x) + \gamma - \|S\tau^*(\Pi_h\hat{u}-\hat{u})\|_{L^\infty(\Omega')} - \|(S_h-S)\tau^*\Pi_h\hat{u}\|_{L^\infty(\Omega')}.\end{aligned}$$

The first L^∞-error, arising by the discretization of control, was estimated in the first part of this work, see [25, Lemma 3.6.], and we obtain

$$\|S\tau^*(\Pi_h\hat{u}-\hat{u})\|_{L^\infty(\Omega')} \le ch^2\|\hat{u}\|_{H^1(\Gamma)}.$$

For the second error, induced by the discretization of the state, it is sufficient to use the estimate (3.7) so that, for all $h \le h_0$ with sufficiently small $h_0 < 1$,

$$\|(S_h-S)\tau^*\Pi_h\hat{u}\|_{L^\infty(\Omega')} \le ch^{3/2}\|\Pi_h\hat{u}\|_{L^2(\Gamma)} \le ch^{3/2}\|\hat{u}\|_{L^2(\Gamma)},$$

where we used Lemma 3.6 for the last estimate. Concluding, we infer

$$\gamma_0 := \gamma - c(h_0^2\|\hat{u}\|_{H^1(\Gamma)} + h_0^{3/2}\|\hat{u}\|_{L^2(\Gamma)}) > 0$$

if $h_0 < 1$ is chosen sufficiently small. ∎

Let us now consider the optimality conditions for problem $(\mathrm{P}_h^\varepsilon)$ using a Lagrange multiplier approach for the mixed constraints. The control constraints are still treated by an admissible set:

$$U_{h,ad}^L := \{u \in U_h : \quad u_a \le u \le u_b \text{ a.e. on } \Gamma\}. \tag{3.15}$$

As mentioned in the introduction, the Lagrange multipliers associated to mixed pointwise control-state constraints are proper functions as for instance shown in [1], [31], and [33]. By adapting the techniques of [31] to the discrete case, one obtains the following result:

PROPOSITION 3.9. *Let $\varepsilon > 0$ and $h > 0$ be fixed and $(\bar{y}_h^\varepsilon, \bar{u}_h^\varepsilon, \bar{v}_h^\varepsilon) \in V_h \times U_h \times V_h$ be the unique solution of $(\mathrm{P}_h^\varepsilon)$. Then there exist an adjoint state $p_h^\varepsilon \in V_h$ and a Lagrange multiplier $\mu_h^\varepsilon \in L^2(\Omega')$ such that the following optimality system is satisfied:*

$$\bar{y}_h^\varepsilon = S_h(\tau^*\bar{u}_h^\varepsilon + \phi(\varepsilon)E_H^*\bar{v}_h^\varepsilon) \tag{3.16}$$

$$p_h^\varepsilon = S_h^*(\bar{y}_h^\varepsilon - y_d - E_{\Omega'}^*\mu_h^\varepsilon) \tag{3.17}$$

$$(\tau p_h^\varepsilon + \nu\bar{u}_h^\varepsilon, u - \bar{u}_h^\varepsilon)_{L^2(\Gamma)} \ge 0 \quad \forall u \in U_{h,ad}^L \tag{3.18}$$

$$(\phi(\varepsilon)p_h^\varepsilon + \psi(\varepsilon)\bar{v}_h^\varepsilon - \xi(\varepsilon)E_{\Omega'}^*\mu_h^\varepsilon\,,\, v_h)_{L^2(\Omega)} = 0 \quad \forall\, v_h \in V_h \tag{3.19}$$

$$\begin{aligned}(\mu_h^\varepsilon\,,\, y_c - \bar{y}_h^\varepsilon - \xi(\varepsilon)\bar{v}_h^\varepsilon)_{L^2(\Omega')} = 0,\\ \mu_h^\varepsilon \ge 0, \quad \bar{y}_h^\varepsilon \ge y_c - \xi(\varepsilon)\bar{v}_h^\varepsilon \quad \text{a.e. in } \Omega',\end{aligned} \tag{3.20}$$

where $E_{\Omega'} : L^2(\Omega) \to L^2(\Omega')$ denotes the respective restriction operator to Ω' so that $E_{\Omega'}^* : L^2(\Omega') \to L^2(\Omega)$ represents the extension by zero on $\Omega \setminus \Omega'$.

The proposition shows that the Lagrange multiplier μ_h^ε is an element of $L^2(\Omega')$ for fixed $\varepsilon > 0$ and $h > 0$. The next lemma provides a uniform bound w.r.t. h and ε for μ_h^ε in $\mathcal{M}(\overline{\Omega'})$.

LEMMA 3.10. *Let $(\bar{y}_h^\varepsilon, \bar{u}_h^\varepsilon, \bar{v}_h^\varepsilon)$ be the optimal solution of problem $(\mathrm{P}_h^\varepsilon)$. Furthermore, let p_h^ε be the adjoint state and μ_h^ε the Lagrange multiplier such that the optimality system (3.16)-(3.20) is fulfilled. Then, there exists a mesh size $0 < h_0 < 1$ such that the Lagrange multiplier μ_h^ε is uniformly bounded in $\mathcal{M}(\overline{\Omega'})$ for all mesh sizes $0 < h \le h_0$ and $\varepsilon > 0$, i.e., there holds*

$$\|\mu_h^\varepsilon\|_{\mathcal{M}(\overline{\Omega'})} \le C, \tag{3.21}$$

with the constant $C > 0$, independent of the regularization parameter ε and the mesh size h.

Proof. The proof completely follows the lines of [25, Lemma 2.10], where the uniform boundedness of the multipliers w.r.t. ε in the L^1-norm is shown for the continuous case. Nevertheless, we recall the arguments for the convenience of the reader. Throughout the proof, we identify μ_h^ε with the measure that is generated by

$$\int_{\overline{\Omega'}} f(x)\, d\mu_h^\varepsilon := \int_{\Omega'} f(x)\mu_h^\varepsilon(x)\, dx \quad \forall f \in C(\overline{\Omega'}). \tag{3.22}$$

We begin be adding (3.19) and (3.18) which implies

$$\begin{aligned}(E_{\Omega'}^*\mu_h^\varepsilon\,,\, \xi(\varepsilon)(v_h - \bar v_h^\varepsilon) + S_h E_H^*\phi(\varepsilon)(v_h - \bar v_h^\varepsilon) + S_h\tau^*(u_h - \bar u_h^\varepsilon))_{L^2(\Omega)}&\\ \le (\psi(\varepsilon)\bar v_h^\varepsilon + \phi(\varepsilon)E_H S_h^*(\bar y_h^\varepsilon - y_d)\,,\, v_h - \bar v_h^\varepsilon)_{L^2(\Omega)}&\\ + (\nu\bar u_h^\varepsilon + \tau S_h^*(\bar y_h^\varepsilon - y_d)\,,\, u_h - \bar u_h^\varepsilon)_{L^2(\Gamma)},&\end{aligned} \tag{3.23}$$

for all $(u_h, v_h) \in U_{h,ad}^L \times V_h$. Next, we choose $(\Pi_h\hat u,\, 0) \in U_{h,ad}^L \times V_h$ as test function in the above inequality. Note that $\Pi_h\hat u \in U_{h,ad}^L$ by Lemma 3.8. In view of $\bar y_h^\varepsilon = S_h(\tau^*\bar u_h^\varepsilon + \phi(\varepsilon)E_H^*\bar v_h^\varepsilon)$, the left hand side in the previous inequality then results in

$$\begin{aligned}(E_{\Omega'}^*\mu_h^\varepsilon\,,\, \xi(\varepsilon)(-\bar v_h^\varepsilon) + S_h E_H^*\phi(\varepsilon)(-\bar v_h^\varepsilon) + S_h\tau^*(\Pi_h\hat u - \bar u_h^\varepsilon))_{L^2(\Omega)}&\\ = (\mu_h^\varepsilon\,,\, y_c - \bar y_h^\varepsilon - \xi(\varepsilon)\bar v_h^\varepsilon)_{L^2(\Omega)} + (E_{\Omega'}^*\mu_h^\varepsilon\,,\, \hat y_h - y_c)_{L^2(\Omega)}&\\ = (E_{\Omega'}^*\mu_h^\varepsilon\,,\, \hat y_h - y_c)_{L^2(\Omega)},&\end{aligned} \tag{3.24}$$

where we used (3.20) for the last equality. As in Lemma 3.8, we use the notation $\hat y_h = S_h\tau^*\Pi_h\hat u$. Using (3.22), we continue with With the help of Assumption 2.3 and the positivity of the Lagrange multiplier, one derives the estimate

$$\begin{aligned}\|\mu_h^\varepsilon\|_{\mathcal M(\overline{\Omega'})} &= \sup_{\|f\|_{C(\overline{\Omega'})}=1} \Big|\int_{\Omega'} f(x)\mu_h^\varepsilon(x)\, dx\Big|\\ &\le \|\mu_h^\varepsilon\|_{L^1(\Omega')} \le \frac{1}{\gamma_0}\int_{\Omega'} \gamma_0\mu_h^\varepsilon\, dx \le \frac{1}{\gamma_0}(E_{\Omega'}^*\mu_h^\varepsilon\,,\, \hat y_h - y_c)_{L^2(\Omega)},\end{aligned} \tag{3.25}$$

where we applied the positivity of μ_h^ε and Lemma 3.8. By inserting (3.24) and (3.25) in (3.23) with $(u_h, v_h) = (\Pi_h\hat u, 0)$, we conclude

$$\begin{aligned}\gamma_0\|\mu_h^\varepsilon\|_{\mathcal M(\overline{\Omega'})} \le\ & (\psi(\varepsilon)\bar v_h^\varepsilon + \phi(\varepsilon)E_H S_h^*(\bar y_h^\varepsilon - y_d)\,,\, -\bar v_h^\varepsilon)_{L^2(\Omega)}\\ & + (\nu\bar u_h^\varepsilon + \tau S_h^*(\bar y_h^\varepsilon - y_d)\,,\, \Pi_h\hat u - \bar u_h^\varepsilon)_{L^2(\Gamma)}\\ \le\ & -\psi(\varepsilon)\|\bar v_h^\varepsilon\|_{L^2(\Omega)}^2 + (\bar y_h^\varepsilon - y_d\,,\, -S_h E_H^*\phi(\varepsilon)\bar v_h^\varepsilon)_{L^2(\Omega)}\\ & + (\bar y_h^\varepsilon - y_d\,,\, S_h\tau^*(\Pi_h\hat u - \bar u_h^\varepsilon))_{L^2(\Omega)} + \nu(\bar u_h^\varepsilon\,,\, \Pi_h\hat u - \bar u_h^\varepsilon)_{L^2(\Gamma)}\\ \le\ & \|y_d\|_{L^2(\Omega)}\|\bar y_h^\varepsilon\|_{L^2(\Omega)} + \nu\|\bar u_h^\varepsilon\|_{L^2(\Gamma)}\|\hat u_h\|_{L^2(\Gamma)}\\ & + \|\bar y_h^\varepsilon - y_d\|_{L^2(\Omega)}\|\hat y_h\|_{L^2(\Omega)}.\end{aligned}$$

In view of Lemma 3.6 and the continuity of the discrete solution operator, the norms $\|\hat u_h\|_{L^2(\Gamma)}$ and $\|\hat y_h\|_{L^2(\Omega)}$ are bounded by constants independent of h. Finally, the optimality of $(\bar y_\varepsilon,\, \bar u_\varepsilon)$ yields the uniform boundedness of the remaining terms. ∎

3.4. Boundedness of the discrete variables. We proceed by deriving the uniform boundedness of the discrete adjoint state p_h^ε and the optimal discrete control $\bar u_h^\varepsilon$ in $H^1(\Gamma)$ w.r.t. ε and h. The first result in this section is an auxiliary result that is necessary for deriving the uniform boundedness of the discrete adjoint state. To this end, we introduce the finite element discretization for PDEs involving measures. Because of $V_h \hookrightarrow W^{1,\infty}(\Omega)$, the discrete version of (2.10) is given by

$$a(z_h, p_h) = \int_{\overline{\Omega'}} z_h\, d\mu \quad \forall z_h \in V_h. \tag{3.26}$$

Since a is coercive on V_h, there clearly exists a unique solution $p_h \in V_h$ to the above equation.

LEMMA 3.11. *Let $\mu \in \mathcal{M}(\overline{\Omega'})$ be given and let $p \in W^{1,s}(\Omega)$ denote the solution of* (2.10) *while $p_h \in V_h$ is the unique solution of* (3.26)*. Then, there is a positive constant c, independent of h, and a mesh size $h_0 < 1$ such that*

$$\|p - p_h\|_{L^2(\Gamma)} \leq ch^{3/2}\|\mu\|_{\mathcal{M}(\overline{\Omega'})} \tag{3.27}$$

is valid for all $0 < h \leq h_0$.

Proof. The arguments are based on a modification of the idea of Casas in [7, Theorem 3]. First, we introduce first the following dual problems for given $f \in L^2(\Gamma)$:

$$a(w, z) = \int_\Gamma f z \, ds \quad \forall z \in W^{1,s}(\Omega), \qquad a(w_h, z_h) = \int_\Gamma f z_h \, ds \quad \forall z_h \in V_h.$$

According to [19] and [35], the first equation admits a unique solution $w \in W^{1,s'}(\Omega)$ and, thanks to the density of $H^1(\Omega) \hookrightarrow W^{1,s}(\Omega)$, we have $w = S\tau^* f$. Since the second equation is equivalent to $w_h = S_h\tau^* f$, Theorem 3.3 applies for the dual problem, and we deduce

$$\|w - w_h\|_{L^\infty(\Omega')} \leq ch^{3/2}\|f\|_{L^2(\Gamma)}.$$

for $0 < h \leq h_0$, where $h_0 < 1$ is sufficiently small. Using this estimate together with Galerkin orthogonality, one obtains

$$\begin{aligned}
\|\tau(p - p_h)\|_{L^2(\Gamma)} &= \sup_{f \in L^2(\Gamma)} \frac{\left|\int_\Gamma (p - p_h) f \, ds\right|}{\|f\|_{L^2(\Gamma)}} \\
&= \sup_{f \in L^2(\Gamma)} \frac{|a(p - p_h, w)|}{\|f\|_{L^2(\Gamma)}} \\
&= \sup_{f \in L^2(\Gamma)} \frac{|a(p, w - w_h)|}{\|f\|_{L^2(\Gamma)}} \\
&= \sup_{f \in L^2(\Gamma)} \frac{\left|\int_{\overline{\Omega'}} (w - w_h) d\mu\right|}{\|f\|_{L^2(\Gamma)}} \\
&\leq \sup_{f \in L^2(\Gamma)} \frac{\|w - w_h\|_{L^\infty(\Omega')}\|\mu\|_{\mathcal{M}(\overline{\Omega'})}}{\|f\|_{L^2(\Gamma)}} \\
&\leq c\, h^{3/2}\|\mu\|_{\mathcal{M}(\overline{\Omega'})},
\end{aligned}$$

which is the assertion. ∎

Next, we introduce an interpolation operator on the boundary of Ω. We recall that the domain is polygonally or polyhedrally bounded such that $\Gamma = \cup_{j=1}^{n_\Gamma} \bar{e}_j$, where each e_j is an open planar polygon or an open line segment that does not overlap any other e_j. Based on the nodal interpolation operator I_h, we define

$$I_h^\Gamma(z|_\Gamma) = (I_h z)|_\Gamma \quad \text{for all } z \in W^{m,p}(\Omega),\ mp > d.$$

Due to this definition, we use the same notation for both operators. Moreover, the standard interpolation error estimate such as

$$\|z - I_h z\|_{L^2(\Gamma)} \leq ch\|z\|_{H^1(\Gamma)} \tag{3.28}$$

is valid for all $z \in H^1(\Gamma)$.

LEMMA 3.12. *Let $p_h^\varepsilon \in V_h$ be the associated discrete adjoint state in the optimality system* (3.16)-(3.20)*. Then, there is positive constant C, independent of ε and h, such that*

$$\|p_h^\varepsilon\|_{H^1(\Gamma)} \leq C. \tag{3.29}$$

Proof. Let p^ε be defined by $p^\varepsilon = S^*(\bar{y}_h^\varepsilon - y_d - \mu_h^\varepsilon)$. Thus, p^ε solves

$$\begin{aligned} -\Delta\, p^\varepsilon + p^\varepsilon &= \bar{y}_h^\varepsilon - y_d - \chi_{\overline{\Omega'}}^*\, \mu_h^\varepsilon && \text{in } \Omega \\ \partial_n p^\varepsilon &= 0 && \text{on } \Gamma, \end{aligned}$$

where μ_h^ε is the measure associated to the Lagrange multiplier via (3.22). Thus, by construction, p_h^ε is the finite element discretization of p^ε. By the same arguments that led to [25, Corollary 2.8] in combination with Lemma 3.10, we have

$$\|p^\varepsilon\|_{H^1(\Gamma)} \le C \tag{3.30}$$

with a constant $C > 0$, independent of ε and h. The triangle inequality then yields

$$\|p_h^\varepsilon\|_{H^1(\Gamma)} \le \|p_h^\varepsilon - I_h p^\varepsilon\|_{H^1(\Gamma)} + \|I_h p^\varepsilon\|_{H^1(\Gamma)},$$

where, as above, I_h denotes the nodal interpolation operator. Due to the stability of the interpolation operator and (3.30), the second term is bounded. For the first term, we continue with applying an inverse estimate, where we refer to [2]:

$$\begin{aligned} \|p_h^\varepsilon - I_h p^\varepsilon\|_{H^1(\Gamma)} &\le ch^{-1}\|p_h^\varepsilon - I_h p^\varepsilon\|_{L^2(\Gamma)} \\ &\le ch^{-1}(\|p_h^\varepsilon - p^\varepsilon\|_{L^2(\Gamma)} + \|p^\varepsilon - I_h p^\varepsilon\|_{L^2(\Gamma)}) \end{aligned}$$

Due to (3.21), Lemma 3.11, and (3.28), the previous term can be bounded by a constant independent of h and ε. Hence, the discrete adjoint state p_h^ε is uniformly bounded in $H^1(\Gamma)$. ∎

In view of (2.7), the discrete optimal control $\bar{u}_h^\varepsilon$ can be interpreted as the L^2-projection of $-p_h^\varepsilon/\nu$ on the discrete admissible set $U_{ad,h}^L$. In the first part of this work, we investigated this projection with regard to the stability in $H^1(\Gamma)$, see [25, Section 3.3]. Analogously to [25, Lemma 3.11], one obtains

LEMMA 3.13. *Let $\bar{u}_h^\varepsilon \in U_{ad,h}^L$ be the discrete optimal control determined by the optimality system* (3.16)-(3.20). *Then, there exists a positive constant C, independent of h and ε, such that*

$$\|\bar{u}_h^\varepsilon\|_{H^1(\Gamma)} \le C$$

is satisfied.

Finally, we derive an a-priori bound for the discrete state $\tilde{y}_h := S_h\tau^*\bar{u}_h^\varepsilon$ in the space of Lipschitz continuous functions $C^{0,1}(\overline{\Omega'})$, which is needed in subsequent estimates.

LEMMA 3.14. *Let $\tilde{y}_h = S_h\tau^*\bar{u}_h^\varepsilon$, where $\bar{u}_h^\varepsilon$ is the optimal control of problem* (P_h^ε). *Then, there exists a positive constant C, independent of h and ε, such that*

$$\|\tilde{y}_h\|_{C^{0,1}(\overline{\Omega'})} \le C. \tag{3.31}$$

Proof. We introduce the continuous counterpart $\tilde{y} := S\tau^*\bar{u}_h^\varepsilon$ to $\tilde{y}_h$. Due to Proposition 2.1 and Sobolev embeddings, we obtain the estimate

$$\|\tilde{y}\|_{C^{0,1}(\overline{\Omega'})} \le c\|\tilde{y}\|_{W^{2,\infty}(\Omega')} \le c\|\tilde{y}\|_{L^2(\Omega)} \le c\|\bar{u}_h^\varepsilon\|_{L^2(\Gamma)}. \tag{3.32}$$

For the rest of the proof, we suppose that Ω' is a union of elements of $\mathcal{T}_h$. If this it not the case, the arguments can easily be adapted as done in the proof of Theorem 3.3. By use of the nodal interpolation $I_h\tilde{y}$, it turns out that

$$\|\tilde{y}_h\|_{C^{0,1}(\overline{\Omega'})} = \|\tilde{y}_h - I_h\tilde{y} + I_h\tilde{y}\|_{C^{0,1}(\bar{\Omega}')} \le \|\tilde{y}_h - I_h\tilde{y}\|_{C^{0,1}(\bar{\Omega}')} + c\|\bar{u}_h^\varepsilon\|_{L^2(\Gamma)}.$$

Due to the optimality of $\bar{u}_h^\varepsilon$, the last term is bounded by the objective functional of (P_h^ε). A standard inverse estimate and the triangle inequality yield for the first term

$$\begin{aligned} \|\tilde{y}_h - I_h\tilde{y}\|_{C^{0,1}(\overline{\Omega'})} &\le ch^{-1}\|\tilde{y}_h - I_h\tilde{y}\|_{L^\infty(\Omega')} \\ &\le ch^{-1}(\|\tilde{y}_h - \tilde{y}\|_{L^\infty(\Omega')} + \|\tilde{y} - I_h\tilde{y}\|_{L^\infty(\Omega')}). \end{aligned}$$

By means of an interpolation error estimate, see e.g., [5], and (3.7), the previous term can be bounded by a constant independent of h and ε. ∎

4. Convergence analysis. In this section we establish an error estimate between the solution of the original problem (P) and solution of the discretized and regularized problem (P_h^ε). The strategy is based on the respective optimality conditions in variational form and the so-called two-way feasibility, i.e., the construction of suitable feasible controls for (P) and (P_h^ε).

4.1. Multiplier-free optimality conditions. Here, we state optimality conditions for the problems (P) and (P_h^ε) respectively, where no Lagrange multiplier occurs. To this end, the admissible sets for problem (P) and (P_h^ε) respectively, are now defined by

$$U_{ad} = \{u \in L^2(\Gamma) |\, u_a \le u \le u_b \text{ a.e. on } \Gamma;\ (S\tau^* u)(x) \ge y_c(x) \text{ a.e. in } \Omega'\}$$

and

$$\begin{aligned} V_{ad}^{\varepsilon,h} := \{&(u_h, v_h) \in U_h \times V_h \,|\, u_a \le u_h(x) \le u_b \text{ a.e. on } \Gamma, \\ &S_h(\tau^* u_h + \phi(\varepsilon) E_H^* v_h)(x) \ge y_c(x) - \xi(\varepsilon) v_h(x) \text{ a.e. in } \Omega'\}. \end{aligned}$$

The necessary and sufficient optimality conditions for both problems that go without Lagrange multipliers, are formulated in the following lemma.

LEMMA 4.1. *Let* $(\bar y, \bar u)$ *and* $(\bar y_h^\varepsilon, \bar u_h^\varepsilon, \bar v_h^\varepsilon)$ *be the optimal solutions of problem* (P) *and* (P_h^ε)*, respectively. The optimality conditions are given by*

$$(\tau \bar p + \nu \bar u, u - \bar u)_{L^2(\Gamma)} \ge 0 \quad \forall u \in U_{ad}, \tag{4.1}$$

and

$$(\tau \bar p_h^\varepsilon + \nu \bar u_h^\varepsilon, u - \bar u_h^\varepsilon)_{L^2(\Gamma)} + (\phi(\varepsilon) E_H \bar p_h^\varepsilon + \psi(\varepsilon) \bar v_h^\varepsilon, v - \bar v_h^\varepsilon)_{L^2(\Omega)} \ge 0 \quad \forall (u, v) \in V_{ad}^{\varepsilon,h}, \tag{4.2}$$

with the associated adjoint states $\bar p = S^*(\bar y - y_d)$ *and* $\bar p_h^\varepsilon = S_h^*(\bar y_h^\varepsilon - y_d)$.

Notice that $\bar p$ and $\bar p_h^\varepsilon$, respectively, differ from the adjoint states p and p_h^ε as defined above, since no Lagrange multiplier occur in the right hand sides of the corresponding adjoint equations. Note moreover that the variational inequalities in the regularized and discretized case cannot be decoupled as in (2.6)–(2.8) and (3.16)–(3.20). We proceed with an estimate that directly results from the above stated optimality conditions.

THEOREM 4.2. *Let* $(\bar y, \bar u)$ *and* $(\bar y_h^\varepsilon, \bar u_h^\varepsilon, \bar v_h^\varepsilon)$ *be the optimal solutions of* (P) *and* (P_h^ε)*, respectively. For all* $u^\delta \in U_{ad}$ *and* $(u_h^\sigma, 0) \in V_{ad}^{\varepsilon,h}$*, there holds*

$$\begin{aligned} &\frac{\nu}{2} \|\bar u - \bar u_h^\varepsilon\|_{L^2(\Gamma)}^2 + \frac{1}{2} \|\bar y - \bar y_h^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\psi(\varepsilon)}{2} \|\bar v_h^\varepsilon\|_{L^2(\Omega)}^2 \\ &\quad \le c\Big((\tau \bar p + \nu \bar u\,,\, u^\delta - \bar u_h^\varepsilon)_{L^2(\Gamma)} + (\tau \bar p_h^\varepsilon + \nu \bar u_h^\varepsilon\,,\, u_h^\sigma - \bar u)_{L^2(\Gamma)} + \frac{(\phi(\varepsilon))^2}{\psi(\varepsilon)} + h^3 \Big) \end{aligned} \tag{4.3}$$

for a certain constant $c > 0$*, independent of* h *and* ε.

The proof of Theorem 4.2 follows by straight forward computations from the variational inequalities in Lemma 4.1 and is depicted in detail in Appendix A.

4.2. Construction of feasible controls. Now we turn to the construction of suitable feasible controls u^δ and u_h^σ for (P) and (P_h^ε) that are close to the optimal control of the respective other problem. Once the existence of such feasible controls is established, Theorem 4.2 allows for the derivation of the final error estimate in Section 4.2. To this end let us consider the violation of the mixed control-state constraints in (P_h^ε) by the control $(\Pi_h \bar u, \bar v = 0)$. We introduce the following violation function

$$d[(\Pi_h \bar u, \bar v = 0), (\mathrm{P}_h^\varepsilon)] := (y_c - S_h \tau^* \Pi_h \bar u)_+ = \max\{0, y_c - S_h \tau^* \Pi_h \bar u\}. \tag{4.4}$$

LEMMA 4.3. *There is a mesh size* $0 < h_0 < 1$ *such that the maximal violation* $\|d[(\Pi_h \bar u, 0), (\mathrm{P}_h^\varepsilon)]\|_{L^\infty(\Omega')}$ *of* $(\Pi_h \bar u, 0)$ *w.r.t.* (P_h^ε) *can be estimated by*

$$\|d[(\Pi_h \bar u, 0), (\mathrm{P}_h^\varepsilon)]\|_{L^\infty(\Omega')} \le c h^2 |\log h| \tag{4.5}$$

for all $0 < h \le h_0$, where the constant $c > 0$ is independent of h and ε.
Proof. Using the triangle inequality and $\bar{y} = S\tau^*\bar{u}$, we find

$$\begin{aligned}\|d[(\Pi_h\bar{u},0),(P_\varepsilon^h)]\|_{L^\infty(\Omega')} &= \|(y_c - S\tau^*\bar{u} + S\tau^*(\bar{u}-\Pi_h\bar{u}) + (S-S_h)\tau^*\Pi_h\bar{u})_+\|_{L^\infty(\Omega')}\\ &\le \|(y_c-\bar{y})_+\|_{L^\infty(\Omega')} + \|S\tau^*(\bar{u}-\Pi_h\bar{u})\|_{L^\infty(\Omega')}\\ &\quad + \|(S-S_h)\tau^*\Pi_h\bar{u}\|_{L^\infty(\Omega')}.\end{aligned}$$

Thanks to the feasibility of $\bar{y}$ for problem (P), the first term vanishes. The second term is estimated by

$$\|S\tau^*(\bar{u}-\Pi_h\bar{u})\|_{L^\infty(\Omega')} \le ch^2\|\bar{u}\|_{H^1(\Gamma)}$$

similarly as in the proof of Lemma 3.8. Note that $\bar{u}$ belongs to $H^1(\Gamma)$, see Proposition 2.6. The last term is estimated by (3.8), i.e.

$$\|(S-S_h)\tau^*\Pi_h\bar{u}\|_{L^\infty(\Omega')} \le ch^2|\log h|\|\Pi_h\bar{u}\|_{H^{1/2}(\Gamma)}$$

for all $0 < h \le h_0$, where $0 < h_0 < 1$ is chosen sufficiently small. The stability of the quasi-interpolation operator by Lemma 3.6 and the boundedness of $\bar{u}$ in $H^{1/2}(\Gamma)$ by Proposition 2.6 yield the assertion. ∎
We continue with the construction of a feasible control u_h^σ for problem (P_h^ε).
LEMMA 4.4. *Let Assumption 2.3 be satisfied. Then, for all sufficiently small mesh sizes h, the control $(u_h^\sigma, 0)$ with $u_h^\sigma := (1-\sigma)\Pi_h\bar{u} + \sigma\Pi_h\hat{u}$ is feasible for (P_h^ε) provided that*

$$1 \ge \sigma \ge \sigma_h := \frac{\|d[(\Pi_h\bar{u},0),(P_\varepsilon^h)]\|_{L^\infty(\Omega')}}{\|d[(\Pi_h\bar{u},0),(P_\varepsilon^h)]\|_{L^\infty(\Omega')} + \gamma_0}. \tag{4.6}$$

The proof is completely along the lines of [25, Lemma 4.4.]. In order to construct a feasible control for the original problem (P), we next consider the violation of the optimal discretized and regularized control $\bar{u}_h^\varepsilon$ with respect to the pure state constraints. Hence, we define the violation function by

$$d[\bar{u}_h^\varepsilon,(P)] := (y_c - S\tau^*\bar{u}_h^\varepsilon)_+. \tag{4.7}$$

The next lemma is essential for the estimation of the maximal violation of $\bar{u}_h^\varepsilon$ w.r.t. problem (P).
LEMMA 4.5. *[26, Lemma 3.2] Let f be a uniformly bounded function in $C^{0,1}(\bar{\Omega})$, then there exist a positive constant $c > 0$ such that*

$$\|f\|_{L^\infty(\Omega)} \le c\|f\|_{L^2(\Omega)}^{2/(2+d)}.$$

LEMMA 4.6. *Let $0 < h \le h_0$, where $h_0 < 1$ is chosen sufficiently small. Then the maximal violation $\|d[\bar{u}_h^\varepsilon,(P)]\|_{L^\infty(\Omega')}$ of $\bar{u}_h^\varepsilon$ w.r.t. problem (P) fulfills*

$$\|d[\bar{u}_h^\varepsilon,(P)]\|_{L^\infty(\Omega')} \le c\left((\xi(\varepsilon)+\phi(\varepsilon))^{2/(2+d)}\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)}^{2/(2+d)} + h^2|\log h|\right) \tag{4.8}$$

with a constant $c > 0$ that is independent of ε and h.
Proof. The first step is done by the use of the triangle inequality:

$$\|d[\bar{u}_h^\varepsilon,(P)]\|_{L^\infty(\Omega')} \le \|(y_c - S_h\tau^*\bar{u}_h^\varepsilon)_+\|_{L^\infty(\Omega')} + \|(S_h-S)\tau^*\bar{u}_h^\varepsilon\|_{L^\infty(\Omega')}. \tag{4.9}$$

According to Lemma 3.14, the function $\tilde{y}_h = S_h\tau^*\bar{u}_h^\varepsilon$ belongs to $C^{0,1}(\overline{\Omega'})$ and is uniformly bounded w.r.t. ε and h. Since we required $y_c \in C^{0,1}(\bar{\Omega})$, the function $(y_c - \tilde{y}_h)_+$ is uniformly bounded in $C^{0,1}(\overline{\Omega'})$ with respect to ε and h. By means of Lemma 4.5, it turns out that

$$\begin{aligned}\|(y_c-\tilde{y}_h)_+\|_{L^\infty(\Omega')} &\le c\|(y_c-\tilde{y}_h)_+\|_{L^2(\Omega')}^{2/(2+d)}\\ &\le c\left(\|(y_c - S_h\tau^*\bar{u}_h^\varepsilon - S_hE_2^*\phi(\varepsilon)\bar{v}_h^\varepsilon)_+\|_{L^2(\Omega')} + \|(S_hE_H^*\phi(\varepsilon)\bar{v}_h^\varepsilon)_+\|_{L^2(\Omega')}\right)^{2/(2+d)}\\ &= c\left(\|(y_c-\bar{y}_h^\varepsilon)_+\|_{L^2(\Omega')} + \|(S_hE_H^*\phi(\varepsilon)\bar{v}_h^\varepsilon)_+\|_{L^2(\Omega')}\right)^{2/(2+d)}.\end{aligned}$$

The feasibility of $(\bar{y}_h^\varepsilon, \bar{u}_h^\varepsilon, \bar{v}_h^\varepsilon)$ for $(\mathrm{P}_h^\varepsilon)$ and the continuity of the discrete solution operator S_h yield

$$\|(y_c - \tilde{y}_h)_+\|_{L^\infty(\Omega')} \leq c\,(\xi(\varepsilon) + \phi(\varepsilon))^{2/(2+d)}\,\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)}^{2/(2+d)}.$$

Due to Lemma 3.13, the control $\bar{u}_h^\varepsilon$ belongs to $H^1(\Gamma)$ and is uniformly bounded with respect to ε and h. Thus, we use the estimate (3.8) for the second term in (4.9) such that

$$\|(S_h - S)\tau^*\bar{u}_h^\varepsilon\|_{L^\infty(\Omega')} \leq ch^2|\log h|$$

with a positive constant c, independent of ε and h. This completes the proof. ∎

The construction of a feasible control for problem (P) can be easily adapted from the first part of this work, see [25, Lemma 4.7.].

LEMMA 4.7. *Let the Assumption 2.3 be satisfied. Then, for every $\varepsilon > 0$ the control $u^\delta := (1-\delta)\bar{u}_h^\varepsilon + \delta\hat{u}$ is feasible for* (P) *for all $\delta \in [\delta_\varepsilon, 1]$, where δ_ε is given by*

$$\delta_\varepsilon := \frac{\|d[\bar{u}_h^\varepsilon, (P)]\|_{L^\infty(\Omega')}}{\|d[\bar{u}_h^\varepsilon, (P)]\|_{L^\infty(\Omega')} + \gamma}. \tag{4.10}$$

4.3. A priori error estimates. In this section, we present the main result of this paper. It deals with an error estimate between the original problem (P) and the discretized and regularized one $(\mathrm{P}_h^\varepsilon)$ based on the construction of feasible controls in the previous section.

THEOREM 4.8. *Let $(\bar{y}, \bar{u})$ and $(\bar{y}_h^\varepsilon, \bar{u}_h^\varepsilon, \bar{v}_h^\varepsilon)$ be the optimal solution of* (P) *and* $(\mathrm{P}_h^\varepsilon)$*, respectively. Then, there exists a positive constant c, independent of the mesh size h and ε, and a mesh size $h_0 < 1$ such that*

$$\begin{aligned} &\frac{\nu}{2}\,\|\bar{u} - \bar{u}_h^\varepsilon\|_{L^2(\Gamma)}^2 + \frac{1}{2}\|\bar{y} - \bar{y}_h^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\psi(\varepsilon)}{2}\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)}^2 \\ &\qquad\leq c\left(\left(\xi(\varepsilon) + \phi(\varepsilon)\right)^{2/(2+d)}\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)}^{2/(2+d)} + \frac{\phi(\varepsilon)^2}{\psi(\varepsilon)} + h^2|\log h|\right) \end{aligned} \tag{4.11}$$

is satisfied for all mesh sizes $0 < h \leq h_0$ and regularization parameters $\varepsilon > 0$.

Proof. With the previous results at hand, the underlying arguments are similar to the proof of [25, Theorem 5.1]. For convenience of the reader, we recall the main arguments. The proof is based on the estimate (4.3) given in Theorem 4.2 and the constructed feasible controls. We start with choosing $u_\delta \in U_{ad}$ as defined in Lemma 4.7 with the specific choice $\delta := \delta_\varepsilon$ given in (4.10). Moreover, for constants $c \geq 1/\gamma$, we obtain

$$\delta_\varepsilon \leq c\|d[\bar{u}_h^\varepsilon, (P)]\|_{L^\infty(\Omega')}.$$

Due to the estimate (4.8), there exists a mesh size $h_0 < 1$ such that

$$\begin{aligned} (\tau\bar{p} + \nu\bar{u}\,,\, u^\delta - \bar{u}_h^\varepsilon)_{L^2(\Gamma)} &\leq \delta\|\tau\bar{p} + \nu\bar{u}\|_{L^2(\Gamma)}\|\hat{u} - \bar{u}_h^\varepsilon\|_{L^2(\Gamma)} \\ &\leq c\|\tau\bar{p} + \nu\bar{u}\|_{L^2(\Gamma)}|\Gamma||u_b - u_a|\|d[\bar{u}_h^\varepsilon, (P)]\|_{L^\infty(\Omega')} \\ &\leq c((\xi(\varepsilon) + \phi(\varepsilon))^{2/(2+d)}\,\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)}^{2/(2+d)} + h^2|\log h|) \end{aligned}$$

for all $h \leq h_0$. The optimality of $\bar{u}$ yield the boundedness of the term $\|\tau\bar{p} + \nu\bar{u}\|_{L^2(\Gamma)}$. We proceed with the choice $(u_h^\sigma, 0) \in V_{ad,h}^{\varepsilon,2}$ given by Lemma 4.4 for $\sigma := \sigma_h$ defined in (4.6). Similarly to the estimate of δ_ε above, we obtain with (4.5)

$$\sigma_h \leq c\|d[(\Pi_h\bar{u}, 0), (\mathrm{P}_h^\varepsilon)]\|_{L^\infty(\Omega')} \leq ch^2|\log h|$$

for all mesh sizes $0 < h \leq h_0$, where $h_0 < 1$ is chosen sufficiently small. We continue with

$$\begin{aligned} (\tau\bar{p}_h^\varepsilon + \nu\bar{u}_h^\varepsilon\,,\, u_h^\sigma - \bar{u})_{L^2(\Gamma)} &= \sigma_h(\tau\bar{p}_h^\varepsilon + \nu\bar{u}_h^\varepsilon\,,\, \Pi_h\hat{u} - \Pi_h\bar{u})_{L^2(\Gamma)} + \\ &\quad (\tau\bar{p}_h^\varepsilon + \nu\bar{u}_h^\varepsilon\,,\, \Pi_h\bar{u} - \bar{u})_{L^2(\Gamma)} \\ &\leq ch^2|\log h|\|\tau\bar{p}_h^\varepsilon + \nu\bar{u}_h^\varepsilon\|_{L^2(\Gamma)}\|\Pi_h(\hat{u} - \bar{u})\|_{L^2(\Gamma)} + \\ &\quad \|\tau\bar{p}_h^\varepsilon + \nu\bar{u}_h^\varepsilon\|_{H^1(\Gamma)}\|\Pi_h\bar{u} - \bar{u}\|_{H^1(\Gamma)^*}. \end{aligned}$$

Thanks to Lemma 3.12 and Lemma 3.13, $\|\bar{u}_h^\varepsilon\|_{H^1(\Gamma)}$ and $\|\bar{p}_h^\varepsilon\|_{H^1(\Gamma)}$ are bounded by constants independent of h and ε. Due to Proposition 2.6, $\bar{u}$ is also an element of $H^1(\Gamma)$. Hence the stability properties of Π_h (see Lemma 3.6) and Lemma 3.7 finally yield

$$(\tau\bar{p}_h^\varepsilon + \nu\bar{u}_h^\varepsilon\,,\, u_h^\sigma - \bar{u})_{L^2(\Gamma)} \le ch^2|\log h|.$$

Summarizing all, we obtain the assertion. ∎

As seen in the above theorem, an $L^2(\Omega)$-estimate of the discrete virtual control is necessary for completion. First, we introduce a coupling of the mesh size and the parameter functions, similarly to the first part of this work, see [25, Assumption 5.2.].

ASSUMPTION 4.9. *The parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$ are chosen such that*

$$\frac{\phi(\varepsilon)+\xi(\varepsilon)}{\sqrt{\psi(\varepsilon)}} \le (h|\log h|^{1/2})^{1+d}. \tag{4.12}$$

Thanks to this assumption, one derives an error estimate for the discrete virtual control $\bar{v}_h^\varepsilon$. The proof given in [25, Corollary 5.3] completely carries over to the fully discretized case discussed here.

COROLLARY 4.10. *Let the assumptions of Theorem 4.8 be fulfilled. Furthermore, suppose that Assumption 4.9 is satisfied. Then, there exist a positive constant c, independent of h and ε, and a mesh size $h_0 < 1$ such that*

$$\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)} \le c\frac{h|\log h|^{1/2}}{\sqrt{\psi(\varepsilon)}} \tag{4.13}$$

is fulfilled for all mesh sizes $0 < h \le h_0$.

With the previous estimate at hand, we are in the position to state our final error estimate. It immediately results from Theorem 4.8 and Corollary 4.10.

THEOREM 4.11. *Let $(\bar{y}, \bar{u})$ and $(\bar{y}_h^\varepsilon, \bar{u}_h^\varepsilon, \bar{v}_h^\varepsilon)$ be the optimal solution of* (P) *and* $(\mathrm{P}_h^\varepsilon)$*, respectively. Moreover, let Assumption 4.9 be satisfied. Then, there exist a positive constant c, independent of ε and h, such that*

$$\|\bar{u} - \bar{u}_h^\varepsilon\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h^\varepsilon\|_{L^2(\Omega)} \le ch|\log h|^{1/2} \tag{4.14}$$

is fulfilled for all mesh sizes $0 < h \le h_0$, provided that $h_0 < 1$ is chosen sufficiently small.

5. Error estimate for the purely discretized problem. In this section, we consider the discretized and regularized problem $(\mathrm{P}_h^\varepsilon)$ for a specific choice of the parameter functions $\psi(\varepsilon)$, $\phi(\varepsilon)$ and $\xi(\varepsilon)$:

$$\psi(\varepsilon) \equiv 1, \quad \phi(\varepsilon) \equiv 0, \quad \xi(\varepsilon) \equiv 0. \tag{5.1}$$

Notice that this setting clearly fulfills Assumption 4.12, but not the initial Assumption 1.2. With these choice for the parameter functions, the virtual control does not longer occur in the discretized state equation and in the inequality constraints. It is easily seen that the minimization of the objective functional in $(\mathrm{P}_h^\varepsilon)$ can then be discussed separately for u_h^ε and v_h^ε, and we obtain $\bar{v}_h^\varepsilon \equiv 0$ for every $\varepsilon > 0$ and every $h > 0$. Hence, for all $\varepsilon > 0$ and $h > 0$, $(\mathrm{P}_h^\varepsilon)$ is equivalent to:

$$\left.\begin{aligned} \min_{(y_h,u_h)\in V_h\times U_h} \quad & J(y_h,u_h) := \frac{1}{2}\|y_h - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|^2_{L^2(\Gamma)} \\ & y_h = S_h(\tau^* u_h) \\ & u_a \le u_h(x) \le u_b \quad \text{a.e. on } \Gamma \\ & y_h(x) \ge y_c(x) \quad \text{a.e. in } \Omega'. \end{aligned}\right\} \tag{P_h}$$

REMARK 5.1. *Problem* (P_h) *is the purely discrete analogon to the original optimal control problem* (P) *without any regularization. Hence, with the setting* (5.1) *for the parameter functions, the discretization* *of* (P) *is a special case of the discretized and regularized problem* (P_h^ε)*. As will be seen in the sequel, the analysis for* (P_h^ε) *and the associated error estimates therefore also apply to* (P_h).

Due to Lemma 3.8, there exists a feasible control for the problem (P_h). Consequently, the existence and uniqueness of an optimal solution, denoted by ($\bar{y}_h$, $\bar{u}_h$), is obtained by standard arguments. The derivation of first-order conditions for (P_h) is much more delicate as in case of (P_h^ε). Since $\xi(\varepsilon) = 0$, (P_h) is an optimal control problem with pointwise state constraints. Thus we expect the Lagrange multipliers associated to these constraints to be measures as in case of (P). Again, the optimality conditions for (P_h) can be deduced by means of the first-order analysis developed in [8], cf. also [17, Lemma 2.1]. The constraint qualifications, required to obtain a qualified optimality system, are guaranteed by Lemma 3.8 which says that $\Pi_h\hat{u}$ represents a Slater point for the constraints in (P_h). The control constraints are again treated by the admissible set $U^L_{h,ad}$ defined in (3.15). Hence, similarly to Theorem 2.4, we obtain the Karush-Kuhn-Tucker conditions for (P_h).

Proposition 5.2. *Suppose that* ($\bar{y}_h$, $\bar{u}_h$) $\in V_h \times U_h$ *is the unique solution of* (P_h)*. Then there exist an adjoint state* $p_h \in V_h$ *and a Lagrange multiplier* $\mu_h \in \mathcal{M}(\overline{\Omega'})$ *so that the optimality system*

$$a(\bar{y}_h, z_h) = \int_\Gamma \bar{u}_h z_h \, ds \quad \forall z_h \in V_h \tag{5.2}$$

$$a(z_h, p_h) = \int_\Omega (\bar{y}_h - y_d) z_h \, dx - \int_{\Omega'} z_h d\mu_h \quad \forall z_h \in V_h \tag{5.3}$$

$$(\tau p_h + \nu \bar{u}_h \, , \, u - \bar{u}_h)_{L^2(\Gamma)} \geq 0, \quad \forall u \in U^L_{h,ad} \tag{5.4}$$

$$\begin{aligned} &\int_{\overline{\Omega'}} (\bar{y}_h - y_c) d\mu_h = 0, \quad \bar{y}_h(x) \geq y_c(x) \quad a.e \text{ in } \Omega' \\ &\int_{\overline{\Omega'}} \varphi d\mu_h \geq 0 \quad \forall \varphi \in C(\overline{\Omega'}), \quad \varphi(x) \geq 0 \quad \forall x \in \overline{\Omega'}. \end{aligned} \tag{5.5}$$

is fulfilled.

We observe that the Lagrange multiplier associated to the state constraints in (P_h) is indeed only a regular Borel measure. In contrast to this, the Lagrange multiplier is a proper function in case of (P_h^ε) provided that Assumption 1.2 holds, i.e., in particular $\xi(\varepsilon) > 0$, see Proposition 3.9. Hence, the error analysis for (P_h) differs from the one for (P_h^ε). However, in the previous sections, we only used the uniform boundedness of the Lagrange multipliers in $\mathcal{M}(\overline{\Omega'})$. This in particular concerns Sections 3.3 and 3.4. The overall error analysis of (P_h^ε) therefore carries over to (P_h) provided that the assertion of Lemma 3.10, i.e., the uniform boundedness of the multipliers in $\mathcal{M}(\overline{\Omega'})$, also holds in case of (P_h). The corresponding result is stated in the next lemma.

Lemma 5.3. *Let* $(\bar{y}_h, \bar{u}_h) \in V_h \times U_h$ *be the optimal solution of problem* (P_h)*. Furthermore, let* $p_h \in V_h$ *be an adjoint state and* μ_h *a Lagrange multiplier such that the optimality system* (5.2)-(5.5) *is fulfilled. Then, there exist a constant* $C > 0$*, independent of* h*, such that*

$$\|\mu_h\|_{\mathcal{M}(\overline{\Omega'})} \leq C.$$

Proof. In principle, the proof is analogous to the one of Lemma 3.10. Moreover, it is similar to the proofs of [17, Theorem 2.3] and [28, Lemma 2.10]. Instead of (3.23), the variational inequality (5.4) implies

$$(\tau p_h \, , \, \Pi_h\hat{u} - \bar{u}_h)_{L^2(\Gamma)} + \nu(\bar{u}_h \, , \, \Pi_h\hat{u} - \bar{u}_h)_{L^2(\Gamma)} \geq 0, \tag{5.6}$$

where we have inserted $\Pi_h\hat{u} \in U^L_{h,ad}$ as test function. Moreover, analogously to (3.24), the weak formulations (5.2) and (5.3) yield

$$
\begin{aligned}
(\tau p_h\,,\,\Pi_h\hat{u}-\bar{u}_h)_{L^2(\Gamma)} &= a(\hat{y}_h-\bar{y}_h,p_h)\\
&= (\bar{y}_h-y_d\,,\,\hat{y}_h-\bar{y}_h)_{L^2(\Omega)}-\int\limits_{\overline{\Omega'}}(\hat{y}_h-\bar{y}_h)d\mu_h\\
&= (\bar{y}_h-y_d\,,\,\hat{y}_h-\bar{y}_h)_{L^2(\Omega)}-\int\limits_{\overline{\Omega'}}(\hat{y}_h-y_c)d\mu_h,
\end{aligned}
\tag{5.7}
$$

where we used the complementary slackness conditions (5.5). The analogon to (3.25) is then given by

$$
\|\mu_h\|_{\mathcal{M}(\overline{\Omega'})}\le\frac{1}{\gamma_0}\int\limits_{\overline{\Omega'}}(\hat{y}_h-y_c)d\mu_h \tag{5.8}
$$

and follows again from Lemma 3.8 and the positivity of μ_h. Inserting (5.6) and (5.7) in (5.8) finally yields

$$
\begin{aligned}
\gamma_0\|\mu_h\|_{\mathcal{M}(\overline{\Omega'})}\le{}&\nu\|\bar{u}_h\|_{L^2(\Gamma)}\|\Pi_h\hat{u}\|_{L^2(\Gamma)}+\|\bar{y}_h\|_{L^2(\Omega)}\|y_d\|_{L^2(\Omega)}\\
&+\|\bar{y}_h-y_d\|_{L^2(\Omega)}\|\hat{y}_h\|_{L^2(\Omega)},
\end{aligned}
$$

where the uniform boundedness of the right hand side follows from optimality of $(\bar{y}_h,\bar{u}_h)$ and the stability of Π_h. □

As indicated above, the rest of the theory developed for $(\mathrm{P}_h^\varepsilon)$ immediately carries over to problem (P_h) by setting $\phi(\varepsilon)\equiv\xi(\varepsilon)\equiv 0$ and $\bar{v}_h^\varepsilon\equiv 0$ in the respective estimates. In conclusion, we arrive at the following finite element error estimate for (P_h):

THEOREM 5.4. *Let $(\bar{y},\bar{u})$ and $(\bar{y}_h,\bar{u}_h)$ be the optimal solutions of* (P) *and* (P_h)*, respectively. Then, there exist a positive constant $c>0$, independent of h, and a sufficiently mesh size $h_0<1$ such that*

$$
\|\bar{u}-\bar{u}_h\|_{L^2(\Gamma)}+\|\bar{y}-\bar{y}_h\|_{L^2(\Omega)}\le ch|\log h|^{1/2} \tag{5.9}
$$

is fulfilled for all mesh sizes $0<h\le h_0$.

REMARK 5.5. *In view of the previous result, a regularization of problem* (P) *does not seem to be necessary. However, some type of regularization is needed for several reasons. The adjoint state of problem* (P) *is in general only uniquely determined on the boundary in the case of boundary control. Consequently, a direct application of an active set strategy leads to singular matrices in the algorithm. Moreover, there is no convergence theory available in the case of pure state constraints. In the virtual control approach we have the following situation. It is easy to see that the dual variables are uniquely determined. Moreover, the convergence theory of the semi-smooth Newton method can be directly applied to the regularized problems of the virtual control approach. That means that the regularized problems can be solved efficiently by such a method. Of course, one can solve the unregularized discretized problem by an interior point approach or by the path-following technique of Hintermüller and Kunisch [21]. However, these methods represent also a certain type of regularization techniques.*

6. Numerical example. Our aim is to study the validity of the regularization and discretization error estimates derived in Section 4.3. To this end, we construct an optimal solution for the following optimal control problem with pointwise state constraints and control constraints acting on the boundary:

$$
\left.
\begin{aligned}
\min\quad J(y,u):=\frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)}+\frac{\nu}{2}\|u-u_d\|^2_{L^2(\Gamma)}&\\
-\Delta y+y=f\qquad\quad \text{in }\Omega&\\
\partial_n y=u+g\qquad\quad \text{on }\Gamma&\\
u_a\le u(x)\le u_b\quad \text{a.e. on }\Gamma&\\
y(x)\ge y_c(x)\quad \text{a.e. in }\Omega',&
\end{aligned}
\right\}
\tag{PT}
$$

with given functions y_d, $f\in L^2(\Omega)$, $g\in L^2(\Gamma)$ and $u_d\in H^1(\Gamma)$. Let $\Omega=(0,1)^2$ be the unit square and let $\Omega'=(0.25,0.75)^2$ be an inner square of Ω. We note that the additional functions $f\in L^2(\Omega)$, $g\in$

$L^2(\Gamma)$ and $u_d \in H^1(\Gamma)$ do not influence the theory of the previous sections. By means of appropriate transformations, the problem (PT) can be converted to a problem of type (P).
We are interested in an example, where the Lagrange multiplier μ associated with the state constraints is only a measure along a curve in the domain Ω'. First, we choose

$$\bar{y}(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2), \quad x \in \Omega$$

as the optimal state. Hence, the state equation implies

$$f(x_1, x_2) = (2\pi^2 + 1)\sin(\pi x_1)\sin(\pi x_2), \quad x \in \Omega.$$

In order to fulfill the boundary condition, we define first

$$\tilde{u}(x_1, x_2) = \partial_n \bar{y} = -\pi(\sin(\pi x_1) + \sin(\pi x_2)), \quad x \in \Gamma.$$

The optimal control is given by the pointwise projection on $[u_a, u_b]$:

$$\bar{u}(x_1, x_2) = P_{[u_a,u_b]}\left(\tilde{u}(x_1, x_2)\right), \quad x \in \Gamma.$$

With the help of $g = \tilde{u} - \bar{u}$, the boundary condition of the state equation is satisfied. The lower state constraint y_c is chosen by

$$y_c(x) = \begin{cases} C & , \bar{y}(x) > C \\ 2\bar{y}(x) - C & , \bar{y}(x) \leq C, \end{cases}$$

with the constant $C = 0.9$. Due to this choice, the constraint is only active along the curve $\bar{y} = C$. This implies that the associated Lagrange multiplier is a line-measure concentrated on the curve $\bar{y} = C$. In order to achieve this property, the adjoint state is defined with a kink along this curve:

$$p(x_1, x_2) = \begin{cases} \nu\cos(\pi x_1)\cos(\pi x_2) + C_1(\bar{y}(x_1, x_2) - C), & \bar{y}(x_1, x_2) \geq C \\ \nu\cos(\pi x_1)\cos(\pi x_2), & \bar{y}(x_1, x_2) < C \end{cases}$$

with a constant $C_1 = 0.1$. It is easy to verify that $\bar{p}$ satisfies the homogenous Neumann boundary condition. In the regular parts of the domain Ω the desired state y_d is evaluated by

$$y_d = \Delta p - p + \bar{y}.$$

Finally, the function u_d is defined by

$$u_d(x) = \frac{1}{\nu}\bar{p}(x)|_\Gamma + \tilde{u}(x).$$

In all computations the Tikhonov parameter is chosen by $\nu = 1$. Moreover, the bounds u_a and u_b are set to $u_a = -3$ and $u_b = -0.4$. We will use the discrete framework introduced in Section 3, i.e., we use a regular and uniform triangulation of the domain Ω. Furthermore, all functions were discretized by piecewise linear finite elements. The regularized and discretized analogons ($\mathrm{PT}_h^\varepsilon$) to (P) were solved by a primal-dual active set method, see e.g. [3], [4] or [27]. The numerical solutions of the problems ($\mathrm{PT}_h^\varepsilon$) are denoted by $(\cdot)_h^\varepsilon$. The Figures 6.1–6.5 present the numerical solution of (PT) for a fixed mesh size $h = 0.005$ and a regularization parameter $\varepsilon = 0.05$ connected with the following choice of parameter functions

$$\psi(\varepsilon) \equiv 1, \quad \phi(\varepsilon) = \varepsilon, \quad \xi(\varepsilon) = \varepsilon.$$

Notice that the control in Figure 6.1 is shown only on one part of the boundary. As one can see, the Lagrange multiplier and the virtual control exhibit some irregularities, especially in the active regions around the curve $\{x \in \Omega \,|\, \bar{y}(x) = C\}$.

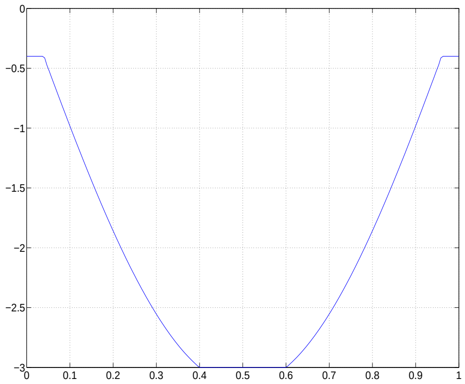

FIG. 6.1. *Control* u_ε

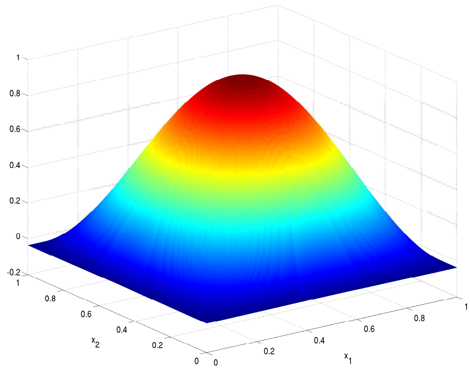

FIG. 6.2. *State* y_ε

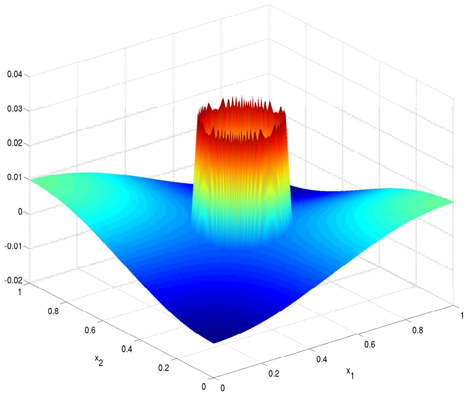

FIG. 6.3. *Virtual control* v_ε

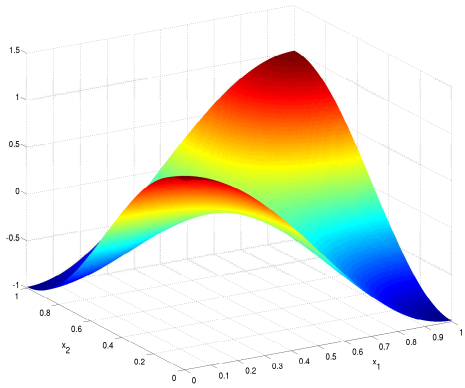

FIG. 6.4. *Adjoint state* p_ε

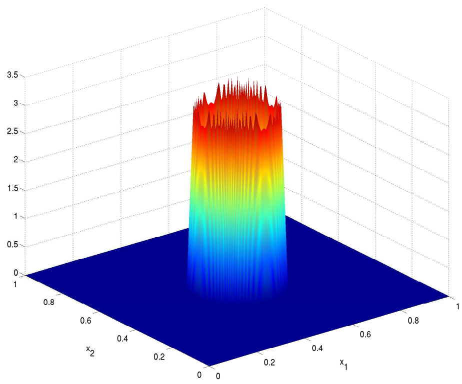

FIG. 6.5. *Lagrange multiplier* μ_ε

Next, we observe the regularization and discretization error for the following choice of parameter functions:

$$\phi(\varepsilon) \equiv 1, \quad \xi(\varepsilon) \equiv 1, \quad \psi(\varepsilon) = \varepsilon^{-2}.$$

To fulfill Assumption 4.9, we have to couple the regularization parameter ε and the mesh size h as follows:

$$\varepsilon \sim h^3 |\log h|^{3/2}.$$

Based on this setting, the results of Theorem 4.11 and Corollary 4.10 predict the following convergence behavior for u and v

$$\|\bar{u} - u_h^\varepsilon\|_{L^2(\Gamma)} = \mathcal{O}(h|\log h|^{1/2}) \quad \text{and} \quad \|v_h^\varepsilon\|_{L^2(\Omega)} = \mathcal{O}(h^4 |\log h|^2).$$

The numerical results are presented in Figure 6.6. Note that we omitted the logarithmic parts in the

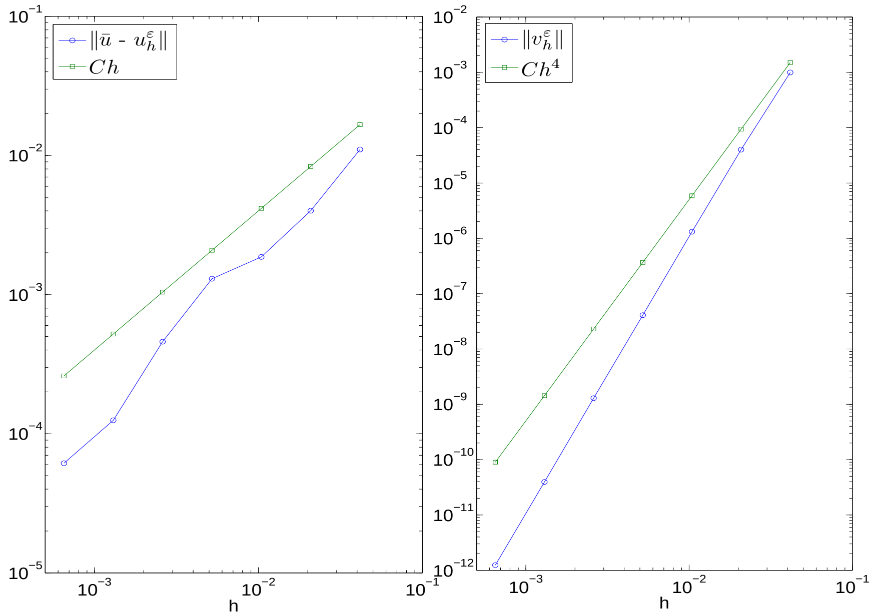

FIG. 6.6. $\|\bar{u} - u_h^\varepsilon\|_{L^2(\Gamma)}$ *and* $\|v_h^\varepsilon\|_{L^2(\Omega)}$

h	ε	$\|\bar{u} - \bar{u}_h^\varepsilon\|_{L^2(\Gamma)}$	r_u	$\|\bar{v}_h^\varepsilon\|_{L^2(\Omega)}$	r_v
2^{-4}	$2.8935e-2$	$1.1043e-2$	1.25	$9.9821e-4$	4.93
2^{-5}	$3.6169e-3$	$4.0084e-3$	1.21	$4.0178e-5$	4.99
2^{-6}	$4.5211e-4$	$1.8653e-3$	1.23	$1.3128e-6$	5.00
2^{-7}	$5.6514e-5$	$1.3016e-3$	1.47	$4.0859e-8$	5.00
2^{-8}	$7.0643e-6$	$4.5900e-4$	1.45	$1.2913e-9$	5.01
2^{-9}	$8.8303e-7$	$1.2495e-4$	1.02	$3.9496e-11$	4.99
2^{-10}	$1.1038e-7$	$6.1408e-5$	–	$1.2387e-12$	–

TABLE 6.1
Errors and experimental order of convergence

particular reference curves. Furthermore, the experimental orders of convergence with respect to h are presented in Table 6.1. We observe a higher order convergence in the numerical test than expected by the theoretical predictions. This especially concerns the virtual control rather than the error in the boundary control. Up to now there is no theoretical explanation for this observation, and we feel that a significantly different analysis is necessary to explain this effect. Such a difference between theoretically predicted and numerically observed convergence rates is also recorded in many other contributions on finite element error analysis for state-constrained problems. We only mention the numerical results in [17].

Appendix A. Proof of Theorem 4.2.

Before proving the assertion of Theorem 4.2, we will establish two auxiliary results.

LEMMA A.1. *Let the assumptions of Theorem 4.2 be fulfilled. Then, the following estimate*

$$\begin{aligned}(\bar{y} - \bar{y}_h^\varepsilon\,,\, S_h\tau^*(\bar{u}_h^\varepsilon - \bar{u}))_{L^2(\Omega)} = -\|\bar{y} - \bar{y}_h^\varepsilon\|^2 + (\bar{y} - \bar{y}_h^\varepsilon\,,\,(S - S_h)\tau^*\bar{u})_{L^2(\Omega)} \\ + (E_H S_h^*(\bar{y} - \bar{y}_h^\varepsilon)\,,\, -\phi(\varepsilon)\bar{v}_h^\varepsilon)_{L^2(\Omega)}\end{aligned} \tag{A.1}$$

is valid.

Proof. Due to $\bar y = S\tau^*\bar u$ and $\bar y_h^\varepsilon = S_h(\tau^*\bar u_h^\varepsilon + \phi(\varepsilon)E_H^*\bar v_h^\varepsilon)$, one obtains

$$\begin{aligned}(\bar y - \bar y_h^\varepsilon\,,\, S_h\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)} &= (\bar y - \bar y_h^\varepsilon\,,\, S_h(\tau^*\bar u_h^\varepsilon + E_H^*\phi(\varepsilon)\bar v_h^\varepsilon) - S_h\tau^*\bar u)_{L^2(\Omega)}\\ &\quad - (\bar y - \bar y_h^\varepsilon\,,\, S_hE_H^*\phi(\varepsilon)\bar v_h^\varepsilon)_{L^2(\Omega)}\\ &= (\bar y - \bar y_h^\varepsilon\,,\, \bar y_h^\varepsilon - \bar y)_{L^2(\Omega)} + (\bar y - \bar y_h^\varepsilon\,,\, \bar y - S_h\tau^*\bar u)_{L^2(\Omega)}\\ &\quad + (E_HS_h^*(\bar y - \bar y_h^\varepsilon)\,,\, -\phi(\varepsilon)\bar v_h^\varepsilon)_{L^2(\Omega)}\\ &= -\|\bar y - \bar y_h^\varepsilon\|^2 + (\bar y - \bar y_h^\varepsilon\,,\, (S - S_h)\tau^*\bar u)_{L^2(\Omega)}\\ &\quad + (E_HS_h^*(\bar y - \bar y_h^\varepsilon)\,,\, -\phi(\varepsilon)\bar v_h^\varepsilon)_{L^2(\Omega)},\end{aligned}$$

which is the assertion. ∎

LEMMA A.2. *Suppose that the assumptions of Theorem 4.2 are satisfied. Then, there holds*

$$\begin{aligned}(\tau(\bar p - \bar p_h^\varepsilon)\,,\, \bar u_h^\varepsilon - \bar u)_{L^2(\Gamma)} &= -\|\bar y - \bar y_h^\varepsilon\|^2 + (\tau(S^* - S_h^*)(\bar y - y_d)\,,\, \bar u_h^\varepsilon - \bar u)_{L^2(\Gamma)}\\ &\quad + (\bar y - \bar y_h^\varepsilon\,,\, (S - S_h)\tau^*\bar u)_{L^2(\Omega)}\\ &\quad + (E_HS_h^*(\bar y - \bar y_h^\varepsilon)\,,\, -\phi(\varepsilon)\bar v_h^\varepsilon)_{L^2(\Omega)},\end{aligned}$$

where $\bar p$ and $\bar p_h^\varepsilon$ are the associated adjoint states defined in Lemma 4.1.

Proof. Due to the definitions

$$\begin{aligned}\bar p &= S^*(S\tau^*\bar u - y_d)\\ \bar p_h^\varepsilon &= S_h^*(S_h(\tau^*\bar u_h^\varepsilon + E_H^*\phi(\varepsilon)\bar v_h^\varepsilon) - y_d)\end{aligned}$$

of the adjoint states, we continue with

$$\begin{aligned}&(\tau(\bar p - \bar p_h^\varepsilon)\,,\, \bar u_h^\varepsilon - \bar u)_{L^2(\Gamma)}\\ &\quad = -(\tau(S^* - S_h^*)y_d\,,\, \bar u_h^\varepsilon - \bar u)_{L^2(\Gamma)}\\ &\qquad + (\tau S^*S\tau^*\bar u - \tau S_h^*(S_h(\tau^*\bar u_h^\varepsilon + \phi(\varepsilon)E_H^*\bar v_h^\varepsilon))\,,\, \bar u_h^\varepsilon - \bar u)_{L^2(\Gamma)}.\end{aligned}$$

The last term in the right hand side can be rewritten as:

$$\begin{aligned}&(S\tau^*\bar u\,, S\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)} - (S_h(\tau^*\bar u_h^\varepsilon + \phi(\varepsilon)E_H^*\bar v_h^\varepsilon)\,,\, S_h\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)}\\ &\quad = (S\tau^*\bar u\,,\, S\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)} - (S\tau^*\bar u\,,\, S_h\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)}\\ &\qquad + (S\tau^*\bar u\,,\, S_h\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)} - (S_h(\tau^*\bar u_h^\varepsilon + \phi(\varepsilon)E_H^*\bar v_h^\varepsilon)\,,\, S_h\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)}\\ &\quad = (S\tau^*\bar u\,,\, (S - S_h)\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)}\\ &\qquad + (S\tau^*\bar u - S_h(\tau^*\bar u_h^\varepsilon + E_H^*\phi(\varepsilon)\bar v_h^\varepsilon)\,,\, S_h\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)}\\ &\quad = (\bar y\,,\, (S - S_h)\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)} + (\bar y - \bar y_h^\varepsilon\,,\, S_h\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)}.\end{aligned}$$

By using Lemma A.1 for the last term, we conclude

$$\begin{aligned}&(\tau(\bar p - \bar p_h^\varepsilon)\,,\, \bar u_h^\varepsilon - \bar u)_{L^2(\Gamma)}\\ &\quad = -(\tau(S^* - S_h^*)y_d\,,\, \bar u_h^\varepsilon - \bar u)_{L^2(\Gamma)} + (\bar y\,,\, (S - S_h)\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)} - \|\bar y - \bar y_h^\varepsilon\|^2\\ &\qquad + (\bar y - \bar y_h^\varepsilon\,,\, (S - S_h)\tau^*\bar u)_{L^2(\Omega)} + (E_HS_h^*(\bar y - \bar y_h^\varepsilon)\,,\, -\phi(\varepsilon)\bar v_h^\varepsilon)_{L^2(\Omega)}.\end{aligned}$$

The first two terms in the previous right hand side simplify to

$$\begin{aligned}&(\bar y\,,\, (S - S_h)\tau^*(\bar u_h^\varepsilon - \bar u))_{L^2(\Omega)} - (\tau(S^* - S_h^*)y_d\,,\, \bar u_h^\varepsilon - \bar u)_{L^2(\Gamma)}\\ &\qquad\qquad = (\tau(S^* - S_h^*)(\bar y - y_d)\,,\, \bar u_h^\varepsilon - \bar u)_{L^2(\Gamma)}.\end{aligned}$$

This completes the proof. ∎

Now, we are in the position to prove the Theorem 4.2.

Proof of Theorem 4.2. We start with the variational inequalities of (P) and (P_h^ε) for $u := u^\delta \in U_{ad}$ and $(u,v) := (u_h^\sigma, 0) \in V_{ad,h}^{\varepsilon,2}$, see (4.1) and (4.2), respectively. Adding both inequalities yields together with straight forward computations

$$\begin{aligned} \nu \|\bar u - \bar u_h^\varepsilon\|_{L^2(\Gamma)}^2 + \psi(\varepsilon)\|\bar v_h^\varepsilon\|^2 \le (\tau\bar p + \nu\bar u\,,\, u^\delta - \bar u_h^\varepsilon)_{L^2(\Gamma)} + (\tau\bar p_h^\varepsilon + \nu\bar u_h^\varepsilon\,,\, u_h^\sigma - \bar u)_{L^2(\Gamma)} \\ + (\tau(\bar p - \bar p_h^\varepsilon)\,,\, \bar u_h^\varepsilon - \bar u)_{L^2(\Gamma)} + (\phi(\varepsilon)E_H\bar p_h^\varepsilon\,,\, -\bar v_h^\varepsilon)_{L^2(\Omega)} \end{aligned}$$

for all $u^\delta \in U_{ad}$ and $(u_h^\sigma, 0) \in V_{ad,h}^{\varepsilon,2}$. It remains to consider the last two terms of the previous inequality. Thanks to Lemma A.2, we obtain:

$$\begin{aligned} \nu \|\bar u - &\bar u_h^\varepsilon\|_{L^2(\Gamma)}^2 + \|\bar y - \bar y_h^\varepsilon\|_{L^2(\Omega)}^2 + \psi(\varepsilon)\|\bar v_h^\varepsilon\|_{L^2(\Omega)}^2 \\ &\le (\tau\bar p + \nu\bar u\,,\, u^\delta - \bar u_h^\varepsilon)_{L^2(\Gamma)} + (\tau\bar p_h^\varepsilon + \nu\bar u_h^\varepsilon\,,\, u_h^\sigma - \bar u)_{L^2(\Gamma)} \\ &\quad + (\tau(S^* - S_h^*)(\bar y - y_d)\,,\, \bar u_h^\varepsilon - \bar u)_{L^2(\Gamma)} + (\bar y - \bar y_h^\varepsilon\,,\, (S - S_h)\tau^*\bar u)_{L^2(\Omega)} \\ &\quad + (E_H S_h^*(\bar y - \bar y_h^\varepsilon)\,,\, -\phi(\varepsilon)\bar v_h^\varepsilon)_{L^2(\Omega)} + (\phi(\varepsilon)E_H\bar p_h^\varepsilon\,,\, -\bar v_h^\varepsilon)_{L^2(\Omega)}. \end{aligned} \tag{A.2}$$

Using $\bar p_h^\varepsilon = S_h^*(\bar y_h^\varepsilon - y_d)$, the last two terms simplify to

$$(E_H S_h^*(\bar y - \bar y_h^\varepsilon) + E_H\bar p_h^\varepsilon\,,\, -\phi(\varepsilon)\bar v_h^\varepsilon)_{L^2(\Omega)} = (E_H S_h^*(\bar y - y_d)\,,\, -\phi(\varepsilon)\bar v_h^\varepsilon)_{L^2(\Omega)}.$$

Moreover, we apply Young's inequality such that

$$(E_H S_h^*(\bar y - y_d)\,,\, -\phi(\varepsilon)\bar v_h^\varepsilon)_{L^2(\Omega)} \le \frac{\phi(\varepsilon)^2}{2\psi(\varepsilon)}\|E_H S_h^*(\bar y - y_d)\|_{L^2(\Omega)}^2 + \frac{\psi(\varepsilon)}{2}\|\bar v_h^\varepsilon\|_{L^2(\Omega)}^2.$$

Again using Young's inequality to the third and the fourth term in (A.2), we derive

$$\begin{aligned} \frac{\nu}{2}\|\bar u - \bar u_h^\varepsilon\|_{L^2(\Gamma)}^2 &+ \frac12\|\bar y - \bar y_h^\varepsilon\|_{L^2(\Omega)}^2 + \frac{\psi(\varepsilon)}{2}\|\bar v_h^\varepsilon\|_{L^2(\Omega)}^2 \\ &\le (\tau\bar p + \nu\bar u\,,\, u^\delta - \bar u_h^\varepsilon)_{L^2(\Gamma)} + (\tau\bar p_h^\varepsilon + \nu\bar u_h^\varepsilon\,,\, u_h^\sigma - \bar u)_{L^2(\Gamma)} \\ &\quad + \frac{1}{2\nu}\|\tau(S^* - S_h^*)(\bar y - y_d)\|_{L^2(\Gamma)}^2 + \frac12\|(S - S_h)\tau^*\bar u\|_{L^2(\Omega)}^2 \\ &\quad + \frac{\phi(\varepsilon)^2}{2\psi(\varepsilon)}\|E_H S_h^*(\bar y - y_d)\|_{L^2(\Omega)}^2. \end{aligned}$$

Standard finite element error estimates now yield

$$\frac{1}{2\nu}\|\tau(S^* - S_h^*)(\bar y - y_d)\|_{L^2(\Gamma)}^2 + \frac12\|(S - S_h)\tau^*\bar u\|_{L^2(\Omega)}^2 \le ch^3(\|\bar y - y_d\|_{L^2(\Omega)} + \|\bar u\|_{L^2(\Gamma)}),$$

cf. also Theorem 3.2 (i). Due to optimality, the remaining norms are bounded by the objective of the original problem (P). Finally, the embedding operator E_H and the adjoint of the discrete solution operator are linear and continuous such that the term $\|E_H S_h^*(\bar y - y_d)\|_{L^2(\Omega)}$ is bounded by constant independent of h and ε. □

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