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für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Quasi-static contact problem with finitely many degrees of freedom and dry friction

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submitted: June 9, 2008

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No. 1336
Berlin 2008

2000 *Mathematics Subject Classification.* 74M10,74M15,74B20,74H20.

Key words and phrases. Coulomb friction, varying friction coefficient, unilateral contact, quasistatic, energetic formulation, nonlinear elasticity, existence, finite-dimensional.

Research partially supported by DFG via SFB404 “Multifield Problems”, subproject C7, and by EU via the Research Training Network “Smart Systems” (HPRN-CT-2002-00284).

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Abstract

A quasi-static contact problem is considered for a non-linear elastic system with finitely many degrees of freedom. Coulomb's law is used to model friction and the friction coefficient may be anisotropic and may vary along the surface of the rigid obstacle. Existence is established following a time-incremental minimization problem. Friction is artificially decreased to resolve the discontinuity arising from making and losing contact.

1 Introduction

The question of existence for quasi-static contact problems in elasticity with dry (Coulomb) friction reached a peak in the literature in the 1990s. The first existence results for the continuous problem of an elastic body were due to Klarbring, Mikelić and Shillor [KMS91], where still regularized problems are considered. It was Andersson [And00] and Rocca [Roc01] who solved the complete model on the continuous level. Andersson [And99] also solved the problem on the discrete level for N particles. Touzaline at al. still extended the continuous theory to non-linear elastic bodies [ToT07]. But most authors have started to expand their models principally including thermal effects, see [EcJ01, ChA04]. The energetic approach, which we use as a method, has been recently adapted such that thermal effects can be included, see [MiP07, Mie07]. Other extensions include non-coercivity of the elastic energy [RiA06], wear [SST04] and damage [HSS01].

In this article we revisit the classical quasi-static contact problem with dry friction and extend the problem to non-linear elasticity, similar to Touzaline at al., but we restrict ourself to the discrete level only. In [ScM07] we considered a non-linear elastic system with three degrees of freedom (dof). Further friction is allowed to be anisotropic and the friction coefficient may vary along the flat obstacle surface. In this article we extend this result to finitely many dof and use significantly weaker assumptions to establish existence. These improved assumptions allow us to extend the existence result to situations of curved obstacles, which will be done in forthcoming work, see also [Sch08].

The structure of the article is as follows. In Section 2 we introduce the model and a problem formulation, which is based on energies only. In the Section 3 we present the main idea: a simplified problem that avoids the discontinuity of the frictional terms, which is one of the main difficulties. In the Section 4 existence is established for simplified problems following the energetic approach introduced by Mielke and Theil [MiT99, Mie05]. In Section 5 we solve the original problem by passing to the limit in the simplified problems.

2 Modeling

In this section we present the mechanical model, fix the notation and present the main result.

We consider an elastic system, which is determined by the position of N particles thus having

$3N$ degrees of freedom. The state of the system is described by the vector

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \cdots \\ z_{3N} \end{pmatrix} = \begin{pmatrix} z^{(1)} \\ \cdots \\ z^{(N)} \end{pmatrix} \quad \in \mathbb{R}^{3N} \cong \left(\mathbb{R}^3\right)^N .$$

For $j = 1, \dots, N$ we denote by $z^{(j)} = \begin{pmatrix} z_{3j-2} & z_{3j-1} & z_{3j} \end{pmatrix}^\top$ the state of the j-th particle. Each single particle cannot penetrate a rigid obstacle and thus is restricted to an admissible set of the form

$$\mathcal{S} := \left\{ z \in \mathbb{R}^3 \ : \ z_3 \geq 0 \right\},$$

i.e., we claim $z^{(j)} \in \mathcal{S}$ for $j = 1, \dots, N$. Further, to keep notation simple, it is convenient to introduce for $j = 1, \dots, N$ the half-spaces

$$\mathcal{A}^{(j)} := \left\{ z \in \mathbb{R}^{3N} \ : \ z_{3j} \geq 0 \right\}.$$

With the help of these sets we define the admissible set $\mathcal{A} \subset \mathbb{R}^{3N}$ for the whole system in three equivalent ways via

$$\begin{aligned} \mathcal{A} &:= \left\{ z \in \mathbb{R}^{3N} \ : \ z_{3j} \geq 0 \quad \text{for all } j = 1, \dots, N \right\} \\ &= \left\{ z \in \mathbb{R}^{3N} \ : \ z^{(j)} \in \mathcal{S} \quad \text{for all } j = 1, \dots, N \right\} = \mathcal{S}^N \\ &= \bigcap_{j \in \{1, \dots, N\}} \mathcal{A}^{(j)}. \end{aligned} \tag{N0}$$

As announced we model the problem in terms of energies. The stored elastic energy of the system is described by an energy functional $\mathcal{E}$ and we assume

$$\mathcal{E} \in \mathrm{C}^2 \left([0, T] \times \mathcal{A}, \mathbb{R} \right). \tag{N1}$$

Also friction is modeled in terms of energy and we first describe the energy, which is dissipated by a single particle $z^{(j)}$. For $j = 1, \dots, N$ we denote by

$$\sigma^{(j)}(t, z) := \partial_{z_{3j}} \mathcal{E}(t, z)$$

the normal force with which the j-th particle is pressed onto the obstacle. For the roughness, which affects each single particle, we introduce matrices of friction $\mathrm{M}^{(j)}$. The dependence on an index j indicates that different particles might consist of different materials and thus behave different while sliding along the same obstacle. We denote by $\nu := \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}^\top$ the normal vector and by $\partial \mathcal{S} = \left\{ z \in \mathbb{R}^3 \ : \ z_3 = 0 \right\}$ the obstacle surface. For $j = 1, \dots, N$ we assume the matrix of friction M^j to satisfy

$$\mathrm{M}^{(j)} \in \mathrm{C}^1 \left(\partial \mathcal{S}, \mathbb{R}^{3 \times 3} \right) \quad \text{with} \quad \mathrm{M}^{(j)}(z) \nu = 0 \text{ for all } z \in \partial \mathcal{S}. \tag{N2}$$

According to Coulomb's law the frictional force is proportional to normal force times roughness. This motivates the definition of a single dissipation functional $\Psi^{(j)} : [0, T] \times \mathcal{A} \times \mathbb{R}^3 \to [0, \infty)$ via

$$\Psi^{(j)}(t, z, v) := \begin{cases} \sigma^{(j)}(t, z)_+ \, \| \mathrm{M}^{(j)}(z^{(j)}) v \| & \text{if } z_{3j} = 0, \\ 0 & \text{if } z_{3j} > 0, \end{cases} \tag{2.1}$$

with $\sigma^{(j)}(t, z)_+ := \max\{0, \sigma^{(j)}(t, z)\}$ and $\| \cdot \|$ being the usual Euclidian norm. Thus for $\|v\| \ll 1$ the expression $\Psi^{(j)}(t, z, v)$ is a rough approximation of the energy lost due to friction for a particle sliding instantaneously from $z^{(j)}$ to $z^{(j)}+v$ at time t. It is only an approximation since along the sliding vector $v \in \mathbb{R}^3$ we assume the frictional force to be constant.

The dissipation functional $\Psi : [0,T] \times \mathcal{A} \times \mathbb{R}^{3N} \to [0,\infty)$ for the whole system is now defined via

$$\Psi(t,z,v) := \sum_{j=1}^{N} \Psi^{(j)}(t,z,v^{(j)}).$$

After having introduced the energy and dissipation functional we now present our problem to solve.

Problem 2.1 *For given initial state $Z_0 \in \mathcal{A}$ find a positive time $T_* \in (0,T]$ and a solution $z \in \mathrm{W}^{1,\infty}([0,T_*],\mathcal{A})$ such that the initial condition $z(0) = Z_0$ is satisfied and such that for all $t \in [0,T_*]$ the following two conditions hold*

$$\mathcal{E}(t,z(t)) \le \mathcal{E}(t,y) + \Psi(t,z(t),y-z(t)) \quad \textit{for all } y \in \mathcal{A} \quad \textit{and} \tag{S}$$

$$\mathcal{E}(t,z(t)) + \int_0^t \Psi(s,z(s),\dot z(s))\,\mathrm{d}s = \mathcal{E}(0,z(0)) + \int_0^t \partial_s\mathcal{E}(s,z(s))\,\mathrm{d}s. \tag{E}$$

We briefly motivate these two energetic conditions, see also [MiT04]. The energetic stability claims that at time t our system is in such a position $z(t)$ that switching instantaneously to any other admissible position $y \in \mathcal{A}$ dissipates more energy due to friction than is released by the elastic system. The equality (E) is an energy balance law and claims that the stored energy at time t plus the dissipated energy up to time t is the same than the energy we start with plus the energy put into the system due to changing external forces.

A direct consequence of the problem formulation is that we have to claim the stability condition (S) for the initial value Z_0 at time $t = 0$, i.e.

$$\mathcal{E}(0,Z_0) \le \mathcal{E}(0,y) + \Psi(0,Z_0,y-Z_0) \quad \text{for all } y \in \mathcal{A}. \tag{N3}$$

Further let us introduce the characteristic function $\mathcal{X}_{\mathcal{A}}(z) := 0$ for $z \in \mathcal{A}$ and $\mathcal{X}_{\mathcal{A}}(z) := +\infty$ for $z \notin \mathcal{A}$. If we denote by '∂ ' the subdifferential and by $\mathbb{R}^{3N^*} \cong \mathbb{R}^{1\times 3N}$ the dual space of $\mathbb{R}^{3N}$ then the above (S)&(E)-formulation, which has to hold for all $t \in [0,T_*]$, is equivalent to claim the following differential inclusion or force balance for almost all $t \in [0,T_*]$

$$0 \in \underbrace{\mathrm{D}\mathcal{E}(t,z(t))}_{\text{elastic force}} + \underbrace{\partial_v\Psi(t,z(t),\dot z(t))}_{\text{frictional forces}} + \underbrace{\partial\mathcal{X}_{\mathcal{A}}(z(t))}_{\text{constraint forces}} \quad \subset \mathbb{R}^{3N^*}. \tag{DI}$$

For a proof see [MiT04].

To present the last assumption and the existence theorem we have to distinguish three different 'types' of particles. Depending on the initial values 0 and Z_0 we define the index sets:

$$\begin{aligned}
\mathcal{I} &:= \{1,\dots,N\},\\
\mathcal{I}_{\mathrm{c}} &:= \left\{ j \in \mathcal{I} \,:\, \sigma^{(j)}(0,Z_0) > 0 \right\},\\
\mathcal{I}_{\mathrm{f}} &:= \left\{ j \in \mathcal{I} \,:\, (Z_0)_{3j} > 0 \right\} \quad \text{and}\\
\mathcal{I}_{\mathrm{s}} &:= \mathcal{I} \setminus (\mathcal{I}_{\mathrm{c}} \cup \mathcal{I}_{\mathrm{f}}).
\end{aligned}$$

Due to the initial assumption (N3) we find $\mathcal{I}_{\mathrm{c}} \cap \mathcal{I}_{\mathrm{f}} = \emptyset$ and the sets $\mathcal{I}_{\mathrm{c}}, \mathcal{I}_{\mathrm{f}}$ and $\mathcal{I}_{\mathrm{s}}$ are pairwise disjoint. Here, the index 'c' stands for *contact*, 'f' for *friction-free* and 's' stands for *switching.*

Due to (N3) we have for all $j \in \mathcal{I}_{\mathrm{c}}$ that the corresponding particle $(Z_0)^{(j)} \in \mathbb{R}^3$ is in contact, i.e. $({Z_0}^{(j)})_3 = 0$. Since it is pressed onto the obstacle with a positive normal force $\sigma^{(j)}(0, Z_0) > 0$ we expect it to remain in contact for a short time span. Hence, the index set $\mathcal{I}_{\mathrm{c}}$ represents all particles, which are in contact and are expected to remain in contact. Further, the index 'f' denotes *friction-free*. By definition we find $\Psi^{(j)}(0, Z_0, \cdot) \equiv 0$ for $j \in \mathcal{I}_{\mathrm{f}}$ and since $(Z_0)_{3j} > 0$ holds we also expect the particle to remain out of contact and thus the index set $\mathcal{I}_{\mathrm{f}}$ denotes all particles, which start friction-free and are expected to remain friction-free for a small time span. The index 's' shall represent the word *switching* since for $j \in \mathcal{I}_{\mathrm{s}} = \left\{ j \in \mathcal{I} \;:\; \sigma^{(j)}(0, Z_0) = 0, (Z_0)_{3j} = 0 \right\}$ it is possible that the particle switches for arbitrary small time spans infinitely many times between a contact situation with friction and a friction-free situation.

For given index set $\mathcal{K} \subset \mathcal{I}$ (the case $\mathcal{K} = \emptyset$ is allowed!) we define the admissible set for the sliding vectors via $\mathbb{S}^{3N-1}_{\mathcal{K}} := \left\{ v \in \mathbb{R}^{3N} \;:\; \|v\| = 1, v_{3j} = 0, j \in \mathcal{K} \right\} = \left\{ v \in \mathbb{S}^{3N-1} \;:\; v_{3j} = 0, j \in \mathcal{K} \right\}$ and the constants $\alpha_{\mathcal{K}} > 0$ and $\mathrm{q}_{\mathcal{K}} > 0$ via

$$\alpha_{\mathcal{K}} := \min \left\{ v^\top \mathrm{H}(0, Z_0) v \;:\; v \in \mathbb{S}^{3N-1}_{\mathcal{K}} \right\}, \tag{2.2}$$

$$\begin{aligned} \mathrm{q}_{\mathcal{K}} := \max \Big\{ \sum_{j \in \mathcal{K}} & \left| \mathrm{D}\sigma^{(j)}(0, Z_0) u \right| \, \|\mathrm{M}^{(j)}({Z_0}^{(j)}) v^{(j)}\| \\ & + \sigma^{(j)}(0, Z_0)_+ \, \|\mathrm{DM}^{(j)}({Z_0}^{(j)})[v^{(j)}, u^{(j)}]\| \;:\; u, v \in \mathbb{S}^{3N-1}_{\mathcal{K}} \Big\}. \end{aligned} \tag{2.3}$$

For $\mathcal{K}_1 \subset \mathcal{K}_2$ we find $\mathbb{S}^{3N-1}_{\mathcal{K}_1} \supset \mathbb{S}^{3N-1}_{\mathcal{K}_2}$ and $\alpha_{\mathcal{K}_1} \leq \alpha_{\mathcal{K}_2}$ while no monotonicity holds for $\mathrm{q}_{\mathcal{K}}$. With increasing index set $\mathcal{K}$ the number of summands increases but the values of the single summands decrease.

Theorem 2.2 (Existence) *Let us assume* (N0)–(N3). *For all index sets $\mathcal{I}_{\mathrm{c}} \subset \mathcal{K} \subset \mathcal{I}_{\mathrm{c}} \cup \mathcal{I}_s$ we further assume*

$$\alpha_{\mathcal{K}} > \mathrm{q}_{\mathcal{K}}. \tag{N4}$$

Then there exists a time $T_* > 0$ and a solution $z \in \mathrm{W}^{1,\infty}\left([0, T_*], \mathcal{A}\right)$ of Problem 2.1.

We prove existence in Theorem 5.2 below.

The assumption (N4) is composed by 2^n single assumptions with $n = \#\mathcal{I}_{\mathrm{s}}$. For example we face 2^N assumptions if all particles are in grazing contact, i.e. $\mathcal{I}_rms = \mathcal{I}$. This system of assumptions in not reduceable as simple examples indicate, see [Sch08].

In the foregoing paper [ScM07] the case $N = 1$ was considered under the strong assumption $\alpha_\emptyset > \mathrm{q}_{\{1\}}$. Due to $\alpha_{\{1\}} \geq \alpha_\emptyset$ this strong assumption implies (N4), which is composed by $\alpha_\emptyset > 0$ and $\alpha_{\{1\}} > \mathrm{q}_{\{1\}}$. The present weaker assumption (N4) is a major step in deriving conditions for existence that are necessary and sufficient. While the strong assumption $\alpha_\emptyset > \mathrm{q}_{\{1\}}$ links normal displacements ($v_3 \neq 0$) with frictional effects the improved assumption $\alpha_{\{1\}} > \mathrm{q}_{\{1\}}$ controls the change of elastic and frictional forces only in tangential directions $v_3 = 0$. This is needed if one wants to understand what happens if the boundary is curved and thus tangential directions change, see [Sch08].

The following section presents the central idea, which is needed for this improvement.

3 Decreasing the friction: a simplified problem

In this section we introduce a restricted admissible set and a new dissipation functional, which helps to avoid the discontinuity of Ψ. This will lead us to a simplified problem.

3.1 Extending the assumptions to some neighborhood

In this subsection we extend the Assumption (N4) to some neighborhood of of the initial values $(0, Z_0)$. The index 'e' stands for *extended*. The parameter $l_{\mathrm{e}} > 0$ denotes half of the edge length of the cuboid

$$\mathcal{Q}_{l_{\mathrm{e}}}(Z_0) := \left\{ z \in \mathbb{R}^{3N} \,:\, |(z - Z_0)_n| \le l_{\mathrm{e}}, n = 1, \dots, 3N \right\}.$$

We favor a cuboid over a ball as neighborhood since the cuboid can be described as an intersection of half-spaces. This will make the formula for corresponding constraint forces $\partial \mathcal{X}_{\mathcal{Q}_{l_{\mathrm{e}}}}$ easier. For a given index set $\mathcal{K} \subset \mathcal{I}$ we introduce the convex set

$$\mathcal{A}_{\mathcal{K},\mathrm{e}} := \mathcal{A} \cap \left(\bigcap_{j \in \mathcal{K}} \partial \mathcal{A}^{(j)} \right) \cap \mathcal{Q}_{l_{\mathrm{e}}}(Z_0).$$

On physical grounds it is intuitively clear that particles, which are pressed onto the obstacle remain locally in contact. This motivated the restriction to the boundary for some index set $\mathcal{K}$. In Figure 1 we present this idea for an elastic system in its initial configuration $Z_0 \in \mathbb{R}^{3N}$

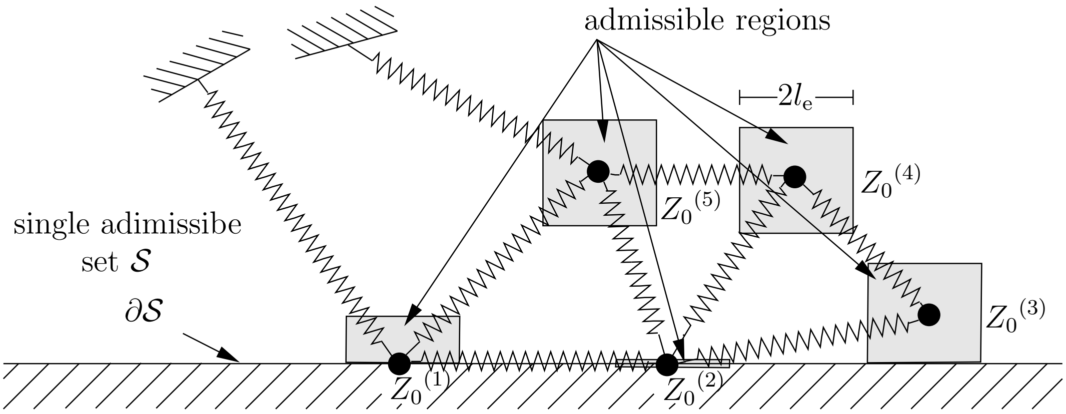

Figure 1: Admissible regions for $z \in \mathcal{A}_{\mathcal{K},\mathrm{e}} \subset \mathbb{R}^{3N}$ with $\mathcal{K} = \{2\}$ and $N = 5$

satisfying $\sigma^{(1)}(0, Z_0) = 0$ and $\sigma^{(2)}(0, Z_0) > 0$. If $z \in \mathcal{A}_{\mathcal{K},\mathrm{e}}$ holds then each single particle $z^{(j)} \in \mathcal{S} \subset \mathbb{R}^3$ is constrained to its grey box or plane.

We denote the extension time $T_{\mathrm{e}} \in (0, T]$ and introduce the cylinder $\mathcal{C}_{\mathcal{K},\mathrm{e}} := [0, T_{\mathrm{e}}] \times \mathcal{A}_{\mathcal{K},\mathrm{e}}$. The following constants are extended versions of the constants $\alpha_{\mathcal{K}} > 0$ and $\mathrm{q}_{\mathcal{K}} > 0$ defined in (2.2) and (2.3). They coincide if we choose $T_{\mathrm{e}} = l_{\mathrm{e}} = 0$.

$$\alpha_{\mathcal{K},\mathrm{e}} := \min \left\{ v^\top \mathrm{H}(t, z) v \,:\, v \in \mathbb{S}^{3N-1}_{\mathcal{K}}, (t, z) \in \mathcal{C}_{\mathcal{K},\mathrm{e}} \right\}, \tag{3.1}$$

$$\begin{aligned} \mathrm{q}_{\mathcal{K},\mathrm{e}} := \max_{u,v \in \mathbb{S}^{3N-1}_{\mathcal{K}}} \Big\{ \sum_{j \in \mathcal{K}} \left\| \mathrm{D}\sigma^{(j)} u \right\|_{\mathrm{L}^\infty(\mathcal{C}_{\mathcal{K},\mathrm{e}}, \mathbb{R})} \|\mathrm{M}^{(j)}(\cdot^{(j)}) v^{(j)}\|_{\mathrm{L}^\infty(\mathcal{A}_{\mathcal{K},\mathrm{e}}, \mathbb{R}^3)} \quad (3.2) \\ + \|\sigma^{(j)}_+\|_{\mathrm{L}^\infty(\mathcal{C}_{\mathcal{K},\mathrm{e}}, \mathbb{R})} \|\mathrm{D}\mathrm{M}^{(j)}(\cdot^{(j)})[v^{(j)}, u^{(j)}]\|_{\mathrm{L}^\infty(\mathcal{A}_{\mathcal{K},\mathrm{e}}, \mathbb{R}^3)} \Big\}. \end{aligned}$$

Due to the regularity of $\mathcal{E}$ and $\mathrm{M}^{(j)}$, see (N1) and (N2), and Assumption (N4) there exist $l_{\mathrm{e}} > 0$ and $T_{\mathrm{e}} > 0$ such that for all $\mathcal{K} \subset \mathcal{I}$ with $\mathcal{I}_{\mathrm{c}} \subset \mathcal{K} \subset \mathcal{I}_{\mathrm{c}} \cup \mathcal{I}_{\mathrm{s}}$ we find

$$\alpha_{\mathcal{K},\mathrm{e}} > \mathrm{q}_{\mathcal{K},\mathrm{e}}. \tag{3.3}$$

We now chose $T_{\mathrm{e}} \in (0, T]$ and $l_{\mathrm{e}} > 0$ such that (3.3) holds and fix this pair $(T_{\mathrm{e}}, l_{\mathrm{e}})$ for the rest of this article.

3.2 Decreasing the friction

In this subsection we present a simplified problem, see Problem 3.1. We present three simplifications that help to avoid the discontinuity of the dissipation functional Ψ, which is the major difficulty in the existence proof.

The first simplification is that we restrict ourself in the rest of the section to the set

$$\mathcal{A}_{\mathrm{e}} := \mathcal{A}_{\mathcal{I}_{\mathrm{c}},\mathrm{e}} = \mathcal{A} \cap \left(\bigcap_{j \in \mathcal{I}_{\mathrm{c}}} \partial \mathcal{A}^{(j)} \right) \cap \mathcal{Q}_{l_{\mathrm{e}}}(Z_0).$$

For $j \in \mathcal{I}_{\mathrm{c}}$ the single dissipation functional $\Psi^{(j)}$ is continuous on $\mathcal{A}_{\mathrm{e}}$. The motivation for this simplification is the physical intuition that particles, which are initially pressed onto the obstacle remain locally, i.e. for a short time span, in contact.

Second for $j \in \mathcal{I}_{\mathrm{f}}$ we neglect all functionals $\Psi^{(j)}$ and thus avoid their discontinuity. The motivation is that particles, which are out of contact, remain locally out of contact and thus are locally friction-free.

The third simplification consists in introducing a parameter $\gamma > 0$ and for $j \in \mathcal{I}_{\mathrm{s}}$ we define the j-th dissipation functional via

$$\Psi^{(j)}_{\gamma}(t, z, v) \mapsto \begin{cases} \left(\sigma^{(j)}(t, z) - \gamma\right)_{+} \|\mathrm{M}^{(j)}(z^{(j)})v\| & \text{if } z_{3j} = 0, \\ 0 & \text{if } z_{3j} > 0. \end{cases} \tag{3.4}$$

Since this is the most important simplification we motivate it in more detail. Due to physical intuition it is clear that the j-th particle makes and loses contact exactly when $\sigma^{(j)}(t, z(t)) = 0$ holds. Thus in the moments of making and losing contact there is no friction. This shows that $\Psi^{(j)}$, as defined in (2.1), behaves continuous along the path of a physical reasonable solution. The problem in approximating these solutions is that the moments of making and losing contact are a priori unknown. But we will be able to construct for arbitrary small $\gamma > 0$ approximative solutions, which make and lose contact for normal forces between 0 and γ. Thus the altered dissipation $\Psi^{(j)}_{\gamma}$ behaves continuous with respect to good approximations. See also Figure 2.

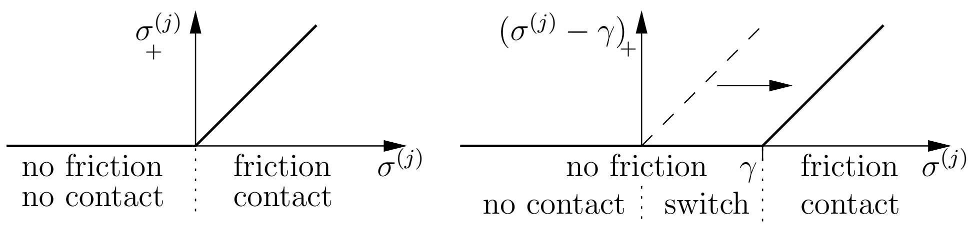

Figure 2: The γ-trick: Physical vs. approximative situation

Summarizing we obtain a new dissipation functional $\Psi_{\gamma} : [0, T] \times \mathcal{A}_{\mathrm{e}} \times \mathbb{R}^{3N} \to [0, \infty)$ with

$$\Psi_{\gamma}(t, z, v) := \sum_{j \in \mathcal{I}_{\mathrm{c}}} \Psi^{(j)}(t, z, v^{(j)}) + \sum_{j \in \mathcal{I}_{\mathrm{s}}} \Psi^{(j)}_{\gamma}(t, z, v^{(j)}). \tag{3.5}$$

We now introduce our simplified problem:

Problem 3.1 *Find a function $z_\gamma \in \mathrm{W}^{1,\infty}([0,T_\mathrm{e}],\mathcal{A}_\mathrm{e})$ such that the initial condition $z_\gamma(0) = Z_0$ is satisfied and such that for all times $t \in [0,T_\mathrm{e}]$ we have*

$$\mathcal{E}(t, z_\gamma(t)) \le \mathcal{E}(t, y) + \Psi_\gamma(t, z_\gamma(t), y - z_\gamma(t)) \quad \text{for all } y \in \mathcal{A}_\mathrm{e} \quad \text{and} \tag{S$_\gamma$}$$

$$\mathcal{E}(t, z_\gamma(t)) + \int_0^t \Psi_\gamma(s, z_\gamma(s), \dot z_\gamma(s))\,\mathrm{d}s = \mathcal{E}(0, Z_0) + \int_0^t \partial_s\mathcal{E}(s, z_\gamma(s))\,\mathrm{d}s. \tag{E$_\gamma$}$$

4 Solving simplified problems

In this section we assume the parameter $\gamma > 0$ to be given and construct a solution of Problem 3.1. The proof is based on an incremental scheme, which was introduced by Mielke & Theil [MiT99]. See also [Mie05, MiR07].

4.1 Approximative solutions

Definition 4.1 (Incremental Problem and approximative solutions)
For the given initial value $z_0 = Z_0 \in \mathcal{A}_\mathrm{e}$ and a partition $\Pi :\ 0 = t_0 < t_1 < \cdots < t_{N_\Pi} = T_\mathrm{e}$ the Incremental Problem *consists in finding incrementally $z_1, z_2, z_3, \ldots, z_{N_\Pi}$ such that for $k = 1, \ldots, N_\Pi$*

$$z_k = \operatorname{argmin}\left\{\mathcal{E}(t_k, y) + \Psi_\gamma(t_{k-1}, z_{k-1}, y - z_{k-1}) \ :\ y \in \mathcal{A}_\mathrm{e}\right\}. \tag{IP$_\gamma$}$$

Here 'argmin' denotes the set of all minimizers.

For the sequence $(\Pi^{(n)})_{n\in\mathbb{N}}$ of equi-distant partitions $\Pi^{(n)} := \left\{t_k = \frac{k}{n}\mathrm{T}_\mathrm{e} \ :\ k = 0, \ldots, n\right\}$ we define the sequence of approximative solutions *$(z_\gamma^{(n)})_{n\in\mathbb{N}} \subset \mathrm{W}^{1,\infty}([0,T_\mathrm{e}],\mathcal{A}_\mathrm{e})$ as the piecewise linear interpolation between the values $(z_k^{(n)})_{k=1,\ldots,n}$ of the solution of the corresponding Incremental Problem* (IP$_\gamma$)*, i.e.*

$${z_\gamma}^{(n)}(t) := z_{k-1}^{(n)} + \frac{t - t_{k-1}^{(n)}}{t_k^{(n)} - t_{k-1}^{(n)}}(z_k^{(n)} - z_{k-1}^{(n)}) \quad \text{for } t \in [t_{k-1}^{(n)}, t_k^{(n)}] \text{ and } k = 1, \ldots, n.$$

Theorem 4.2 (Existence & uniqueness of approximative solutions)
We assume (N0)–(N4)*. Then for any equi-distant sequence of partitions $(\Pi^{(n)})_{n\in\mathbb{N}}$ of $[0,T_\mathrm{e}]$ the corresponding sequence of approximative solutions $({z_\gamma}^{(n)})_{n\in\mathbb{N}} \subset \mathrm{W}^{1,\infty}([0,T_\mathrm{e}],\mathcal{A}_\mathrm{e})$ of* (IP$_\gamma$) *exists and is unique.*

Proof: By construction the set $\mathcal{A}_\mathrm{e}$ is closed an convex and the dissipation functional Ψ_γ is convex with respect to its third variable. Further, the function $\mathcal{E}(t,\cdot)$ is strictly convex on $\mathcal{A}_\mathrm{e}$ for all $t \in [0,T_\mathrm{e}]$. This is a consequence of estimate (3.3). Thus, the result follows from Weierstrass' extreme principle. ∎

4.2 Convergence

In this subsection we show that the existence of a convergent subsequence for the sequence of approximative solutions $(z_\gamma^{(n)})_{n\in\mathbb{N}} \subset \mathrm{W}^{1,\infty}([0,T_\mathrm{e}],\mathcal{A}_\mathrm{e})$, see Theorem 4.8. The proof is based on the Arzela-Ascoli theorem and to be able to apply it, it is sufficient to establish the existence of a uniform Lipschitz constant $\mathrm{c}_\mathrm{lip} > 0$, which neither depends on n nor on the choice of $\gamma > 0$.

The following two constants are Lipschitz constants with respect to time for the elastic and frictional forces $\mathrm{D}\mathcal{E}$ and $\partial\Psi_\gamma$. We introduce the cuboid $\mathcal{C}_\mathrm{e} := [0, T_\mathrm{e}] \times \mathcal{A}_\mathrm{e}$ and define

$$
\begin{aligned}
\mathrm{c}_1 &:= \frac{1}{2} \left\|\partial_t \mathrm{D}\mathcal{E}\right\|_{\mathrm{L}^\infty(\mathcal{C}_\mathrm{e},\mathbb{R}^{3N*})}, \\
\mathrm{c}_2 &:= \sum_{j\in\mathcal{I}_\mathrm{c}\cup\mathcal{I}_\mathrm{s}} \left\|\partial_t \sigma^{(j)}\right\|_{\mathrm{L}^\infty(\mathcal{C}_\mathrm{e},\mathbb{R})} \left\|\mathrm{M}^{(j)}\right\|_{\mathrm{L}^\infty(\partial\mathcal{S}\cap\mathcal{Q}_{l_\mathrm{e}}(Z_0{}^{(j)}),\mathbb{R}^{3\times 3})} \qquad \text{and} \\
\mathrm{c}_\mathrm{e} &:= \mathrm{c}_1 + \mathrm{c}2.
\end{aligned} \tag{4.1}
$$

Note that the constants are independent of the parameter $\gamma > 0$. Our aim will be to establish the Lipschitz constant $\mathrm{c}_\mathrm{lip} := \sup\left\{\frac{\mathrm{c}_\mathrm{e}}{\alpha_{\mathcal{K},\mathrm{e}} - \mathrm{q}_{\mathcal{K},\mathrm{e}}} \;:\; \mathcal{I}_\mathrm{c} \subset \mathcal{K} \subset \mathcal{I}_\mathrm{c} \cup \mathcal{I}_\mathrm{s}\right\}$. We shorten notation as follows. For $\mathcal{I}_\mathrm{c} \subset \mathcal{K} \subset \mathcal{I}_\mathrm{c} \cup \mathcal{I}_\mathrm{s}$ we define

$$
\begin{aligned}
\mathcal{A}_{\mathcal{K},\mathrm{e}} &:= \mathcal{A}_\mathrm{e} \cap \Big(\bigcap_{j\in\mathcal{K}} \partial\mathcal{A}^{(j)}\Big) = \mathcal{A}_\mathrm{e} \cap \Big(\bigcap_{j\in\mathcal{K}\cap\mathcal{I}_\mathrm{s}} \partial\mathcal{A}^{(j)}\Big) \quad \text{and} \\
\Psi_{\mathcal{K},\gamma} &:= \sum_{j\in\mathcal{I}_\mathrm{c}} \Psi^{(j)} + \sum_{j\in\mathcal{K}\cap\mathcal{I}_\mathrm{s}} \Psi_\gamma^{(j)}.
\end{aligned}
$$

In the extreme cases $\mathcal{K} = \mathcal{I}_\mathrm{c}$ and $\mathcal{K} = \mathcal{I}_\mathrm{c} \cup \mathcal{I}_\mathrm{s}$ we simply write $\Psi_{\mathcal{I}_\mathrm{c}}$ and Ψ_γ as hitherto.

Lemma 4.3 (Lipschitz-continuity of the dissipation functional)
For arbitrary $\gamma > 0$ and given times $t, \tau \in [0, T_\mathrm{e}]$, states $z, \tilde{z} \in \mathcal{A}_{\mathcal{K},\mathrm{e}}$ and $v \in \mathbb{S}_\mathcal{K}^{3N-1}$ we have

$$
\begin{aligned}
|\Psi_{\mathcal{K},\gamma}(t, z, v) - \Psi_{\mathcal{K},\gamma}(\tau, z, v)| &\le \mathrm{c}_2 |t - \tau| \quad \textit{and} \\
|\Psi_{\mathcal{K},\gamma}(t, z, v) - \Psi_{\mathcal{K},\gamma}(t, \tilde{z}, v)| &\le \mathrm{q}_{\mathcal{K},\mathrm{e}} \|z - \tilde{z}\|.
\end{aligned}
$$

Proof: We first observe $\left|(\sigma^{(j)}(t,z) - \gamma)_+ - (\sigma^{(j)}(\tau,z) - \gamma)_+\right| \le \left|\sigma^{(j)}(t,z) - \sigma^{(j)}(\tau,z)\right|$, which helps to explain why c_2 and $\mathrm{q}_{\mathcal{K},\mathrm{e}}$ can be chosen independent of γ. The first estimate now follows directly by definition of c_2. For the second estimate we consider only the difference of single dissipation functionals. We label all data with a tilde if the data depends on $\tilde{z}$ and write in a symbolic way

$$
\begin{aligned}
\left|\Psi_\gamma^{(j)} - \tilde{\Psi}_\gamma^{(j)}\right| &\le |\sigma^{(j)} - \tilde{\sigma}^{(j)}| \|\mathrm{M}^{(j)}\| + \left(\tilde{\sigma}^{(j)} - \gamma\right)_+ \|\mathrm{M}^{(j)} - \tilde{\mathrm{M}}^{(j)}\| \\
&\le \Bigg(\left\|\mathrm{D}\sigma^{(j)} u\right\|_{\mathrm{L}^\infty(\mathcal{C}_{\mathcal{K},\mathrm{e}},\mathbb{R})} \|\mathrm{M}^{(j)}(\cdot^{(j)}) v^{(j)}\|_{\mathrm{L}^\infty(\mathcal{A}_{\mathcal{K},\mathrm{e}},\mathbb{R}^3)} \\
&\qquad + \|\sigma_+^{(j)}\|_{\mathrm{L}^\infty(\mathcal{C}_{\mathcal{K},\mathrm{e}},\mathbb{R})} \|\mathrm{D}\mathrm{M}^{(j)}(\cdot^{(j)})[v^{(j)}, u^{(j)}]\|_{\mathrm{L}^\infty(\mathcal{A}_{\mathcal{K},\mathrm{e}},\mathbb{R}^3)}\Bigg) \|z - \tilde{z}\|
\end{aligned}
$$

with the vector $u = \frac{z - \tilde{z}}{\|z - \tilde{z}\|} \in \mathbb{S}_\mathcal{K}^{3N-1}$. Summing up we find the constant $\mathrm{q}_{\mathcal{K},\mathrm{e}}$, see (3.2). ■

The following recursive estimate was established in this context in [MiR07].

Lemma 4.4 (Recursive estimate) *Let* (N0)–(N2), (N4) *hold and assume an index set $\mathcal{I}_\mathrm{c} \subset \mathcal{K} \subset (\mathcal{I}_\mathrm{c} \cup \mathcal{I}_\mathrm{s})$, the times $t_0, t_1, t_2 \in [0, T_\mathrm{e}]$ and the state $z_0 \in \mathcal{A}_{\mathcal{K},\mathrm{e}}$ to be given. We define first z_1 and then z_2 via*

$$
\begin{aligned}
z_1 &= \operatorname{argmin}\{\mathcal{E}(t_1, y) + \Psi_{\mathcal{K},\gamma}(t_0, z_0, y - z_0) \;:\; y \in \mathcal{A}_{\mathcal{K},\mathrm{e}}\} \\
z_2 &= \operatorname{argmin}\{\mathcal{E}(t_2, y) + \Psi_{\mathcal{K},\gamma}(t_1, z_1, y - z_1) \;:\; y \in \mathcal{A}_{\mathcal{K},\mathrm{e}}\}
\end{aligned}
$$

then we find the recursive estimate

$$
\alpha_{\mathcal{K},\mathrm{e}} \|z_2 - z_1\| \le \mathrm{c}_\mathrm{e} \max\{|t_2 - t_1|, |t_1 - t_0|\} + \mathrm{q}_{\mathcal{K},\mathrm{e}} \|z_1 - z_0\| \tag{4.2}
$$

with the constants $\alpha_{\mathcal{K},\mathrm{e}}, \mathrm{q}_{\mathcal{K},\mathrm{e}}$ and c_e as defined in (3.1), (3.2) *and* (4.1).

Proof: For $k=1,2$ we reformulate the above minimizing problems using the characteristic function $\mathcal{X}_{\mathcal{A}_{\mathcal{K},\mathrm{e}}}(z) := 0$ for $z \in \mathcal{A}_{\mathcal{K},\mathrm{e}}$ and $\mathcal{X}_{\mathcal{A}_{\mathcal{K},\mathrm{e}}}(z) := +\infty$ for $z \notin \mathcal{A}_{\mathcal{K},\mathrm{e}}$ as follows

$$z_k = \operatorname{argmin}\left\{\mathcal{E}(t_k,y) + \Psi_{\mathcal{K},\gamma}(t_{k-1},z_{k-1},y-z_{k-1}) + \mathcal{X}_{\mathcal{A}_{\mathcal{K},\mathrm{e}}}(y) \,:\, y \in \mathbb{R}^{3N}\right\}.$$

Denoting by ∂_y the subdifferential with respect to y this is equivalent to

$$-\mathrm{D}\mathcal{E}(t_k,z_k) \in \partial_y\left\{\Psi_{\mathcal{K},\gamma}(t_{k-1},z_{k-1},\cdot-z_{k-1}) + \mathcal{X}_{\mathcal{A}_{\mathcal{K},\mathrm{e}}}(\cdot)\right\}(z_k).$$

Here we used the convexity of $\mathcal{E}(t,\cdot)$ on $\mathcal{A}_{\mathcal{K},\mathrm{e}}$, see (3.3) and the *Moreau-Rockafellar*-Theorem A.3. By definition of the subdifferential this is for $y \in \mathcal{A}_{\mathcal{K},\mathrm{e}}$ equivalent to

$$-\mathrm{D}\mathcal{E}(t_k,z_k)(y-z_k) \le \Psi_{\mathcal{K},\gamma}(t_{k-1},z_{k-1},y-z_{k-1}) - \Psi_{\mathcal{K},\gamma}(t_{k-1},z_{k-1},z_k-z_{k-1}). \tag{4.3}$$

This was already the crucial estimate. Next we introduce the function $y(s) := z_k + s(y-z_k)$ and applying the fundamental theorem of calculus twice we deduce

$$\mathcal{E}(t_k,y) - \mathcal{E}(t_k,z_k) = \int_0^1\int_0^r \langle y-z_k, \mathrm{H}(t_k,y(s))(y-z_k)\rangle \mathrm{d}s\mathrm{d}r + \mathrm{D}\mathcal{E}(t_k,z_k)(y-z_k).$$

The definition of the constant $\alpha_{\mathcal{K},\mathrm{e}} > 0$ in (3.1) was chosen such that we can estimate the double integral from below by the term $\frac{1}{2}\alpha_{\mathcal{K},\mathrm{e}}\|y-z_k\|^2$ for $y \in \mathcal{A}_{\mathcal{K},\mathrm{e}}$. Together with the estimate (4.3) this shows

$$\frac{\alpha_{\mathcal{K},\mathrm{e}}}{2}\|y-z_k\|^2 \le \mathcal{E}(t_k,y) - \mathcal{E}(t_k,z_k) + \Psi_{\mathcal{K},\gamma}(t_{k-1},z_{k-1},y-z_{k-1}) - \Psi_{\mathcal{K},\gamma}(t_{k-1},z_{k-1},z_k-z_{k-1}).$$

Applying this estimate twice, once for $k=1$ and $y=z_2$ and once for $k=2$ and $y=z_1$, we find

$$\alpha_{\mathcal{K},\mathrm{e}}\|z_2-z_1\|^2 \le \mathcal{E}(t_1,z_2) - \mathcal{E}(t_1,z_1) + \mathcal{E}(t_2,z_1) - \mathcal{E}(t_2,z_2) + S_\Psi$$

with $S_\Psi := \Psi_{\mathcal{K},\gamma}(t_1,z_1,z_1-z_1) - \Psi_{\mathcal{K},\gamma}(t_1,z_1,z_2-z_1) + \Psi_{\mathcal{K},\gamma}(t_0,z_0,z_2-z_0) - \Psi_{\mathcal{K},\gamma}(t_0,z_0,z_1-z_0)$. We introduce the function $t(r) := t_2 + r(t_1-t_2)$. Applying again the fundamental theorem of calculus twice we reformulate the four energy terms as a double integral, which we estimate from above with the help of the constant $\mathrm{c}_1 = \frac{1}{2}\|\partial_t\mathrm{D}\mathcal{E}\|_{\mathrm{L}^\infty(\mathcal{C}_\mathrm{e},\mathbb{R}^{3N*})}$, i.e.

$$\begin{aligned}\sum_{j,k=1}^{2}(-1)^{j+k+1}\mathcal{E}(t_j,z_k) &= \int_0^1\int_0^1 \partial_t\mathrm{D}\mathcal{E}(t(r),y(s))(z_2-z_1)\,\mathrm{d}s(t_1-t_2)\,\mathrm{d}r\\ &\le \mathrm{c}_1\|z_2-z_1\||t_2-t_1|.\end{aligned}$$

In the rest of the proof we derive the following estimate for the four remaining dissipational terms

$$S_\Psi \le \big(\mathrm{c}_2|t_1-t_0| + \mathrm{q}_{\mathcal{K},\mathrm{e}}\|z_1-z_0\|\big)\|z_2-z_1\|. \tag{4.4}$$

Before we prove this last estimate we want to point out that the combination of the last three estimates leads us to our desired recursive estimate. To establish (4.4) we use $\Psi_{\mathcal{K},\gamma}(t_1,z_1,z_1-z_1) = \Psi_{\mathcal{K},\gamma}(t_1,z_1,0) = 0$ and the triangle inequality giving $\Psi_{\mathcal{K},\gamma}(t_0,z_0,z_2-z_0) - \Psi_{\mathcal{K},\gamma}(t_0,z_0,z_1-z_0) \le \Psi_{\mathcal{K},\gamma}(t_0,z_0,z_2-z_1)$. Summarizing we find

$$S_\Psi \le \Psi_{\mathcal{K},\gamma}(t_0,z_0,z_2-z_1) - \Psi_{\mathcal{K},\gamma}(t_1,z_1,z_2-z_1).$$

Note that (4.2) is trivial for $z_2 = z_1$. Thus we assume $\|z_2-z_1\| > 0$, introduce the vector $v := \frac{z_2-z_1}{\|z_2-z_1\|} \in \mathbb{S}_{\mathcal{K}}^{3N-1}$ and rewrite the above estimate as follows $S_\Psi \le \big(\Psi_{\mathcal{K},\gamma}(t_0,z_0,v) -$

$\Psi_{\mathcal{K},\gamma}(t_1, z_1, v)\big)\|z_2 - z_1\|$. We expand the difference of the right hand side to $\Psi_{\mathcal{K},\gamma}(t_0, z_0, v) - \Psi_{\mathcal{K},\gamma}(t_1, z_0, v) + \Psi_{\mathcal{K},\gamma}(t_1, z_0, v) - \Psi_{\mathcal{K},\gamma}(t_1, z_1, v)$. But the constants c_2 and $\mathrm{q}_{\mathcal{K},\mathrm{e}}$ in (4.1) and (3.2) where chosen exactly such that the first difference is estimated by $\mathrm{c}_2|t_1 - t_0|$ and the second difference by $\mathrm{q}_{\mathcal{K},\mathrm{e}}\|z_1 - z_0\|$. ■

The following Lemma is only an auxiliary but technically useful result. It will be used several times in the crucial Theorem 4.7.

Lemma 4.5 *We assume* (N0)–(N4). *Further let* $t_0, t \in [0, T_\mathrm{e}]$, $z_0 \in \mathcal{A}_\mathrm{e}$ *and* $\mathcal{I}_* \subset \mathcal{I}_\mathrm{s}$ *be given. Then for* $j_* \in \mathcal{I}_*$ *the following two conditions are equivalent:*

$$z = \operatorname{argmin}\left\{\mathcal{E}(t, y) + \Psi_\gamma(t_0, z_0, y - z_0) \ : \ y \in \mathcal{A}_\mathrm{e} \cap \bigcap_{j \in \mathcal{I}_*} \partial\mathcal{A}^{(j)}\right\}, \ \sigma^{(j_*)}(t, z) > 0; \tag{4.5}$$

$$z = \operatorname{argmin}\left\{\mathcal{E}(t, y) + \Psi_\gamma(t_0, z_0, y - z_0) \ : \ y \in \mathcal{A}_\mathrm{e} \cap \bigcap_{j \in \mathcal{I}_* \setminus \{j_*\}} \partial\mathcal{A}^{(j)}\right\}, \ \sigma^{(j_*)}(t, z) > 0. \tag{4.6}$$

Proof: To keep notation short we introduce the cuboids $\mathcal{Q} := \mathcal{A}_\mathrm{e} \cap \bigcap_{j \in \mathcal{I}_*} \partial\mathcal{A}^{(j)}$ and $\mathcal{Q}_* := \mathcal{A}_\mathrm{e} \cap \bigcap_{j \in \mathcal{I}_* \setminus \{j_*\}} \partial\mathcal{A}^{(j)}$. Using the *Moreau-Rockafellar*-theorem A.3 we reformulate the minimizing problems in (4.5) and (4.6) as differential inclusions, i.e.

$$\begin{aligned} &0 \in \mathrm{D}\mathcal{E}(t, z) + \partial_v \Psi_\gamma(t_0, z_0, y - z_0) + \partial\mathcal{X}_\mathcal{Q}(z) \quad \subset \mathbb{R}^{3N^*}, \\ &0 \in \mathrm{D}\mathcal{E}(t, z) + \partial_v \Psi_\gamma(t_0, z_0, y - z_0) + \partial\mathcal{X}_{\mathcal{Q}^*}(z) \quad \subset \mathbb{R}^{3N^*} \end{aligned}$$

with the characteristic function $\mathcal{X}_\mathcal{Q}(z) := 0$ for $z \in \mathcal{Q}$ and $\mathcal{X}_\mathcal{Q}(z) = +\infty$ for $z \notin \mathcal{Q}$. The inclusions correspond to force balances and they differ only in the formula for the constraint forces $\partial\mathcal{X}_\mathcal{Q}$ and $\partial\mathcal{X}_{\mathcal{Q}^*}$. For a general cuboid $\tilde{\mathcal{Q}} = \big\{z \in \mathbb{R}^{3N} \ : \ a_j \le z_j \le b_j, j = 1, \ldots, 3N\big\}$ we represent the constraint force component-wise with the help of the set-valued function $\mathcal{F}(x) := \{0\}$ for $x > 0$ and $\mathcal{F}(0) := (-\infty, 0]$. We find $(\partial\mathcal{X}_{\tilde{\mathcal{Q}}}(z))_j = \mathcal{F}(z_j - a_j) - \mathcal{F}(b_j - z_j)$ for $j = 1, \ldots, 3N$. Thus the above inclusions differ only in the $3j^*$-th column. For $\mathcal{Q}$ we have $z_{3j^*} = a_{3j^*} = b_{3j^*} = 0$ and thus $(\partial\mathcal{X}_\mathcal{Q})_{3j^*}(z) = \mathbb{R}$ while for $\mathcal{Q}_*$ we find $a_{3j^*} = 0$, $b_{3j^*} = l_\mathrm{e}$ and $(\partial\mathcal{X}_{\mathcal{Q}_*})_{3j^*}(z) = \mathcal{F}(z_{3j^*} - 0) - \mathcal{F}(l_\mathrm{e} - z_{3j^*})$. For the elastic and dissipational forces we find by definition $(\mathrm{D}\mathcal{E})_{3j^*} \equiv \sigma^{j^*}$ and $(\partial_v \Psi)_{3j^*} \equiv \{0\}$, since the dissipational functional is invariant along the sliding direction e_{3j^*}, see (N2) . Summarizing, the two above differential inclusions differ only in the $3j^*$-th columns, which read

$$\begin{aligned} &0 \in \sigma^{j^*}(t, z) + \mathcal{F}(z_{3j^*}) - \mathcal{F}(l_\mathrm{e} - z_{3j^*}) \quad \text{for (4.5)}, \\ &0 \in \sigma^{j^*}(t, z) + \mathbb{R} \quad \text{for (4.6)}. \end{aligned}$$

Having these inclusions at hand the equivalence between (4.5) and (4.6) is easy to check using the additional information $\sigma^{j^*}(t, z) > 0$. ■

Corollary 4.6 *Assume* $\mathcal{I}_\mathrm{c} \subset \mathcal{K} \subset \mathcal{I}_\mathrm{c} \cup \mathcal{I}_\mathrm{s}$ *and* $\mathcal{I}_\mathrm{c} \subset \mathcal{K}^* \subset \mathcal{I}_\mathrm{c} \cup \mathcal{I}_\mathrm{s}$ *and further let* $\sigma^{(j)}(t_0, z_0) < \gamma$ *for all* $j \in (\mathcal{I}_\mathrm{c} \cup \mathcal{I}_\mathrm{s}) \setminus (\mathcal{K} \cup \mathcal{K}^*)$. *We define* $\mathcal{H} := (\mathcal{K} \cup \mathcal{K}^*) \setminus (\mathcal{K} \cap \mathcal{K}^*)$ *then the following conditions are equivalent:*

$$\begin{aligned} &z = \operatorname{argmin}\Big\{\mathcal{E}(t, y) + \Psi_{\mathcal{K}^*,\gamma}(t_0, z_0, y - z_0) \ : \ y \in \mathcal{A}_{\mathcal{K}^*,\gamma}\Big\}, \ \sigma^{(j)}(t, z) > 0, \ j \in \mathcal{H}; \\ &z = \operatorname{argmin}\Big\{\mathcal{E}(t, y) + \Psi_{\mathcal{K},\gamma}(t_0, z_0, y - z_0) \ : \ y \in \mathcal{A}_{\mathcal{K},\gamma}\Big\}, \ \sigma^{(j)}(t, z) > 0, \ j \in \mathcal{H}. \end{aligned}$$

Proof: The additional assumption $\sigma^{(j)} < \gamma$ implies $\Psi_\gamma^{(j)} \equiv 0$ and allows us to replace $\Psi_{\mathcal{K}^*,\gamma}$ by Ψ_γ. Using the above lemma recursively we replace first $\mathcal{A}_{\mathcal{K}^*}$ by $\mathcal{A}_{\mathcal{K}^*\cup\mathcal{K}}$ and then by $\mathcal{A}_{\mathcal{K}}$. Again due to $\sigma^{(j)} < \gamma$ we replace in the last step $\Psi_{\mathcal{K}^*}$ by $\Psi_{\mathcal{K}}$. ■

For a given partition $\Pi : 0 = t_0 < t_1 < \cdots < t_{N_\Pi} = T_\mathrm{e}$ we define the fineness $f_\Pi := \{|t_k - t_{k-1}| : k = 1, \ldots, N_\Pi\}$.

Theorem 4.7 (Lipschitz continuity for (IP_γ)**)** *Let Assumptions* (N0)–(N4) *hold then there exists a Lipschitz constant* $\mathrm{c}_\mathrm{lip} > 0$ *and* $\mathrm{C}_{\mathit{fine}} > 0$*, independent of the choice of* γ*, such that the following holds: If* $\gamma > 0$ *and the partition* Π *of* $[0, T_\mathrm{e}]$ *satisfies*

$$f_\Pi \leq \mathrm{C}_{\mathit{fine}}\gamma,$$

then the solution $(z_k)_{k=0,\ldots,N_\Pi}$ *of* (IP_γ) *satisfies the uniform Lipschitz estimate*

$$\|z_k - z_{k-1}\| \leq \mathrm{c}_\mathrm{lip}|t_k - t_{k-1}| \quad \textit{for } k = 1, \ldots, N_\Pi. \tag{4.7}$$

Proof: We define the Lipschitz constant $\mathrm{c}_\mathrm{lip} > 0$ via

$$\mathrm{c}_\mathrm{lip} := \max\left\{\frac{\mathrm{c}_1 + \mathrm{c}_2}{\alpha_{\mathcal{K},\mathrm{e}} - \mathrm{q}_{\mathcal{K},\mathrm{e}}} : \mathcal{I}_\mathrm{c} \subset \mathcal{K} \subset \mathcal{I}_\mathrm{c} \cup \mathcal{I}_\mathrm{s}\right\}, \tag{4.8}$$

which is independent of γ. See (3.1), (3.2) and (4.1) for the definition of the constants involved. The definition of the upper limit C_fine of the fineness of the partition we find in (4.11) below.

The key argument of this proof is the recursive estimate of Lemma 4.4. For the lemma we have to show that two consecutive solutions z_k, z_{k+1} solve similar minimizing problems. To simplify this task we introduce an auxiliary incremental problem. We put $z_0 := Z_0$ and define for $k = 1, \ldots, N_\Pi$ incrementally first the set $\mathcal{I}_{k-1} \subset \mathcal{I}_\mathrm{s}$ and then the state $z_k \in \mathcal{A}_\mathrm{e}$ via

$$\mathcal{I}_{k-1} := \left\{j \in \mathcal{I}_\mathrm{s} : \sigma^{(j)}(t_{k-1}, z_{k-1}) \geq \frac{\gamma}{2}\right\} \tag{4.9}$$

and

$$z_k = \operatorname{argmin}\left\{\mathcal{E}(t_k, y) + \Psi_\gamma(t_{k-1}, z_{k-1}, y - z_{k-1}) : y \in \mathcal{A}_\mathrm{e} \cap \bigcap_{j \in \mathcal{I}_{k-1}} \partial\mathcal{A}^{(j)}\right\}, \tag{4.10}$$

Physically we restrict a single particle $z^{(j)} \in \mathbb{R}^3$, $j \in \mathcal{I}_\mathrm{s}$ to the boundary when it was pressed onto the boundary with a big normal force $\sigma^{(j)} \geq \gamma/2$ in the previous step. Our first aim is to prove that all these particles remain pressed onto the obstacle if the time step is chosen small enough, see (4.12).

Let us assume for a moment that the auxiliary solution $(z_k)_{k=1,\ldots,N_\Pi}$ satisfies the Lipschitz estimate (4.7) for $k = 1, \ldots, N$, i.e.

$$\|z_k - z_{k-1}\| \leq \mathrm{c}_\mathrm{lip}|t_k - t_{k-1}|.$$

We define the constant $\mathrm{C}_\sigma := \sup\left\{\|\partial_t\sigma^{(j)}\|_{\mathrm{L}^\infty(\mathcal{C}_\mathrm{e},\mathbb{R})} + \|\mathrm{D}\sigma^{(j)}\|_{\mathrm{L}^\infty(\mathcal{C}_\mathrm{e},\mathbb{R}^{3N^*})} : j \in \mathcal{I}\right\}$ such that for all $k = 1, \ldots, N_\Pi$ and $j \in \mathcal{I}$ the following estimate holds $|\sigma^{(j)}(t_k, z_k) - \sigma^{(j)}(t_{k-1}, z_{k-1})| \leq \mathrm{C}_\sigma(|t_k - t_{k-1}| + \|z_k - z_{k-1}\|) \leq \mathrm{C}_\sigma(1 + \mathrm{c}_\mathrm{lip})f_\Pi$. Let us choose

$$f_\Pi < \mathrm{C}_\mathrm{fine}\gamma \quad \text{and} \quad \mathrm{C}_\mathrm{fine} := \frac{1}{2\mathrm{C}_\sigma(1 + \mathrm{c}_\mathrm{lip})}. \tag{4.11}$$

This implies $|\sigma^{(j)}(t_k, z_k) - \sigma^{(j)}(t_{k-1}, z_{k-1})| < \gamma/2$ for small fineness f_Π. But for $j \in \mathcal{I}_{k-1}$ the estimate $\sigma^{(j)}(t_{k-1}, z_{k-1}) \geq \gamma/2$ holds and thus we find that the particles remain pressed onto the obstacle, i.e.

$$\sigma^{(j)}(t_k, z_k) > 0 \quad \text{for all } j \in \mathcal{I}_{k-1}. \tag{4.12}$$

As a consequence the auxiliary solution coincides with the solution of the Incremental Problem (IP_γ). For this we apply Lemma 4.5 and replace the set $\mathcal{A}_\mathrm{e} \cap_{j\in\mathcal{I}_{k-1}} \partial\mathcal{A}^{(j)}$ by $\mathcal{A}_\mathrm{e}$. Summarizing we have shown that if the solution $(z_k)_{k=1,\cdots,N_\Pi}$ of the auxiliary problem (4.10) satisfies for all $k = 1, \cdots, N_\Pi$ the Lipschitz estimate (4.7) then it coincides with the solution of the Incremental Problem (IP_γ) for small time steps.

In the second part of the proof we show the Lipschitz estimate (4.7) for the auxiliary solution. The proof is carried out by induction. To be able to apply the key argument, the recursive estimate of Lemma 4.4, we have to show that two consecutive solutions z_k and z_{k+1} solve minimizing problems, which are formulated with respect to the same dissipation and admissible set. See for this (4.13) and (4.14) or (4.16) and (4.17) below.

We start our induction and consider the initial values $t_0 = 0$ and $z_0 = Z_0$. For $j \notin \mathcal{I}_\mathrm{c}$ we have by definition $\Psi^{(j)}(t_0, z_0, \cdot) \equiv 0$. Thus we can replace Ψ by $\Psi_{\mathcal{I}_\mathrm{c}}$ in the initial Assumption (N3) and find

$$z_0 = \operatorname{argmin}\left\{\mathcal{E}(t_0, z) + \Psi_{\mathcal{I}_\mathrm{c}}(t_0, z_0, z - z_0) \;:\; z \in \mathcal{A}_\mathrm{e}\right\}.$$

By a similar argument we can replace Ψ_γ by $\Psi_{\mathcal{I}_\mathrm{c}} + \sum_{j\in\mathcal{I}_{k-1}} \Psi_\gamma^{(j)}$ in the auxiliary problem (4.10). And since $\mathcal{I}_0 = \emptyset$ holds we find in the definition of z_0 and z_1 the same minimizing set $\mathcal{A}_\mathrm{e}$ and the same dissipation functional $\Psi_{\mathcal{I}_\mathrm{c}}$, i.e.

$$z_0 = \operatorname{argmin}\left\{\mathcal{E}(t_0, z) + \Psi_{\mathcal{I}_\mathrm{c}}(t_0, z_0, z - z_0) \;:\; z \in \mathcal{A}_\mathrm{e}\right\} \quad \text{and} \tag{4.13}$$

$$z_1 = \operatorname{argmin}\left\{\mathcal{E}(t_1, z) + \Psi_{\mathcal{I}_\mathrm{c}}(t_0, z_0, z - z_0) \;:\; z \in \mathcal{A}_\mathrm{e}\right\}. \tag{4.14}$$

We are now formally in the setting of Lemma 4.4 and we deduce

$$\|z_1 - z_0\| \leq \frac{\mathrm{c}_1 + \mathrm{c}_2}{\alpha_{\mathcal{I}_\mathrm{c},\mathrm{e}}}|t_1 - t_0| \leq \frac{\mathrm{c}_1 + \mathrm{c}_2}{\alpha_{\mathcal{I}_\mathrm{c},\mathrm{e}} - \mathrm{q}_{\mathcal{I}_\mathrm{c},\mathrm{e}}}|t_1 - t_0| \leq \mathrm{c}_\mathrm{lip}|t_1 - t_0|.$$

This completes the proof of the induction start.

For the induction step we assume for given $k \in \{1, \cdots, N_\Pi - 1\}$ that the Lipschitz estimate (4.7) holds and we will prove

$$\|z_{k+1} - z_k\| \leq \mathrm{c}_\mathrm{lip}|t_{k+1} - t_k|. \tag{4.15}$$

As mentioned above we can replace the dissipation function Ψ_γ by $\Psi_{(\mathcal{I}_\mathrm{c}\cup\mathcal{I}_k),\gamma}$ in (4.10). For this we introduce the index sets $\mathcal{K}^* := \mathcal{I}_\mathrm{c} \cup \mathcal{I}_{k-1}$ and $\mathcal{K} := \mathcal{I}_\mathrm{c} \cup \mathcal{I}_k$ and thus find

$$z_k = \operatorname{argmin}\left\{\mathcal{E}(t_k, z) + \Psi_{\mathcal{K}^*,\gamma}(t_{k-1}, z_{k-1}, z - z_{k-1}) \;:\; y \in \mathcal{A}_{\mathcal{K}^*,\mathrm{e}}\right\},$$
$$z_{k+1} = \operatorname{argmin}\left\{\mathcal{E}(t_{k+1}, z) + \Psi_{\mathcal{K},\gamma}(t_k, z_k, z - z_k) \;:\; y \in \mathcal{A}_{\mathcal{K},\mathrm{e}}\right\}.$$

Our aim is to replace the index set $\mathcal{K}^*$ by $\mathcal{K}$ in the definition of z_k with the help of Corollary 4.6. We find for all $j \in \mathcal{H} := (\mathcal{I}_{k-1} \cup \mathcal{I}_k) \setminus (\mathcal{I}_{k-1} \cap \mathcal{I}_k)$ that the normal force $\sigma^{(j)}$ crosses the level $\gamma/2$, i.e. $\left(\sigma^{(j)}(t_{k-1}, z_{k-1}) - \gamma/2\right)\left(\sigma^{(j)}(t_k, z_k) - \gamma/2\right) \leq 0$. Note that the normal force is formulated with respect to the the states z_{k-1}, z_k for which the Lipschitz estimate (4.7) already holds. As in the first part of the proof we find $|\sigma^{(j)}(t_k, z_k) - \sigma^{(j)}(t_{k-1}, z_{k-1})| < \gamma/2$ for $f_\Pi < \mathrm{C}_\mathrm{fine}\gamma$ with $\mathrm{C}_\mathrm{fine} > 0$ as defined in (4.11). Together with the crossing of the level this proves on the one hand

$$0 < \sigma^{(j)}(t_k, z_k) \qquad \text{for all } j \in \mathcal{H}.$$

On the other hand, for $j \in \mathcal{I}_\mathrm{s} \backslash \mathcal{I}_k = (\mathcal{I}_\mathrm{c} \cup \mathcal{I}_\mathrm{s}) \backslash \mathcal{K}$ we have $\sigma^{(j)}(t_k, z_k) < \gamma/2$ and we deduce $\sigma^{(j)}(t_{k-1}, z_{k-1}) < \gamma$. And since the same estimate holds by definition for $j \in \mathcal{I}_\mathrm{s} \backslash \mathcal{I}_{k-1}$ we find

$$\sigma^{(j)}(t_{k-1}, z_{k-1}) < \gamma \qquad \text{for all } j \in (\mathcal{I}_\mathrm{c} \cup \mathcal{I}_\mathrm{s}) \backslash (\mathcal{K}^* \cap \mathcal{K}).$$

These estimates on the normal forces allow us to apply Corollary 4.6 and to replace $\mathcal{K}^*$ by $\mathcal{K}$, i.e.

$$z_k = \operatorname{argmin} \left\{ \mathcal{E}(t_k, z) + \Psi_{\mathcal{K},\gamma}(t_{k-1}, z_{k-1}, z - z_{k-1}) \;:\; y \in \mathcal{A}_{\mathcal{K},\mathrm{e}} \right\}, \tag{4.16}$$

$$z_{k+1} = \operatorname{argmin} \left\{ \mathcal{E}(t_{k+1}, z) + \Psi_{\mathcal{K},\gamma}(t_k, z_k, z - z_k) \;:\; y \in \mathcal{A}_{\mathcal{K},\mathrm{e}} \right\}. \tag{4.17}$$

We next apply Lemma 4.4 and using $|t_{k+1} - t_k| = |t_k - t_{k-1}|$ we find

$$\|z_{k+1} - z_k\| \le \frac{\mathrm{c_e}}{\alpha_{\mathcal{K},\mathrm{e}}} |t_{k+1} - t_k| + \frac{\mathrm{q}_{\mathcal{K},\mathrm{e}}}{\alpha_{\mathcal{K},\mathrm{e}}} \|z_k - z_{k-1}\|$$

By estimate (4.7) we deduce $\|z_{k+1} - z_k\| \le \left(\frac{\mathrm{c_e}}{\alpha_{\mathcal{K},\mathrm{e}}} + \frac{\mathrm{q}_{\mathcal{K},\mathrm{e}}}{\alpha_{\mathcal{K},\mathrm{e}}} \mathrm{c_{lip}} \right) |t_{k+1} - t_k|$. It remains to prove $\left(\frac{\mathrm{c_e}}{\alpha_{\mathcal{K},\mathrm{e}}} + \frac{\mathrm{q}_{\mathcal{K},\mathrm{e}}}{\alpha_{\mathcal{K},\mathrm{e}}} \mathrm{c_{lip}} \right) \le \mathrm{c_{lip}}$. This is equivalent to $\mathrm{c_e} \le (\alpha_{\mathcal{K},\mathrm{e}} - \mathrm{q}_{\mathcal{K},\mathrm{e}}) \mathrm{c_{lip}}$. Due to the extended convexity assumption $\alpha_{\mathcal{K},\mathrm{e}} > \mathrm{q}_{\mathcal{K},\mathrm{e}}$ the inequality does not change if we divide by $(\alpha_{\mathcal{K},\mathrm{e}} - \mathrm{q}_{\mathcal{K},\mathrm{e}}) > 0$. By the definition of $\mathrm{c_{lip}}$ in (4.8) we find

$$\|z_{k+1} - z_k\| \le \mathrm{c_{lip}} |t_{k+1} - t_k|.$$

We concluded the induction step. Hence, the solution $(z_k)_{k=1,\ldots,N_\Pi}$ of the Auxiliary Problem (4.10) satisfies the Lipschitz estimate (4.7). In the first part of the proof we have shown that this implies that the auxiliary solution coincides with the solution of (IP_γ). ■

Theorem 4.8 (Convergence) *Let* (N0)–(N4) *hold, then the the sequence of approximative solutions has a convergent subsequence* $(z_\gamma^{(n_j)})_{j \in \mathrm{N}} \subset \mathrm{W}^{1,\infty}([0, T_\mathrm{e}], \mathcal{A}_\mathrm{e})$*, i.e.*

$$\|z_\gamma^{(n_j)} - z_\gamma\|_{\mathrm{L}^\infty([0,T_\mathrm{e}],\mathbb{R}^{3N})} \to 0 \quad \text{for } j \to \infty \tag{4.18}$$

for some $z_\gamma \in \mathrm{W}^{1,\infty}([0, T_\mathrm{e}], \mathcal{A}_\mathrm{e})$*. The subsequence satisfies the uniform estimate*

$$\|z_\gamma^{(n_j)}(t) - z_\gamma^{(n_j)}(\tau)\| \le \mathrm{c_{lip}} |t - \tau| \quad \text{for all } t, \tau \in [0, T_\mathrm{e}] \text{ and } j \in \mathbb{N} \tag{4.19}$$

and with the Lipschitz constant $\mathrm{c_{lip}}$ *being independent of* γ*, see* (4.8)*.*

Proof: The result directly follows from the previous Theorem 4.7 and the Arzela-Ascoli theorem. ■

4.3 Existence

In this section we prove that the limit function $z_\gamma \in \mathrm{W}^{1,\infty}([0, T_\mathrm{e}], \mathcal{A}_\mathrm{e})$ of Theorem 4.8 represents a solution of the simplified problem. In literature this step is well known if the dissipation functional is continuous with respect to z, for example see [ScM07]. Thus it is sufficient to show Ψ_γ to be continuous on a set $\mathcal{D}$, which contains all discrete solutions. For $j \in \mathcal{I}_\mathrm{s}$ we introduce the closed domain

$$\mathcal{D}^{(j)} := \left\{ (t, z) \in [0, T_\mathrm{e}] \times \mathcal{A}_\mathrm{e} \;:\; \sigma^{(j)}(t, z) \le 0 \text{ or } z^{(j)} \in \partial \mathcal{S} \right\}$$

and define the general domain $\mathcal{D}$ via

$$\mathcal{D} := ([0,T_{\mathrm{e}}] \times \mathcal{A}_{\mathrm{e}}) \cap \left(\bigcap_{j\in\mathcal{I}_{\mathrm{s}}} \mathcal{D}^{(j)} \right). \tag{4.20}$$

Lemma 4.9 (Continuity of $\Psi_\gamma^{(j)}$) *Let the Assumptions* (N0)–(N2) *hold. Then each discrete solution* $(z_k)_{k=0,\dots,n}$ *of the Incremental Problem* (IP_γ) *satisfies*

$$(t_k, z_k) \in \mathcal{D} \qquad \textit{for all } k = 0,\dots,n.$$

For $j \in \mathcal{I}_{\mathrm{s}}$ we find for single dissipation functionals

$$\Psi_\gamma^{(j)} \in \mathrm{C}(\mathcal{D}^{(j)} \times \mathbb{R}^{3N}, [0,\infty)) \quad \textit{for all } j = 1,\dots,N.$$

Proof: For the inclusion it is sufficient to prove $(t_k, z_k) \in \mathcal{D}^{(j)}$ for all $j \in \mathcal{I}_{\mathrm{s}}$. For this we assume $z_k^{(j)} \notin \partial\mathcal{S}$ or equivalently $(z_k)_{3j} > 0$. Thus, for all $\lambda \in [0,(z_k)_{3j}]$, we may choose $y := z_k - \lambda e_{3j} \in \mathcal{A}_{\mathrm{e}}$ as a test state in (IP_γ). Due to Assumption (N2) the dissipation is invariant along the sliding direction $v = e_{3j}$ and we find $0 \le \mathcal{E}(t_k, z_k - \lambda e_{3j}) - \mathcal{E}(t_k, z_k)$. Dividing by $\lambda > 0$ and taking the limit $\lambda \to 0$ we find $\sigma^{(j)}(t_k, z_k) \le 0$ and thus $(t_k, z_k) \in \mathcal{D}^{(j)}$.

For the second part we recall the definition

$$\Psi_\gamma^{(j)}(t,z,v) = \begin{cases} \left(\sigma^{(j)}(t,z) - \gamma\right)_+ \|\mathrm{M}^{(j)}(z^{(j)})v\| & \text{if } z^{(j)} \in \partial\mathcal{S}, \\ 0 & \text{if } z^{(j)} \in \mathrm{int}\mathcal{S}. \end{cases}$$

The function $\Psi_\gamma^{(j)}$ is upper semi-continuous and positive. Thus it is sufficient to consider a sequence $(t_n, z_n, v_n)_{n\in\mathbb{N}} \subset \mathcal{D}^{(j)} \times \mathbb{R}^3$ with a limit $(t,z,v) \in \mathcal{D}^{(j)} \times \mathbb{R}^3$ that satisfies $\Psi_\gamma^{(j)}(t,z,v) > 0$. This implies $z^{(j)} \in \partial\mathcal{S}$ and $\sigma^{(j)}(t,z) > \gamma$. Since $\sigma^{(j)}$ is continuous there exists $n_0 \in \mathbb{N}$ such that $\sigma^{(j)}(t_n, z_n) > 0$ for all $n \ge n_0$. By the definition of $\mathcal{D}^{(j)}$ this implies $z_n^{(j)} \in \partial\mathcal{S}$ for all $n \ge n_0$. Thus for $n \ge n_0$ we find $\Psi_\gamma^{(j)}(t_n, z_n, v_n)$ and $\Psi_\gamma^{(j)}(t,z,v)$ being both defined by the continuous function $\left(\sigma^{(j)}(t,z) - \gamma\right)_+ \|\mathrm{M}^{(j)}(z^{(j)}v\|$. ∎

Since all approximative solutions satisfy $z_\gamma^{(n)}(0) = Z_0$ the convergence (4.18) implies the initial condition

$$z_\gamma(0) = Z_0. \tag{4.21}$$

From the above continuity result 4.9 and Lemma 3.12 in [ScM07] we derive the stability.

Lemma 4.10 (Stability of the limit function z) *If the Assumptions* (N0)–(N4) *hold then the limit function* $z_\gamma \in \mathrm{W}^{1,\infty}([0,T_{\mathrm{e}}], \mathcal{A}_{\mathrm{e}})$ *of Theorem 4.8 is stable for all* $t \in [0,T_{\mathrm{e}}]$, *i.e.*

$$\mathcal{E}(t, z_\gamma(t)) \le \mathcal{E}(t,y) + \Psi_\gamma\left(t, z_\gamma(t), y - z_\gamma(t)\right) \quad \textit{for all } y \in \mathcal{A}_{\mathrm{e}}. \tag{S}$$

A direct consequence of the stability (S) of the limit function $z_\gamma \in \mathrm{W}^{1,\infty}([0,T_{\mathrm{e}}], \mathcal{A}_{\mathrm{e}})$ shown in the previous Lemma 4.10 is the following lower energy estimate, see [Mie05] Proposition 5.7 for a first proof of this fact.

Lemma 4.11 (Lower energy estimate) *Let* (N0)–(N4) *hold and assume the limit function* $z \in \mathrm{W}^{1,\infty}([0,T_\mathrm{e}],\mathcal{A}_\mathrm{e})$ *to satisfy for all times* $t \in [0,T_\mathrm{e}]$ *the stability condition* (S). *Then we find the lower energy estimate*

$$\mathcal{E}(t_2,z(t_2)) + \int_{t_1}^{t_2} \Psi_\gamma\big(\tau,z(\tau),\dot z(\tau)\big)\ d\tau \geq \mathcal{E}(t_1,z(t_1)) + \int_{t_1}^{t_2} \partial_t\mathcal{E}(\tau,z(\tau))\ d\tau$$

for all $t_1,t_2 \in [0,T_\mathrm{e}]$ *with* $t_1 \leq t_2$.

Before we prove the upper energy estimate we recall that the lower energy energy estimate of Lemma 4.11 is equivalent to the monotonicity of the function

$$f(t) := \mathcal{E}(t,z(t)) + \int_0^t \Psi_\gamma\big(\tau,z(\tau),\dot z(\tau)\big)\ \mathrm{d}\tau - \int_0^t \partial_t\mathcal{E}(\tau,z(\tau))$$

on the interval $[0,T_\mathrm{e}]$. The next lemma states the opposite estimate but only for the initial and end time, i.e. $f(T_\mathrm{e}) \leq f(0)$. Together with the monotonicity this proves the equality $f(t) = f(0)$ for all $t \in [0,T_\mathrm{e}]$. Recalling the definition of f it is easy to see that this equality is equivalent to the desired energy equality (E).

For the following result see Lemma 4.9 above and Lemma 3.14 in [ScM07].

Lemma 4.12 (upper energy estimate) *We assume* (N0)–(N4), *then the limit function* z *satisfies the upper energy estimate*

$$\mathcal{E}(T_\mathrm{e},z(T_\mathrm{e})) + \int_0^{T_\mathrm{e}} \Psi_\gamma\big(\tau,z(\tau),\dot z(\tau)\big)\ \mathrm{d}\tau \leq \mathcal{E}(0,Z_0) + \int_0^{T_\mathrm{e}} \partial_t\mathcal{E}(\tau,z(\tau))d\tau. \tag{4.22}$$

Summarizing the results 4.10–4.12 we deduce the following theorem.

Theorem 4.13 (Existence for Problem 3.1) *Let us assume* (N0)–(N4). *Then there exists a constant* $\mathrm{c}_\mathrm{lip} > 0$ *independent of* γ *and a solution* $z_\gamma \in \mathrm{W}^{1,\infty}([0,T_\mathrm{e}],\mathcal{A}_\mathrm{e})$ *of Problem 3.1 satisfying for all* $\tau_1,\tau_2 \in [0,T_\mathrm{e}]$

$$\|z_\gamma(\tau_2) - z_\gamma(\tau_1)\| \leq \mathrm{c}_\mathrm{lip}|\tau_2-\tau_1| \quad \textit{and} \quad (t,z_\gamma(t)) \in \mathcal{D} \tag{4.23}$$

with $\mathcal{D}$ *as defined in* (4.20).

5 Solving the problem

In this section we solve the N particle Problem 2.1. We construct a solution considering a sequence of simplified solutions z_{γ_n} whose parameters converge to zero, i.e $\gamma_n \to 0$ for $n \to 0$.

We start with a technical lemma.

Lemma 5.1 (Lower sequentially semi-continuity) *Let* $z, z^{(n)} \in \mathrm{W}^{1,\infty}([0,T_\mathrm{e}],\mathcal{A}_\mathrm{e})$ *satisfy*

$$\|z^{(n)} - z\|_{\mathrm{C}^0([0,T_\mathrm{e}],\mathbb{R}^{3N})} \to 0,\ (n\to\infty) \quad \textit{and} \quad \|\dot z^{(n)}\|_{\mathrm{L}^\infty([0,T_\mathrm{e}],\mathbb{R}^{3N})} \leq \mathrm{c}_\mathrm{lip}. \tag{5.1}$$

with $(t,z(t)) \in \mathcal{D}$ *for all* $t \in [0,T_\mathrm{e}]$. *Then for* $t \in [0,T_\mathrm{e}]$ *we find*

$$\int_0^t \Psi_\gamma(\tau,z(\tau),\dot z(\tau))\mathrm{d}\tau \leq \liminf_{n\to\infty} \int_0^t \Psi_\gamma(\tau,z(\tau),\dot z^{(n)}(\tau))\mathrm{d}\tau.$$

Proof: The assumptions in (5.1) assure the sequence $(z^{(n)})_{n\in\mathbb{N}}$ to be uniformly bounded in $\mathrm{W}^{1,\infty}([0,T_\mathrm{e}],\mathbb{R}^{3N})$. Thus, for all $1\le p<\infty$ there exists a subsequence, which we still denote by $(z^{(n)})_{n\in\mathbb{N}}$, such that

$$z^{(n)} \rightharpoonup z \quad \text{in } \mathrm{W}^{1,p}([0,T_\mathrm{e}],\mathbb{R}^{3N}).$$

By a standard result of the direct calculus of variations, see [Dac08] the operator $I(u) := \int_0^t \Psi_\gamma(t,z(t),\dot u(t))\mathrm{d}t$ is sequentially weakly lower semi-continuous on $\mathrm{W}^{1,p}([0,T_\mathrm{e}],\mathbb{R}^{3N})$ if the function $g(t,v) := \Psi_\gamma(t,z(t),v)$ is positive, continuous and convex with respect to $v\in\mathbb{R}^{3N}$. ∎

Theorem 5.2 *We assume* (N0)–(N4). *Then there exists a time* $T_*\in(0,T_\mathrm{e}]$ *and a solution* $z\in\mathrm{W}^{1,\infty}([0,T_*],\mathcal{A})$ *of Problem 2.1, i.e. for all* $t\in[0,T_*]$ *we have*

$$\mathcal{E}(t,z(t)) \le \mathcal{E}(t,y)+\Psi(t,z(t),y-z(t)) \quad \textit{for all } y\in\mathcal{A} \quad \textit{and} \tag{S}$$

$$\mathcal{E}(t,z(t)) + \int_0^t \Psi(s,z(s),\dot z(s))\,\mathrm{d}s = \mathcal{E}(0,z(0)) + \int_0^t \partial_s\mathcal{E}(s,z(s))\,\mathrm{d}s. \tag{E}$$

Proof: We consider a sequence of parameters $(\gamma_n)_{n\in\mathbb{N}}\subset(0,\infty)$ with $\gamma_n\to 0$ for $n\to\infty$. We denote by $(z_{\gamma_n})_{n\in\mathbb{N}}\subset\mathrm{W}^{1,\infty}([0,T_\mathrm{e}],\mathcal{A}_\mathrm{e})$ the sequence of solutions of the corresponding simplified Problems 3.1. The Existence Theorem 4.13 shows the existence of the solutions and provides the uniform Lipschitz estimate (4.23), i.e.

$$\|z_{\gamma_n}(\tau_2)-z_{\gamma_n}(\tau_1)\| \le \mathrm{c_{lip}}|\tau_2-\tau_1| \quad \text{and} \quad (\tau,z_{\gamma_n}(\tau))\in\mathcal{D}$$

for all $\tau_1,\tau_2,\tau\in[0,T_\mathrm{e}]$, $n\in\mathbb{N}$ and $\mathcal{D}$ as defined in (4.20). Since $\mathrm{c_{lip}}>0$ is independent of $n\in\mathbb{N}$ and all solutions satisfy the same initial condition $z_{\gamma_n}(0)=Z_0$ we can apply the Arzela-Ascoli theorem and we extract a convergent subsequence, which we still denote by $(z_{\gamma_n})_{n\in\mathbb{N}}\subset\mathrm{W}^{1,\infty}([0,T_\mathrm{e}],\mathcal{A}_\mathrm{e})$. Thus there exists a limit function $z\in\mathrm{W}^{1,\infty}([0,T_\mathrm{e}],\mathcal{A}_\mathrm{e})$ such that

$$\|z_{\gamma_n}-z\|_{\mathrm{L}^\infty([0,T_\mathrm{e}],\mathcal{A}_\mathrm{e})}\to 0 \quad \text{for } n\to\infty. \tag{5.2}$$

In the following we prove that z is our desired solution. But let us present a further convergence result first. There exists $\mathrm{c_{te}}>0$ such that for all $j\in\mathcal{I}$, $z\in\mathcal{A}_\mathrm{e}$ and $v\in\mathbb{R}^{3N}$ we have

$$\left|\Psi_\gamma^{(j)}(t,z,v)-\Psi^{(j)}(t,z,v)\right| \le \mathrm{c_{te}}\|v\|\gamma. \tag{5.3}$$

By definition this difference equals $\left|\big(\sigma^{(j)}(t,z)-\gamma\big)_+ - \sigma^{(j)}(t,z)_+\right| \|\mathrm{M}^{(j)}(z^{(j)})v\|$. Finally we define $\mathrm{c_{te}} := \sup\left\{\|\mathrm{M}^{(j)}(z^{(j)})\| \,:\, j\in\mathcal{I}, z\in\mathcal{A}_\mathrm{e}\right\}$ such that the estimate (5.3) follows directly.

We next show that z satisfies an (S)&(E)-formulation. For all $n\in\mathbb{N}$ and $t\in[0,T_\mathrm{e}]$ we have

$$\mathcal{E}(t,z_{\gamma_n}(t)) \le \mathcal{E}(t,y)+\Psi_{\gamma_n}(t,z_{\gamma_n}(t),y-z_{\gamma_n}(t)) \quad \text{for all } y\in\mathcal{A}_\mathrm{e} \quad \text{and} \tag{S_n}$$

$$\mathcal{E}(t,z_{\gamma_n}(t)) + \int_0^t \Psi_{\gamma_n}(s,z_{\gamma_n}(s),\dot z_{\gamma_n}(s))\,\mathrm{d}s = \mathcal{E}(0,Z_0)) + \int_0^t \partial_s\mathcal{E}(s,z_{\gamma_n}(s))\,\mathrm{d}s. \tag{E_n}$$

Passing to the limit we see due to (5.2) and (N1) that we can replace z_{γ_n} by z in all energy terms. Concerning the dissipational terms we replace Ψ_{γ_n} by $\Psi_{\mathrm{c,s}} := \sum_{j\in\mathcal{I}_\mathrm{c}\cup\mathcal{I}_\mathrm{s}}\Psi^{(j)}$. Note that we have $\Psi_\gamma-\Psi_{\mathrm{c,s}} = \sum_{j\in\mathcal{I}_\mathrm{s}}\big(\Psi_\gamma^{(j)}-\Psi^{(j)}\big)$. The dissipation in ($\mathrm{S}_n$) is estimated as follows (we drop any dependence on t to keep the formula short), for $j\in\mathcal{I}_\mathrm{s}$ we have

$$\begin{aligned}|\Psi_{\gamma_n}^{(j)}(z_{\gamma_n},y-z_{\gamma_n})-\Psi^{(j)}(z,y-z)| &\le |\Psi_{\gamma_n}^{(j)}(z_{\gamma_n},y-z_{\gamma_n})-\Psi^{(j)}(z_{\gamma_n},y-z_{\gamma_n})| \\ &\quad + |\Psi^{(j)}(z_{\gamma_n},y-z_{\gamma_n})-\Psi^{(j)}(z,y-z)|.\end{aligned}$$

The first difference converges due to (5.3). For the second difference we exploit $(t, z_{\gamma_n}(t)) \in \mathcal{D}$ and $\Psi^{(j)} \in \mathrm{C}(\mathcal{D} \times \mathbb{R}^{3N}, \mathbb{R})$, see Lemma 4.9. Hence the uniform convergence (5.2) is sufficient to pass to the limit and to find for all $t \in [0, 0]$ the stability condition

$$\mathcal{E}(t, z(t)) \leq \mathcal{E}(t, y) + \Psi_{\mathrm{c,s}}(t, z(t), y - z(t)) \quad \text{for all } y \in \mathcal{A}_{\mathrm{e}}. \tag{S$_{\mathrm{c,s}}$}$$

Applying Lemma 4.11 we deduce for all $t \in [0, T_{\mathrm{e}}]$ the lower energy estimate

$$\mathcal{E}(t, z(t)) + \int_0^t \Psi_{\mathrm{c,s}}(s, z(s), \dot{z}(s))\,\mathrm{d}s \geq \mathcal{E}(0, Z_0)) + \int_0^t \partial_s \mathcal{E}(s, z(s))\,\mathrm{d}s.$$

It remains us to derive the upper energy estimate and our starting point is the equation (E$_n$). We replace the integrand $\Psi_{\gamma_n}(s, z_{\gamma_n}(s), \dot{z}_{\gamma_n}(s))$ by $\Psi_{\mathrm{c,s}}(s, z(s), \dot{z}_{\gamma_n}(s))$ doing the same estimates as above. For this note that $\Psi^{(j)}$ is uniformly continuous on $\mathcal{D} \times \mathcal{B}_{\mathrm{c_{lip}}}(0)$. Taking the $\liminf$ in (E$_n$), see also Lemma 5.1, we find for the dissipational integral

$$\int_0^t \Psi_{\mathrm{c,s}}(s, z(s), \dot{z}(s))\,\mathrm{d}s \leq \liminf_{n \to \infty} \int_0^t \Psi_{\mathrm{c,s}}(s, z(s), \dot{z}_{\gamma_n}(s))\,\mathrm{d}s.$$

We thus established the upper energy estimate and together with the above lower estimate this proves

$$\mathcal{E}(t, z(t)) + \int_0^t \Psi_{\mathrm{c,s}}(s, z(s), \dot{z}(s))\,\mathrm{d}s = \mathcal{E}(0, Z_0)) + \int_0^t \partial_s \mathcal{E}(s, z(s))\,\mathrm{d}s. \tag{E$_{\mathrm{c,s}}$}$$

All in all, we have shown that the function z satisfies an (S$_{\mathrm{c,s}}$)&(E$_{\mathrm{c,s}}$)-formulation.

In this last paragraph we shorten the time interval and choose a new final time $0 < T_* \leq T_{\mathrm{e}}$. We first choose T_* such that the particles $z^{(j)}, j \in \mathcal{I}_{\mathrm{f}}$, which are initially out of contact remain out of contact up to a time T_*. This allows us to replace $\Psi_{\mathrm{c,s}}$ by Ψ. Thus we established (E) while in the stability condition (S$_{\mathrm{c,s}}$) we find still have to replace $\mathcal{A}_{\mathrm{e}} = \mathcal{A} \cap \mathcal{Q}_{l_{\mathrm{e}}} \cap (\bigcap_{j \in \mathcal{I}_{\mathrm{c}}} \partial \mathcal{A}^{(j)})$ by the set $\mathcal{A}$. Up to now $z(t)$ solves the convex minimizing problem $z = \operatorname{argmin}\{\mathcal{E}(t, y) + \Psi(t, z(t), y - z(t)) \,:\, y \in \mathcal{A}_{\mathrm{e}}\}$. Choosing $T_* \leq \min\{l_{\mathrm{e}}/(2\mathrm{c_{lip}}), T_{\mathrm{e}}\}$ we find $z(t) \in \mathrm{int}\mathcal{Q}_{l_{\mathrm{e}}}(Z_0)$ for all $t \in [0, T_*]$ and thus drop the set $\mathcal{Q}_{l_{\mathrm{e}}}$ in the definition of $\mathcal{A}_{\mathrm{e}}$ and replace the set $\mathcal{A}_{\mathrm{e}}$ by $\mathcal{A}^* := \mathcal{A} \cap \bigcap_{j \in \mathcal{I}_{\mathrm{c}}} \partial \mathcal{A}^{(j)}$ in (S$_{\mathrm{c,s}}$). To replace $\mathcal{A}^*$ by $\mathcal{A}$ we use the *Moreau-Rockafellar*-Theorem A.3 to rewrite the stability conditions (S$_{\mathrm{c,s}}$) and (S) as inclusions

$$0 \in \mathrm{D}\mathcal{E}(t, z(t)) + \partial_v \Psi(t, z(t), 0) + \partial \mathcal{X}_{\mathcal{A}^*}(z(t)) \quad \subset \mathbb{R}^{3N^*}.$$

A direct calculus shows that the constraint forces $\partial \mathcal{X}_{\mathcal{A}^*}$ and $\partial \mathcal{X}_{\mathcal{A}}$ differ only in the $3j$-th columns for $j \in \mathcal{I}_{\mathrm{c}}$. The corresponding column in the above force balance law reads

$$0 \in \sigma^{(j)}(t, z(t)) + \{0\} + \mathbb{R}^*.$$

The $\{0\}$ corresponds to $\partial_v \Psi$ since dissipation is invariant in sliding directions, which are normal to the obstacle, see (N2). Further, for $j \in \mathcal{I}_{\mathrm{c}}$ we have by definition $\sigma^{(j)}(0, Z_0) > 0$ and we further shorten $T_* \in (0, T_{\mathrm{e}}]$ such that $\sigma^{(j)}(t, z(t)) \geq 0$ holds for all $j \in \mathcal{I}_{\mathrm{c}}$ and $t \in [0, T_*]$. This allows us to replace $\mathbb{R}^*$ by $(-\infty, 0]$ or equivalently $\partial \mathcal{X}_{\mathcal{A}^*}$ by $\partial \mathcal{X}_{\mathcal{A}}$. Applying once again Theorem A.3 we find (S) for all $t \in [0, T_*]$. ∎

A Subdifferential calculus

In the Appendix we present some results from convex analysis, which are reused in the whole article.

Definition A.1 (Subdifferential) *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ for $z \in \mathbb{R}^n$ we define*

$$\partial f(z) := \Big\{ v^* \in \mathbb{R}^{n^*} \ : \ v^*(y-z) \le f(y) - f(z) \text{ for all } y \in \mathbb{R}^n \Big\}.$$

We call the set $\partial f(z) \subset \mathbb{R}^{n^*}$ the subdifferential of f in z.

We now present two classical results from convex analysis. Both can be found in the book [EkT76], see Proposition 5.3 and 5.6 there.

Theorem A.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and assume for $z \in \mathbb{R}^n$ the Gateaux-derivative $\mathrm{D}f(z) \in \mathbb{R}^{n^*}$ to exist then we have*

$$\partial f(z) = \{\mathrm{D}f(z)\}.$$

We present the following result in a simplified form. It was established by Moreau [Mor66] and Rockafellar [Roc66].

Theorem A.3 (Moreau-Rockafellar 1966) *For $j = 1, 2$ let $f_j : \mathbb{R}^n \to (-\infty, \infty]$ be a convex, lower semi-continuous function. If $\max\{f_1(z), f_2(z)\} < \infty$ holds for some point $z \in \mathbb{R}^n$ and f_1 is continuous in z, then we have:*

$$\partial(f_1 + f_2)(z) = \partial f_1(z) + \partial f_2(z).$$

References

[And99] L.-E. Andersson. Quasistatic fricional contact problem with finitely many degrees of freedom. LiTH-MAT-R-1999-22, Department of Mathematics, 1999.

[And00] L.-E. Andersson. Existence results for quasistatic contact problems with coulomb friction. *Appl. Math. Optim.*, 42(2), 169–202, 2000.

[ChA04] O. Chau and B. Awbi. Quasistatic thermoviscoelastic frictional contact problem with damped response. *Appl. Anal.*, 83(6), 635–648, 2004.

[Dac08] B. Dacorogna. *Direct methods in the calculus of variations*, volume 78 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2008.

[EcJ01] C. Eck and J. Jarušek. On the thermal aspect of dynamic contact problems. In *Proceedings of Partial Differential Equations and Applications (Olomouc, 1999)*, volume 126, pages 337–352, 2001.

[EkT76] I. Ekeland and R. Temam. *Convex analysis and variational problems.* North-Holland Publishing Co., Amsterdam, 1976. Translated from the French, Studies in Mathematics and its Applications, Vol. 1.

[HSS01] W. Han, M. Shillor, and M. Sofonea. Variational and numerical analysis of a quasistatic viscoelastic problem with normal compliance, friction and damage. *J. Comput. Appl. Math.*, 137(2), 377–398, 2001.

[KMS91] A. Klarbring, A. Mikelić, and M. Shillor. A global existence result for the quasistatic frictional contact problem with normal compliance. In *Unilateral problems in structural analysis, VI (Capri, 1989)*, pages 85–111. Internat. Ser. Numer. Math. 101. Birkhäuser, Basel, 1991.

[Mie05] A. Mielke. Evolution in rate-independent systems (ch.6). In C.Dafermos and E.Feireisl, editors, *Handbook of Differential Equations, Evolutionary Equations*, volume 2, pages 461–559. Elsevier B.V., 2005.

[Mie07] A. Mielke. A model for temperature-induced phase transformations in finite-strain elasticity. *IMA J. Appl. Math.*, 72(5), 644–658, 2007.

[MiP07] A. Mielke and A. Petrov. Thermally driven phase transformation in shape-memory alloys. *Adv. Math. Sci. Appl.*, 17(2), 667–685, 2007.

[MiR07] A. Mielke and R. Rossi. Existence and uniqueness results for a class of rate-independent hysteresis problems. *Math. Models Methods Appl. Sci.*, 17(1), 81–123, 2007.

[MiT99] A. Mielke and F. Theil. A mathematical model for rate–independent phase transformations with hysteresis. In H.-D. Alber, R. Balean, and R. Farwig, editors, *Proceedings of the Workshop on "Models of Continuum Mechanics in Analysis and Engineering"*, pages 117–129. Shaker–Verlag, 1999.

[MiT04] A. Mielke and F. Theil. On rate-independent hysteresis models. *NoDEA Nonlinear Differential Equations Appl.*, 11(2), 151–189, 2004. Accepted July 2001.

[Mor66] J. Moreau. *Fonctionnelles convexes.* Séminaire sur les Équations aux Dérivées Partielles (1966–1967). II. Collège de France, Paris, 1966.

[RiA06] A. Rietz and L.-E. Andersson. Noncoercive incremental friction problems for discrete systems. *ZAMM Z. Angew. Math. Mech.*, 86(12), 932–950, 2006.

[Roc66] R. T. Rockafellar. Characterization of the subdifferentials of convex functions. *Pacific J. Math.*, 17, 497–510, 1966.

[Roc01] R. Rocca. Existence of approximation of a solution to quasi-static problem signorini problem with local friction. *Internat. J. Engrg. Sci.*, 39, 1233–1255, 2001.

[Sch08] F. Schmid. *Analysis for geometrically non-linear models for contact with dry friction.* PhD thesis, Humboldt Universität Berlin, 2008.

[ScM07] F. Schmid and A. Mielke. Existence results for a contact problem with varying friction coefficient and nonlinear forces. *ZAMM Z. Angew. Math. Mech.*, 87(8-9), 616–631, 2007.

[SST04] M. Shillor, M. Sofonea, and J. J. Telega. Quasistatic viscoelastic contact with friction and wear diffusion. *Quart. Appl. Math.*, 62(2), 379–399, 2004.

[ToT07] A. Touzaline and D. E. Teniou. A quasistatic unilateral contact problem with friction for nonlinear elastic materials. *Math. Model. Anal.*, 12(4), 497–514, 2007.