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Inhomogeneous Dependence Modelling with Time Varying Copulae

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Abstract

Measuring dependence in a multivariate time series is tantamount to modelling its dynamic structure in space and time. In the context of a multivariate normally distributed time series, the evolution of the covariance (or correlation) matrix over time describes this dynamic. A wide variety of applications, though, requires a modelling framework different from the multivariate normal. In risk management the non-normal behaviour of most financial time series calls for non-Gaussian dependence. The correct modelling of non-Gaussian dependences is therefore a key issue in the analysis of multivariate time series. In this paper we use copulae functions with *adaptively estimated* time varying parameters for modelling the distribution of returns, free from the usual normality assumptions. Further, we apply copulae to estimation of *Value-at-Risk* (VaR) of portfolios and show their better performance over the *RiskMetrics* approach, a widely used methodology for VaR estimation.

1 INTRODUCTION

Time series of financial data are high dimensional and have typically a non-Gaussian behavior. The standard modelling approach based on properties of the multivariate normal distribution therefore often fails to reproduce the stylized facts (i.e. fat tails, asymmetry) observed in returns from financial assets.

A correct understanding of the time varying multivariate (conditional) distribution of returns is vital to many standard applications in finance like portfolio selection, asset pricing and Value-at-Risk calculation. Empirical evidence from asymmetric return distributions have been reported in the recent literature. Longin and Solnik (2001) investigate the distribution of joint extremes from international equity returns and reject multivariate normality in their lower orthant, Ang and Chen (2002) test for conditional correlation asymmetries in U.S. equity data, rejecting multivariate normality at daily, weekly and monthly frequencies, Hu (2006) models the distribution of index returns with mixtures of copulae, finding asymmetries in the dependence structure across markets. For a concise survey on stylized empirical facts from financial returns see Cont (2001) and Granger (2003).

Modelling distributions with copulae has drawn attention from many researchers as it avoids the procrustean bedöf normality assumptions, producing better fits of the empirical characteristics of financial returns. A natural extension is to apply copulae in a dynamic framework with conditional distributions modelled by copulae with time varying parameters. The question though is how to steer the time varying copulae parameters. This question is exactly in the focus of this paper.

A possible approach is to estimate the parameter from structurally invariant periods. There is a broad field of econometric literature on structural breaks. Tests for unit-root in macroeconomic series against stationarity with structural break at a known change point have been investigated by Perron (1989) and for unknown change point by Zivot and Andrews (1992), Stock (1994) and Hansen (2001); Andrews (1993) tests for parameter instability in nonlinear models; Andrews and Ploberger (1994) construct asymptotic optimal tests for multiple structural breaks. In a different set up, Quintos et al. (2001) test for constant tail index coefficient in Asian equity data against break at unkwnown point.

Time varying copulae and structural breaks are combined in Patton (2006). The dependence structure across exchange rates is modelled with time varying copulae with parameter specified to evolve as an ARMA type process. Tests for structural break in the ARMA coefficients at known change point are performed and strong evidence of break is found. In a similar fashion, Rodriguez (2007) models the dependence across sets of Asian and Latin American stock indexes using time varying copula where the parameter follows regime-switching dynamics. Common to these papers is that they employ a fixed (parametric) structure for the pattern of changes in the copula parameter.

In this paper we follow a semiparametric approach, since we are not specifying the parameter changing scheme. We rather *locally* select the time varying copula parameter. The choice is performed via an *adaptive estimation* under the assumption of local homogeneity: for every time point there

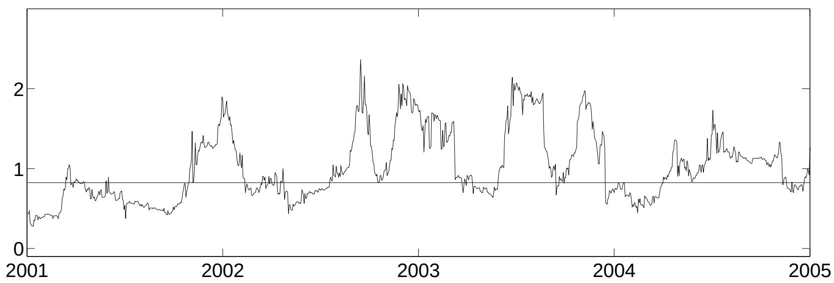

Figure 1: Time varying dependence parameter and global parameter (horizontal line) estimated with Clayton copula. Portfolio of stocks from Allianz, Münchener Rückversicherung, BASF, Bayer, DaimlerChrysler and Volkswagen

exists an interval of time homogeneity in which the copula parameter can be well approximated by a constant. This interval is recovered from the data using local change point analysis. This does not imply that the model follows a change point structure: the adaptive estimation also applies when the parameter smoothly varies from one value to another, see Spokoiny (2007).

Figure 1 shows the time varying copula parameter determined by our procedure for a portfolio composed of daily prices of six German equities and the "global" copula parameter, shown by a constant horizontal line. The absence of parametric specification for time variations in the dependence structure - its dynamics is adaptively obtained from the data - allows for flexibility in estimating dependence shifts across time.

The obtained time varying dependence structure can be used in financial engineering applications, the most prominent being the calculation of the Value-at-Risk (VaR) of a portfolio. Using copulae with adaptively estimated dependence parameters we estimate the VaR from DAX portfolios over time. As benchmark procedure we choose *RiskMetrics*, a widely used methodology based on conditional normal distributions with a GARCH specification for the covariance matrix. Backtesting underlines the improved performance of the proposed *adaptive time varying copulae fitting*.

This paper is organized as follows: Section 2 presents the basic copulae definitions, Section 3 discusses the VaR and its estimation procedure. The adaptive copula estimation is exposed in Section 4 and applied on simulated data in Section 5. In Section 6 the VaR from DAX portfolios is estimated based on adaptive time varying copulae. The estimation performance is compared with the *RiskMetrics* approach by means of backtesting.

2 COPULAE

Copulae merge marginal into joint distributions, providing a natural way for measuring the dependence structure between random variables. Copulae are present in the literature since Sklar (1959), although related concepts originate in Hoeffding (1940) and Fréchet (1951), and have been widely studied in the statistical literature, see Joe (1997), Nelsen (1998) and Mari and Kotz (2001). Applications of copulae in finance, insurance and econometrics have been investigated in Embrechts et al. (2002), Embrechts et al. (2003a), Franke et al. (2004) and Patton (2004) among others. Cherubini et al. (2004) and McNeil et al. (2005) provide an overview of copulae for practical problems in finance and insurance.

Assuming absolutely continuous distributions and continuous marginals throughout this paper, we have from Sklar's theorem that for a d-dimensional distribution function F with marginal cdf's $F_1, \dots, F_d$ there exists a unique copula $C : [0,1]^d \to [0,1]$ satisfying

$$F(x_1, \dots, x_d) = C\{F_1(x_1), \dots, F_d(x_d)\} \tag{2.1}$$

for every $x = (x_1, \dots, x_d)^\top \in \mathbb{R}^d$. Conversely, for a random vector $X = (X_1, \dots, X_d)^\top$ with cdf F_X the copula of X may be written as

$$C_X(u_1, \dots, u_d) = F_X\{F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\}$$

where $u_j = F_j(x_j)$, F_j is the cdf of X_j and $F_j^{-1}(\alpha) = \inf\{x_j : F_j(x_j) \geq \alpha\}$ its generalized inverse, $j = 1, \dots, d$. A prominent copula is the Gaussian

$$C_\Psi^{Ga}(u_1, \dots, u_d) = F_Y\{\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d)\} \tag{2.2}$$

where $\Phi(s)$, $s \in \mathbb{R}$ stands for the one-dimensional standard normal cdf, F_Y is the cdf of $Y = (Y_1, \ldots, Y_d)^\top \sim N_d(0, \Psi)$ and Ψ is a correlation matrix. The Gaussian copula represents the *dependence structure* of the multivariate normal distribution. In contrast, the Clayton copula given by

$$C_\theta(u_1, \ldots, u_d) = \left\{ \left(\sum_{j=1}^{d} u_j^{-\theta} \right) - d + 1 \right\}^{-\theta^{-1}} \tag{2.3}$$

for $\theta > 0$, expresses asymmetric dependence structures.

The dependence at upper and lower orthants of a copula C may be expressed by the upper and lower tail dependence coefficients

$$\begin{aligned} \lambda_U &= \lim_{u \to 0} \frac{\widehat{C}(u, \ldots, u)}{u} \\ \lambda_L &= \lim_{u \to 0} \frac{C(u, \ldots, u)}{u} \end{aligned}$$

where $u \in (0, 1]$ and $\widehat{C}$ is the survival copula of C, see Joe (1997) and Embrechts et al. (2003b). While Gaussian copulae are asymptotically independent at the tails $(\lambda_L = \lambda_U = 0)$, the d-dimensional Clayton copulae exhibit lower tail dependence $(\lambda_L = d^{-1/\theta})$ but are asymptotically independent at the upper tail $(\lambda_U = 0)$. Joe (1997) provides a summary of diverse copula families and detailed description of their properties.

For estimating the copula parameter, consider a sample $\{x_t\}_{t=1}^T$ of realizations from X where the copula of X belongs to a parametric family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$. Using (2.1), the log-likelihood reads as

$$L(\theta; x_1, \ldots, x_T) = \sum_{t=1}^{T} \log c\{F_1(x_{t,1}), \ldots, F_d(x_{t,d}); \theta\} + \sum_{t=1}^{T} \sum_{j=1}^{d} \log f_j(x_{t,j}).$$

where $c(u_1, \ldots, u_d) = \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \ldots \partial u_d}$ is the density of the copula C and f_j is the probability density function of F_j. The *canonical maximum likelihood* estimator $\widehat{\theta}$ maximizes the *pseudo log-likelihood* with *empirical* marginal cdf's

$$\widetilde{L}(\theta) = \sum_{t=1}^{T} \log c\{\widehat{F}_1(x_{t,1}), \ldots, \widehat{F}_d(x_{t,d}); \theta\}$$

where

$$\widehat{F}_j(s) = \frac{1}{T+1} \sum_{k=1}^{T} \mathbf{1}_{\{x_{k,j} \leq s\}} \tag{2.4}$$

for $j = 1, \ldots, d$. Note that $\widehat{F}_j$ differs from the usual empirical cdf by the denominator $T+1$. This ensures that $\{\widehat{F}_1(x_{t,1}), \ldots, \widehat{F}_d(x_{t,d})\}^\top \in (0,1)^d$ and avoids infinite values the copula density may take on the boundary of the unit cube, see McNeil et al. (2005). Joe (1997), Cherubini et al. (2004) and Chen and Fan (2006) provide a detailed exposition of inference methods for copulae.

3 VALUE-AT-RISK AND COPULAE

The dependence (over time) between asset returns is especially important in risk management since the *profit and loss (P&L) function* determines the Value-at-Risk. More precisely, the Value-at-Risk of a portfolio is determined by the multivariate distribution of risk factor increments. If $w = (w_1, \ldots, w_d)^\top \in \mathbb{R}^d$ denotes a portfolio of positions on d assets and $S_t = (S_{t,1}, \ldots, S_{t,d})^\top$ a non-negative random vector representing the prices of the assets at time t, the value V_t of the portfolio w is given by

$$V_t = \sum_{j=1}^{d} w_j S_{t,j}.$$

The random variable

$$L_t = (V_t - V_{t-1}) \tag{3.1}$$

called *profit and loss (P&L) function*, expresses the change in the portfolio value between two subsequent time points. Defining the *log-returns* $X_t = (X_{t,1}, \ldots, X_{t,d})^\top$ where $X_{t,j} = \log S_{t,j} - \log S_{t-1,j}$ and $\log S_{0,j} = 0$, $j = 1, \ldots, d$, (3.1) can be written as

$$L_t = \sum_{j=1}^{d} w_j S_{t-1,j} \left\{\exp(X_{t,j}) - 1\right\}. \tag{3.2}$$

The cdf of L_t is given by $F_{t,L_t}(x) = P_t(L_t \leq x)$. The *Value-at-Risk* at level α from a portfolio w is defined as the α-quantile from F_{t,L_t}:

$$\text{VaR}_t(\alpha) = F^{-1}_{t,L_t}(\alpha). \tag{3.3}$$

It follows from (3.2) that F_{t,L_t} depends on the specification of the d-dimensional distribution of the risk factors X_t. Thus, modelling their distribution over time is essential for obtaining the quantiles (3.3).

The *RiskMetrics* technique, a widely used methodology for VaR estimation, assumes that risk factors X_t follow a conditional multivariate normal distribution $\mathcal{L}(X_t|\mathcal{F}_{t-1}) = N(0, \Sigma_t)$, where $\mathcal{F}_{t-1} = \sigma(X_1, \ldots, X_{t-1})$ is the σ-field generated by the first $t-1$ observations, and estimates the covariance matrix Σ_t for one-period return as

$$\widehat{\Sigma}_t = \lambda \widehat{\Sigma}_{t-1} + (1-\lambda) X_{t-1} X_{t-1}^{\top} \tag{3.4}$$

where the parameter λ is the so-called *decay factor*. $\lambda = 0.94$ provides the best backtesting results for daily returns according to J.P.Morgan/Reuters (1996). In the copulae based approach one first corrects the contemporaneous mean and volatility in the log-returns process:

$$X_{t,j} = \mu_{t,j} + \sigma_{t,j}\varepsilon_{t,j} \tag{3.5}$$

where $\mu_{t,j} = E[X_{t,j} \mid \mathcal{F}_{t-1}]$ is the conditional mean and $\sigma^2_{t,j} = E[(X_{t,j} - \mu_{t,j})^2 \mid \mathcal{F}_{t-1}]$ the conditional variance of $X_{t,j}$. The standardised innovations $\varepsilon_t = (\varepsilon_{t,1}, \ldots, \varepsilon_{t,d})^{\top}$ have joint cdf F_{ε_t} given by

$$F_{\varepsilon_t}(x_1, \ldots, x_d) = C_\theta\{F_{t,1}(x_1), \ldots, F_{t,d}(x_d)\} \tag{3.6}$$

where $F_{t,j}$ is the cdf of $\varepsilon_{t,j}$ and C_θ is a *copula* belonging to a parametric family $\mathcal{C} = \{C_\theta, \theta \in \Theta\}$. For details on the above model specification see Chen and Fan (2006) and Chen et al. (2006). For the Gaussian copula with Gaussian marginals we recover the conditional Gaussian *RiskMetrics* framework.

To obtain the Value-at-Risk in this set up, the dependence parameter and cdf's from residuals are estimated from a sample of log-returns and used to generate P&L Monte Carlo samples. Their quantiles at different levels are the estimators for the Value-at-Risk, see Embrechts et al. (2002).

The whole procedure can be summarized as follows, see Härdle et al. (2002) and Giacomini and Härdle (2005): for a portfolio $w \in \mathbb{R}^d$ and a sample $\{x_{t,j}\}_{t=1}^T$, $j = 1, \ldots, d$ of log-returns, the Value-at-Risk at level α is estimated according to the following steps

1. determination of innovations $\{\widehat{\varepsilon}_t\}_{t=1}^T$ by e.g. deGARCHing
2. specification and estimation of marginal cdf's $F_j(\widehat{\varepsilon}_j)$
3. specification of a parametric copula family $\mathcal{C}$ and estimation of the dependence parameter θ
4. generation of Monte Carlo sample of innovations ε and losses L
5. estimation of $\widehat{\mathrm{VaR}}(\alpha)$, the empirical α-quantile of F_L.

4 MODELLING WITH TIME VARYING COPULAE

Very similar to the *RiskMetrics* procedure, one can perform a *moving (fixed length) window* estimation of the copula parameter. This procedure though does not fine tune local changes in dependences. In fact, the cdf F_{ε_t} from (3.6) is modelled as $F_{t,\varepsilon_t} = C_{\theta_t}\{F_{t,1}(\cdot), \ldots, F_{t,d}(\cdot)\}$ with probability measure P_{θ_t}. The moving window of fixed width will estimate a θ_t for each t, but has clear limitations. The choice of a small window results in a high pass filtering and hence, in a very unstable estimate with huge variability. The choice of a large window leads to a poor sensitivity of the estimation procedure and to a high delay in the reaction to changes in dependence measured by the parameter θ_t.

In order to choose an interval of homogeneity we employ a local parametric fitting approach as introduced by Polzehl and Spokoiny (2006), Belomestny and Spokoiny (2007) and Spokoiny (2007). The basic idea is to select for each time point t_0 an interval $I_{t_0} = [t_0 - m_{t_0}, t_0]$ of length m_{t_0} in such a way that the time varying copula parameter θ_t can be well approximated by a constant value θ. The question is of course how to select m_{t_0} in an online situation from historical data. The aim should be to select I_{t_0} as close as possible to the so-called öraclechoice interval. The "oracle" choice is defined as the largest interval $I = [t_0 - m^*_{t_0}, t_0]$, for which the *small modelling bias condition (SMB)*:

$$\Delta_I(\theta) = \sum_{t \in I} \mathcal{K}(P_{\theta_t}, P_\theta) \leq \Delta \tag{4.1}$$

for some $\Delta \geq 0$ holds. Here θ is constant and

$$\mathcal{K}(P_\vartheta, P_{\vartheta'}) = E_\vartheta \log \frac{p(y, \vartheta)}{p(y, \vartheta')}$$

denotes the *Kullback-Leibler divergence*. In such an oracle choice interval, the parameter $\theta_{t_0} = \theta_t|_{t=t_0}$ can be "optimally" estimated from $I = [t_0 - m^*_{t_0}, t_0]$. The error and risk bounds are calculated in Spokoiny (2007). It is important to mention that the concept of local parametric approximation allows to treat in a unified way the case of "switching regime" models with spontaneous changes of parameters and the "smooth transition" case when the parameter varies smoothly in time.

The "oracle" choice of the interval of homogeneity depends of course on the unknown time varying copula parameter θ_t. The next Section presents an adaptive (data driven) procedure which "mimics" the "oracle" in the sense that it delivers the same accuracy of estimation as the "oracle" one. The trick is to find the largest interval in which the hypothesis of a local constant copula parameter is supported. The *Local Change Point (LCP)* detection procedure, originates from Mercurio and Spokoiny (2004) and sequentially tests the hypothesis: θ_t is constant (i.e. $\theta_t = \theta$) within some interval I (local parametric assumption).

The LCP procedure for a given point t_0 starts with a family of nested intervals $I_0 \subset I_1 \subset I_2 \subset \ldots \subset I_K = I_{K+1}$ of the form $I_k = [t_0 - m_k, t_0]$. The sequence m_k determines the length of these interval c̈andidates; see Subsection (4.2). Every interval I_k leads to an estimate $\widetilde{\theta}_k$ of the copula parameter θ_{t_0}. The procedure selects one interval $\widehat{I}$ out of the given family and therefore, the corresponding estimate $\widehat{\theta} = \widetilde{\theta}_{\widehat{I}}$.

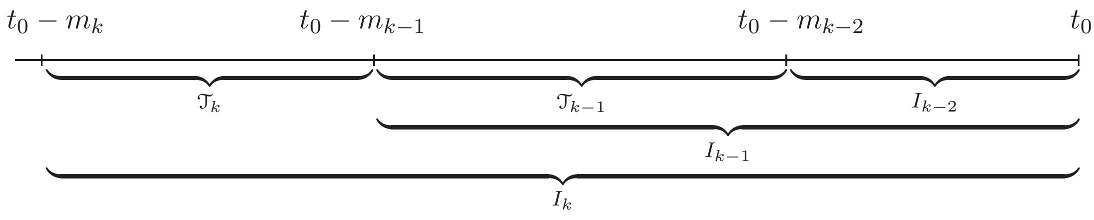

Figure 2: Choice of the intervals I_k and $\mathcal{T}_k$

The idea of the procedure is to sequentially screen each interval $\mathcal{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$ and check each point $\tau \in \mathcal{T}_k$ as a possible change point location, see Subsection (4.1) for more details. The family of intervals I_k and $\mathcal{T}_k$ are illustrated in Figure 2. The interval I_k is accepted if no change point is detected within $\mathcal{T}_1, \ldots, \mathcal{T}_k$. If the hypothesis of homogeneity is rejected for an interval-candidate I_k the procedure stops and selects the latest accepted interval. The formal description reads as follows:

Start the procedure with $k = 1$ and

1. test the hypothesis $H_{0,k}$ of no structural changes within $\mathcal{T}_k$ using the larger testing interval I_{k+1};
2. if no change points were found in $\mathcal{T}_k$, then I_k is accepted. Take the next interval $\mathcal{T}_{k+1}$ and repeat the previous step until homogeneity is rejected or the largest possible interval $I_K = [t_0 - m_K, t_0]$ is accepted;
3. if $H_{0,k}$ is rejected for $\mathcal{T}_k$, the estimated interval of homogeneity is the last accepted interval $\widehat{I} = I_{k-1}$.
4. if the largest possible interval I_K is accepted we take $\widehat{I} = I_K$.

We estimate the copula dependence parameter θ at time instant t_0 from observations in $\widehat{I}$, assuming the homogeneous model within $\widehat{I}$, i.e. we define $\widehat{\theta}_{t_0} = \widetilde{\theta}_{\widehat{I}}$. We also denote by $\widehat{I}_k$ the largest accepted interval after k steps of the algorithm and by $\widehat{\theta}_k$ the corresponding estimate of the copula parameter.

It is worth mentioning that the objective of the described estimation algorithm is not to detect the points of change for the copula parameter, but rather to determine the current dependence structure from historical data by selecting an interval of time homogeneity. This distinguishes our approach from other procedures for estimating a time varying parameter by change point detection. A visible advantage of our approach is that it equally applies to the case of spontaneous changes in the dependence structure and in the case of smooth transition in the copula parameter. The obtained dependence structure can be used for different purposes in financial engineering, the most prominent being the calculation of the VaR, see also Section 6.

The theoretical results from Spokoiny and Chen (2007) and Spokoiny (2007) indicate that the proposed procedure provides the rate optimal estimation of the underlying parameter when this smoothly varies with time. It has also been shown that the procedure is very sensitive to structural breaks and provides the minimal possible delay in detection of changes, where the delay depends on the size of change in terms of Kullback-Leibler divergence.

4.1 Test of homogeneity against a change point alternative

In the homogeneity test against change point alternative we want to check every point of an interval $\mathcal{T}$ (recall Figure 2), here called *tested* interval, on a possible change in the dependence structure at this moment. To perform this check, we assume a larger *testing* interval I of form $I = [t_0 - m, t_0]$, so that $\mathcal{T}$ is an internal subset within I. The null hypothesis H_0 means that $\forall t \in I$, $\theta_t = \theta$, i.e., the observations in I follow the model with dependence parameter θ. The alternative hypothesis H_1 claims that $\exists \tau \in \mathcal{T}$ such that $\theta_t = \theta_1$ for $t \in J = [\tau, t_0]$ and $\theta_t = \theta_2 \neq \theta_1$ for $t \in J^c = [t_0 - m, \tau)$, i.e. the parameter θ changes spontaneously in some point $\tau \in \mathcal{T}$. Figure 3 depicts I, $\mathcal{T}$ and the subintervals J and J^c determined by the point $\tau \in \mathcal{T}$.

Let $L_I(\theta)$ be the log-likelihood and $\widetilde{\theta}_I$ the maximum likelihood estimate for the interval I. The log-likelihood functions corresponding to H_0 and H_1 are $L_I(\theta)$ and $L_J(\theta_1) + L_{J^c}(\theta_2)$ respectively.

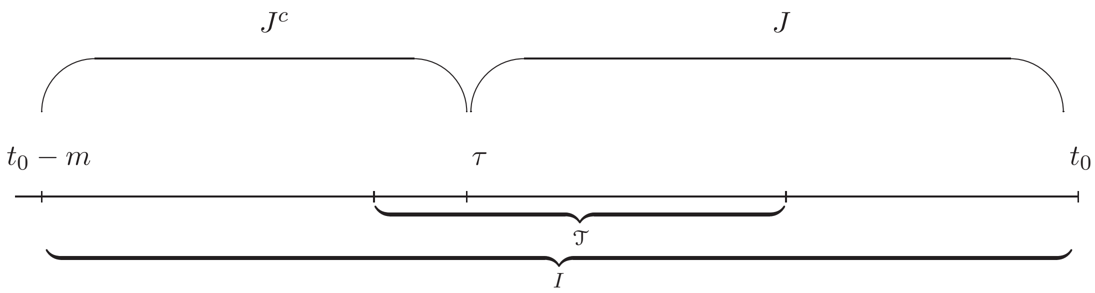

Figure 3: Testing interval I, tested interval $\mathfrak{T}$, subintervals J and J^c for a point $\tau \in \mathfrak{T}$

The likelihood ratio test for the single change point with known fixed location τ can be written as

$$\begin{aligned} T_{I,\tau} &= \max_{\theta_1,\theta_2} \{L_J(\theta_1) + L_{J^c}(\theta_2)\} - \max_{\theta} L_I(\theta) \\ &= L_J(\widetilde{\theta}_J) + L_{J^c}(\widetilde{\theta}_{J^c}) - L_I(\widetilde{\theta}_I). \end{aligned}$$

The test statistic for unknown change point location is defined as

$$T_I = \max_{\tau \in \mathfrak{T}} T_{I,\tau}.$$

The change point test compares this test statistic with a critical value $\mathfrak{z}_I$ which may depend on the interval I. One rejects the hypothesis of homogeneity if $T_I > \mathfrak{z}_I$.

4.2 Parameters of the LCP procedure

In order to apply the LCP testing procedure for local homogeneity, we have to specify some parameters. This includes: selection of interval candidates I_k, or equivalently, of the tested intervals $\mathfrak{T}_k$ and choice of respective critical values $\mathfrak{z}_k$. One possible parameter set that has been succesfully employed in simulations is presented below.

Selection of interval candidates I_k and internal points $\mathfrak{T}_k$: it is useful to take the set of numbers m_k defining the length of I_k and $\mathfrak{T}_k$ in form of a geometric grid. We fix the value m_0 and define $m_k = [m_0 c^k]$ for $k = 1, 2, \ldots, K$ and $c > 1$ where $[x]$ means the integer part of x. We set $I_k = [t_0 - m_k, t_0]$ and $\mathfrak{T}_k = [t_0 - m_k, t_0 - m_{k-1}]$ for $k = 1, 2, \ldots, K$, see Figure 2.

Choice of the critical values $\mathfrak{z}_k$. The algorithm is in fact a multiple testing procedure. Mercurio and Spokoiny (2004) suggested to select the critical value $\mathfrak{z}_k$ to provide the overall first type error probability of rejecting the hypothesis of homogeneity in the homogeneous situation. Here, we follow another proposal from Spokoiny and Chen (2007) which focuses on estimation losses caused by the "false alarm" - in our case obtaining a too small homogeneity interval - rather than on its probability.

In the homogeneous situation with $\theta_t \equiv \theta^*$ for all $t \in I_{k+1}$, the desirable behavior of the procedure is that after the first k steps the selected interval $\widehat{I}_k$ coincides with I_k and the corresponding estimate $\widehat{\theta}_k$ coincides with $\widetilde{\theta}_k$, that means there is no "false alarm". In the contrary, in case of "false alarm" the selected interval $\widehat{I}_k$ is smaller than I_k, and hence, the corresponding estimate $\widehat{\theta}_k$ has larger variability than $\widetilde{\theta}_k$. This means that the "false alarm" at the early steps of the procedure is more critical than at the final steps, as it may lead to selecting an estimate with very high variance. The difference between $\widehat{\theta}_k$ and $\widetilde{\theta}_k$ can naturally be measured by the value $L_{I_k}(\widetilde{\theta}_k, \widehat{\theta}_k) = L_{I_k}(\widetilde{\theta}_k) - L_{I_k}(\widehat{\theta}_k)$ normalized by the risk $\mathfrak{R}(\theta^*)$ of the non-adaptive estimate $\widetilde{\theta}_k$:

$$\mathfrak{R}(\theta^*) = \max_{k \geq 1} \boldsymbol{E}_{\theta^*} \big| L_{I_k}(\widetilde{\theta}_k, \theta^*) \big|^{1/2}.$$

The conditions we impose read as:

$$\boldsymbol{E}_{\theta^*} \big| L_{I_k}(\widetilde{\theta}_k, \widehat{\theta}_k) \big|^{1/2} \leq \rho \mathfrak{R}(\theta^*), \qquad k = 1, \ldots, K, \qquad \theta^* \in \Theta. \tag{4.2}$$

The critical values $\mathfrak{z}_k$ are selected as minimal values providing these constraints. In total we have K conditions to select K critical values $\mathfrak{z}_1, \ldots, \mathfrak{z}_K$. The values $\mathfrak{z}_k$ can be sequentially selected by Monte Carlo simulation where one simulates under $H_0 : \theta_t = \theta^*$, $\forall t \in I_K$. The parameter ρ defines how conservative the procedure is. Small ρ leads to larger critical values and hence to a conservative and non-sensitive procedure while an increase in ρ results in more sensitiveness at cost of stability. For details, see Spokoiny and Chen (2007) or Spokoiny (2007).

	$\theta^* = 0.5$			$\theta^* = 1.0$			$\theta^* = 1.5$		
k	$\rho = 0.2$	$\rho = 0.5$	$\rho = 1.0$	$\rho = 0.2$	$\rho = 0.5$	$\rho = 1.0$	$\rho = 0.2$	$\rho = 0.5$	$\rho = 1.0$
1	3.64	3.29	2.88	3.69	3.29	2.84	3.95	3.49	2.96
2	3.61	3.14	2.56	3.43	2.91	2.35	3.69	3.02	2.78
3	3.31	2.86	2.29	3.32	2.76	2.21	3.34	2.80	2.09
4	3.19	2.69	2.07	3.04	2.57	1.80	3.14	2.55	1.86
5	3.05	2.53	1.89	2.92	2.22	1.53	2.95	2.65	1.49
6	2.87	2.26	1.48	2.92	2.17	1.19	2.83	2.04	0.94
7	2.51	1.88	1.02	2.64	1.82	0.56	2.62	1.79	0.31
8	2.49	1.72	0.35	2.33	1.39	0.00	2.35	1.33	0.00
9	2.18	1.23	0.00	2.03	0.81	0.00	2.10	0.60	0.00
10	0.92	0.00	0.00	0.82	0.00	0.00	0.79	0.00	0.00

Table 1: Critical values $\mathfrak{z}_k(\rho, \theta^*)$ for $m_0 = 20$ and $c = 1.25$. Clayton copula, based on 5000 simulations

5 SIMULATED EXAMPLES

In this Section we apply the LCP procedure on simulated data with dependence structure given by the Clayton copula. We generate sets of 6 dimensional data with a sudden jump in the dependence parameter given by

$$\theta_t = \begin{cases} \vartheta_a & \text{if} \quad -390 \le t \le 10 \\ \vartheta_b & \text{if} \quad 10 < t \le 210 \end{cases}$$

for different values of $(\vartheta_a, \vartheta_b)$: one of them is fixed at 0.1 (close to independence) while the other is set to larger values.

The LCP procedure is implemented with the family of interval candidates in form of a geometric grid defined by $m_0 = 20$ and $c = 1.25$. The critical values, selected according to (4.2) for different ρ and θ^*, are displayed in Table 1. The choice of θ^* has negligible influence in the critical values for fixed ρ, therefore we use $\mathfrak{z}_1, \ldots, \mathfrak{z}_K$ obtained with $\theta^* = 1.0$. Based on our experience, see Spokoiny and Chen (2007) and Spokoiny (2007), the default choice for ρ is 0.5.

Figure 4 shows the pointwise median and quantiles of the estimated parameter $\widehat{\theta}_t$ for distinct values of $(\vartheta_a, \vartheta_b)$ based on 100 simulations. The *detection delay* δ at rule $r \in [0, 1]$ to jump of

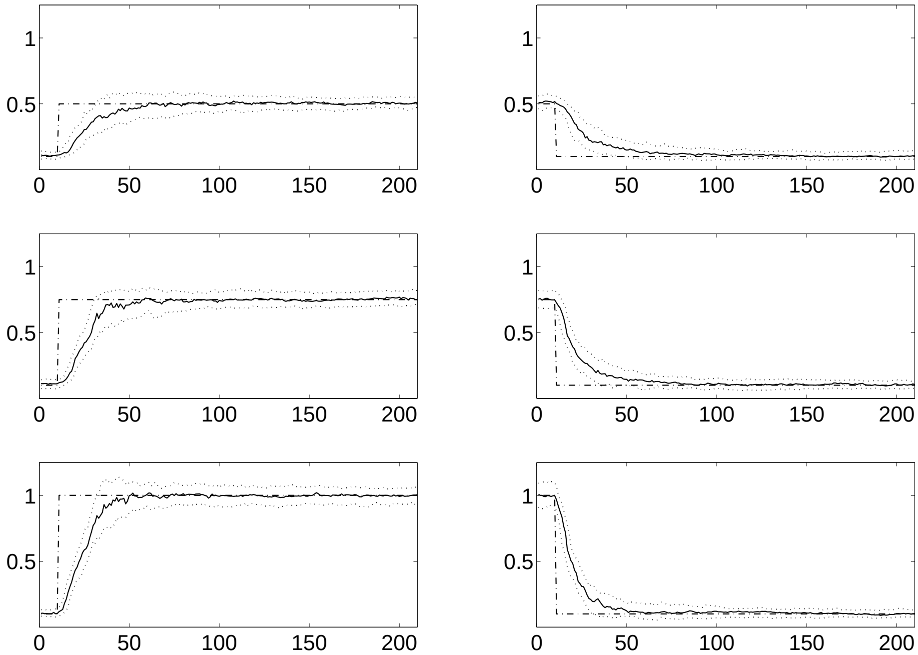

Figure 4: Pointwise median (full), 0.25 and 0.75 quantiles (dotted) from $\widehat{\theta}_t$. True parameter θ_t (dashed) with $\vartheta_a = 0.10$, $\vartheta_b = 0.50$, 0.75 and 1.00 (left, top to bottom) and $\vartheta_b = 0.10$, $\vartheta_a = 0.50$, 0.75 and 1.00 (right, top to bottom). Based on 100 simulations from Clayton copula, estimated with LCP, $m_0 = 20$, $c = 1.25$ and $\rho = 0.5$

size $\gamma = \theta_t - \theta_{t-1}$ at t is expressed by

$$\delta(t, \gamma, r) = \min\{u \geq t : \widehat{\theta}_u = \theta_{t-1} + r\gamma\} - t \tag{5.1}$$

and represents the number of steps necessary for the estimated parameter to reach the r-fraction of a jump in the true parameter.

Detection delays are proportional to the probability of error of type II, i.e. the probability of accepting homogeneity in case of a jump. Thus, tests with higher power correspond to lower delays δ. Moreover, as the Kullback-Leibler divergences for upward and downward jumps are proportional to the power of the respective homogeneity tests, larger divergences result in faster jump detections.

The descriptive statistics for detection delays to jumps at $t = 11$ for different values of $(\vartheta_a, \vartheta_b)$ are

$(\vartheta_a, \vartheta_b)$	r	mean	std dev.	max	min
	0.25	9.06	7.28	56	0
$(0.50, 0.10)$	0.50	13.64	9.80	60	0
	0.75	21.87	14.52	89	3
	0.25	5.16	4.24	21	0
$(0.75, 0.10)$	0.50	8.85	5.55	25	0
	0.75	16.72	10.37	64	3
	0.25	4.47	2.94	12	0
$(1.00, 0.10)$	0.50	7.94	4.28	22	0
	0.75	14.79	7.38	62	5
	0.25	8.94	6.65	36	0
$(0.10, 0.50)$	0.50	14.21	9.06	53	0
	0.75	21.43	12.15	68	0
	0.25	9.00	4.80	25	0
$(0.10, 0.75)$)	0.50	14.30	5.96	40	3
	0.75	21.00	10.97	75	6
	0.25	7.39	3.67	19	0
$(0.10, 1.00)$	0.50	13.10	4.13	22	2
	0.75	20.13	7.34	55	10

Table 2: Statistics for detection delay δ calculated as in (5.1) at rule r, based on 100 simulations from Clayton copula, $m_0 = 20$, $c = 1.25$ and $\rho = 0.5$

in Table 2. The mean detection delay decreases with $\gamma = \vartheta_b - \vartheta_a$ and are higher for downward than for upward jumps. Figure 5 shows that for Clayton copulae the Kullback-Leibler divergence is higher for upward than for downward jumps. Figure 6 displays the mean detection delays against jump size for upward and downward jumps.

The LCP procedure is also applied on simulated data with smooth transition in the dependence parameter given by

$$\theta_t = \begin{cases} \vartheta_a & \text{if} \quad -350 \le t \le 50 \\ \vartheta_a + \frac{t-50}{100}(\vartheta_b - \vartheta_a) & \text{if} \quad 50 < t \le 150 \\ \vartheta_b & \text{if} \quad 150 < t \le 350 \end{cases}$$

Figure 7 depicts the pointwise median and quantiles of the estimated parameter $\widehat{\theta}_t$ and the true parameter θ_t for $(\vartheta_a, \vartheta_b)$ set to $(0.10, 1.00)$ and $(1.00, 0.10)$.

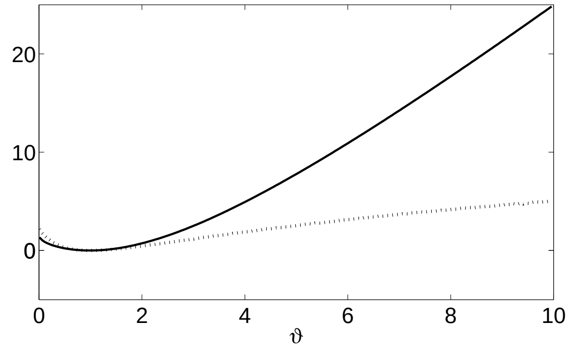

Figure 5: Kullback-Leibler divergences $\mathcal{K}(0.10, \vartheta)$ (full) and $\mathcal{K}(\vartheta, 0.10)$ (dashed), Clayton copula

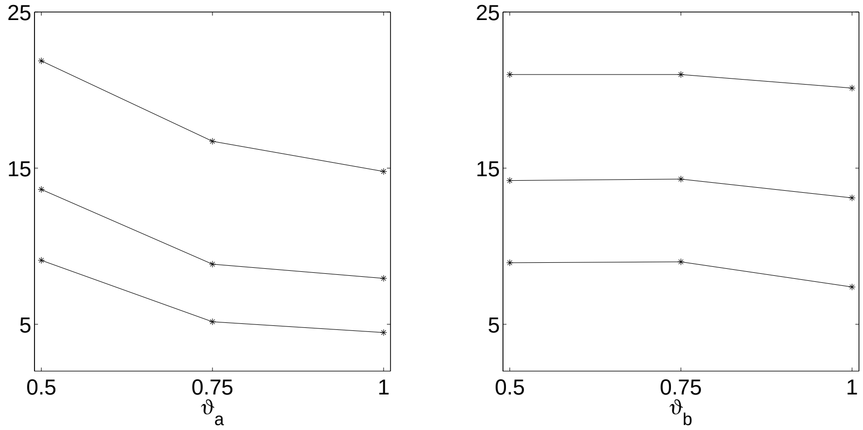

Figure 6: Mean detection delays (dots) at rule $r = 0.75$, 0.50 and 0.25 from top to bottom. Left: $\vartheta_b = 0.10$ (upward jump). Right: $\vartheta_a = 0.10$ (downward jump), based on 100 simulations from Clayton copula, $m_0 = 20$, $c = 1.25$ and $\rho = 0.5$

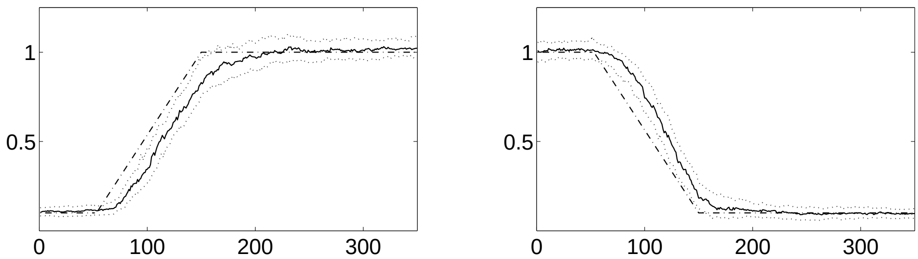

Figure 7: Pointwise median (full), 0.25 and 0.75 quantiles (dotted) from $\widehat{\theta}_t$ and true parameter θ_t (dashed) with $\vartheta_a = 0.10$ and $\vartheta_b = 1.00$ (left) and $\vartheta_a = 1.00$ and $\vartheta_b = 0.10$ (right). Based on 100 simulations from Clayton copula, estimated with LCP, $m_0 = 20$, $c = 1.25$ and $\rho = 0.5$

6 EMPIRICAL RESULTS

In this Section the Value-at-Risk from German stock portfolios is estimated based on time varying copulae and *RiskMetrics* (RM) approaches. The time varying copula parameters are selected by Local Change Point (LCP) and moving window (MW) procedures. Backtesting is used to evaluate the performances of the three methods in VaR estimation.

Two groups of 6 stocks listed on DAX are used to compose the portfolios. Stocks from group 1 belong to three different industries: automotive (Volkswagen and DaimlerChrysler), insurance (Allianz and Münchener Rückversicherung) and chemical (Bayer and BASF) while group 2 is composed of stocks from six industries: electrical (Siemens), energy (E.ON), metallurgical (ThyssenKrupp), airlines (Lufthansa), pharmaceutical (Schering) and chemical (Henkel). The portfolio values are calculated using 1270 observations, from 01.01.2000 to 31.12.2004, of the daily stock prices (data available in http://sfb649.wiwi.hu-berlin.de/fedc).

The selected copula belongs to the Clayton family (2.3). Clayton copulae have a natural interpretation and are well advocated in risk management applications. In line with the stylized facts for financial returns, Clayton copulae are asymmetric and present lower tail dependence, modelling joint extreme events at lower orthants with higher probability than Gaussian copulae for the same correlation, see McNeil et al. (2005). This fact is essential for VaR calculations and is illustrated by the ratio between (2.2) and (2.3) for off-diagonal elements of Ψ set to 0.25 and $\theta = 0.5$. For the quantiles $u_i = 0.05$, $i = 1, \dots, 6$ the ratio $C_{\Psi}^{Ga}(u_1, \dots, u_6)/C_\theta(u_1, \dots, u_6)$ equals 2.3×10^{-2} while for the 0.01 quantiles it equals 1.3×10^{-3}.

The VaR estimation follows the steps described in Section 3. In the *RiskMetrics* approach the log-returns X_t are assumed conditionally normal distributed with zero mean and covariance matrix following a GARCH specification with fixed decay factor $\lambda = 0.94$ as in (3.4).

In the time varying copulae estimation the log-returns are modelled as in (3.5) where the innovations ε_t have cdf

$$F_{t,\varepsilon_t}(x_1, \dots, x_d) = C_{\theta_t}\{F_{t,1}(x_1), \dots, F_{t,d}(x_d)\}$$

j	p-values LB		p-values ARCH	
	group 1	group 2	group 1	group 2
1	0.33	0.52	0.15	0.04
2	0.13	0.35	0.15	0.98
3	0.21	0.08	0.34	0.72
4	0.99	0.05	0.10	0.18
5	0.90	0.07	0.91	0.77
6	0.28	0.81	0.28	0.94

Table 3: p-values from Ljung-Box (LB) and ARCH tests on residuals $\widehat{\varepsilon}_{t,j}$, $j = 1, \ldots, 6$ for groups 1 and 2

and C_θ is the Clayton copula. The *univariate* log-returns $X_{t,j}$ corresponding to stock j are devolatized according to *RiskMetrics*, i.e. with zero conditional means and conditional variances $\sigma^2_{t,j}$ estimated by the univariate version of (3.4) with decay factor equal to 0.94. We note that this choice sets the same specification for the dynamics of the univariate returns across all methods (RM, MW and LCP), making their performances in VaR estimation comparable. Moreover, as the means from daily returns are clearly dominated by the variances and are approximately independent on the available information sets, see Jorion (1995), Fleming et al. (2001) and Christoffersen and Diebold (2006), their specification is very unlikely to cause a perceptible bias in the estimated variances and dependence parameters. Therefore the zero mean assumption is, as pointed out by Kim et al. (1999), as good as any other choice. Daily returns are also modelled with zero conditional means in Fan and Gu (2003) and Härdle et al. (2003) among others.

The GARCH specification (3.4) with $\lambda = 0.94$ optimizes variance forecasts across a large number of assets, J.P.Morgan/Reuters (1996), and is widely used in the financial industry. Different choices for the decay factor (like 0.85 or 0.98) result in negligible changes (about 3%) in the estimated dependence parameter.

The p-values from Ljung-Box (LB) test for serial correlation and from ARCH test for heteroscedasticity effects in the obtained residuals $\widehat{\varepsilon}_{t,j}$ are in Table 3. Normality is rejected by Jarque-Bera test with p-values approximately 0.00 for all residuals in both groups. The empirical cdf's of residuals as defined in (2.4) are used for the copula estimation.

In the MW approach the size of the estimating window is fixed as 250 days corresponding to one business year, the same size is used in e.g. Fan and Gu (2003); for the LCP procedure, following Subsection 4.2, we set the family of interval candidates as a geometric grid with $m_0 = 20$ and $c = 1.25$ and $\rho = 0.5$. We have chosen these parameters from our experience in simulations, for details on robustness of the reported results with respect to the choice of m_0 and c refer to Spokoiny (2007).

The performance of the VaR estimation is evaluated based on backtesting. At each time t the estimated Value-at-Risk at level α for a portfolio w is compared with the realization l_t of the corresponding P&L function, see (3.2), an *exceedance* occuring for each l_t smaller than $\widehat{\mathrm{VaR}}_t(\alpha)$. The ratio of the number of exceedances to the number of observations gives the *exceedance ratio*

$$\widehat{\alpha}_w(\alpha) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}_{\{l_t < \widehat{\mathrm{VaR}}_t(\alpha)\}}$$

As the first 250 observations are used for estimation, $T = 1020$. The difference between $\widehat{\alpha}$ and the desired level α is expressed by the *relative exceedance error*

$$e_w = \frac{\widehat{\alpha}_w - \alpha}{\alpha}.$$

We compute exceedance ratios and relative exceedance errors to levels $\alpha = 0.05$ and 0.01 for a set $\mathcal{W} = \{w^*, w_n; n = 1, \dots, 100\}$ of portfolios where each $w_n = (w_{n,1}, \dots, w_{n,6})^\top$ is a realization of a random vector uniformly distributed on $\mathcal{S} = \{(x_1, \dots, x_6) \in \mathbb{R}^6 : \sum_{i=1}^6 x_i = 1, x_i \geq 0.1\}$ and $w^* = \frac{1}{6}\mathbb{I}_6$, with $\mathbb{I}_d$ denoting the $(d \times 1)$ vector of ones, is the equally weighted portfolio. The degree of diversification of a portfolio can be measured based on the majorization pre-ordering on $\mathcal{S}$, see Marshall and Olkin (1979), i.e. a portfolio w_a is more diversified than portfolio w_b if $w_a \prec w_b$. Under the majorization pre-ordering the vector w^* satisfies $w^* \preceq w$ for all $w \in \mathcal{S}$, therefore the equally weighted portfolio is the most diversified portfolio from $\mathcal{W}$, see Ibragimov and Walden (2007).

The average relative exceedance error over portfolios and the corresponding standard deviation

$$A_{\mathcal{W}} = \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} e_w$$

$$D_{\mathcal{W}} = \left\{ \frac{1}{|\mathcal{W}|} \sum_{w \in \mathcal{W}} (e_w - A_{\mathcal{W}})^2 \right\}^{\frac{1}{2}}$$

are used to evaluate the performances of the time varying copulae and *RiskMetrics* methods in VaR estimation.

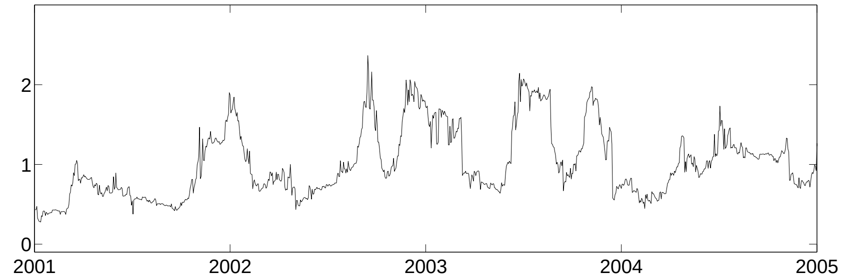

Figure 8: Estimated copula parameter $\widehat{\theta}_t$ for group 1, LCP method, $m_0 = 20$, $c = 1.25$ and $\rho = 0.5$, Clayton copula

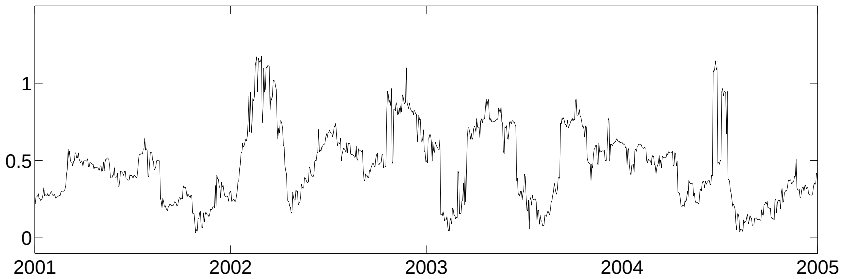

Figure 9: Estimated copula parameter $\widehat{\theta}_t$ for group 2, LCP method, $m_0 = 20$, $c = 1.25$ and $\rho = 0.5$, Clayton copula

The dependence parameter estimated with LCP for stocks from group 1 and 2 are shown in figures 8 and 9. The different industry concentrations in each group are reflected in the higher parameter values obtained for group 1. The P&L and the VaR at level 0.05 estimated with LCP, MW and RM methods for the equally weighted portfolio w^* are in Figure 10 (group 1) and 11 (group 2). Ex-

	RM		MW		LCP		
			α				
	5.00	1.00	5.00	1.00	5.00	1.00	$(\times 10^{-2})$
$\widehat{\alpha}_{w^*}$	6.11	1.48	5.62	0.59	5.52	0.69	
$\widehat{\alpha}_{w_1}$	5.91	1.38	5.42	0.49	5.42	0.69	$(\times 10^{-2})$
$\widehat{\alpha}_{w_2}$	6.40	1.28	5.91	0.49	5.71	0.59	
$A_{\mathcal{W}}$	0.23	0.45	0.11	-0.49	0.11	-0.36	
$D_{\mathcal{W}}$	0.04	0.14	0.06	0.08	0.06	0.10	

Table 4: Exceedance ratios for portfolios w^*, w_1 and w_2, average and standard deviation from relative exceedance errors across levels and methods, group 1

ceedance ratios for portfolios w^*, w_1 and w_2, average relative exceedance errors and corresponding standard deviations across methods and levels are shown in Table 4 (group 1) and 5 (group 2).

Based on the exceedance errors the LCP procedure outperforms the MW (second best) and RM methods in VaR estimation in group 1. At level 0.05 the average error associated with copula methods are about half the error from RM estimation for nearly the same standard deviation. At level 0.01 the LCP average error is the smallest in absolute value and copula methods present less standard deviations. At this level copula methods overestimate VaR and RM underestimates it. While overestimation of VaR means that a financial institution would be requested to keep *more* capital aside than necessary to guarantee the desired confidence level, underestimation means that *less* capital is reserved and the desired level is not guaranteed. Therefore, from the regulatory point of view overestimation is preferred to underestimation. In the less concentrated group 2, LCP outperforms MW and RM at level 0.05 presenting the smallest average error in magnitude for nearly the same value of $D_{\mathcal{W}}$. At level 0.01 copula methods overestimate and RM underestimates the VaR by about 60%.

It is interesting to note the effect of portfolio diversification on the exceedance errors for group 1 and level 0.01. The errors decrease with increasing portfolio diversification for copulae methods but

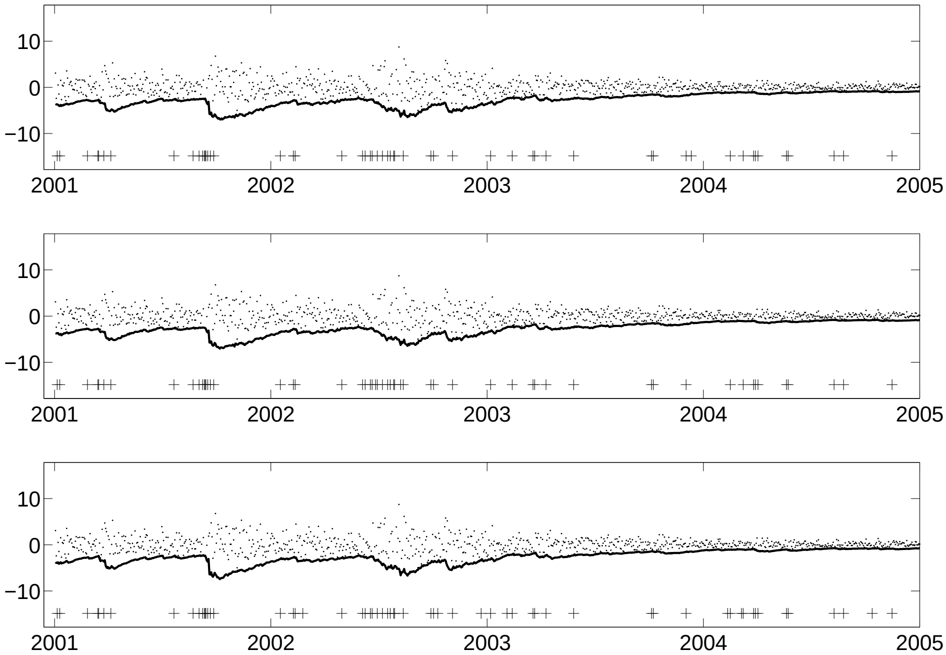

Figure 10: P&L (dots), Value-at-Risk at level $\alpha = 0.05$ (line), exceedances (crosses), estimated with LCP (above), MW (middle) and RM (below), for equally weighted portfolio w^*, group 1

become larger under the RM estimation. For another groups and levels the diversification effects are not clear. Refer to Ibragimov (2007) and Ibragimov and Walden (2007) for details on the effects of portfolio diversification under heavy-tailed distributions in risk management.

7 CONCLUSION

In this paper we modelled the dependence structure from German equity returns using time varying copulae with *adaptively estimated parameters*. In contrast to Patton (2006) and Rodriguez (2007), neither did we specify the dynamics nor assumed regime switching models for the copula parameter. The parameter choice was performed under the local homogeneity assumption with homogeneity intervals recovered from the data through local change point analysis.

We used time varying Clayton copulae, which are asymmetric and present lower tail dependence, to

	RM		MW		LCP		
			α				
	5.00	1.00	5.00	1.00	5.00	1.00	$(\times 10^{-2})$
$\widehat{\alpha}_{w^*}$	5.42	1.58	4.53	0.39	4.53	0.30	
$\widehat{\alpha}_{w_1}$	5.81	1.77	5.02	0.39	5.02	0.39	$(\times 10^{-2})$
$\widehat{\alpha}_{w_2}$	5.62	1.58	5.12	0.39	5.22	0.30	
$A_{\mathcal{W}}$	0.16	0.57	−0.10	−0.65	−0.09	−0.65	
$D_{\mathcal{W}}$	0.04	0.16	0.06	0.09	0.06	0.08	

Table 5: Exceedance ratios for portfolios w^*, w_1 and w_2, average and standard deviation from relative exceedance errors across levels and methods, group 2

estimate the Value-at-Risk from portfolios of two groups of German securities, presenting different levels of industry concentration. *RiskMetrics*, a widely used methodology based on multivariate normal distributions was chosen as benchmark for comparison. Based on backtesting the adaptive copula achieved the best VaR estimation performance in both groups, with average exceedances errors mostly small in magnitude and corresponding to sufficient capital reserve for covering losses at the desired levels.

The better VaR estimates provided by Clayton copulae indicate that the dependence structure from German equities may contain nonlinearities and asymmetries, like e.g. stronger dependence at lower tails than at upper tails, that can not be captured by the multivariate normal distribution. This asymmetry translates into extremely negative returns being more correlated than extremely positive returns. Thus, our results for the German equities resemble those from Longin and Solnik (2001), Ang and Chen (2002) and Patton (2006) for international markets, U.S. equities and deutsche mark / japanese yen exchange rates, where empirical evidence for asymmetric dependences with increasing correlations in market downturns were found.

Furthermore, in the non-Gaussian framework - with nonlinearities and asymmetries taken into consideration through the use of Clayton copulae - the adaptive estimation produces better VaR

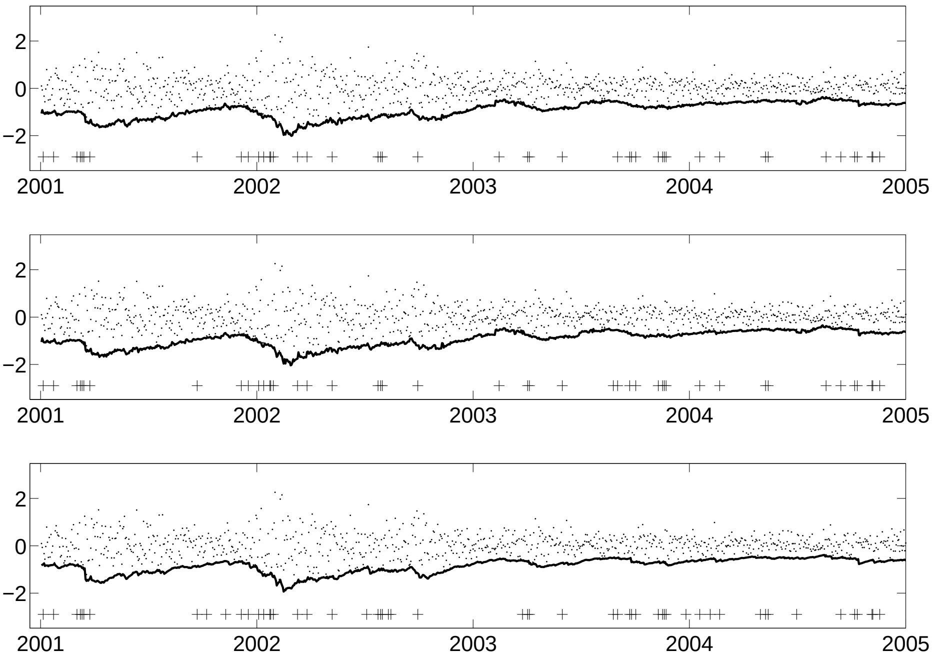

Figure 11: P&L (dots), Value-at-Risk at level $\alpha = 0.05$ (line), exceedances (crosses), estimated with LCP (above), MW (middle) and RM (below) for equally weighted portfolio w^*, group 2

fits than the moving window estimation. The high sensitive adaptive procedure can capture local changes in the dependence parameter that are not detected by the estimation with a scrolling window of fixed size.

The main advantage of using time varying copulae to model dependence dynamics is that the normality assumption is not needed. With the proposed *adaptively estimated* time varying copulae *neither* normality assumption *nor* specification for the dependence dynamics are necessary. Hence, the method provides more flexibilty in modelling dependences between markets and economies over time.

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