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Rate independent Kurzweil processes

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Abstract

The Kurzweil integral technique is applied to a class of rate independent processes with convex energy and discontinuous inputs. We prove existence, uniqueness, and continuous data dependence of solutions in BV spaces. It is shown that in the context of elastoplasticity, the Kurzweil solutions coincide with natural limits of viscous regularizations when the viscosity coefficient tends to zero. The discontinuities produce an additional positive dissipation term, which is not homogeneous of degree one.

Introduction

As an extension of [6], we propose here the Kurzweil integral approach to rate independent processes in a reflexive Banach space X that may formally be described by the inclusion

$$0 \in \partial_\xi E(t, \xi(t)) + \partial M_{K(t)}(\dot{\xi}(t)), \tag{0.1}$$

where E is an energy functional and $M_{K(t)}$ is a dissipation potential represented by the Minkowski functional of a moving convex closed set $K(t)$. Recall that the Minkowski functional $M_{\tilde{K}} : X \to [0, \infty]$ of a convex closed set $\tilde{K} \subset X$ containing 0 is defined as

$$M_{\tilde{K}}(x) = \inf \left\{ s > 0 \, : \, \frac{1}{s} x \in \tilde{K} \right\} . \tag{0.2}$$

Inclusion (0.1) can be considered as a constitutive law of nonlinear elastoplasticity with or without hardening/softening. The energetic method for solving such problems has been developed in [10] under the hypothesis that the dependence $t \mapsto E(t, \xi)$ for fixed ξ is absolutely continuous and K is fixed. An extension to moving state dependent sets K has been done in [9] as an energetic reformulation of the quasivariational inequality considered in [2]. The results of [6] were stated in terms of the Young integral in the case that K is independent of t, and E is quadratic in ξ and regulated (cf. Definition 1.7) in t. Since the Young integral is a special case of the Kurzweil integral (see [7]) and the Kurzweil calculus is simpler, we decided for the latter and show that the Kurzweil integral setting (2.8)–(2.10) explained below allows to remove some restrictions on E and K, and solve a more general problem in the space of left continuous functions of bounded variation. It is true, however, that our technique does not cover the whole range of problems treated in [10], in particular further constraints on the state space or nonstrictly convex energies.

The solution is constructed first for piecewise constant inputs; the general case then follows from the convergence properties of the Kurzweil integral. If we reformulate the problem in the energetic setting of [9, 10], it turns out that the dissipation is no longer homogeneous of degree one as in the continuous case, but additional dissipation terms related to the discontinuities occur. For a quadratic energy E, this dissipation is quadratic and can be obtained as a limit of the viscous dissipation as the viscosity parameter tends to zero. We propose an example (Example 4.2) showing that this additional dissipation cannot be neglected.

The following text is divided into four sections. In Section 1, we give a brief overview of the Kurzweil theory of integration as presented in [13]. The main results are stated in Section 2. Section 3 is devoted to the existence and uniqueness proof in the general case. In Section 4, we prove the viscous approximation result for quadratic energies.

1 The Kurzweil integral

In this section we recall the definition and some basic properties of the Kurzweil integral introduced in [8] as a framework for solving ODEs with singular right hand sides. We cite most of the results without proof, and an interested reader can find more information also in [5, 7, 14, 15].

The basic concept in the Kurzweil integration theory is that of a δ*-fine partition.* Consider a nondegenerate closed interval $[a,b] \subset \mathbb{R}$, and denote by $\mathcal{D}_{a,b}$ the set of all divisions of the form

$$d = \{t_0, \dots, t_m\}, \quad a = t_0 < t_1 < \dots < t_m = b. \tag{1.1}$$

With a division $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ we associate *partitions* D defined as

$$D = \big\{(\tau_j, [t_{j-1}, t_j]) : \ j = 1, \dots, m\big\}; \quad \tau_j \in [t_{j-1}, t_j] \ \ \forall j = 1, \dots, m. \tag{1.2}$$

We define the set

$$\Gamma(a,b) := \big\{\delta : [a,b] \to \mathbb{R} : \ \delta(t) > 0 \text{ for every } t \in [a,b]\big\}. \tag{1.3}$$

An element $\delta \in \Gamma(a,b)$ is called a *gauge*. For $t \in [a,b]$ and $\delta \in \Gamma(a,b)$ we denote

$$I_\delta(t) \ := \ (t - \delta(t), t + \delta(t))\,. \tag{1.4}$$

Definition 1.1. ([13]) *Let $\delta \in \Gamma(a,b)$ be a given gauge. A partition D of the form (1.2) is said to be δ-*fine *if for every $j = 1, \dots, m$ we have*

$$\tau_j \in [t_{j-1}, t_j] \subset I_\delta(\tau_j),$$

and the following implications hold:

$$\tau_j = t_{j-1} \Rightarrow j = 1, \quad \tau_j = t_j \Rightarrow j = m.$$

The set of all δ-fine partitions is denoted by $\mathcal{F}_\delta(a,b)$.

It is easy to see that $\mathcal{F}_\delta(a,b)$ is nonempty for every $\delta \in \Gamma(a,b)$; this follows e.g. from [5, Lemma 1.2].

Consider a reflexive Banach space X endowed with a norm $|x|$ for $x \in X$. The duality between X and its dual X^* will be denoted by $\langle \cdot, \cdot \rangle$, and $|\cdot|_*$ will be the dual norm in X^*. For given functions $f : [a,b] \to X^*$, $g : [a,b] \to X$, and a partition D of the form (1.2) we define the *Kurzweil integral sum* $K_D(f,g)$ by the formula

$$K_D(f,g) = \sum_{j=1}^{m} \langle f(\tau_j), g(t_j) - g(t_{j-1}) \rangle \, . \tag{1.5}$$

Definition 1.2. *Let $f : [a,b] \to X^*$ and $g : [a,b] \to X$ be given. We say that $J \in \mathbb{R}$ is the* Kurzweil integral *over $[a,b]$ of f with respect to g and denote*

$$J = \int_a^b \langle f(t), \mathrm{d}g(t) \rangle \, , \tag{1.6}$$

if for every $\varepsilon > 0$ there exists $\delta \in \Gamma(a,b)$ such that for every $D \in \mathcal{F}_\delta(a,b)$ we have

$$|J - K_D(f,g)| \le \varepsilon \, . \tag{1.7}$$

Using the fact that the implication

$$\delta \le \min\{\delta_1, \delta_2\} \implies \mathcal{F}_\delta(a,b) \subset \mathcal{F}_{\delta_1}(a,b) \cap \mathcal{F}_{\delta_2}(a,b) \tag{1.8}$$

holds for every $\delta, \delta_1, \delta_2 \in \Gamma(a,b)$, we easily check that the value of J in Definition 1.2 is uniquely determined.

We list below in Propositions 1.3, 1.4 some standard properties common to most integral concepts.

Proposition 1.3. *Let $f, f_1, f_2 : [a,b] \to X^*$, $g, g_1, g_2 : [a,b] \to X$ be any functions. Then the following implications hold.*

(i) If $\int_a^b \langle f_1(t), \mathrm{d}g(t) \rangle$, $\int_a^b \langle f_2(t), \mathrm{d}g(t) \rangle$ exist, then $\int_a^b \langle f_1(t) + f_2(t), \mathrm{d}g(t) \rangle$ exists and

$$\int_a^b \langle f_1(t) + f_2(t), \mathrm{d}g(t) \rangle = \int_a^b \langle f_1(t), \mathrm{d}g(t) \rangle + \int_a^b \langle f_2(t), \mathrm{d}g(t) \rangle \, . \tag{1.9}$$

(ii) If $\int_a^b \langle f(t), \mathrm{d}g_1(t) \rangle$, $\int_a^b \langle f(t), \mathrm{d}g_2(t) \rangle$ exist, then $\int_a^b \langle f(t), \mathrm{d}(g_1 + g_2)(t) \rangle$ exists and

$$\int_a^b \langle f(t), \mathrm{d}(g_1 + g_2)(t) \rangle = \int_a^b \langle f(t), \mathrm{d}g_1(t) \rangle + \int_a^b \langle f(t), \mathrm{d}g_2(t) \rangle \, . \tag{1.10}$$

(iii) If $\int_a^b \langle f(t), \mathrm{d}g(t) \rangle$ exists, then $\int_a^b \langle \lambda f(t), \mathrm{d}g(t) \rangle$, $\int_a^b \langle f(t), \mathrm{d}\lambda g(t) \rangle$ exist for every constant $\lambda \in \mathbb{R}$ and

$$\int_a^b \langle \lambda f(t), \mathrm{d}g(t) \rangle = \int_a^b \langle f(t), \mathrm{d}\lambda g(t) \rangle = \lambda \int_a^b \langle f(t), \mathrm{d}g(t) \rangle \, . \tag{1.11}$$

Proposition 1.4. *Let $f : [a,b] \to X^*$, $g : [a,b] \to X$ be given functions and let $s \in (a,b)$ be given.*

(i) Assume that $\int_a^b \langle f(t), \mathrm{d}g(t)\rangle$ *exists. Then* $\int_a^s \langle f(t), \mathrm{d}g(t)\rangle$ *,* $\int_s^b \langle f(t), \mathrm{d}g(t)\rangle$ *exist.*

(ii) Assume that $\int_a^s \langle f(t), \mathrm{d}g(t)\rangle$ *,* $\int_s^b \langle f(t), \mathrm{d}g(t)\rangle$ *exist. Then* $\int_a^b \langle f(t), \mathrm{d}g(t)\rangle$ *exists and*

$$\int_a^b \langle f(t), \mathrm{d}g(t)\rangle = \int_a^s \langle f(t), \mathrm{d}g(t)\rangle + \int_s^b \langle f(t), \mathrm{d}g(t)\rangle . \tag{1.12}$$

In order to preserve the consistency of (1.12) also in the limit cases $s = a$ and $s = b$, we set

$$\int_s^s \langle f(t), \mathrm{d}g(t)\rangle = 0 \qquad \forall s \in [a,b],\ \forall f : [a,b] \to X^*,\ g : [a,b] \to X. \tag{1.13}$$

Let us recall some typical formulas. We denote by χ_Ω the characteristic function of a set $\Omega \subset [0,T]$.

Proposition 1.5. *For every* $g : [a,b] \to X$ *,* $a \le r \le s \le b$ *and* $v \in X^*$ *we have*

(i) $\int_a^b \left\langle v\chi_{\{s\}}(t), \mathrm{d}g(t)\right\rangle = \langle v, g\rangle(s+) - \langle v, g\rangle(s-)$,

(ii) $\int_a^b \left\langle v\chi_{(r,s)}(t), \mathrm{d}g(t)\right\rangle = \langle v, g\rangle(s-) - \langle v, g\rangle(r+)$,

provided the limits on the right-hand sides exist, with the convention $\langle v, g\rangle(a-) = \langle v, g(a)\rangle$ *,* $\langle v, g\rangle(b+) = \langle v, g(b)\rangle$ *.*

Proposition 1.6. *For every* $f : [a,b] \to X^*$ *,* $a \le r \le s \le b$ *and* $v \in X$ *we have*

(i) $$\int_a^b \left\langle f(t), \mathrm{d}\big(v\chi_{\{s\}}\big)(t)\right\rangle = \begin{cases} 0 & s \in (a,b), \\ -\langle f(a), v\rangle & s = a, \\ \langle f(b), v\rangle & s = b. \end{cases}$$

(ii) $\int_a^b \left\langle f(t), \mathrm{d}\big(v\chi_{(r,s)}\big(t\big)\right\rangle = \langle f(r) - f(s), v\rangle$.

We now introduce the concept of *regulated functions*, which goes back to [1].

Definition 1.7. *Let* Y *be a Banach space with norm* $|\cdot|_Y$ *. We say that a function* $f : [a,b] \to Y$ *is* regulated *if for every* $t \in [a,b]$ *there exist both one-sided limits* $f(t+), f(t-) \in Y$ *with the convention* $f(a-) = f(a)$, $f(b+) = f(b)$ *.*

We denote by $G(a,b;Y)$ the set of all regulated functions $f : [a,b] \to Y$, and by $G_L(a,b;Y)$ and $G_R(a,b;Y)$ the space of left continuous and right continuous regulated functions on $[a,b]$, respectively. The space $BV(a,b;Y)$ of all functions of bounded variation with values in Y is included in $G(a,b;Y)$. As an important example of regulated functions, let us mention *step functions* w of the form

$$w(t) := \sum_{k=0}^{m} \hat{c}_k \chi_{\{t_k\}}(t) + \sum_{k=1}^{m} c_k \chi_{(t_{k-1},t_k)}(t), \quad t \in [a,b], \tag{1.14}$$

where $d = \{t_0, \ \ldots, t_m\} \in \mathcal{D}_{a,b}$ is a given division, and $\hat{c}_0, \ldots, \hat{c}_m, c_1, \ldots, c_m$ are given elements from Y. We further set $BV_L(a,b;Y) = BV(a,b;Y) \cap G_L(a,b;Y)$, and $BV_R(a,b;Y) = BV(a,b;Y) \cap G_R(a,b;Y)$. On $G(a,b;Y)$ we introduce a norm $\|\cdot\|_{[a,b]}$ by

$$\|f\|_{[a,b]} := \sup\{|f(\tau)|_Y : \ \tau \in [a,b]\}. \tag{1.15}$$

Lemma 1.8.

(i) Every regulated function is bounded.

(ii) The space $G(a,b;Y)$ *is complete and non-separable with respect to the norm* $\|\cdot\|_{[a,b]}$.

(iii) Given $C > 0$, *the set* $V_C = \{g \in BV(a,b;Y) : \ \mathrm{Var}_{[a,b]} g \le C\}$ *is closed in* $G(a,b;Y)$.

(iv) For every $f \in G(a,b;Y)$ *and* $\varepsilon > 0$ *there exists a step function* w *of the form (1.14) such that* $\|f - w\|_{[a,b]} \le \varepsilon$, $w(t) \in \cup_{\tau\in[a,b]}\{f(\tau)\}$ *for every* $t \in [a,b]$, *and* $\mathrm{Var}_{[a,b]}\, w \le \mathrm{Var}_{[a,b]}\, f$.

Theorem 1.9. *If* $f \in G(a,b;X^*)$ *and* $g \in BV(a,b;X)$ *or* $f \in BV(a,b;X^*)$ *and* $g \in G(a,b;X)$, *then* $\int_a^b \langle f(t), \mathrm{d}g(t)\rangle$ *exists and satisfies the estimate*

$$\left|\int_a^b \langle f(t), \mathrm{d}g(t)\rangle\right| \le \min\left\{\|f\|_{[a,b]} \operatorname*{Var}_{[a,b]} g,\ \left(|f(a)|_* + |f(b)|_* + \operatorname*{Var}_{[a,b]} f\right)\|g\|_{[a,b]}\right\}. \tag{1.16}$$

The following identity explains the motivation for a Kurzweil solution to the process (0.1) defined in (2.8)–(2.10) below.

Proposition 1.10. *If* $f \in G(a,b;X^*)$ *and* $g \in W^{1,1}(a,b;X)$, *then*

$$\int_a^b \langle f(t), \mathrm{d}g(t)\rangle = (L)\int_a^b \langle f(t), \dot g(t)\rangle\,\mathrm{d}t\,,$$

where (L) *denotes the Lebesgue integral.*

The next Proposition 1.11 plays a key role in the construction of a solution to (0.1).

Proposition 1.11. *Consider* $f, f_n \in G(a,b;X^*)$ *and* $g, g_n \in BV(a,b;X)$ *for* $n \in \mathbb{N}$ *such that*

$$\lim_{n\to\infty}\|f - f_n\|_{[a,b]} = 0, \quad \lim_{n\to\infty}\|g - g_n\|_{[a,b]} = 0, \quad \sup_{n\in\mathbb{N}} \operatorname*{Var}_{[a,b]} g_n = C < \infty.$$

Then

$$\int_a^b \langle f(t), \mathrm{d}g(t)\rangle = \lim_{n\to\infty}\int_a^b \langle f_n(t), \mathrm{d}g_n(t)\rangle\,. \tag{1.17}$$

The integration by parts formula for the Kurzweil integral contains additional jump terms and reads as follows. The proof is the same as for the Young integral in [6, Theorem 3.14].

Proposition 1.12. *For every* $f \in G(a,b;X^*)$ *and* $g \in BV(a,b;X)$ *we have*

$$\begin{aligned}\int_a^b \langle f(t), \mathrm{d}g(t)\rangle + \int_a^b \langle g(t), \mathrm{d}f(t)\rangle \ &= \ \langle f(b), g(b)\rangle - \langle f(a), g(a)\rangle \\ &+ \sum_{t\in[a,b]} \big(\langle f(t) - f(t-), g(t) - g(t-)\rangle - \langle f(t+) - f(t), g(t+) - g(t)\rangle\big).\end{aligned}$$

Note that only countably many points t enter into the sum, which is finite due to the bounded variation of g.

For a continuously differentiable mapping $E_0 : X \to \mathbb{R}$, the following integration formula holds.

Corollary 1.13. *For every $g \in BV(a,b;X)$ we have*

$$\int_a^b \langle E_0'(g(t+)), \mathrm{d}g(t)\rangle = E_0(g(b)) - E_0(g(a)) + \sum_{t\in[a,b]} \Delta(g(t+), g(t-)) \,, \tag{1.18}$$

where E_0' is the Fréchet derivative of E_0 and

$$\Delta(\xi,\eta) := \langle E_0'(\xi), \xi - \eta\rangle - E_0(\xi) + E_0(\eta) \qquad \textit{for } \ \xi,\eta \in X \,.$$

Indeed, this can directly be checked for every step function w of the form (1.14) using Propositions 1.3–1.6, which yield, after setting $c_0 = \hat{c}_0$, $c_{m+1} = \hat{c}_m$, that

$$\begin{aligned} \int_a^b \langle E_0'(w(t+)), \mathrm{d}w(t)\rangle &= \langle E_0'(\hat{c}_m), \hat{c}_m\rangle - \langle E_0'(c_1), \hat{c}_0\rangle + \sum_{k=1}^{m} \langle E_0'(c_k) - E_0'(c_{k+1}), c_k\rangle \\ &= \sum_{k=1}^{m+1} \langle E_0'(c_k), c_k - c_{k-1}\rangle \\ &= E_0(\hat{c}_m) - E_0(\hat{c}_0) + \sum_{k=1}^{m+1} \Delta(c_k, c_{k-1}), \end{aligned}$$

which is precisely (1.18). If g is an arbitrary BV-function, then it suffices to use the approximation and convergence argument of Lemma 1.8 (iv) and Proposition 1.11.

2 Statement of the problem and main results

In addition to X, X^*, consider further Banach spaces U, V endowed with norms $|\cdot|_U$, $|\cdot|_V$, respectively, and their closed subsets $U_0 \subset U, V_0 \subset V$ playing the role of parameter sets. By $\mathrm{Lin}\,(X \to X^*)$ we denote the space of continuous linear mappings from X to X^*, endowed with the norm $\|\cdot\|$. For $\gamma > 0$, we denote by $\mathrm{Sym}_\gamma\,(X \to X^*)$ the set of all $F \in \mathrm{Lin}\,(X \to X^*)$ such that

$$\langle F\xi, \eta\rangle = \langle F\eta, \xi\rangle \,, \quad \langle F\xi, \xi\rangle \geq \gamma\, |\xi|^2 \qquad \forall \xi,\eta \in X \,. \tag{2.1}$$

Indeed, if $\mathrm{Sym}_\gamma\,(X \to X^*)$ is nonempty, then X can be considered as a Hilbert space endowed with the scalar product $\langle \xi,\eta\rangle_F = \langle F\xi, \eta\rangle$ with some fixed $F \in \mathrm{Sym}_\gamma\,(X \to X^*)$.

We are given a family $K(v) \subset X$ of convex closed sets depending on a parameter $v \in V_0$, and assume that $0 \in K(v)$ for all $v \in V_0$. The *polar set* $K^*(v) \subset X^*$ of $K(v)$ is defined as

$$K^*(v) = \{y \in X^* : \langle y, \xi\rangle \leq 1 \ \ \forall \xi \in K(v)\} \,. \tag{2.2}$$

Since $K(v)$ is convex, closed, and contains 0, we have $(K^*(v))^* = K(v)$. This and other convex analysis concepts and results used here can be found in [12] and [3, Chapter 2].

To measure the distance between sets in X^*, we define the *Hausdorff distance* $d_H(A,B)$ of the sets $A, B \subset X^*$ as

$$d_H(A,B) \;=\; \max\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\}\,,$$

where $\operatorname{dist}(a,B) = \inf\{|a-b|_* : b \in B\}$ etc. For each $v \in V_0$, we define the *projection* $Q_v(x)$ of an element $x \in X^*$ onto $K^*(v)$ as the set of all $z \in K^*(v)$ such that

$$|x - z|_* \;=\; \min\{|x - z'|_* : z' \in K^*(v)\}\,. \tag{2.3}$$

For $v_1, v_2 \in V_0$ we obviously have the implication

$$x \in K^*(v_1), z \in Q_{v_2}x \implies |x - z|_* \le d_H(K^*(v_1), K^*(v_2))\,. \tag{2.4}$$

We will assume in the sequel that there exists a constant $C_H > 0$ such that

$$d_H(K^*(v_1), K^*(v_2)) \le C_H\,|v_1 - v_2|_V \quad \forall v_1, v_2 \in V_0\,. \tag{2.5}$$

Assume that $E : U_0 \times X \to \mathbb{R}$ is a functional, which with each $u \in U_0$ and $\xi \in X$ associates the *stored energy* corresponding to u and ξ. The *conjugate energy functional* $E^* : U_0 \times X^* \to \mathbb{R}$ is defined by the Legendre transform

$$E^*(u,y) = \sup_{\xi \in X}\{\langle y, \xi\rangle - E(u,\xi)\} \qquad \text{for } (u,y) \in U_0 \times X^*\,. \tag{2.6}$$

We assume the following hypothesis to hold.

Hypothesis 2.1. *Let $\partial_\xi E : U_0 \times X \to X^*$, $\partial_\xi^2 E : U_0 \times X \to \mathrm{Lin}\,(X \to X^*)$ denote the first and the second partial Fréchet derivatives of E with respect to ξ.*

(i) There exists a constant $L > 0$ such that for every $u_1, u_2 \in U_0$ and $\xi \in X$ we have

$$|\partial_\xi E(u_1,\xi) - \partial_\xi E(u_2,\xi)|_* \le L\,|u_1 - u_2|_U\,.$$

(ii) There exists a constant $\gamma > 0$ such that $\partial_\xi^2 E(u,\xi) \in \mathrm{Sym}_\gamma\,(X \to X^*)$ for every $(u,\xi) \in U_0 \times X$.

(iii) For every $R > 0$ there exists $C(R) > 0$ such that for all $u_1, u_2 \in U_0$ and $\xi_1, \xi_2 \in X$, $|\xi_i| \le R$ for $i = 1, 2$, we have

$$\|\partial_\xi^2 E(u_1,\xi_1) - \partial_\xi^2 E(u_2,\xi_2)\| \le C(R)\,(|u_1 - u_2|_U + |\xi_1 - \xi_2|)\,.$$

As a consequence of Hypothesis 2.1, we see that both $E(u,\cdot)$ and $E^*(u,\cdot)$ are strictly convex and twice continuously differentiable. As a classical property of the Legendre transform, we have

$$x = \partial_\xi E(u,\xi) \iff \xi = \partial_x E^*(u,x). \tag{2.7}$$

The symmetry of $\partial_\xi^2 E$ in (ii) follows indeed from the continuity property (iii).

The Kurzweil integral setting of Problem (0.1) is defined as follows.

Problem 2.2. *For given input functions* $u \in BV_L(0,T;U_0)$, $v \in BV_L(0,T;V_0)$ *and initial condition* $x_0 \in K^*(v_0)$, *we look for a function* $\xi \in BV_L(0,T;X)$ *such that*

$$x(t) \ := \ \partial_\xi E(u(t),\xi(t)) \in K^*(v(t)) \qquad \forall t \in [0,T]\,, \tag{2.8}$$

$$\partial_\xi E(u(0),\xi(0)) = x_0\,, \tag{2.9}$$

$$\int_0^T \langle x(t+) - y(t), \mathrm{d}\xi(t)\rangle \ \le \ 0 \tag{2.10}$$

for every $y \in G(0,T;X^*)$ *such that* $y(t) \in K^*(v(t+))$ *for every* $t \in [0,T]$.

Note that every solution to Problem 2.2 has the property

$$\int_s^t \langle x(\tau+) - y(\tau), \mathrm{d}\xi(\tau)\rangle \ \le \ 0 \tag{2.11}$$

for every test function y as in Theorem 2.3 and for every $0 \le s < t \le T$. Indeed, it suffices to set

$$\tilde y(\tau) = \begin{cases} y(\tau) & \text{for} \ \ \tau \in [s,t)\,, \\ x(\tau+) & \text{for} \ \ \tau \in [0,s) \cup [t,T]\,, \end{cases}$$

and check, by virtue of Propositions 1.4–1.5 and the left continuity of ξ, that

$$\begin{aligned} 0 \ &\ge \ \int_0^T \langle x(\tau+) - \tilde y(\tau), \mathrm{d}\xi(\tau)\rangle \ = \ \int_0^T \left\langle \chi_{[s,t)}(\tau)(x(\tau+) - y(\tau)), \mathrm{d}\xi(\tau)\right\rangle \\ &= \ \int_s^t \langle x(\tau+) - y(\tau), \mathrm{d}\xi(\tau)\rangle + \int_0^s \left\langle \chi_{\{s\}}(\tau)(x(\tau+) - y(\tau)), \mathrm{d}\xi(\tau)\right\rangle \\ &\quad - \int_s^t \left\langle \chi_{\{t\}}(\tau)(x(\tau+) - y(\tau)), \mathrm{d}\xi(\tau)\right\rangle \\ &= \ \int_s^t \langle x(\tau+) - y(\tau), \mathrm{d}\xi(\tau)\rangle\,. \end{aligned}$$

Proposition 1.10 enables us to understand the relation between (2.8)–(2.10) and (0.1). In fact, we can formally rewrite (2.8)–(2.10) as

$$\dot\xi(t) \in \partial I_{-K^*(v(t))}(-\partial_\xi E(u(t),\xi(t)))\,, \tag{2.12}$$

where $I_{\tilde K}$ is the indicator function of an arbitrary set $\tilde K$, and this is in turn equivalent to

$$-\partial_\xi E(u(t),\xi(t)) \in \partial M_{-K(v(t))}(\dot\xi(t))\,, \tag{2.13}$$

which is precisely (0.1) with $K(t)$ replaced by $-K(v(t))$.

We prove the following existence and uniqueness result.

Theorem 2.3. *Let Hypothesis 2.1 and inequality (2.5) be fulfilled. Then for every* $u \in BV_L(0,T;U_0)$, $v \in BV_L(0,T;V_0)$, *and* $x_0 \in K^*(v_0)$, *Problem 2.2 has a unique* *solution* $\xi \in BV_L(0,T;X)$. *Moreover, for every* $D > 0$ *there exists* $C_D > 0$ *such that for all input functions* u_i, v_i, $i = 1,2$, *such that*

$$\|u_i\|_{[0,T]} + \|v_i\|_{[0,T]} + \mathrm{Var}_{[0,T]}\, u_i + \mathrm{Var}_{[0,T]}\, v_i \;\le\; D\,, \quad i = 1,2\,,$$

the solutions ξ_1 *and* ξ_2 *corresponding to* u_1, v_1 *and* u_2, v_2 *and to initial conditions* x_1^0, x_2^0, *respectively, satisfy the inequality*

$$\|\xi_1 - \xi_2\|^2_{[0,T]} \le C_D \left(|x_1^0 - x_2^0|^2_* + \|u_1 - u_2\|_{[0,T]} + \|v_1 - v_2\|_{[0,T]} \right) . \tag{2.14}$$

Let now $K_0 \subset X$ be a fixed convex closed set containing 0. We define the K_0-*variation* of a function $\xi : [0,T] \to X$ on an interval $[s,t] \subset [0,T]$ by the formula

$$\mathop{\mathrm{Var}_{K_0}}_{[s,t]} \xi = \sup \sum_{i=1}^{p} M_{K_0}(\xi(\sigma_i) - \xi(\sigma_{i-1}))\,,$$

the supremum being taken over all divisions $s = \sigma_0 < \sigma_1 < \cdots < \sigma_p = t$. For a left continuous function ξ, an equivalent definition reads

$$\mathop{\mathrm{Var}_{K_0}}_{[s,t]} \xi \;=\; \sup \left\{ \int_s^t \langle y(\tau), \mathrm{d}\xi(\tau) \rangle : y \in G(s,t;X^*), y(\tau) \in K_0^* \;\; \forall \tau \in [s,t] \right\} . \tag{2.15}$$

Consider the special case of Problem 2.2, where E is of the form $E(u,\xi) = E_0(\xi) - \langle u, \xi \rangle$ for $u \in U := X^*$ and $\xi \in X$, and $K(v) = -K_0$. According to [10], the energetic solution to (2.13) with an absolutely continuous input u is defined by the *stability condition*

$(\mathcal{S})$ $\quad E(u(t),\xi(t)) \;\le\; E(u(t),\eta) + M_{K_0}(\eta - \xi(t)) \qquad \text{a.e.} \;\; \forall \eta \in X\,,$

and by the *energy inequality*

$(\mathcal{E})$ $\quad E(u(t),\xi(t)) - E(u(s),\xi(s)) + \mathop{\mathrm{Var}_{K_0}}_{[s,t]} \xi \;\le\; -(L) \int_s^t \langle \xi(\tau), \dot{u}(\tau) \rangle \, \mathrm{d}\tau$

for all $0 \le s < t \le T$,

where the right hand side corresponds to the energy supply, $\mathrm{Var}_{K_0}\, \xi$ is the dissipation, and the symbol (L) denotes again the Lebesgue integral. For differentiable energies, condition $(\mathcal{S})$ is equivalent to the inclusion $\partial_\xi E(u(t),\xi(t)) \in -K_0^*$, which is precisely (2.8).

Let $0 \le s < t \le T$ be arbitrarily chosen. For $\xi \in BV_L(0,T;X)$ and $u \in BV_L(0,T;X^*)$ we have, by Proposition 1.12 and Corollary 1.13, that

$$\int_s^t \langle u(\tau+), \mathrm{d}\xi(\tau) \rangle + \int_s^t \langle \xi(\tau), \mathrm{d}u(\tau) \rangle \;=\; \langle u(t), \xi(t) \rangle - \langle u(s), \xi(s) \rangle \,, \tag{2.16}$$

and

$$\int_s^t \langle \partial_\xi E_0(\xi(\tau+)), \mathrm{d}\xi(\tau) \rangle \;=\; E_0(\xi(t)) - E_0(\xi(s)) + \sum_{\tau \in [s,t]} \Delta(\xi(\tau+), \xi(\tau))\,, \tag{2.17}$$

with

$$\Delta(\xi,\eta) \;=\; \langle \partial_\xi E_0(\xi), \xi-\eta\rangle - E_0(\xi) + E_0(\eta) \;\geq\; \frac{\gamma}{2}|\xi-\eta|^2 \qquad \forall \xi,\eta\in X\,.$$

Using (2.15), we can take the supremum in (2.11) over all regulated functions y with values in $-K_0^*$ and obtain

$$\int_s^t \langle x(\tau+), \mathrm{d}\xi(\tau)\rangle + \operatorname*{Var}_{[s,t]}{}_{K_0}\,\xi \;\leq\; 0\,.$$

Since $x(\tau+)$ also belongs to $-K_0^*$, we have in fact the identity

$$\int_s^t \langle x(\tau+), \mathrm{d}\xi(\tau)\rangle + \operatorname*{Var}_{[s,t]}{}_{K_0}\,\xi \;=\; 0\,. \tag{2.18}$$

From identities (2.16)–(2.18) we derive for the process described by (2.8)–(2.10) the energy balance equation in the form

$$E(u(t),\xi(t)) - E(u(s),\xi(s)) + \sum_{\tau\in[s,t]} \Delta(\xi(\tau+),\xi(\tau)) + \operatorname*{Var}_{[s,t]}{}_{K_0}\,\xi = -\int_s^t \langle \xi(\tau), \mathrm{d}u(\tau)\rangle\;. \tag{2.19}$$

Conversely, the energy inequality

$$E(u(T),\xi(T)) - E(u(0),\xi(0)) + \sum_{\tau\in[0,T]} \Delta(\xi(\tau+),\xi(\tau)) + \operatorname*{Var}_{[0,T]}{}_{K_0}\,\xi \leq -\int_0^T \langle \xi(\tau), \mathrm{d}u(\tau)\rangle \tag{2.20}$$

implies (2.10) by virtue of (2.15)–(2.17).

If we compare (2.19) or (2.20) with the condition $(\mathcal{E})$, we see that in addition to the homogeneous dissipation $\mathrm{Var}_{K_0}\,\xi$ of degree 1, there is in the discontinuous case a nonhomogeneous jump dissipation $\sum \Delta(\xi(\tau+),\xi(\tau))$. We show below in Example 4.2 that it cannot be omitted.

As an even more special case, we assume now that X is a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $|\xi| = \sqrt{\langle\xi,\xi\rangle}$, $U = X$, and $K_0\subset X$ is a *bounded* convex closed set containing 0. Then there exists $r>0$ such that

$$B_r(0)\subset K_0^*\,, \tag{2.21}$$

where $B_r(0)$ is the ball centered at 0 with radius r.

Let us consider the energy functional

$$E(u,\xi) = \frac{1}{2}|\xi|^2 - \langle u,\xi\rangle\;. \tag{2.22}$$

For a given initial condition $x_0\in -K_0^*$, Problem 2.2 then has the form

$$x(t) \;:=\; \xi(t) - u(t) \in -K_0^* \qquad \forall t\in[0,T]\,, \tag{2.23}$$

$$\xi(0) = u(0) + x_0\,, \tag{2.24}$$

$$\int_0^T \langle u(t+) - \xi(t+) - y(t), \mathrm{d}\xi(t)\rangle \;\geq\; 0 \tag{2.25}$$

for every $y\in G(0,T;X)$ such that $y(t)\in K_0^*$ for every $t\in[0,T]$, which can formally be written similarly to (2.12)–(2.13) as

$$\partial M_{K_0}(\dot\xi(t)) + \xi(t) \ni u(t) \quad \Longleftrightarrow \quad \dot\xi(t) \in \partial I_{K_0^*}(u(t) - \xi(t)) \,. \tag{2.26}$$

We compare the solution ξ to (2.23)–(2.25) with the solution ξ_ε to the regularized problem

$$\partial M_{K_0}(\dot\xi_\varepsilon(t)) + \varepsilon\,\dot\xi_\varepsilon(t) + \xi_\varepsilon(t) \ni u(t) \tag{2.27}$$

with $\varepsilon > 0$ and the same initial condition

$$\xi_\varepsilon(0) = u(0) + x_0 \,. \tag{2.28}$$

In mechanical interpretation, (2.26) is the constitutive relation of a parallel elasto-plastic model, where u stands for the dimensionless stress, ξ is the strain, K_0^* is the admissible plastic stress domain, and its boundary ∂K_0^* is the yield surface. Inclusion (2.27) can again be interpreted as a parallel viscoelastoplastic constitutive relation between the dimensionless stress u and strain ξ, with a viscosity coefficient ε.

Theorem 2.4. *Let $u \in G_L(0,T;X)$ and $x_0 \in -K_0^*$ be given. Then problem (2.23)–(2.25) admits a unique solution $\xi \in BV_L(0,T;X)$, problem (2.27)–(2.28) admits a unique solution $\xi_\varepsilon \in W^{1,\infty}(0,T;X)$ for every $\varepsilon > 0$, and we have*

$$\lim_{\varepsilon\to 0+} |\xi_\varepsilon(t) - \xi(t)| = 0 \qquad \forall t \in [0,T] \,. \tag{2.29}$$

Moreover, for every $0 \le s < t \le T$ we have

$$\lim_{\varepsilon\to 0+} \left(\varepsilon \int_s^t |\dot\xi_\varepsilon(\tau)|^2 \mathrm{d}\tau + \operatorname*{Var}_{[s,t]}{}_{K_0}\, \xi_\varepsilon \right) = \frac{1}{2} \sum_{\tau\in[s,t]} |\xi(\tau+) - \xi(\tau)|^2 + \operatorname*{Var}_{[s,t]}{}_{K_0}\, \xi \,. \tag{2.30}$$

In Theorem 2.4, we do not have to assume that u has bounded variation. This is due to the regularizing property of the nonempty interior condition (2.21), see [6]. It would be interesting to establish a similar result for the general system (2.8)–(2.10).

We focus here on the case that u is allowed to be discontinuous. It cannot be expected that the convergence $\xi_\varepsilon \to \xi$ is uniform, since all ξ_ε are continuous, while the discontinuities of u give rise to discontinuities of ξ.

The right hand side of (2.30) is the rate independent dissipation as in (2.19), while the left hand side is the dissipation of the approximating process (2.27). We see that the second order jump dissipation can be interpreted as the remainder of the viscous one when the viscosity coefficient ε tends to zero.

Theorem 2.3 will be proved in the next section, the proof of Theorem 2.4 is postponed to Section 4.

3 Proof of Theorem 2.3

Consider first step functions u and v of the form

$$u(t) = u_0\,\chi_{\{0\}}(t) + \sum_{k=1}^{m} u_k\,\chi_{(t_{k-1},t_k]}(t)\,, \tag{3.1}$$

$$v(t) = v_0\,\chi_{\{0\}}(t) + \sum_{k=1}^{m} v_k\,\chi_{(t_{k-1},t_k]}(t)\,, \tag{3.2}$$

where $u_0,\dots,u_m \in U_0$ and $v_0,\dots,v_m \in V_0$ are given, and $0 = t_0 < t_1 < \cdots < t_m = T$ is a division of the interval $[0,T]$. By virtue of Propositions 1.5–1.6, the function

$$\xi(t) = \xi_0\,\chi_{\{0\}}(t) + \sum_{k=1}^{m} \xi_k\,\chi_{(t_{k-1},t_k]}(t)\,, \tag{3.3}$$

is a solution to Problem 2.2 if and only if

$$x_k = \partial_\xi E(u_k,\xi_k) \in K^*(v_k) \qquad \text{for } k = 0,1,\dots,m\,, \tag{3.4}$$

$$\langle x_k - y, \xi_k - \xi_{k-1}\rangle \le 0 \qquad \text{for every } y \in K^*(v_k)\,, k = 1,\dots,m\,. \tag{3.5}$$

By (2.7), we have

$$x_k = \partial_\xi E(u_k,\xi_k) \iff \xi_k = \partial_x E^*(u_k,x_k). \tag{3.6}$$

For $k = 0$, this gives the initial value ξ_0. For $k \ge 1$, we check that x_k satisfies (3.4)–(3.5) if and only if it is the (unique) solution of the minimization problem

$$x_k = \operatorname{argmin}\left(x \mapsto E^*(u_k,x) - \langle x,\xi_{k-1}\rangle + I_{K^*(v_k)}(x)\right)\,. \tag{3.7}$$

Indeed, if (3.7) holds, then $x_k \in K^*(v_k)$, and

$$E^*(u_k,x_k) - \langle x_k,\xi_{k-1}\rangle \le E^*(u_k,x_k+\alpha(y-x_k)) - \langle x_k+\alpha(y-x_k),\xi_{k-1}\rangle \tag{3.8}$$

for all $y \in K^*(v_k)$ and $\alpha \in (0,1]$. This yields, letting α tend to $0+$, that

$$\langle \partial_x E^*(u_k,x_k) - \xi_{k-1}, x_k - y\rangle \le 0\,, \tag{3.9}$$

which is precisely (3.4)–(3.5). The inverse implication (3.4)–(3.5) $\Rightarrow$ (3.7) follows from the convexity of $E^*(u_k,\cdot)$.

We will make repeated use of the following "discrete Gronwall lemma".

Lemma 3.1. *Let $g_k \in X$ and $F_k \in \mathrm{Sym}_\gamma(X \to X^*)$ be given for $k \in \mathbb{N}\cup\{0\}$. Let there exist constants $B, M \ge 0$ such that*

$$\sum_{k=1}^{\infty} \|F_k - F_{k-1}\| \le B\,,$$

and

$$\sum_{j=1}^{k} \langle F_j g_j, g_j - g_{j-1} \rangle \;\le\; M \quad \forall k \in \mathbb{N}.$$

Then there exists a constant $C > 0$ *depending only on* B *and* γ*, such that*

$$|g_k|^2 \le C(\langle F_0 g_0, g_0 \rangle + M)\,.$$

Proof of Lemma 3.1. Using the elementary identity

$$\begin{aligned} &\langle F_k\, g_k, g_k - g_{k-1} \rangle - \frac{1}{2}\langle F_k\, g_k, g_k \rangle + \frac{1}{2}\langle F_{k-1}\, g_{k-1}, g_{k-1} \rangle \\ &\quad = \frac{1}{2}\langle F_k\,(g_k - g_{k-1}), g_k - g_{k-1} \rangle + \frac{1}{2}\langle (F_{k-1} - F_k)\, g_{k-1}, g_{k-1} \rangle \end{aligned}$$

we obtain

$$\frac{1}{2}\langle F_k\, g_k, g_k \rangle - \frac{1}{2}\langle F_0\, g_0, g_0 \rangle \;\le\; M + \frac{1}{2}\sum_{j=1}^{k} \langle (F_j - F_{j-1})\, g_{j-1}, g_{j-1} \rangle \quad \forall k \in \mathbb{N}\,. \tag{3.10}$$

For $j \in \mathbb{N}$ set

$$\beta_j = \| F_j - F_{j-1} \|\,.$$

Then

$$\frac{\beta_{k+1}}{2}\langle F_k\, g_k, g_k \rangle \;\le\; \beta_{k+1}\left(M + \frac{1}{2}\langle F_0\, g_0, g_0 \rangle + \frac{1}{\gamma}\sum_{j=1}^{k} \frac{\beta_j}{2}\langle F_{j-1} g_{j-1}, g_{j-1} \rangle \right). \tag{3.11}$$

Put

$$r_k = \sum_{j=1}^{k+1} \frac{\beta_j}{2}\langle F_{j-1}\, g_{j-1}, g_{j-1} \rangle \quad \text{for} \quad k \in \mathbb{N} \cup \{0\}\,.$$

Then

$$r_k - r_{k-1} \;\le\; \beta_{k+1}\left(M + \frac{1}{2}\langle F_0\, g_0, g_0 \rangle + \frac{1}{\gamma} r_{k-1} \right) \qquad \text{for} \quad k \in \mathbb{N}\,.$$

Set

$$A_k = \prod_{j=1}^{k+1} \left(1 + \frac{\beta_j}{\gamma} \right),$$

i. e. $A_k \ge 1$ for all k. Then for all $k \in \mathbb{N}$ we have

$$\frac{r_k}{A_k} - \frac{r_{k-1}}{A_{k-1}} \;\le\; \frac{\beta_{k+1}}{A_k}\left(M + \frac{1}{2}\langle F_0\, g_0, g_0 \rangle \right),$$

hence

$$\begin{aligned} r_k \;&\le\; A_k \left(r_0 + \left(M + \frac{1}{2}\langle F_0\, g_0, g_0 \rangle \right) \sum_{j=1}^{k} \frac{\beta_{j+1}}{A_j} \right) \\ &\le\; A_k \left(r_0 + B\left(M + \frac{1}{2}\langle F_0\, g_0, g_0 \rangle \right) \right). \end{aligned} \tag{3.12}$$

Note that

$$\log A_k \;=\; \sum_{j=1}^{k} \log\left(1+\frac{\beta_j}{\gamma}\right) \;\le\; \frac{1}{\gamma}\sum_{j=1}^{\infty}\beta_j \;\le\; \frac{B}{\gamma}\,, \qquad r_0 \;\le\; \frac{B}{2}\,\langle F_0\, g_0, g_0\rangle\ .$$

The assertion now follows directly from (3.12). ∎

We now use the above result to prove the following "Gronwall-Kurzweil" lemma.

Lemma 3.2. *Let $g \in BV_L(0,T;X)$ and $F \in BV_L(0,T;\mathrm{Sym}_\gamma(X \to X^*))$ be given such that $g(0)=0$. Assume that*

$$\int_0^t \langle F(\tau+)g(\tau+), \mathrm{d}g(\tau)\rangle \;\le\; 0 \qquad \forall t\in[0,T]\,. \tag{3.13}$$

Then $g(t)=0$ for all $t\in[0,T]$.

The relation between Lemma 3.2 and the classical Gronwall lemma can easily be seen if F and g are absolutely continuous. Then we may rewrite (3.13) using Proposition 1.10 as

$$\frac{1}{2}\langle F(t)g(t), g(t)\rangle - \frac{1}{2}(L)\int_0^t \left\langle \dot F(\tau)g(\tau), g(\tau)\right\rangle \mathrm{d}\tau = (L)\int_0^t \langle F(\tau)g(\tau), \dot g(\tau)\rangle\,\mathrm{d}\tau \le 0$$

with $\dot F$ in $L^1(0,T;\mathrm{Lin}\,(X\to X^*))$.

Proof of Lemma 3.2. It suffices to prove that $g(T)=0$. Let $\varepsilon>0$ be arbitrarily given. By Lemma 1.8 (iv), we find step functions of the form

$$\bar g(t) \;=\; g_0\,\chi_{\{0\}}(t) + \sum_{k=1}^{m} g_k\,\chi_{(t_{k-1},t_k]}(t)\,, \tag{3.14}$$

$$\bar F(t) \;=\; F_0\,\chi_{\{0\}}(t) + \sum_{k=1}^{m} F_k\,\chi_{(t_{k-1},t_k]}(t)\,, \tag{3.15}$$

analogous to (3.1)–(3.2) and such that, taking into account Theorem 1.9,

$$g_0 = 0\,,\quad g_m = g(T)\,,$$

$$\sup_{t\in[0,T]} |g(t)-\bar g(t)| < \varepsilon\,,\qquad \sup_{t\in[0,T]} \|F(t)-\bar F(t)\| < \varepsilon\,,$$

$$\mathrm{Var}_{[0,T]}\,\bar g \le \mathrm{Var}_{[0,T]}\,g\,,\quad \mathrm{Var}_{[0,T]}\,\bar F = \sum_{k=1}^{m}\|F_k - F_{k-1}\| \le \mathrm{Var}_{[0,T]}\,F\,,$$

$$\int_0^t \left\langle \bar F(\tau+)\bar g(\tau+), \mathrm{d}\bar g(\tau)\right\rangle \;\le\; \varepsilon \qquad \forall t\in[0,T]\,.$$

For all $k=1,\dots,m$ we have

$$\int_0^{t_k} \left\langle \bar F(\tau+)\bar g(\tau+), \mathrm{d}\bar g(\tau)\right\rangle \;=\; \sum_{j=1}^{k}\langle F_j g_j, g_j - g_{j-1}\rangle \;\le\; \varepsilon\,.$$

By Lemma 3.1, we have $|g_m|^2 \le C\varepsilon$. Since ε is arbitrary, we obtain the assertion. ■

As a next step, we compare the solutions $\xi_k^{(i)}$ of the form (3.3) corresponding to different input sequences $u_0^{(i)}, u_1^{(i)}, u_2^{(i)}, \dots \in U_0$ and $v_0^{(i)}, v_1^{(i)}, v_2^{(i)}, \dots \in V_0$, and different initial conditions $x_0^{(i)}$, $i = 1,2$. We do not specify the lengths and consider possibly infinite sequences. We will assume that there exist constants $C_U, C_V > 0$ such that

$$\sum_{k=1}^{\infty} |u_k^{(i)} - u_{k-1}^{(i)}|_U \le C_U\,, \quad \sum_{k=1}^{\infty} |v_k^{(i)} - v_{k-1}^{(i)}|_V \le C_V \quad \text{for } i = 1,2\,. \tag{3.16}$$

In the inequalities (3.5) for $\xi_k^{(i)}$, we choose $y \in Q_{v_k^{(i)}}(x_{k-1}^{(i)})$, and obtain, using Hypothesis 2.1 (i) and (2.4)–(2.5), that

$$\begin{aligned} \gamma\,|\xi_k^{(i)} - \xi_{k-1}^{(i)}|^2 &\le \left\langle \partial_\xi E(u_k^{(i)}, \xi_k^{(i)}) - \partial_\xi E(u_k^{(i)}, \xi_{k-1}^{(i)}), \xi_k^{(i)} - \xi_{k-1}^{(i)} \right\rangle \\ &\le \left\langle x_k^{(i)} - x_{k-1}^{(i)}, \xi_k^{(i)} - \xi_{k-1}^{(i)} \right\rangle + L\,|u_k^{(i)} - u_{k-1}^{(i)}|_U\,|\xi_k^{(i)} - \xi_{k-1}^{(i)}| \\ &\le \left\langle y - x_{k-1}^{(i)}, \xi_k^{(i)} - \xi_{k-1}^{(i)} \right\rangle + L\,|u_k^{(i)} - u_{k-1}^{(i)}|_U\,|\xi_k^{(i)} - \xi_{k-1}^{(i)}| \\ &\le \Big(L\,|u_k^{(i)} - u_{k-1}^{(i)}|_U + C_H\,|v_k^{(i)} - v_{k-1}^{(i)}|_V \Big)\, |\xi_k^{(i)} - \xi_{k-1}^{(i)}|\,, \end{aligned} \tag{3.17}$$

hence,

$$|\xi_k^{(i)} - \xi_{k-1}^{(i)}| \le \frac{L}{\gamma}\,|u_k^{(i)} - u_{k-1}^{(i)}|_U + \frac{C_H}{\gamma}\,|v_k^{(i)} - v_{k-1}^{(i)}|_V\,. \tag{3.18}$$

In particular,

$$\sum_{k=1}^{\infty} |\xi_k^{(i)} - \xi_{k-1}^{(i)}| \le \frac{L}{\gamma}\,C_U + \frac{C_H}{\gamma}\,C_V\,, \quad \sup_{k\in\mathbb{N}} |\xi_k^{(i)}| \le |\xi_0^{(i)}| + \frac{L}{\gamma}\,C_U + \frac{C_H}{\gamma}\,C_V \quad \text{for } i = 1,2\,. \tag{3.19}$$

We now set

$$R = \max\{|\xi_0^{(1)}|, |\xi_0^{(2)}|\} + \frac{L}{\gamma}\,C_U + \frac{C_H}{\gamma}\,C_V\,. \tag{3.20}$$

In the following estimate, we proceed similarly, choosing $y \in Q_{v_k^{(1)}}(x_k^{(2)})$ and $y \in Q_{v_k^{(2)}}(x_k^{(1)})$ in inequality (3.5) for $\xi_k^{(1)}, \xi_k^{(2)}$, respectively. Summing up the two resulting inequalities, we obtain from (2.4)–(2.5) that

$$\left\langle x_k^{(1)} - x_k^{(2)}, \xi_k^{(1)} - \xi_{k-1}^{(1)} - \xi_k^{(2)} + \xi_{k-1}^{(2)} \right\rangle \le C_H\,|v_k^{(1)} - v_k^{(2)}|_V \sum_{i=1}^{2} \left|\xi_k^{(i)} - \xi_{k-1}^{(i)}\right|. \tag{3.21}$$

Note that the difference $x_k^{(1)} - x_k^{(2)}$ can be written as

$$x_k^{(1)} - x_k^{(2)} = \Big(\partial_\xi E(u_k^{(1)}, \xi_k^{(1)}) - \partial_\xi E(u_k^{(1)}, \xi_k^{(2)})\Big) + \Big(\partial_\xi E(u_k^{(1)}, \xi_k^{(2)}) - \partial_\xi E(u_k^{(2)}, \xi_k^{(2)})\Big)\,,$$

where

$$\partial_\xi E(u_k^{(1)}, \xi_k^{(1)}) - \partial_\xi E(u_k^{(1)}, \xi_k^{(2)}) = \int_0^1 \left\langle \partial_\xi^2 E(u_k^{(1)}, \xi_k^{(2)} + s(\xi_k^{(1)} - \xi_k^{(2)})), \xi_k^{(1)} - \xi_k^{(2)} \right\rangle \mathrm{d}s \,.$$

We define the mapping $F_k \in \mathrm{Lin}\,(X \to X^*)$ by the formula

$$F_k \;=\; \int_0^1 \partial_\xi^2 E(u_k^{(1)}, \xi_k^{(2)} + s(\xi_k^{(1)} - \xi_k^{(2)}))\,\mathrm{d}s \,. \tag{3.22}$$

The above computations and Hypothesis 2.1 (i) yield

$$\begin{aligned} &\left\langle F_k\,(\xi_k^{(1)} - \xi_k^{(2)}), \xi_k^{(1)} - \xi_k^{(2)} - \xi_{k-1}^{(1)} + \xi_{k-1}^{(2)} \right\rangle \\ &\qquad \le\; \left(L\,|u_k^{(1)} - u_k^{(2)}|_U + C_H\,|v_k^{(1)} - v_k^{(2)}|_V \right) \sum_{i=1}^{2} \left| \xi_k^{(i)} - \xi_{k-1}^{(i)} \right| . \end{aligned} \tag{3.23}$$

To simplify the notation, we introduce the sequences

$$\begin{aligned} w_k &= |u_k^{(1)} - u_k^{(2)}|_U + |v_k^{(1)} - v_k^{(2)}|_V \\ \alpha_k &= \max_{i=1,2}(|u_k^{(i)} - u_{k-1}^{(i)}|_U + |v_k^{(i)} - v_{k-1}^{(i)}|_V) \\ g_k &= \xi_k^{(1)} - \xi_k^{(2)} \,. \end{aligned}$$

By virtue of (3.18), inequality (3.23) can be written in the form

$$\langle F_k\, g_k, g_k - g_{k-1} \rangle \;\le\; C_1\, \alpha_k\, w_k \tag{3.24}$$

for every $k \in \mathbb{N}$ with a constant $C_1 > 0$. Hypothesis 2.1 (ii)–(iii) together with (3.20) imply that we may use Lemma 3.1 to conclude that there exists a constant $C_2 > 0$ such that $|g_k|^2 \le C_2(|g_0|^2 + \sup_{j\in\mathbb{N}} w_j)$. In other words, we have the inequality

$$\sup_{k\in\mathbb{N}} |\xi_k^{(1)} - \xi_k^{(2)}|^2 \;\le\; C_3 \left(|x_0^{(1)} - x_0^{(2)}|_*^2 + \sup_{k\in\mathbb{N}} \left(|u_k^{(1)} - u_k^{(2)}|_U + |v_k^{(1)} - v_k^{(2)}|_V \right) \right) \tag{3.25}$$

with a suitable constant $C_3 > 0$.

We are now ready to finish the proof of Theorem 2.3. Let $u \in BV_L(0,T;U_0)$, $v \in BV_L(0,T;V_0)$, and $x_0 \in K^*(v_0)$ be arbitrarily given. We first prove the uniqueness. Let ξ_1, ξ_2 be two solutions with the expected regularity, and set $x_i(t) = \partial_\xi E(u(t), \xi_i(t)) \in K(v(t))$ for $i = 1,2$ and $t \in [0,T]$. We may set $y(\tau) = x_{1-i}(\tau+)$ in the equation (2.11) for ξ_i on the interval $[0,t]$, and obtain

$$\begin{aligned} 0 \;&\ge\; \int_0^t \langle x_1(\tau+) - x_2(\tau+), \mathrm{d}(\xi_1 - \xi_2)(\tau) \rangle \\ &=\; \int_0^t \langle F(\tau+)\,(\xi_1(\tau+) - \xi_2(\tau+)), \mathrm{d}(\xi_1 - \xi_2)(\tau) \rangle \end{aligned} \tag{3.26}$$

with

$$F(\tau) \;=\; \int_0^1 \partial_\xi^2 E(u(\tau), \xi_2(\tau) + s(\xi_1(\tau) - \xi_2(\tau)))\,\mathrm{d}s\,,$$

and it suffices to use Lemma 3.2 and Hypothesis 2.1 (iii) to obtain $\xi_1 = \xi_2$.

To prove the existence, we use Lemma 1.8 (iv) to find sequences of step functions $u^{(n)} \in BV_L(0,T;U_0)$, $v^{(n)} \in BV_L(0,T;V_0)$ such that $u^{(n)}(0) = u(0)$, $v^{(n)}(0) = v(0)$, $\mathrm{Var}_{[0,T]}\, u^{(n)} \le \mathrm{Var}_{[0,T]}\, u$, $\mathrm{Var}_{[0,T]}\, v^{(n)} \le \mathrm{Var}_{[0,T]}\, v$, and

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |u^{(n)}(t) - u(t)|_U = 0\,, \quad \lim_{n\to\infty} \sup_{t\in[0,T]} |v^{(n)}(t) - v(t)|_V = 0\,.$$

We know by (3.18) that the corresponding solutions $\xi^{(n)} \in BV_L(0,T;X)$ have uniformly bounded variation. Let $n, n' \in \mathbb{N}$ be arbitrarily chosen indices. By inserting additional division points t_j, if necessary, we may assume that $u^{(n)}, v^{(n)}, u^{(n')}, v^{(n')}$ are of the form (3.1)–(3.2) with the same division $0 = t_0 < t_1 < \cdots < t_m = T$. It follows from (3.25) that

$$\sup_{t\in[0,T]} |\xi^{(n)}(t) - \xi^{(n')}(t)|^2 \;\le\; C_3 \left(\sup_{t\in[0,T]} \left(|u^{(n)}(t) - u^{(n')}(t)|_U + |v^{(n)}(t) - v^{(n')}(t)|_V \right) \right) \tag{3.27}$$

with a constant C_3 independent of n. Hence, $\{\xi^{(n)}\}$ is a Cauchy sequence with respect to the sup-norm and admits a uniform limit $\xi \in BV_L(0,T;X)$. Using the continuity of $\partial_\xi E$ and Proposition 1.11, we may pass to the limit as $n \to \infty$ in (2.8)–(2.10) for $\xi^{(n)}$, and check that ξ is the desired solution. The Hölder property (2.14) of the solution mapping follows immediately from (3.25).

4 Proof of Theorem 2.4

In the Hilbert framework, the projection $Q_{K_0^*} : X \to K_0^*$ analogous to (2.3) can be characterized as

$$z = Q_{K_0^*}(x) \iff \begin{cases} z \in K_0^* \\ \langle x - z, z - \tilde{z}\rangle \;\ge\; 0 \quad \forall \tilde{z} \in K_0^*\,. \end{cases} \tag{4.1}$$

We denote $P_{K_0^*}(x) = x - Q_{K_0^*}(x)$, and recall that for every $x \in X$ and $\alpha \ge 0$, the projection has the property

$$Q_{K_0^*}(\alpha x + (1-\alpha) Q_{K_0^*}(x)) \;=\; Q_{K_0^*}(x)\,, \tag{4.2}$$

or equivalently

$$P_{K_0^*}(Q_{K_0^*}(x) + \alpha P_{K_0^*}(x)) \;=\; \alpha P_{K_0^*}(x)\,. \tag{4.3}$$

Note also the following easy relation between the Minkowski functional M_{K_0} and the projection $Q_{K_0^*}$:

$$\forall x, y \in X:\; x \in \partial M_{K_0}(y) \iff x = Q_{K_0^*}(x + y)\,. \tag{4.4}$$

We see in particular that $\partial M_{K_0}(y) \subset K_0^*$ for every $y \in X$. Moreover, for every $y \in X$ we have

$$x \in \partial M_{K_0}(y) \implies \langle x, y\rangle = M_{K_0}(y) = \sup_{z\in K_0^*} \langle z, y\rangle \tag{4.5}$$

Since M_{K_0} is 1-homogeneous, we may rewrite (2.27) as

$$u(t) - \xi_\varepsilon(t) - \varepsilon\,\dot{\xi}_\varepsilon(t) \ \in \ \partial M_{K_0}(\varepsilon\dot{\xi}_\varepsilon(t)), \tag{4.6}$$

which is, by virtue of (4.4), in turn equivalent to

$$\varepsilon\,\dot{\xi}_\varepsilon(t) \ = \ P_{K_0^*}(u(t) - \xi_\varepsilon(t)). \tag{4.7}$$

The existence and uniqueness of a global absolutely continuous solution ξ_ε to (4.7) follows from the Lipschitz continuity of the mapping $P_{K_0^*}$. Furthermore, by (2.21) and [6, Proposition 2.2 and Theorem 2.4], for every $u \in G_L(0,T;X)$ there exists a unique solution $\xi \in BV_L(0,T;X)$ to (2.23)–(2.25). As in the previous section, the convergence analysis starts with left continuous step functions of the form

$$u(t) = u_0\,\chi_{\{0\}}(t) + \sum_{k=1}^{m} u_k\,\chi_{(t_{k-1},t_k]}(t), \tag{4.8}$$

where $u_0, u_1, \ldots, u_m$ are given elements of X, and $0 = t_0 < t_1 < \cdots < t_m = T$ is a division of the interval $[0,T]$. If u is as in (4.8), then, by [6, Proposition 4.3], the unique solution ξ of (2.23)–(2.25) has also the form of (4.8), more specifically

$$\xi(t) = \xi_0\,\chi_{\{0\}}(t) + \sum_{k=1}^{m} \xi_k\,\chi_{(t_{k-1},t_k]}(t), \tag{4.9}$$

where

$$\xi_0 = u_0 + x_0,\ \xi_k = \xi_{k-1} + P_{K_0^*}(u_k - \xi_{k-1}) \quad \text{for} \ \ k = 1,\ldots,m. \tag{4.10}$$

This is in fact nothing but the classical Moreau formula (see [11]) for time-discrete approximations of a sweeping process. Here, however, it provides the *exact solution* for piecewise constant inputs.

We first prove the following result.

Lemma 4.1. *Let u be as in (4.8), let ξ be given by (4.9), and let $\xi_\varepsilon \in W^{1,\infty}(0,T;X)$ be the solution to (2.27)–2.28) for $\varepsilon > 0$. Then*

$$\lim_{\varepsilon\to 0+} |\xi_\varepsilon(t) - \xi(t)| = 0 \qquad \forall t \in [0,T]. \tag{4.11}$$

Proof. Let us denote

$$\xi_k^\varepsilon = \xi_\varepsilon(t_k) \qquad \text{for} \ \ k = 0,1,\ldots,m. \tag{4.12}$$

For $t \in (t_{k-1}, t_k]$, Eq. (4.7) has the form

$$\varepsilon\,\dot{\xi}_\varepsilon(t) \ = \ P_{K_0^*}(u_k - \xi_\varepsilon(t)), \quad \xi_\varepsilon(t_{k-1}) = \xi_{k-1}^\varepsilon. \tag{4.13}$$

We claim that the solution of (4.13) can be represented in closed form as

$$\xi_\varepsilon(t) \ = \ \xi^\varepsilon_{k-1} + \left(1 - \mathrm{e}^{-\frac{t-t_{k-1}}{\varepsilon}}\right) P_{K_0^*}(u_k - \xi^\varepsilon_{k-1}) \qquad \text{for } \ t \in [t_{k-1}, t_k]\,. \tag{4.14}$$

Indeed, assuming (4.14), we have by (4.3) that

$$\begin{aligned} P_{K_0^*}(u_k - \xi_\varepsilon(t)) \ &= \ P_{K_0^*}\left(Q_{K_0^*}(u_k - \xi^\varepsilon_{k-1}) + \mathrm{e}^{-\frac{t-t_{k-1}}{\varepsilon}} P_{K_0^*}(u_k - \xi^\varepsilon_{k-1})\right) \\ &= \ \mathrm{e}^{-\frac{t-t_{k-1}}{\varepsilon}} P_{K_0^*}(u_k - \xi^\varepsilon_{k-1})\,, \end{aligned}$$

hence (4.13) holds.

It suffices to prove the convergence (4.11) only for $t = T$ (the process is causal!). In other words, we have to check that

$$\lim_{\varepsilon\to 0+} |\xi^\varepsilon_m - \xi_m| \ = \ 0\,. \tag{4.15}$$

To this end, we set for $k = 1, \dots, m$

$$z_k = u_k - \xi_k\,, \ z^\varepsilon_k = u_k - \xi^\varepsilon_k\,, \ e^\varepsilon_k = \mathrm{e}^{-\frac{t_k - t_{k-1}}{\varepsilon}}\,, \ z^\varepsilon_0 = -x_0 = u_0 - \xi_0. \tag{4.16}$$

We then have for all $k = 1, \dots, m$ that

$$z_k = Q_{K_0^*}(z_{k-1} + u_k - u_{k-1}), \quad z^\varepsilon_k = Q_{K_0^*}(z^\varepsilon_{k-1} + u_k - u_{k-1}) + e^\varepsilon_k P_{K_0^*}(z^\varepsilon_{k-1} + u_k - u_{k-1})\,. \tag{4.17}$$

This yields in particular that

$$z^\varepsilon_k - z^\varepsilon_{k-1} + (1 - e^\varepsilon_k) P_{K_0^*}(z^\varepsilon_{k-1} + u_k - u_{k-1}) = u_k - u_{k-1}\,. \tag{4.18}$$

On the other hand, from (4.17) and (4.3) it follows that

$$P_{K_0^*}(z^\varepsilon_k) = e^\varepsilon_k P_{K_0^*}(z^\varepsilon_{k-1} + u_k - u_{k-1})\,, \tag{4.19}$$

hence

$$z^\varepsilon_k - z^\varepsilon_{k-1} + \frac{1 - e^\varepsilon_k}{e^\varepsilon_k} P_{K_0^*}(z^\varepsilon_k) = u_k - u_{k-1}\,. \tag{4.20}$$

We have $\left\langle P_{K_0^*}(z), z\right\rangle \geq 0$ for every $z \in X$. Testing Eq. (4.20) by z^ε_k, we thus obtain

$$|z^\varepsilon_k| \leq |z^\varepsilon_{k-1} + u_k - u_{k-1}| \leq |z^\varepsilon_{k-1}| + |u_k - u_{k-1}| \tag{4.21}$$

and, in particular,

$$|z^\varepsilon_k| \leq |x_0| + \mathop{\mathrm{Var}}_{[0,T]} u \tag{4.22}$$

for every $k = 1, \dots, m$. Both $Q_{K_0^*}$ and $P_{K_0^*}$ are nonexpansive mappings, $P_{K_0^*}(0) = 0$. Using (4.17) and (4.22), we thus have

$$|z^\varepsilon_k - z_k| \leq |z^\varepsilon_{k-1} - z_{k-1}| + (|x_0| + 2 \mathop{\mathrm{Var}}_{[0,T]} u)\, e^\varepsilon_k\,. \tag{4.23}$$

Summing up over k we obtain the final estimate

$$|z_m^\varepsilon - z_m| \le (|x_0| + 2 \operatorname*{Var}_{[0,T]} u) \sum_{k=1}^m e_k^\varepsilon\,, \tag{4.24}$$

and (4.15) follows. ∎

To prove Theorem 2.4, we fix a sequence $\{u^{(n)}\}$ of left continuous step functions of the form (4.8) and such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |u^{(n)}(t) - u(t)| \;=\; 0\,, \tag{4.25}$$

and denote by $\xi^{(n)}$, $\xi_\varepsilon^{(n)}$ the respective solutions to (2.23)–(2.25) and (2.27)–(2.28), with u replaced by $u^{(n)}$. By [6], there exists a constant $C > 0$ independent of n such that

$$\operatorname*{Var}_{[0,T]} \xi^{(n)} \le C \qquad \lim_{n\to\infty} \sup_{t\in[0,T]} |\xi^{(n)}(t) - \xi(t)| = 0\,. \tag{4.26}$$

To estimate the total variation of $\xi_\varepsilon^{(n)}$, we use a similar argument as in [6] that goes back to Section 19.2 of the pioneering Krasnosel'skii and Pokrovskii monograph [4]. As mentined on p. 261 of the Russian edition, this part of the book was written by Alexander Vladimirov. We fix a division $0 = s_0 < s_1 < \cdots < s_\ell = T$ such that

$$s_{j-1} < \tau < t \le s_j \implies |u(t) - u(\tau)| < \frac{r}{2}\,, \tag{4.27}$$

where r is as in (2.21). Let now u^* be an arbitrary left continuous regulated function such that

$$\sup_{t\in[0,T]} |u^*(t) - u(t)| \;\le\; \frac{r}{6}\,, \tag{4.28}$$

and let ξ_ε^* be the solution to (2.27)–(2.28) corresponding to u^*. We have

$$\left\langle \dot\xi_\varepsilon^*(t), u^*(t) - \xi_\varepsilon^*(t) - z \right\rangle \;\ge\; \varepsilon|\dot\xi_\varepsilon^*(t)|^2 \quad \text{a.e.} \tag{4.29}$$

for every $z \in K_0^*$. In every interval $(s_{j-1}, s_j]$, we may choose in particular

$$z(t) = \frac{r}{2} p(t) + u^*(t) - u^*(s_{j-1}+)\,, \tag{4.30}$$

where

$$p(t) := \begin{cases} \frac{\dot\xi_\varepsilon^*(t)}{|\dot\xi_\varepsilon^*(t)|} & \text{if } \dot\xi_\varepsilon^*(t) \ne 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For all $t \in (s_{j-1}, s_j]$ we then have $|z(t)| \le r$, hence $z(t) \in K_0^*$, and from (4.29) we obtain

$$\varepsilon|\dot\xi_\varepsilon^*(t)|^2 + \frac{r}{2}|\dot\xi_\varepsilon^*(t)| + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u^*(s_{j-1}+) - \xi_\varepsilon^*(t)|^2 \;\le\; 0 \tag{4.31}$$

a.e. in $(s_{j-1}, s_j]$. Hence,

$$r \operatorname*{Var}_{[s_{j-1},s_j]} \xi_\varepsilon^* + |u^*(s_{j-1}+) - \xi_\varepsilon^*(s_j)|^2 \;\le\; |u^*(s_{j-1}+) - \xi_\varepsilon^*(s_{j-1})|^2 \tag{4.32}$$

for all $j = 1, \ldots, \ell$. Set $x_j^* = u^*(s_j+) - \xi_\varepsilon^*(s_j)$. From (4.32) it follows

$$|x_j^*| \le |u^*(s_{j-1}+) - u^*(s_j+)| + |x_{j-1}^*| \le |u(s_{j-1}+) - u(s_j+)| + |x_{j-1}^*| + \frac{r}{2}, \quad (4.33)$$

hence $|x_j^*| \le C_u$ for all $j = 0, 1, \ldots, \ell$, where $C_u > 0$ is a constant depending only on u and the fixed division $s_0, s_1, \ldots, s_\ell$. Using (4.32) once more, we obtain

$$r \operatorname*{Var}_{[0,T]} \xi_\varepsilon^* \le \ell C_u^2 . \quad (4.34)$$

We now choose $n_0 \in \mathbb{N}$ sufficiently large such that (4.28) holds with $u^* = u^{(n)}$ for all $n \ge n_0$. For all such n we have by virtue of (4.34) that

$$\operatorname*{Var}_{[0,T]} \xi_\varepsilon^{(n)} \le C , \quad (4.35)$$

with a constant $C > 0$ independent of n and ε. Furthermore, the mapping $y \mapsto \partial M_{K_0}(y) + \varepsilon y$ is monotone, hence the equivalent formulation (2.27) of (4.7) yields

$$\left\langle \dot\xi_\varepsilon^{(n)}(t) - \dot\xi_\varepsilon(t), u^{(n)}(t) - u(t) - \xi_\varepsilon^{(n)}(t) + \xi_\varepsilon(t) \right\rangle \ \ge \ 0 \qquad \text{a. e.} \quad , \quad (4.36)$$

that is,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\xi_\varepsilon^{(n)}(t) - \xi_\varepsilon(t)|^2 \le (|\dot\xi_\varepsilon^{(n)}(t)| + |\dot\xi_\varepsilon(t)|)|u^{(n)}(t) - u(t)| \qquad \text{a. e.} \quad (4.37)$$

From (4.34) we conclude that

$$\sup_{t\in[0,T]} |\xi_\varepsilon^{(n)}(t) - \xi_\varepsilon(t)|^2 \le \frac{4\ell C_u^2}{r} \sup_{t\in[0,T]} |u^{(n)}(t) - u(t)| . \quad (4.38)$$

To obtain the convergence (2.29), we have to check that for every $\delta > 0$ and every $t \in [0,T]$ there exists ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$|\xi_\varepsilon(t) - \xi(t)| < \delta . \quad (4.39)$$

This follows immediately from Lemma 4.1 and from the uniform convergences $\xi_\varepsilon^{(n)} \to \xi_\varepsilon$ and $\xi^{(n)} \to \xi$.

It remains to prove the convergence in (2.30). Following (2.19), we can rewrite (2.25) in energetic form for every $0 \le s < t \le T$ as

$$\frac{1}{2}|\xi(t)|^2 - \langle u(t), \xi(t)\rangle - \frac{1}{2}|\xi(s)|^2 + \langle u(s), \xi(s)\rangle + \frac{1}{2} \sum_{\tau\in[s,t]} |\xi(\tau+) - \xi(\tau)|^2 + \operatorname*{Var}_{K_0\,[s,t]} \xi$$

$$= \ - \int_s^t \langle \xi(\tau), \mathrm{d}u(\tau)\rangle \ . \quad (4.40)$$

In the simple case (4.8), the energy balance reads

$$\frac{1}{2}|\xi_k|^2 - \langle u_k, \xi_k\rangle - \frac{1}{2}|\xi_{k-1}|^2 + \langle u_{k-1}, \xi_{k-1}\rangle + \frac{1}{2}|\xi_k - \xi_{k-1}|^2 + M_{K_0}(\xi_k - \xi_{k-1})$$

$$= \ -\langle \xi_{k-1}, u_k - u_{k-1}\rangle \quad (4.41)$$

for every $k = 1, \dots, m$.

We now derive the energy balance for Eq. (4.7). By definition of $P_{K_0^*}$, we have

$$\left\langle \dot{\xi}_\varepsilon(t), u(t) - \xi_\varepsilon(t) - \varepsilon\dot{\xi}_\varepsilon(t) - z \right\rangle \ \geq \ 0 \quad \text{a.e.} \tag{4.42}$$

for every $z \in K_0^*$, which in view of (4.5) yields

$$\left\langle \dot{\xi}_\varepsilon(t), u(t) - \xi_\varepsilon(t) \right\rangle \ = \ \varepsilon|\dot{\xi}_\varepsilon(t)|^2 + M_{K_0}(\dot{\xi}_\varepsilon(t)) \quad \text{a.e.} \quad , \tag{4.43}$$

hence for every $0 \leq s < t \leq T$ we have

$$\frac{1}{2}|\xi_\varepsilon(t)|^2 - \langle u(t), \xi_\varepsilon(t)\rangle - \frac{1}{2}|\xi_\varepsilon(s)|^2 + \langle u(s), \xi_\varepsilon(s)\rangle + \varepsilon \int_s^t |\dot{\xi}_\varepsilon(\tau)|^2 \mathrm{d}\tau + \operatorname*{Var}_{K_0}_{[s,t]} \xi_\varepsilon$$
$$= \ -\int_s^t \langle \xi_\varepsilon(\tau), \mathrm{d}u(\tau)\rangle \ . \tag{4.44}$$

We see that (2.30) follows from (2.29), (4.40), and (4.44), provided we check for every $0 \leq s < t \leq T$ that

$$\forall \delta > 0 \ \ \exists \varepsilon_0 > 0 \ \ \forall \varepsilon < \varepsilon_0 : \quad \int_s^t \langle (\xi_\varepsilon(\tau) - \xi(\tau)), \mathrm{d}u(\tau)\rangle \ < \ \delta \, . \tag{4.45}$$

By (4.35) and Lemma 1.8 (iii), the functions ξ_ε and ξ, as well as their variations $\operatorname{Var}_{[s,t]} \xi_\varepsilon$ and $\operatorname{Var}_{[s,t]} \xi$, are uniformly bounded by a constant C independent of ε. Using Lemma 1.8 (iv), we find a step function w such that $\|u - w\|_{[s,t]} < \delta/(12C)$. By Theorem 1.9, we have

$$\int_s^t \langle (\xi_\varepsilon(\tau) - \xi(\tau)), \mathrm{d}(u - w)(\tau)\rangle \leq 6C \, \|u - w\|_{[s,t]} \leq \frac{\delta}{2} \, .$$

Since w is a step function, we may refer to Proposition 1.6, and conclude from the pointwise convergence $\xi_\varepsilon(\tau) \to \xi(\tau)$ that

$$\lim_{\varepsilon \to 0+} \int_s^t \langle (\xi_\varepsilon(\tau) - \xi(\tau)), \mathrm{d}w(\tau)\rangle \ = \ 0 \, ,$$

whence (4.45) follows. Theorem 2.4 is proved. ■

The following example shows that uniqueness of the solution ξ is lost if the jump dissipation term is omitted in (2.20).

Example 4.2. Consider the simple case $X = \mathbb{R}$, $K_0 = K_0^* = [-1, 1]$, and

$$u(t) \ = \ \begin{cases} u_0 & \text{for} \ \ t \in [0, t_0] \\ 0 & \text{for} \ \ t \in (t_0, T] \end{cases}$$

with some given $t_0 \in (0,T)$ and $u_0 \geq 3$. We look for a left continuous solution ξ to the problem

$$\left.\begin{array}{ll} u(t) - \xi(t) \in K_0^* & \forall t \in [0,T] \\ \frac{1}{2}|\xi(t)|^2 - \langle u(t), \xi(t)\rangle - \frac{1}{2}|\xi(s)|^2 + \langle u(s), \xi(s)\rangle + \operatorname*{Var}_{K_0}\limits_{[s,t]} \xi & \\ \qquad \leq -\int_s^t \langle \xi(\tau), \mathrm{d}u(\tau)\rangle & \forall 0 \leq s < t \leq T\,, \end{array}\right\} \tag{4.46}$$

with initial condition $\xi(0) = u_0$. Every solution ξ is necessarily constant in every interval, where u is constant. Indeed, this follows from the inequality

$$\begin{aligned} \frac{1}{2}|\xi(t)|^2 - \langle u, \xi(t)\rangle - \frac{1}{2}|\xi(s)|^2 + \langle u, \xi(s)\rangle &= \frac{1}{2}\langle \xi(t) - \xi(s), \xi(t) + \xi(s) - u\rangle \\ &\leq M_{K_0}(\xi(t) - \xi(s))\,. \end{aligned}$$

Hence, ξ must have the form

$$\xi(t) = \begin{cases} u_0 & \text{for } t \in [0, t_0] \\ \xi_1 & \text{for } t \in (t_0, T] \end{cases}$$

with $\xi_1 \in [-1,1]$. By a counterpart of (4.41) without the quadratic dissipation term, we see that ξ is a solution of (4.46) if and only if

$$\frac{1}{2}\xi_1^2 - \frac{1}{2}u_0^2 + u_0^2 + |\xi_1 - u_0| \leq u_0^2\,. \tag{4.47}$$

This inequality is satisfied for all $\xi_1 \in [-1,1]$. Hence, we have a continuum of distinct solutions.

The situation is even worse if we replace (4.46) by

$$\left.\begin{array}{ll} u(t) - \xi(t) \in K_0^* & \forall t \in [0,T] \\ \frac{1}{2}|\xi(t)|^2 - \langle u(t), \xi(t)\rangle - \frac{1}{2}|\xi(s)|^2 + \langle u(s), \xi(s)\rangle + \operatorname*{Var}_{K_0}\limits_{[s,t]} \xi & \\ \qquad \leq -\int_s^t \langle \xi(\tau+), \mathrm{d}u(\tau)\rangle\,. & \forall 0 \leq s < t \leq T\,. \end{array}\right\} \tag{4.48}$$

Inequality (4.47) is now replaced by

$$\frac{1}{2}\xi_1^2 - \frac{1}{2}u_0^2 + u_0^2 + |\xi_1 - u_0| \leq u_0\,\xi_1\,, \tag{4.49}$$

which is never satisfied for $\xi_1 \in [-1,1]$. Hence, there exists no solution to Problem (4.48) for $\xi(0) = u_0$.

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