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Attractors for semilinear equations of viscoelasticity with very low dissipation

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Abstract. We analyze a differential system arising in the theory of isothermal viscoelasticity. This system is equivalent to an integrodifferential equation of hyperbolic type with a cubic nonlinearity, where the dissipation mechanism is contained only in the convolution integral, accounting for the past history of the displacement. In particular, we consider here a convolution kernel which entails an extremely weak dissipation. In spite of that, we show that the related dynamical system possesses a global attractor of optimal regularity.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. For $t \in \mathbb{R}^+ = (0, \infty)$, we consider the evolution system arising in the theory of isothermal viscoelasticity [9, 20]

$$
\begin{cases} \partial_{tt} u - \Delta u - \int_0^\infty \mu(s)\Delta\eta(s)ds + g(u) = f, \\ \partial_t \eta = T\eta + \partial_t u, \end{cases} \tag{1.1}
$$

where $u = u(t) : \Omega \times [0, \infty) \to \mathbb{R}$, $\eta = \eta^t(s) : \Omega \times [0, \infty) \times \mathbb{R}^+ \to \mathbb{R}$ and $T = -\partial_s$, supplemented with the boundary and initial conditions

$$
\begin{cases} u(t)|_{\partial\Omega} = \eta^t|_{\partial\Omega} = \eta^t(0) = 0, \\ u(0) = u_0, \quad \partial_t u(0) = v_0, \quad \eta^0(s) = \eta_0(s). \end{cases} \tag{1.2}
$$

Here, $g : \mathbb{R} \to \mathbb{R}$ is a nonlinear term of (at most) cubic growth satisfying some dissipativity conditions, $f : \Omega \to \mathbb{R}$ is an external force, whereas the memory kernel μ is an absolutely continuous summable decreasing (thus nonnegative) function defined on $\mathbb{R}^+$. Problem (1.1)-(1.2) is cast in the so-called *memory setting* (see [5, 6]), and is equivalent to the integrodifferential equation

$$
\partial_{tt} u - (1 + \varsigma)\Delta u + \int_0^\infty \mu(s)\Delta u(t - s)ds + g(u) = f,
$$

where $\varsigma = \int_0^\infty \mu(s)ds > 0$, with boundary condition $u(t)|_{\partial\Omega} = 0$ and initial conditions $u(0) = u_0$, $u(t) = u_0 - \eta_0(-t)$, for $t < 0$, and $\partial_t u(0) = v_0$. We address the reader to [11] for more details on the equivalence of the two formulations. It is known that (1.1)-(1.2) generates a dissipative dynamical system $S(t)$ on the phase space $H^1_0(\Omega) \times L^2(\Omega) \times L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$, the so-called *history space*, since the variable η contains the information on the past history of the system. The asymptotic behavior of $S(t)$ has been investigated quite extensively. For instance, if the first equation contains an extra term of the form $\partial_t u$ (physically, a dynamical friction), then $S(t)$ has a global attractor of optimal regularity [1, 3, 18]. When this term does not appear, as in our case, the existence of the global attractor and its regularity can still be proved, although the dissipation is contained in the memory term only [4, 10]. Clearly, this situation requires a more careful analysis, the dissipation being much weaker. However, all the above results (as well as all the results on the asymptotic behavior of dynamical systems arising from equations with memory) have been proved under the apparently unavoidable condition

$$
\mu'(s) + \delta\mu(s) \le 0, \tag{1.3}
$$

for some $\delta > 0$ and (almost) every $s \in \mathbb{R}^+$. Indeed, even in the linear homogeneous case, (1.3) seemed to play an essential role in establishing exponential stability (see [8, 15, 16]). It is readily seen that (1.3) is equivalent to

$$\mu(s+\sigma) \leq e^{-\delta\sigma}\mu(s),$$

for every $\sigma \geq 0$ and (almost) every $s \in \mathbb{R}^+$. On the other hand, [2] proves that a *necessary* condition in order to have exponential stability in the linear homogeneous case (and, consequently, in order for $S(t)$ to possess at least an absorbing set) is

$$\mu(s+\sigma) \leq Ce^{-\delta\sigma}\mu(s), \tag{1.4}$$

for some $C \geq 1$, $\delta > 0$, every $\sigma \geq 0$ and (almost) every $s \in \mathbb{R}^+$. Nonetheless, between (1.3) and (1.4), there is quite an elbowroom. In particular, (1.3) does not hold when μ is too flat (which corresponds to having zones of very low, or even null, dissipation). An interesting situation from the physical viewpoint, that might not comply with (1.3), but obviously fits (1.4), occurs when μ eventually vanishes. Along this direction, the very recent article [17], focused on the linear homogeneous case, shows that exponential stability is still present when (1.4) holds, but (1.3) is heavily violated. Here, we are able to translate the semigroup approach of [17] in terms of suitable energy functionals, so to extend the analysis to the nonlinear case. This is not, in general, a straightforward fact: there are linear systems (in particular, the one associated with our problem) which can be tackled via semigroup methods, but whose nonlinear counterparts require the introduction of *ad hoc*, and often quite subtle, techniques. In the present work, we establish the existence of a global attractor of optimal regularity for $S(t)$ when μ fulfills the necessary condition (1.4), but under much weaker hypotheses than (1.3). Besides, contrary to [4], the kernel μ will be allowed to blow up at zero. For instance, we can consider the weakly singular kernel

$$\mu(s) = \frac{ke^{-\alpha s}}{s^{1-\beta}},$$

with $k \geq 0$ and $\alpha, \beta > 0$, which has been successfully used to fit experimental data for some real materials. To the best of our knowledge, this is the first result of this kind for nonlinear systems with memory. In fact, this approach can be successfully applied to other low-dissipative models with memory, such as reaction-diffusion equations with a Gurtin-Pipkin conduction law [12].

Plan of the paper. In the next Section 2, we write the assumptions on f, g and μ. In Section 3, we formulate the main theorem, which is proved in Section 4. The remaining sections are devoted to the proofs of Lemma 4.3 and Lemma 4.4 appearing is Section 4.

Notation. We consider the positive operator $A = -\Delta$ acting on $(L^2(\Omega), \langle\cdot,\cdot\rangle, \|\cdot\|)$ with domain $\mathcal{D}[A] = H^2(\Omega) \cap H_0^1(\Omega)$. For $r \in \mathbb{R}$, we denote by $H_r = \mathcal{D}[A^{r/2}]$ the scale of Hilbert spaces generated by A, with the usual inner products $\langle\cdot,\cdot\rangle_{\mathcal{D}[A^{r/2}]} = \langle A^{r/2}\cdot, A^{r/2}\cdot\rangle$, and by $\mathcal{M}_r = L^2_\mu(\mathbb{R}^+; H_{1+r})$ the Hilbert space of square summable functions on $\mathbb{R}^+$ with values in H_{1+r}, with respect to the measure $\mu(s)ds$. To account for the boundary conditions on η, we view $T = -\partial_s$ as the linear operator with domain

$$\mathcal{D}[T] = \big\{\psi = \psi(s) \in \mathcal{M}_0 \,:\, \partial_s\psi \in \mathcal{M}_0,\ \psi(0) = 0\big\},$$

where ∂_s is the distributional derivative with respect to the internal variable s. Then, T is the infinitesimal generator of the right-translation semigroup $R(t)$ on $\mathcal{M}_0$ acting as

$$[R(t)\psi](s) = \begin{cases} 0, & 0 < s \leq t, \\ \psi(s-t), & s > t. \end{cases}$$

Finally, we introduce the product Hilbert spaces

$$\mathcal{H}_r = H_{1+r} \times H_r \times \mathcal{M}_r.$$

Throughout the paper, $c \geq 0$ will denote a *generic* constant (whose value may vary even within the same formula). Any further dependence of c on other quantities will be specified on occurrence. Also, we shall often tacitly use the Poincaré, the Young and the Hölder inequalities, as well as the usual Sobolev embeddings.

2. General Assumptions

Concerning the nonlinearity and the external force, we take $f \in H_0$ independent of time, and $g \in C^2(\mathbb{R})$, with $g(0) = 0$, such that the following growth and dissipation conditions are satisfied:

$$|g''(u)| \leq c\big(1 + |u|\big), \tag{2.1}$$

$$\liminf_{|u|\to\infty} \frac{g(u)}{u} > -\lambda, \tag{2.2}$$

where $\lambda > 0$ is the first eigenvalue of A. Following [4], we decompose g into the sum $g = g_0 + g_1$, where $g_0, g_1 \in C^2(\mathbb{R})$ fulfill

$$|g_0''(u)| \leq c(1 + |u|), \tag{2.3}$$

$$g_0(u)u \geq 0, \tag{2.4}$$

$$g_0'(0) = 0, \tag{2.5}$$

$$|g_1'(u)| \leq c. \tag{2.6}$$

Setting $G(u) = \int_0^u g(y)dy$ and $G_0(u) = \int_0^u g_0(y)dy$, it follows from (2.1)-(2.4) that

$$-(1-\varpi)\|A^{1/2}u\|^2 - c \leq 2\langle G(u), 1\rangle \leq c(1 + \|A^{1/2}u\|^4), \tag{2.7}$$

$$0 \leq 2\langle G_0(u), 1\rangle \leq c(1 + \|A^{1/2}u\|^4), \tag{2.8}$$

for every $u \in H_1$ and some $\varpi > 0$. Concerning instead the memory kernel, we assume that $\mu : \mathbb{R}^+ \to [0, \infty)$ is absolutely continuous, summable and nonincreasing. In particular, μ is differentiable almost everywhere with $\mu' \leq 0$, and it is possibly unbounded in a neighborhood of zero. Without loss of generality, we may (and do) assume that

$$\int_0^\infty \mu(s)ds = 1.$$

This, together with the above assumptions on f and g, is enough to show that problem (1.1)-(1.2) generates a strongly continuous semigroup $S(t)$ on the phase space $\mathcal{H}_0$ (see [4, 18]). For further convenience, we recall that the third component of the solution $S(t)(u_0, v_0, \eta_0) = (u(t), \partial_t u(t), \eta^t)$ has the explicit representation [18]

$$\eta^t(s) = \begin{cases} u(t) - u(t-s), & 0 < s \leq t, \\ \eta_0(s-t) + u(t) - u_0, & s > t. \end{cases} \tag{2.9}$$

We point out that, given $u(t)$, the representation formula (2.9) depends only on the structure of the second equation of (1.1). When $f = g = 0$ (linear homogeneous case), the monotonicity of μ ensures that $S(t)$ is a (linear) contraction semigroup.

Remark 2.1. In fact, as in [2, 17], we could consider without substantial changes in the subsequent analysis more general kernels, allowing μ to have a finite number of jumps, or even an infinite number of jumps, provided that the points where μ has jumps form an increasing sequence.

Definition 2.2. We say that μ is an *admissible kernel* if there exists $\Theta > 0$ such that

$$\int_s^\infty \mu(\sigma) d\sigma \leq \Theta \mu(s), \qquad \forall s \in \mathbb{R}^+. \tag{2.10}$$

Remark 2.3. Note that, in view of the other assumptions on μ, conditions (1.4) and (2.10) are equivalent. Indeed, it is apparent that (1.4) implies (2.10) (just take $\Theta = C/\delta$). Concerning the reverse implication, since μ is positive and monotone nonincreasing, we have, for every $r > 0$,

$$\Theta \mu(s) \geq \int_s^\infty \mu(\sigma) d\sigma \geq \int_s^{s+r} \mu(\sigma) d\sigma \geq r\mu(s+r).$$

Hence, there exists $\varrho < 1$ and $r > 0$ such that

$$\mu(s+r) \leq \varrho \mu(s).$$

Due to the monotonicity of μ, the above inequality readily yields (1.4). Indeed, setting $\sigma = nr + \vartheta$, with $n \in \mathbb{N}$ and $\vartheta \in [0, r)$, we get

$$\mu(s+\sigma) \leq \mu(s+nr) \leq \varrho^n \mu(s) = e^{n \log \varrho} \mu(s) \leq Ce^{-\delta\sigma} \mu(s),$$

with $C = 1/\varrho$ and $\delta = -(\log \varrho)/r$.

Thus, μ is admissible if and only if the semigroup associated with the linear homogeneous system is exponentially stable (see [2]).

3. The Main Theorem

Our main result reads as follows.

Theorem 3.1 (Existence of the global attractor)**.** *Let μ be an admissible kernel. Assume in addition that*

$$\mu'(s) < 0, \qquad \text{for a.e. } s \in \mathbb{R}^+. \tag{3.1}$$

Then $S(t)$ possesses a connected global attractor $\mathcal{A} \subset \mathcal{H}_0$ which coincides with the unstable set of equilibria.

Corollary 3.2 (Regularity of the global attractor)**.** *The global attractor $\mathcal{A}$ is contained and bounded in $\mathcal{H}_1$. Moreover, calling Π the projection of $\mathcal{H}$ onto $\mathcal{M}_0$, we have the additional regularity*

$$\Pi\mathcal{A} \subset \mathcal{D}[T], \quad \sup_{\eta\in\Pi\mathcal{A}} \|T\eta\|_{\mathcal{M}_0} < \infty, \quad \sup_{\eta\in\Pi\mathcal{A},\, s\in\mathbb{R}^+} \|A\eta(s)\| < \infty.$$

Remark 3.3. Hypothesis (3.1) can be relaxed when the nonlinearity g is subcritical, that is, if (2.1) is replaced by

$$|g'(u)| \leq c(1+|u|^\beta), \qquad \beta < 2.$$

More precisely, the above results hold true even if the set $P_0 = \{s \in \mathbb{R}^+ : \mu'(s) = 0\}$ has positive measure not exceeding a certain limit which depends on the physical constants of the system. The exact condition is the same as the one required to have exponential stability of the corresponding linear semigroup (see [17]).

Remark 3.4. If the first equation of (1.1) also contains the dissipative term $\partial_t u$, it is not hard to show, using the techniques of this paper, that Theorem 3.1 and Corollary 3.2 hold *without* hypothesis (3.1). Hence, in that situation, being an admissible kernel is a necessary and sufficient condition in order for the related dynamical system $S(t)$ to possess the global attractor.

4. Proof of the Main Theorem

4.1. **The gradient system.** We begin to establish the following fact.

Proposition 4.1. *The semigroup $S(t)$ is a gradient system on $\mathcal{H}_0$ and the set $\mathcal{S}$ of its equilibria is bounded in $\mathcal{H}_0$.*

Proof. The second assertion is quite immediate. Indeed,

$$\mathcal{S} = \big\{(u_0,0,0) \in \mathcal{H}_0 \,:\, Au_0 + g(u_0) = f\big\},$$

which is bounded on account of the assumptions on f and g. We define the function $\mathcal{L} \in C(\mathcal{H}_0, \mathbb{R})$ as

$$\mathcal{L}(p,q,\psi) = \|(p,q,\psi)\|^2_{\mathcal{H}_0} + 2\langle G(p),1\rangle - 2\langle f,p\rangle.$$

We have to show that $\mathcal{L}$ is a *Lyapunov function*, namely,

(i) $\mathcal{L}(z) \to \infty$ if and only if $\|z\|_{\mathcal{H}_0} \to \infty$,
(ii) $\mathcal{L}(S(t)z)$ is nonincreasing for any $z \in \mathcal{H}_0$,
(iii) if $\mathcal{L}(S(t)z) = \mathcal{L}(z)$ for all $t > 0$, then z is an equilibrium.

Property (i) is apparent in light of (2.7). Indeed,

$$\frac{1}{c}\|z\|^2_{\mathcal{H}_0} - c \leq \mathcal{L}(z) \leq c\|z\|^4_{\mathcal{H}_0} + c, \qquad \forall z \in \mathcal{H}_0, \tag{4.1}$$

for some $c \geq 1$. Next, if $z = (u_0, v_0, \eta_0)$ is a sufficiently regular datum (in particular, $\eta_0 \in \mathcal{D}[T]$), we have (see [4])

$$\frac{d}{dt}\mathcal{L}(S(t)z) = \int_0^\infty \mu'(s)\|A^{1/2}\eta^t(s)\|^2 ds.$$

Hence, choosing $\delta > 0$ small enough such that the set $N = \{s \in \mathbb{R}^+ : \mu'(s) + \delta\mu(s) \leq 0\}$ has positive measure (here we are using (3.1)),

$$\mathcal{L}(S(t)z) \leq \mathcal{L}(z) - \delta \int_0^t \int_N \mu(s)\|A^{1/2}\eta^\tau(s)\|^2 ds d\tau, \qquad \forall t > 0.$$

By density, the inequality holds for every $z \in \mathcal{H}_0$. In particular, (ii) follows. Finally, if $\mathcal{L}(S(t)z) = \mathcal{L}(z)$ for all $t > 0$, then $\eta^t(s) = 0$ for every $t > 0$ and every $s \in N$. From the representation formula (2.9), we learn that $u(t)$ has period s, for every $s \in N$. Since N has positive measure, it follows that $u(t) = u_0$, and therefore $\partial_t u(t) = v_0 = 0$. Using again (2.9), we get

$$\eta^t(s) = \begin{cases} 0, & 0 < s \leq t, \\ \eta_0(s - t), & s > t. \end{cases}$$

To prove (iii), we are left to show that $\eta_0 = 0$. Indeed, the equality $\mathcal{L}(S(t)z) = \mathcal{L}(z)$ now reads

$$\int_0^\infty \mu(s + t)\|A^{1/2}\eta_0(s)\|^2 ds = \int_0^\infty \mu(s)\|A^{1/2}\eta_0(s)\|^2 ds, \qquad \forall t > 0.$$

Since μ vanishes monotonically at infinity, taking the limit $t \to \infty$ in the right-hand side and applying the dominated convergence theorem, we conclude that $\eta_0 = 0$. □

Remark 4.2. Note that we did not use (3.1) in its full strength. Indeed, to obtain the desired conclusion, it is enough to have a set of positive measure on which μ is not constant (cf. Remark 2.1).

4.2. **The semigroup decomposition.** We decompose the solution $S(t)z$ into the sum

$$S(t)z = D(t)z + K(t)z,$$

where $D(t)z = (v(t), \partial_t v(t), \xi^t)$ and $K(t)z = (w(t), \partial_t w(t), \zeta^t)$ solve the problems

$$\begin{cases} \partial_{tt} v + Av + \int_0^\infty \mu(s)A\xi(s)ds + g_0(v) = 0, \\ \partial_t \xi = T\xi + \partial_t v, \\ (v(0), \partial_t v(0), \xi^0) = z \end{cases}$$

and

$$\begin{cases} \partial_{tt} w + Aw + \int_0^\infty \mu(s)A\zeta(s)ds + g(u) - g_0(v) = f, \\ \partial_t \zeta = T\zeta + \partial_t w, \\ (w(0), \partial_t w(0), \zeta^0) = 0. \end{cases}$$

Then, we have

Lemma 4.3. *There exist $\kappa > 0$ and an increasing nonnegative function Q such that*

$$\|D(t)z\|_{\mathcal{H}_0} \leq Q(\|z\|_{\mathcal{H}_0})e^{-\kappa t},$$

for every $t \geq 0$.

Lemma 4.4. *Let $\mathcal{B} \subset \mathcal{H}_0$. Assume that*

$$\sup_{t\geq 0}\sup_{z\in\mathcal{B}} \|S(t)z\|_{\mathcal{H}_0} = C < \infty.$$

Then, $K(t)\mathcal{B} \subset \mathcal{H}_r$, for every $t \geq 0$ and every $r \in [0, \frac{1}{2})$, and there is $M = M(C, r) \geq 0$ such that

$$\sup_{t\geq 0}\sup_{z\in\mathcal{B}} \|K(t)z\|_{\mathcal{H}_r} \leq M.$$

Lemma 4.5. *Let $\mathcal{B} \subset \mathcal{H}_{1/3}$. Assume that*

$$\sup_{t\geq 0}\sup_{z\in\mathcal{B}} \|S(t)z\|_{\mathcal{H}_{1/3}} = C < \infty.$$

Then, $K(t)\mathcal{B} \subset \mathcal{H}_1$, for every $t \geq 0$, and there is $M = M(C) \geq 0$ such that

$$\sup_{t\geq 0}\sup_{z\in\mathcal{B}} \|K(t)z\|_{\mathcal{H}_1} \leq M.$$

The proofs of the three above lemmata will be given in the following sections.

Corollary 4.6. *Let $\mathcal{B} \subset \mathcal{H}_r$, for some $r \in (0, 1]$. Assume that $K(t)\mathcal{B} \subset \mathcal{H}_r$, for every $t \geq 0$, and*

$$\sup_{t\geq 0}\sup_{z\in\mathcal{B}} \|K(t)z\|_{\mathcal{H}_r} = M < \infty.$$

Then, for every $t \geq 0$, $K(t)\mathcal{B}$ belongs to the compact set

$$\mathcal{K}_r^M = \Big\{ z_0 \,:\, \|z_0\|_{\mathcal{H}_r} \leq M,\ \|\partial_s \Pi z_0\|_{\mathcal{M}_{r-1}} \leq M,\ \|A^{(1+r)/2}\Pi z_0(s)\| \leq 2M,\ \Pi z_0(0) = 0 \Big\}.$$

Proof. The compactness of $\Pi\mathcal{K}_r^M$ in $\mathcal{M}_0$ (and, consequently, the compactness of $\mathcal{K}_r^M$ in $\mathcal{H}_0$) is guaranteed by Lemma 5.5 of [18]. From the analogue of (2.9) for ζ^t, we know that

$$\zeta^t(s) = \begin{cases} w(t) - w(t-s), & 0 < s \leq t, \\ w(t), & s > t. \end{cases}$$

This shows that $\zeta^t(0) = 0$ and $\|A^{(1+r)/2}\zeta^t(s)\| \leq 2M$. Besides,

$$\partial_s \zeta^t(s) = \begin{cases} \partial_t w(t-s), & 0 < s \leq t, \\ 0, & s > t. \end{cases}$$

Hence, $\|A^r \partial_s \zeta^t(s)\| \leq M$, which implies that $\|\partial_s \zeta^t\|_{\mathcal{M}_{r-1}} \leq M$. □

4.3. **Proof of Theorem 3.1.** Since $S(t)$ is a gradient system and $\mathcal{S}$ is bounded in $\mathcal{H}_0$, using a general argument that can be found in [13, 14] (see also the Appendix of [4]), the existence of the (connected) global attractor $\mathcal{A}$ coinciding with the unstable set of $\mathcal{S}$ is achieved if we show that

- (a) $D(t)$ decays to zero uniformly on bounded sets,
- (b) for any given $R > 0$, there is a compact set $\mathcal{K} = \mathcal{K}(R) \subset \mathcal{H}_0$ such that $K(t)z \in \mathcal{K}$ for every $t \geq 0$ and every $z \in \mathcal{H}_0$ of norm less than or equal to R.

In that case, $\mathcal{A} \subset \mathcal{K}$, for some $R > 0$ large enough. Point (a) is exactly the content of Lemma 4.3, which says even more than is needed, since the decay is of exponential type. Concerning point (b), due to (4.1) and to the monotonicity of $\mathcal{L}$ along the trajectories, if $\|z\|_{\mathcal{H}_0} \leq R$, then $\|S(t)z\|_{\mathcal{H}_0} \leq C$, for some $C = C(R)$. Hence, given $r \in (0, \frac{1}{2})$, applying Lemma 4.4 (with $\mathcal{B}$ equal to the ball of $\mathcal{H}_0$ of radius R), it follows that $\|K(t)z\|_{\mathcal{H}_r} \leq M$. Therefore, by Corollary 4.6 (with $\mathcal{B}$ equal to the ball of $\mathcal{H}_r$ of radius M), we conclude that $K(t)z \in \mathcal{K}_r^M$.

4.4. **Proof of Corollary 3.2.** At this point, we know (in particular) that $\mathcal{A}$ is bounded in $\mathcal{H}_{1/3}$. Besides, $\mathcal{A}$ is fully invariant for $S(t)$, namely, $S(t)\mathcal{A} = \mathcal{A}$, for every $t \geq 0$. Hence, for every $z \in \mathcal{A}$ and every $t \geq 0$, there exists $z_t \in \mathcal{A}$ such that $z = D(t)z_t + K(t)z_t$. An application of Lemma 4.3 and Lemma 4.5 entails the boundedness of $\mathcal{A}$ in $\mathcal{H}_1$. Finally, Corollary 4.6 yields the desired regularity. Indeed, $\Pi\mathcal{K}_1^M \subset \mathcal{D}[T]$.

Remark 4.7. In fact, by Lemma 4.3 and a slight modification of Lemma 4.4 and Lemma 4.5, together with the transitivity of the exponential attraction property [7], one can show the existence of a regular exponentially attracting set and, in turn, of an exponential attractor of finite fractal dimension, whose basin of exponential attraction is the whole phase space $\mathcal{H}_0$. As a byproduct, the global attractor $\mathcal{A}$ has finite fractal dimension as well. It is also worth observing that the regularity of $\mathcal{A}$ can be increased up to where f and g permit.

5. Some Auxiliary Functionals

We begin with some preliminary work in order to be in a position to prove Lemma 4.3, Lemma 4.4 and Lemma 4.5. We introduce the probability measure $\hat{\mu}$ on $\mathbb{R}^+$ as

$$\hat{\mu}(P) = \int_P \mu(s)ds,$$

for any (measurable) set $P \subset \mathbb{R}^+$. For any $\delta > 0$, we consider the sets

$$P_\delta = \big\{s \in \mathbb{R}^+ : \mu'(s) + \delta\mu(s) > 0\big\} \qquad \text{and} \qquad N_\delta = \big\{s \in \mathbb{R}^+ : \mu'(s) + \delta\mu(s) \leq 0\big\}.$$

Clearly, $P_\delta \cup N_\delta = \mathbb{R}^+$ (except, possibly, a nullset). Besides, on account of (3.1),

$$\lim_{\delta \to 0} \hat{\mu}(P_\delta) = 0.$$

Then, for $\psi \in \mathcal{M}_0$, we denote

$$\mathcal{P}_\delta[\psi] = \int_{P_\delta} \mu(s)\|A^{1/2}\psi(s)\|^2 ds \qquad \text{and} \qquad \mathcal{N}_\delta[\psi] = \int_{N_\delta} \mu(s)\|A^{1/2}\psi(s)\|^2 ds.$$

Observe that $\mathcal{P}_\delta[\psi] + \mathcal{N}_\delta[\psi] = \|\psi\|^2_{\mathcal{M}_0}$. In order to deal with the (possible) singularity of $\mu(s)$ at zero, given any $\nu \in (0, \frac{1}{2})$, we choose $s_* = s_*(\nu) > 0$ such that

$$\int_0^{s_*} \mu(s)ds \leq \frac{\nu}{2},$$

and we introduce the function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\omega(s) = \mu(s_*)\chi_{(0,s_*]}(s) + \mu(s)\chi_{(s_*,\infty)}(s),$$

where χ denotes the characteristic function. Finally, we define the functionals on $\mathcal{H}_0$

$$\Phi^1(p,q,\psi) = -\int_0^\infty \omega(s)\langle q,\psi(s)\rangle ds,$$
$$\Phi^2(p,q,\psi) = \langle q,p\rangle,$$
$$\Psi(p,q,\psi) = \int_0^\infty \bigg(\int_s^\infty \mu(\sigma)\chi_{P_\delta}(\sigma)d\sigma\bigg)\|A^{1/2}(\psi(s)-p)\|^2 ds,$$

and the functional

$$\Phi(p,q,\psi) = \Phi^1(p,q,\psi) + (1-2\nu)\Phi^2(p,q,\psi).$$

In light of (2.10), it is readily seen that

$$0 \le |\Phi^1(p,q,\psi)| + |\Phi^2(p,q,\psi)| + \Psi(p,q,\psi) \le c\|(p,q,\psi)\|^2_{\mathcal{H}_0}. \tag{5.1}$$

We now consider the system

$$\begin{cases} \partial_{tt}p + Ap + \int_0^\infty \mu(s)A\psi(s)ds + k = 0, \\ \partial_t\psi = T\psi + \partial_t p, \end{cases} \tag{5.2}$$

where $k = k(p,t)$ is a suitable nonlinearity. Observe that (5.2) may not generate a strongly continuous semigroup on $\mathcal{H}_0$. Assuming that $(p,\partial_t p,\psi)$ is a sufficiently regular global solution to (5.2) (in particular, $\psi \in \mathcal{D}[T]$), we have

Lemma 5.1. *The following inequality holds:*

$$\begin{aligned}\frac{d}{dt}\Phi^1(p,\partial_t p,\psi) &\le 2\sqrt{\nu}\,\|A^{1/2}p\|^2 - (1-\nu)\|\partial_t p\|^2 \\ &\quad - \frac{\mu(s_*)}{\lambda\nu}\int_0^\infty \mu'(s)\|A^{1/2}\psi(s)\|^2 ds + (2\hat\mu(P_\delta)+\sqrt{\nu}\,)\mathcal{P}_\delta[\psi] + \frac{3}{\nu}\mathcal{N}_\delta[\psi] \\ &\quad + \int_{P_\delta}\mu(s)\langle A^{1/2}p, A^{1/2}\psi(s)\rangle ds + \int_0^\infty \omega(s)\langle k,\psi(s)\rangle ds.\end{aligned}$$

The last term can be conveniently estimated as

$$\int_0^\infty \omega(s)\langle k,\psi(s)\rangle ds \le \|k\|\|\psi\|_{\mathcal{M}_{-1}}.$$

Lemma 5.2. *The following inequality holds:*

$$\begin{aligned}\frac{d}{dt}\Phi^2(p,\partial_t p,\psi) &\le -(1-\nu)\|A^{1/2}p\|^2 + \|\partial_t p\|^2 + \frac{1}{\nu}\mathcal{N}_\delta[\psi] \\ &\quad - \int_{P_\delta}\mu(s)\langle A^{1/2}p, A^{1/2}\psi(s)\rangle ds - \langle k,p\rangle.\end{aligned}$$

The proofs of Lemma 5.1 and Lemma 5.2 can be found in [17], where the same functionals have been introduced to treat the linear homogeneous case. Collecting the above results, and observing that

$$\int_{P_\delta}\mu(s)\langle A^{1/2}p, A^{1/2}\psi(s)\rangle ds \le \alpha\hat\mu(P_\delta)\|A^{1/2}p\|^2 + \frac{1}{4\alpha}\mathcal{P}_\delta[\psi], \qquad \forall\alpha>0, \tag{5.3}$$

we readily obtain

Lemma 5.3. *The following inequality holds:*

$$\frac{d}{dt}\Phi(p,\partial_t p,\psi) \leq -(1-6\sqrt{\nu})\|A^{1/2}p\|^2 - \nu\|\partial_t p\|^2 - \frac{\mu(s_*)}{\lambda\nu}\int_0^\infty \mu'(s)\|A^{1/2}\psi(s)\|^2 ds$$
$$+ 2(\hat{\mu}(P_\delta)+\sqrt{\nu})\mathcal{P}_\delta[\psi] + \frac{4}{\nu}\mathcal{N}_\delta[\psi] + \|k\|\|\psi\|_{\mathcal{M}_{-1}} - (1-2\nu)\langle k,p\rangle.$$

Finally, we have

Lemma 5.4. *The following inequality holds:*

$$\frac{d}{dt}\Psi(p,\partial_t p,\psi) \leq -\frac{1}{2}\mathcal{P}_\delta[\psi] + 2\hat{\mu}(P_\delta)\|A^{1/2}p\|^2.$$

Proof. Using the equality $\partial_t\psi = T\psi + \partial_t p$,

$$\begin{aligned}\frac{d}{dt}\Psi(p,\partial_t p,\psi) &= 2\int_0^\infty \bigg(\int_s^\infty \mu(\sigma)\chi_{P_\delta}(\sigma)d\sigma\bigg)\langle A^{1/2}T\psi(s), A^{1/2}(\psi(s)-p)\rangle ds\\ &= 2\int_0^\infty \bigg(\int_s^\infty \mu(\sigma)\chi_{P_\delta}(\sigma)d\sigma\bigg)\frac{d}{ds}\langle A^{1/2}\psi(s), A^{1/2}p\rangle ds\\ &\quad - \int_0^\infty \bigg(\int_s^\infty \mu(\sigma)\chi_{P_\delta}(\sigma)d\sigma\bigg)\frac{d}{ds}\|A^{1/2}\psi(s)\|^2 ds.\end{aligned}$$

An integration by parts then yields

$$\frac{d}{dt}\Psi(p,\partial_t p,\psi) = -\mathcal{P}_\delta[\psi] + 2\int_{P_\delta}\mu(s)\langle A^{1/2}\psi(s), A^{1/2}p\rangle ds,$$

and, using (5.3), the conclusion follows. ☐

Remark 5.5. The above results continue to hold with $(A^{r/2}p, A^{r/2}\partial_t p, A^{r/2}\psi)$ in place of $(p,\partial_t p,\psi)$. The only difference is that the terms $\|k\|\|\psi\|_{\mathcal{M}_{-1}}$ and $\langle k,p\rangle$ must be replaced by $\|k\|\|A^r\psi\|_{\mathcal{M}_{-1}}$ and $\langle k, A^r p\rangle$, respectively.

6. Proof of Lemma 4.3

Here and in the sequel, all the estimates are performed within a suitable regularization scheme. We define $\mathcal{L}_0 \in C(\mathcal{H}_0,\mathbb{R})$ as

$$\mathcal{L}_0(p,q,\psi) = \|(p,q,\psi)\|^2_{\mathcal{H}_0} + 2\langle G_0(p),1\rangle.$$

For every $\delta > 0$, we have

$$\frac{d}{dt}\mathcal{L}_0(D(t)z) = \int_0^\infty \mu'(s)\|A^{1/2}\xi^t(s)\|^2 ds \leq 0.$$

We now choose an arbitrary z such that $\|z\|_{\mathcal{H}_0} \leq R$. Throughout the end of the proof, the generic constant $c \geq 0$ may depend (increasingly) on R. Hence, on account of (2.3)-(2.5),

$$\|D(t)z\|^2_{\mathcal{H}_0} \leq \mathcal{L}_0(D(t)z) \leq c\|D(t)z\|^2_{\mathcal{H}_0}.$$

Let $\varepsilon \in (0, \frac{1}{2})$ to be specified later, and put $\nu = \varepsilon^2$ (this fixes the corresponding s_*). Then, select $\delta > 0$ small enough such that $\mu(s_*) \leq \lambda/\delta$ and $\hat{\mu}(P_\delta) \leq \varepsilon^2$. Finally, setting $(p(t), \partial_t p(t), \psi^t) = D(t)z$ and $k = g_0(v)$ in (5.2), introduce the functional

$$\mathcal{E}(t) = \frac{1}{\delta}\mathcal{L}_0(D(t)z) + \varepsilon^3\Phi(D(t)z) + \frac{\varepsilon^2}{4}\Psi(D(t)z).$$

For ε small enough,

$$\frac{1}{2}\|D(t)z\|^2_{\mathcal{H}_0} \leq \mathcal{E}(t) \leq c\|D(t)z\|^2_{\mathcal{H}_0}. \tag{6.1}$$

With the above choice of ν and δ, exploiting Lemma 5.3 and Lemma 5.4, and noting that $\langle g_0(v), v\rangle \geq 0$, we obtain the differential inequality

$$\begin{aligned}\frac{d}{dt}\mathcal{E} &\leq -\varepsilon^3\Big(\frac{1}{2} - 6\varepsilon\Big)\|A^{1/2}v\|^2 - \varepsilon^5\|\partial_t v\|^2 - \varepsilon^2\Big(\frac{1}{8} - 4\varepsilon^2\Big)\mathcal{P}_\delta[\xi] \\ &\quad + \Big(\frac{1}{\delta} - \frac{\varepsilon\mu(s_*)}{\lambda}\Big)\int_0^\infty \mu'(s)\|A^{1/2}\xi(s)\|^2 ds + 4\varepsilon\mathcal{N}_\delta[\xi] + \varepsilon^3\|g_0(v)\|\|\xi\|_{\mathcal{M}_{-1}}.\end{aligned}$$

Observe that

$$\begin{aligned}\Big(\frac{1}{\delta} - \frac{\varepsilon\mu(s_*)}{\lambda}\Big)\int_0^\infty \mu'(s)\|A^{1/2}\xi(s)\|^2 ds + 4\varepsilon\mathcal{N}_\delta[\xi] &\leq \frac{1}{2\delta}\int_0^\infty \mu'(s)\|A^{1/2}\xi(s)\|^2 ds + 4\varepsilon\mathcal{N}_\delta[\xi] \\ &\leq -\Big(\frac{1}{2} - 4\varepsilon\Big)\mathcal{N}_\delta[\xi],\end{aligned}$$

while

$$\varepsilon^3\|g_0(v)\|\|\xi\|_{\mathcal{M}_{-1}} \leq c\varepsilon^3\|A^{1/2}v\|\|\xi\|_{\mathcal{M}_0} \leq \frac{\varepsilon^3}{4}\|A^{1/2}v\|^2 + c\varepsilon^3\mathcal{N}_\delta[\xi] + c\varepsilon^3\mathcal{P}_\delta[\xi].$$

Hence,

$$\frac{d}{dt}\mathcal{E} \leq -\varepsilon^3\Big(\frac{1}{4} - 6\varepsilon\Big)\|A^{1/2}v\|^2 - \varepsilon^5\|\partial_t v\|^2 - \varepsilon^2\Big(\frac{1}{8} - c\varepsilon\Big)\mathcal{P}_\delta[\xi] - \Big(\frac{1}{2} - c\varepsilon\Big)\mathcal{N}_\delta[\xi].$$

It is then clear that, up to taking ε small enough (depending on c), we obtain

$$\frac{d}{dt}\mathcal{E}(t) + \varepsilon^5\|D(t)z\|^2_{\mathcal{H}_0} \leq 0,$$

which, together with (6.1) and the Gronwall lemma, yield the desired conclusion. Notice that the obtained decay rate κ depends on c (and thus on R). However, using the semigroup properties, it is immediate to show that it can be fixed independently of R, provided that we enlarge $Q(R)$ accordingly.

7. Proofs of Lemma 4.4 and Lemma 4.5

The proofs of the lemmata lean on the existence of a (weak) dissipation integral. Namely,

Lemma 7.1. *Assume that the hypotheses of Lemma 4.4 hold. Then, for every $\varepsilon > 0$ and every $t \geq \tau \geq 0$,*

$$\int_\tau^t \|\partial_t u(y)\| dy \leq \varepsilon(t-\tau) + K,$$

for some $K = K(C, \varepsilon) \geq 0$.

Proof. In this proof, the generic constant c will depend on the bound C of the norm of $S(t)z$ in $\mathcal{H}_0$. For any fixed $\varepsilon > 0$ (without loss of generality, we assume that $\varepsilon \le 1/2$), choose $\nu = \varepsilon^2$ and $\delta > 0$ such that $\mu(s_*) \le \lambda/\delta$ and $\hat{\mu}(P_\delta) \le \varepsilon^2$. It is apparent that

$$\int_{P_\delta} \mu(s)\langle A^{1/2}u, A^{1/2}\eta(s)\rangle ds \le c\varepsilon$$

and

$$\|g(u) - f\|\|\eta\|_{\mathcal{M}_{-1}} \le \frac{1}{\varepsilon^2}\mathcal{N}_\delta[\eta] + c\varepsilon.$$

Then, setting $(p(t), \partial_t p(t), \psi^t) = S(t)z$ and $k = g(u) - f$ in (5.2), in view of Lemma 5.1 and Remark 5.5, the functional $\Phi^1(S(t)z)$ satisfies the inequality

$$\frac{d}{dt}\Phi^1 \le -\frac{1}{2}\|\partial_t u\|^2 - \frac{\mu(s_*)}{\lambda\varepsilon^2}\int_0^\infty \mu'(s)\|A^{1/2}\eta(s)\|^2 ds + \frac{4}{\varepsilon^2}\mathcal{N}_\delta[\eta] + c\varepsilon.$$

Finally, we define

$$\mathcal{E}(t) = \frac{1}{\delta}\mathcal{L}(S(t)z) + \varepsilon^4\Phi^1(S(t)z),$$

where $\mathcal{L}$ is the Lyapunov function introduced above. Due to (5.1), we have $|\mathcal{E}| \le c/\delta$. Reasoning as in the proof of Lemma 4.3, if ε is small enough (which is clearly not a constraint in view of our aim), we obtain

$$\frac{d}{dt}\mathcal{E} + \frac{\varepsilon^4}{2}\|\partial_t u\|^2 \le c\varepsilon^5.$$

Integrating this inequality over (τ, t), and subsequently applying the Hölder inequality, we reach the desired conclusion. □

7.1. **Proof of Lemma 4.4.** Again, the generic constant c appearing below will depend on the bound C of the norm of $S(t)z$ in $\mathcal{H}_0$. For $r \in [0, \frac{1}{2})$, we introduce the functional

$$\mathcal{Q}_r(t) = \|K(t)z\|^2_{\mathcal{H}_r} + 2\langle g(u(t)) - g_0(v(t)) - f, A^r w(t)\rangle,$$

which satisfies the estimates

$$\frac{1}{2}\mathcal{Q}_r(t) - c \le \|K(t)z\|^2_{\mathcal{H}_r} \le 2\mathcal{Q}_r(t) + c$$

and the differential equality

$$\begin{aligned}
&\frac{d}{dt}\mathcal{Q}_r - \int_0^\infty \mu'(s)\|A^{(1+r)/2}\zeta(s)\|^2 ds \\
&= 2\langle [g_0'(u) - g_0'(v)]\partial_t u, A^r w\rangle + 2\langle g_0'(v)\partial_t w, A^r w\rangle + 2\langle g_1'(u)\partial_t u, A^r w\rangle.
\end{aligned}$$

By virtue of (2.3), (2.5)-(2.6) and the continuous embedding $H^\alpha \hookrightarrow L^{6/(3-2\alpha)}(\Omega)$, we obtain the following estimates:

(7.1)
$$\begin{aligned}
2\langle [g_0'(u) - g_0'(v)]\partial_t u, A^r w\rangle &\le c(1 + \|u\|_{L^6} + \|v\|_{L^6})\|\partial_t u\|\|w\|_{L^{6/(1-2r)}}\|A^r w\|_{L^{6/(1+2r)}} \\
&\le c\|\partial_t u\|\|A^{(1+r)/2}w\|^2,
\end{aligned}$$

$$(7.2)\qquad 2\langle g_0'(v)\partial_t w, A^r w\rangle \le c\| |v| + |v|^2\|_{L^3}\|\partial_t w\|_{L^{6/(3-2r)}}\|A^r w\|_{L^{6/(1+2r)}} \le c\|A^{1/2}v\|\|A^{r/2}\partial_t w\|\|A^{(1+r)/2}w\|,$$

and

$$(7.3)\qquad 2\langle g_1'(u)\partial_t u, A^r w\rangle \le c\|\partial_t u\|\|A^r w\| \le c\|\partial_t u\| + c\|\partial_t u\|\|A^{(1+r)/2}w\|^2.$$

Thus, we readily obtain

$$\frac{d}{dt}\mathcal{Q}_r - \int_0^\infty \mu'(s)\|A^{(1+r)/2}\zeta(s)\|^2 ds \le h + h\mathcal{Q}_r,$$

where we put

$$h(t) = c\|\partial_t u\| + c\|A^{1/2}v\|.$$

For $\varepsilon \in (0, \frac{1}{2})$, we choose $\nu = \varepsilon^2$ and $\delta > 0$ such that $\mu(s_*) \le \lambda/\delta$ and $\hat{\mu}(P_\delta) \le \varepsilon^2$. Setting $(p(t), \partial_t p(t), \psi^t) = A^{r/2}K(t)z$ (here, $A^{r/2}$ is in fact the diagonal matrix whose entries are $A^{r/2}$) and $k = g(u) - g_0(v) - f$ in (5.2), we consider the functional

$$\Upsilon_r(t) = \Phi(A^{r/2}K(t)z) + \Psi(A^{r/2}K(t)z).$$

Applying Lemma 5.3 and Lemma 5.4, together with Remark 5.5 and the immediate control

$$\|g(u) - g_0(v) - f\|\|A^r\zeta\|_{\mathcal{M}_{-1}} - (1 - 2\varepsilon^2)\langle g(u) - g_0(v) - f, A^r w\rangle \le \frac{1}{2}\|A^{(1+r)/2}w\|^2 + \frac{1}{4}\mathcal{P}_\delta[A^{r/2}\zeta] + \mathcal{N}_\delta[A^{r/2}\zeta] + c,$$

we obtain

$$\frac{d}{dt}\Upsilon_r \le -\varepsilon^2\Big(\|A^{(1+r)/2}w\|^2 + \|A^{r/2}\partial_t w\|^2 + \mathcal{P}_\delta[A^{r/2}\zeta]\Big) - \frac{\mu(s_*)}{\lambda\varepsilon^2}\int_0^\infty \mu'(s)\|A^{(1+r)/2}\zeta(s)\|^2 ds + \frac{5}{\varepsilon^2}\mathcal{N}_\delta[A^{r/2}\zeta] + c,$$

provided that ε is small enough. Finally, we introduce the energy

$$\mathcal{W}_r(t) = \frac{1}{\delta}\mathcal{Q}_r(t) + \varepsilon^3\Upsilon_r(t),$$

which fulfills the inequalities (again, if ε is small enough)

$$\frac{1}{c}\mathcal{W}_r(t) - c \le \|K(t)z\|^2_{\mathcal{H}_r} \le c\mathcal{W}_r(t) + c,$$

for some $c \ge 1$ depending on ε. Thus, we reach the desired conclusion if we show that $\mathcal{W}_r(t)$ is bounded for all times. In light of the previous computations, we have

$$\frac{d}{dt}\mathcal{W}_r \le -\varepsilon^5\Big(\|A^{(1+r)/2}w\|^2 + \|A^{r/2}\partial_t w\|^2 + \mathcal{P}_\delta[A^{r/2}\zeta]\Big) + 5\varepsilon\mathcal{N}_\delta[A^{r/2}\zeta] + \Big(1 - \frac{\varepsilon\delta\mu(s_*)}{\lambda}\Big)\int_0^\infty \mu'(s)\|A^{(1+r)/2}\zeta(s)\|^2 ds + h + h\mathcal{W}_r + c.$$

It is then apparent that, provided that we fix ε small, we end up with the inequality

$$\frac{d}{dt}\mathcal{W}_r + \beta\mathcal{W}_r \le h + h\mathcal{W}_r + c,$$

for some $\beta > 0$. Observe also that, by virtue of Lemma 4.3 and Lemma 7.1,

$$\int_\tau^t h(y)dy \le \frac{\beta}{2}(t-\tau) + c.$$

Since $\mathcal{W}_r(0) = 0$, the conclusion follows from a Gronwall-type lemma (see e.g. [4]).

7.2. **Proof of Lemma 4.5.** We basically repeat the proof of Lemma 4.4, setting $r = 1$. In this case, the generic constant c appearing below will depend on the bound C of the norm of $S(t)z$ in $\mathcal{H}_{1/3}$. The only difference here is how we reach the control (7.1), whereas (7.2)-(7.3) remain the same (for $r = 1$). Since

$$|g_0'(u) - g_0'(v)| \le c|w|(1 + |u| + |w|),$$

exploiting the Agmon inequality

$$\|w\|_{L^\infty} \le c\|A^{1/2}w\|^{1/2}\|Aw\|^{1/2} \le c\|Aw\|^{1/2}$$

and the embeddings $H^{4/3} \hookrightarrow L^9(\Omega)$ and $H^{1/3} \hookrightarrow L^{18/7}(\Omega)$, we are led to

$$\begin{aligned}
2\langle [g_0'(u) - g_0'(v)]\partial_t u, Aw\rangle &\le c\|\partial_t u\|\|w\|_{L^\infty}\|Aw\| + c\|u\|_{L^9}\|\partial_t u\|_{L^{18/7}}\|w\|_{L^\infty}\|Aw\| \\
&\quad + c\|w\|_{L^6}^{2/3}\|\partial_t u\|_{L^{18/7}}\|w\|_{L^\infty}^{4/3}\|Aw\| \\
&\le c\|Aw\|^{3/2} + c\|Aw\|^{5/3} \\
&\le \gamma\|Aw\|^2 + \frac{c}{\gamma^5},
\end{aligned}$$

for every $\gamma \in (0,1)$. Thus, for any given $\gamma \in (0,1)$, we conclude that

$$\frac{d}{dt}\mathcal{Q}_1 - \int_0^\infty \mu'(s)\|A^{(1+r)/2}\zeta(s)\|^2 ds \le \gamma\|Aw\|^2 + h + h\mathcal{Q}_1 + \frac{c}{\gamma^5}.$$

We can now proceed exactly as in the proof of Lemma 4.4. Note that the term $\gamma\|Aw\|^2$ is easily controlled, upon fixing γ small enough.

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