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# On calculating the normal cone to a finite union of convex polyhedra

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**Abstract**

The paper provides formulae for calculating the limiting normal cone introduced by Mordukhovich to a finite union of convex polyhedra. In the first part, special cases of independent interest are considered (almost disjoint cones, halfspaces, orthants). The second part focusses on unions of general polyhedra. Due to the local nature of the normal cone, one may restrict considerations without loss of generality to finite unions of polyhedral cones. First, an explicit formula for the normal cone is provided in the situation of two cones. An algorithmic approach is presented along with a refined, more efficient formula. Afterwards, a general formula for the union of N cones is derived. Finally, an application to the stability analysis of a special type of probabilistic constraints is provided.

# 1 Introduction

In the variational geometry of nonconvex sets a distinguished role is played by the (limiting) normal cone introduced in 1976 by Mordukhovich [6]. Its importance in many areas of optimization and optimal control, stability analysis, set-valued analysis etc. can be recognized from the monographs [10] (with emphasis on finite dimensions) and [7] (with an extensive theory in infinite dimensions). Though this cone enjoys a rich calculus, its computation for a concrete nonconvex set can be a challenging task.

In [2] the authors studied the stability of a class of parameter-dependent variational inequalities with a convex polyhedral constraint set $C$. The key step, that enabled them to derive the main results, was the computation of the normal cone to the graph of the standard normal cone mapping $N_C(\cdot)$ associated with $C$. Due to the polyhedrality of $C$, this graph is a union of finitely many convex polyhedra [9]. It possesses a special structure that was extensively exploited in the computation of the normal cone. Nevertheless, the resulting formula is by no means easy to apply (it describes a procedure which is quite involved even in case of very simple sets $C$).

A problem which, at least formally, is not too far from the investigations in [10], arises in the stability analysis of a parameter-dependent constraint set

$$\Gamma(x) = \{y \in \mathbb{R}^m | F(x,y) \in \Lambda\},$$

where again the normal cone to $\Lambda$ plays a key role. Nevertheless, here $\Lambda$ does not necessarily have the structure of gph $N_C$, which prevents a straightforward application of the results from [10].

The aim of this paper is to compute the normal cone to a finite union of convex polyhedral cones. This result can be used then in the stability analysis of $\Gamma$. It may have, however, yet other applications, e.g. in disjunctive programming, and has definitely importance of its own in variational analysis. As expected, also in this case the resulting formula describes a nontrivial procedure, the complexity of which substantially increases with the number of cones. Throughout the paper, we use Motzkin's Theorem of the Alternative as our workhorse which is well suited for desribing the position of the considered point at the boundary of polyhedral sets. Moreover, the focus of our analysis is on polyhedral cones, because only local information is required for the computation of the limiting normal cone and, locally, polyhedra look like cones.

The paper is organized as follows: In Section 2, we collect several basic results extensively used in the sequel. Section 3 provides a compilation of explicit formulae for the normal cone to specially structured unions of polyhedral cones. In Section 4, an explicit formula for the calculation of the limiting normal cone to the union of two arbitrary polyhedral cones is derived in terms of the data of the original cones. This formula may be used for numerical calculations in moderate dimension, but it becomes inefficient soon. Therefore, Section 5 proposes a more efficient variant of the formula along with an algorithmic procedure. The situation with a general finite union is not fully recognized from the case of two cones. Section 6 generalizes the observations obtained so far to the union of $N$ cones. Finally, Section 7 presents an application to the stability analysis of certain probabilistic constrains.

# 2 Preliminaries

We start with the definitions of the main objects in our investigation. For a closed set $\Lambda \subseteq \mathbb{R}^n$ and a point $\bar{x} \in \Lambda$, the *Fréchet normal cone* to $\Lambda$ at $\bar{x} \in \Lambda$ is defined by

$$\hat{N}_\Lambda(\bar{x}) := \{x^* \in \mathbb{R}^n | \, \langle x^*, x - \bar{x} \rangle \leq o(\|x - \bar{x}\|) \quad \forall x \in \Lambda\}.$$

The *(limiting) normal cone* to $\Lambda$ at $\bar{x} \in \Lambda$ results from the Fréchet normal cone in the following way:

$$N_\Lambda(\bar{x}) := \operatorname*{Limsup}_{x \to \bar{x}, x \in \Lambda} \hat{N}_\Lambda(x).$$

The 'Limsup' in the definition above is the upper limit of sets in the sense of Kuratowski-Painlevé, cf. [10]. In this finite-dimensional setting, $\hat{N}_\Lambda(\bar{x})$ is the negative polar cone to the *contingent cone* to $\Lambda$ at $\bar{x}$:

$$T_\Lambda(\bar{x}) := \operatorname*{Limsup}_{t \downarrow 0} \frac{\Lambda - \{\bar{x}\}}{t}.$$

Hence, $\hat{N}_\Lambda(\bar{x})$ coincides with the standard normal cone in the sense of convex analysis whenever $\Lambda$ is convex. At this point, we state a simple observation which will be useful in the sequel:

**Lemma 2.1** *Let $K$ be a closed convex cone. Then, denoting by $K^*$ the negative polar cone of $K$, it holds that*

$$\text{bd}\, K^* = \bigcup_{x \in K \setminus \{0\}} \hat{N}_K(x),$$

*where 'bd' refers to the topological boundary.*

**Proof.** First note that, in the setting of the lemma, $\hat{N}$ coincides with the normal cone of convex analysis. From a well-known representation of the boundary of convex sets (see [5], Prop. 3.1.4 and 3.1.5), it follows that

$$\text{bd}\, K^* = \bigcup_{x \neq 0} \{s^* | x \in \hat{N}_{K^*}(s^*)\}.$$

Since $\hat{N}_{K^*}(s^*) \subseteq K^{**} = K$, we may shrink the union to $x \in K \setminus \{0\}$. Exploiting now the equivalence

$$x \in \hat{N}_{K^*}(s^*) \Longleftrightarrow s^* \in \hat{N}_K(x)$$

(see [10], Example 11.4), the result follows. ■

In this paper, we will be dealing with finite unions of polyhedra and polyhedral cones. Let $P := \cup_{i=1}^N P_i$, where each component $P_i$ is a convex polyhedron. For $x \in P$, denote the *index set of active components* by

$$\mathbb{I}(x) := \{i \in \{1, \ldots, N\} | x \in P_i\}.$$

Clearly, there exists some neighbourhood $\mathcal{V}$ of 0 such that

$$(P - \{x\}) \cap \mathcal{V} = \bigcup_{i \in \mathbb{I}(x)} (P_i - \{x\}) \cap \mathcal{V}.$$

Moreover, for each $i \in \mathbb{I}(x)$, we can associate with $P_i$ the contingent cone $T_{P_i}(x)$ as well as a neighbourhood $\mathcal{U}_i$ of zero such that

$$T_{P_i}(x) \cap \mathcal{U}_i = (P_i - \{x\}) \cap \mathcal{U}_i.$$

Consequently, the polyhedral cone $\Lambda_i := T_{P_i}(x)$ and the neighbourhood $\mathcal{U} := \cap_{i \in \mathbb{I}(x)} \mathcal{U}_i \cap \mathcal{V}$ of zero satisfies

$$(P - \{x\}) \cap \mathcal{U} = \bigcup_{i \in \mathbb{I}(x)} \Lambda_i \cap \mathcal{U}.$$

Thus, for $\Lambda := \cup_{i \in \mathbb{I}(x)} \Lambda_i$, one ends up with $N_P(x) = N_\Lambda(0)$. In other words, it suffices to compute the normal cone to a finite union of polyhedral cones at zero. Given this reduction, we shall focus now on sets

$$\Lambda := \bigcup_{i=1}^N \Lambda_i,$$

where the $\Lambda_i$ are convex polyhedral cones. Due to the polyhedral structure of $\Lambda$, only a finite number of cones can be manifested as $\hat{N}_\Lambda(x)$, $x \in \Lambda$, and, moreover, for $x \neq 0$,

$$\hat{N}_\Lambda(x) = \hat{N}_\Lambda(tx) \quad \forall t > 0.$$

It follows that

$$N_\Lambda(0) = \bigcup_{x\in\Lambda} \hat{N}_\Lambda(x) = \hat{N}_\Lambda(0) \cup \bigcup_{x\in\Lambda\setminus\{0\}} \hat{N}_\Lambda(x) = \bigcap_{i=1}^{N} \Lambda_i^* \cup \bigcup_{x\in\Lambda\setminus\{0\}} \hat{N}_\Lambda(x), \tag{1}$$

where the last equality relies on the identity

$$\hat{N}_\Lambda(0) = [T_\Lambda(0)]^0 = \left[\bigcup_{i=1}^{N} T_{\Lambda_i}(0)\right]^0 = \bigcap_{i=1}^{N} \hat{N}_{\Lambda_i}(0) = \bigcap_{i=1}^{N} \Lambda_i^*.$$

Owing to Lemma 2.1 above, one gets the inclusion

$$\bigcup_{x\in\Lambda\setminus\{0\}} \hat{N}_\Lambda(x) \subseteq \bigcup_{x\in\Lambda\setminus\{0\}} \bigcup_{i=1}^{N} \hat{N}_{\Lambda_i}(x) = \bigcup_{i=1}^{N} \bigcup_{x\in\Lambda\setminus\{0\}} \hat{N}_{\Lambda_i}(x) = \bigcup_{i=1}^{N} \operatorname{bd} \Lambda_i^*.$$

Along with (1), we have the following upper estimate for the limiting normal cone:

$$N_\Lambda(0) \subseteq \bigcap_{i=1}^{N} \Lambda_i^* \cup \bigcup_{i=1}^{N} \operatorname{bd} \Lambda_i^*. \tag{2}$$

# 3 Special Cases

Before turning to the union of polyhedral cones without further structural assumptions, it is worth considering some special cases which are of independent interest and can be analyzed directly.

## 3.1 Almost Disjoint Cones

First, we consider the union of cones which are almost disjoint in the sense that their pairwise intersections reduce to zero. As in this specific situation the polyhedral structure is not essential, we formulate the result for arbitrary convex cones. The resulting formula itself already appears as an inclusion in (2) and so, some derivations in the next proposition are parallel to those preceding (2). However, here we consider a situation which is more general on the one hand, in the sense that non-polyhedral cones are considered and, which is more specific on the other hand in requiring disjoint cones.

**Proposition 3.1** *Let $\Lambda := \cup_{i=1}^N \Lambda_i$ be a finite union of closed convex cones such that $\Lambda_i \cap \Lambda_j = \{0\}$ for $i \neq j$. Then,*

$$N_\Lambda(0) = \bigcap_{i=1}^N \Lambda_i^* \cup \bigcup_{i=1}^N \operatorname{bd} \Lambda_i^*.$$

**Proof.** First, we check the relation

$$N_\Lambda(0) = \bigcup_{x \in \Lambda} \hat{N}_\Lambda(x). \tag{3}$$

If $x^* \in N_\Lambda(0)$, then, by definition, there are sequences $x_n \to 0$, $x_n^* \to x^*$, such that $x_n \in \Lambda$ and $x_n^* \in \hat{N}_\Lambda(x_n)$. If $x_n = 0$, then $x_n^* \in \hat{N}_\Lambda(0)$. Therefore, if $x_n = 0$ holds true for a subsequence, then

$$x^* \in \hat{N}_\Lambda(0) \subseteq \bigcup_{x \in \Lambda} \hat{N}_\Lambda(x)$$

by closedness of the normal cone. Otherwise, $x_n \neq 0$ for all $n$ large enough. Since $\Lambda$ is just a finite union of the $\Lambda_i$'s, we may assume, after passing to subsequences, that, without loss of generality, $x_n \in \Lambda_1$. Now, our assumption guarantees that, locally around $x_n$, $\Lambda$ coincides with $\Lambda_1$, hence, by Lemma 2.1

$$x_n^* \in \hat{N}_\Lambda(x_n) = \hat{N}_{\Lambda_1}(x_n) \subseteq \operatorname{bd} \Lambda_1^*.$$

With $\operatorname{bd} \Lambda_1^*$ being a closed set, it follows again from Lemma 2.1 that

$$x^* \in \operatorname{bd} \Lambda_1^* = \bigcup_{x \in \Lambda_1 \setminus \{0\}} \hat{N}_{\Lambda_1}(x).$$

Consequently, there exists some $x' \in \Lambda_1 \setminus \{0\} \subseteq \Lambda$ such that, with the same argumentation as before,

$$x^* \in \hat{N}_{\Lambda_1}(x') = \hat{N}_\Lambda(x') \subseteq \bigcup_{x \in \Lambda} \hat{N}_\Lambda(x).$$

This establishes the inclusion '$\subseteq$' in (3). For the reverse inclusion, let $x^* \in \hat{N}_\Lambda(x)$ for some $x \in \Lambda$. If $x = 0$, then

$$x^* \in \hat{N}_\Lambda(0) \subseteq N_\Lambda(0).$$

Therefore, we may assume that $x \neq 0$. Then, by our assumption, $x$ belongs to exactly one of the $\Lambda_i$'s, say $x \in \Lambda_1$. Put $x_n := n^{-1}x \in \Lambda_1 \setminus \{0\}$. We derive that

$$x^* \in \hat{N}_\Lambda(x) = \hat{N}_{\Lambda_1}(x) = \hat{N}_{\Lambda_1}(x_n) = \hat{N}_\Lambda(x_n).$$

Since $x_n \to 0$, the definition of the limiting cone yields that $x^* \in N_\Lambda(0)$. This establishes (3).

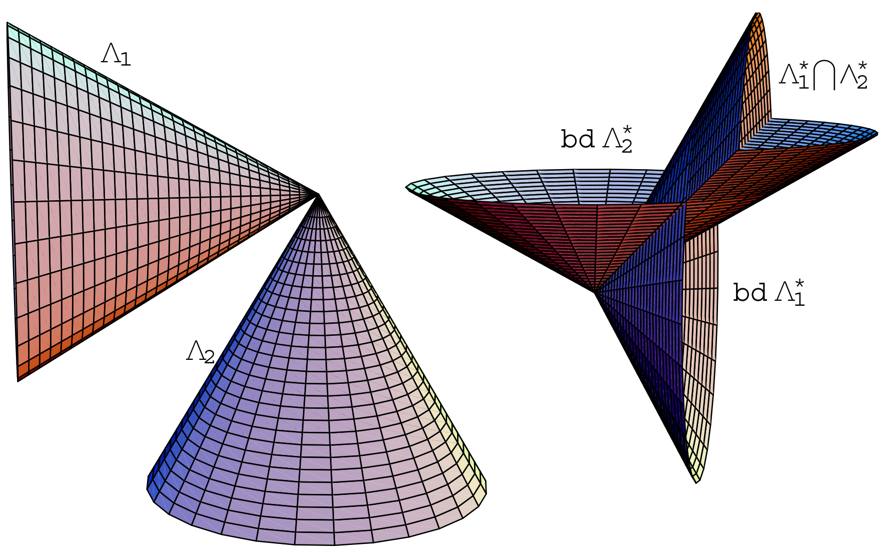
We may continue (3) now as follows:

$$\begin{aligned}
N_\Lambda(0) &= \bigcup_{x\in\Lambda}\hat{N}_\Lambda(x) = \hat{N}_\Lambda(0)\cup\bigcup_{x\in\Lambda\setminus\{0\}}\hat{N}_\Lambda(x) = \bigcap_{i=1}^N \hat{N}_{\Lambda_i}(0)\cup\bigcup_{i=1}^N\bigcup_{x\in\Lambda_i\setminus\{0\}}\hat{N}_\Lambda(x)\\
&= \bigcap_{i=1}^N \Lambda_i^*\cup\bigcup_{i=1}^N\bigcup_{x\in\Lambda_i\setminus\{0\}}\hat{N}_{\Lambda_i}(x) = \bigcap_{i=1}^N \Lambda_i^*\cup\bigcup_{i=1}^N \operatorname{bd}\Lambda_i^*.
\end{aligned}$$

Here, in the next to last equality, we exploited once more the assumption of our proposition, whereas the last equality relies on Lemma 2.1. ■

Proposition 3.1 is illustrated in Figure 1.

Figure 1: Illustration of Proposition 3.1. The left part shows the union of two almost disjoint cones, the right part shows the resulting normal cone evaluated at the origin.




## 3.2 Halfspaces

In this section, we consider unions of halfspaces, i.e.

$$\Lambda := \bigcup_{i=1}^N \Lambda_i,\ \text{where}\ \Lambda_i := \{x\in\mathbb{R}^n |\, \langle c_i, x\rangle \le 0\}\ \text{for some}\ c_i\in\mathbb{R}^n \quad (i=1,\dots,N). \tag{4}$$

**Proposition 3.2** *In (4), assume that the $c_i$ are positively linear independent and that*

$$c_i \notin \text{con}\,\{c_j | j = 1, \ldots, N,\ j \neq i\} \quad (i = 1, \ldots, N), \tag{5}$$

*where 'con' refers to the convex conic hull (i.e., we assume that the description (4) is free of redundance). Then,*

$$N_\Lambda(0) = \bigcup_{i=1}^{N} \mathbb{R}_+\{c_i\} = \bigcup_{i=1}^{N} \Lambda_i^*.$$

**Proof.** Let $j \in \{1, \ldots, N\}$ be arbitrary. We establish a contradiction to the statement, that there exists some $\pi \in \mathbb{R}^N \backslash \{0\}$ such that

$$\sum_{i=1}^{N} \pi_i c_i = 0, \quad \pi_i \geq 0 \quad i \in \{1, \ldots, N\} \backslash \{j\}.$$

Indeed, otherwise we obtain a contradiction either with the assumption on positive linear independence of the $c_i$'s (if $\pi_j \geq 0$) or with (5), because for $\pi_j < 0$ one has

$$c_j = \sum_{i=1, i \neq j}^{N} -\frac{\pi_i}{\pi_j} c_i.$$

By Motzkin's Theorem, the non-existence of a solution to the system above is equivalent to the existence of some $h \in \mathbb{R}^n$ such that

$$\langle c_j, h \rangle = 0; \quad \langle c_i, h \rangle > 0 \quad (i \in \{1, \ldots, N\} \backslash \{j\}).$$

In particular, $h \in \Lambda \backslash \{0\}$ and, locally around $h$, $\Lambda$ coincides with $\Lambda_j$. Thus,

$$\hat{N}_\Lambda(h) = \hat{N}_{\Lambda_j}(h) = \Lambda_j^*.$$

Along with (1), and taking into account that $j$ was arbitrary, we derive that

$$\Lambda_j^* \subseteq \bigcup_{x \in \Lambda \backslash \{0\}} \hat{N}_\Lambda(x) \subseteq N_\Lambda(0) \quad (j = 1, \ldots, N). \tag{6}$$

Conversely, (5) guarantees that

$$\bigcap_{i=1}^{N} \Lambda_i^* = \bigcap_{i=1}^{N} \mathbb{R}_+\{c_i\} = \{0\}.$$

Therefore, the result follows from (2) and (6):

$$N_\Lambda(0) \subseteq \bigcup_{i=1}^{N} \text{bd}\, \Lambda_i^* \subseteq \bigcup_{i=1}^{N} \Lambda_i^* \subseteq N_\Lambda(0).$$

■

The following two examples illustrate necessity of the assumptions in Proposition 3.2:

**Example 3.3** *Let*

$$c_1 = (1,0), \quad c_2 = (0,1), \quad c_3 = (-1,-1).$$

*Then,* $\Lambda = \mathbb{R}^2$ *and the statement of Proposition 3.2 becomes false due to the failure of positive linear independence:*

$$N_\Lambda(0) = \{0\} \neq \bigcup_{i=1}^{3} \mathbb{R}_+\{c_i\}.$$

**Example 3.4** *Let*

$$c_1 = (1,0), \quad c_2 = (0,1), \quad c_3 = (1,1).$$

*Then,* $\Lambda = \mathbb{R}^2 \backslash \operatorname{int} \mathbb{R}^2_+$ *and the statement of Proposition 3.2 becomes false due to the failure of (5):*

$$N_\Lambda(0) = \bigcup_{i=1}^{2} \mathbb{R}_+\{c_i\} \neq \bigcup_{i=1}^{3} \mathbb{R}_+\{c_i\}.$$

*Actually, the inequality defined by* $c_3$ *is redundant here.*

As an application of Proposition 3.2, we consider the set

$$\Omega := \bigcup_{i=1}^{N} \Omega_i, \quad \Omega_i := \{x \in \mathbb{R}^n | f_i(x) \leq 0\} \quad (i = 1, \ldots, N),$$

where the $f_i : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable. We want to compute $N_\Omega(x)$ at a given point $x \in \Omega$ and assume without loss of generality that $f_i(x) = 0$ for $i = 1, \ldots, N$.

**Corollary 3.5** *Assume that the gradients* $\nabla f_i(x)$ *are positively linearly independent and none of them can be represented as a nonnegative linear combination of the others. Then,*

$$N_\Omega(x) = \bigcup_{i=1}^{N} \mathbb{R}_+\{\nabla f_i(x)\}.$$

**Proof.** Being positively linearly independent, the $\nabla f_i(x)$ are nonzero and, the same holds true locally around $x$. In particular,

$$T_\Omega(x) = \bigcup_{i=1}^{N} \Lambda_i, \quad \Lambda_i := T_{\Omega_i}(x) = \{h \in \mathbb{R}^n | \langle \nabla f_i(x), h \rangle \leq 0\} \quad (i = 1, \ldots, N).$$

Now, by virtue of our assumptions, Proposition 3.2 yields that

$$\bigcup_{i=1}^{N} \mathbb{R}_+\{\nabla f_i(x)\} = N_{T_\Omega(x)}(0) \subseteq N_\Omega(x),$$

where the inclusion holds generally true (cf. [10], Prop. 6.27 (a)). On the other hand, as stated above, the $\nabla f_i(y)$ are nonzero for $y$ in a neighbourhood of $x$. Consequently,

$$\hat{N}_{\Omega_i}(y) = \mathbb{R}_+\{\nabla f_i(y)\}$$

for such $y$, whenever $y \in \Omega_i$. Since for all $y \in \Omega$ there is some $i$ such that $y \in \Omega_i$, the definition of the normal cone implies that

$$N_\Omega(x) = \operatorname*{Limsup}_{y \to x, y \in \Omega} \hat{N}_\Omega(y) \subseteq \operatorname*{Limsup}_{y \to x, y \in \Omega} \bigcup_{i=1}^{N} \mathbb{R}_+\{\nabla f_i(y)\} = \bigcup_{i=1}^{N} \mathbb{R}_+\{\nabla f_i(x)\},$$

where the last equality again relies on the fact that the $\nabla f_i(x)$ are nonzero around $x$. ■

Corollary 3.5 improves a result in [4], where the same expression for the normal cone was obtained under the stronger assumption of (full) linear independence of the $\nabla f_i(x)$.

## 3.3 Orthants

We consider the following union of translated orthants

$$P := \bigcup_{i=1}^{N} P_i, \quad P_i := u^i + \mathbb{R}^n_+, \quad u^i \in \mathbb{R}^n \quad (i = 1, \dots, N), \tag{7}$$

which is of interest in disjunctive programming or in optimization problems with probabilistic constraints under discrete distributions (see, e.g., Remark 1 in [1]). In order to compute $N_P(x)$, we shall assume that the $u^i$'s are in *general position*, i.e.,

$$u^i_j \neq u^k_j \quad \forall j \in \{1, \dots, n\} \quad \forall i, k \in \{1, \dots, N\} : i \neq k.$$

For $y \in \mathbb{R}^n$ we introduce the index set of active orthants as

$$\mathbb{I}(y) := \{i \in \{1, \dots, N\} | y \in u^i + \mathbb{R}^n_+\}$$

and the index set of active components with respect to the $i$-th active orthant as

$$I^i(y) := \{j \in \{1, \dots, n\} | y_j = u^i_j\} \quad (i \in \mathbb{I}(y)).$$

**Lemma 3.6** *Let $x \in \operatorname{bd} P$. Then, for each $i \in \mathbb{I}(x)$ there is a sequence $\{y^{(\nu)}\} \in P$ such that $y^{(\nu)} \to_\nu x$, $\mathbb{I}(y^{(\nu)}) = \{i\}$ and $I^i(y^{(\nu)}) = I^i(x)$.*

**Proof.** Let $i \in \mathbb{I}(x)$ be arbitrary. Define $h \in \mathbb{R}^n$ by

$$h_j := \begin{cases} 0 & \text{if } j \in I^i(x) \\ -1 & \text{if } j \in \{1, \dots, n\} \backslash I^i(x) \end{cases}$$

For $\nu \in \mathbb{N}$ put $y^{(\nu)} := x + \nu^{-1} h$. Then, $y_j^{(\nu)} = x_j = u_j^i$ for $j \in I^i(x)$. Moreover, since $x_j > u_j^i$ for $j \in \{1, \dots, n\} \backslash I^i(x)$, one has that $y_j^{(\nu)} = x_j - \nu^{-1} > u_j^i$ for $\nu$ sufficiently large and $j \in \{1, \dots, n\} \backslash I^i(x)$. It follows that $i \in \mathbb{I}(y^{(\nu)})$ and $I^i(y^{(\nu)}) = I^i(x)$. To show that $\mathbb{I}(y^{(\nu)}) = \{i\}$, assume that $k \in \mathbb{I}(y^{(\nu)})$ for some $k \neq i$. Our assumption on the $u^i$'s being in general position implies that $I^i(x) \cap I^k(x) = \varnothing$ (otherwise $x_j = u_j^i = u_j^k$ for some $j$). Consequently, $I^k(x) \subseteq \{1, \dots, n\} \backslash I^i(x)$, and

$$y_j^{(\nu)} = x_j - \nu^{-1} < x_j = u_j^k \quad \forall j \in I^k(x).$$

Now, the assumption $x \in \operatorname{bd} P$ implies that $I^k(x) \neq \varnothing$. Hence, the relation above shows that there really exists some $j$ with $y_j^{(\nu)} < u_j^k$, whence a contradiction to $k \in \mathbb{I}(y^{(\nu)})$. This finishes our proof. ∎

**Proposition 3.7** *Under the assumption of general position, the normal cone to $P$ in (7) calculates as*

$$N_P(x) = \begin{cases} \bigcup\limits_{i \in \mathbb{I}(x)} \operatorname{con} \{-e^j | j \in I^i(x)\} & x \in \operatorname{bd} P \\ \{0\} & x \in \operatorname{int} P \end{cases},$$

*where the $e^j$ refer to the standard unit vectors in $\mathbb{R}^n$.*

**Proof.** The assertion is trivial in case that $x \in \operatorname{int} P$, so let $x \in \operatorname{bd} P$. In case that $|\mathbb{I}(x)| = 1$, say $\mathbb{I}(x) = \{1\}$, then $P$ coincides with $P_1$ locally around $x$. With $P_1$ being a translated orthant, it follows that

$$N_P(x) = N_{P_1}(x) = \operatorname{con} \{-e^j | j \in I^1(x)\},$$

so the assertion holds true. Now, let $|\mathbb{I}(x)| \geq 2$. Referring back to the argumentation in Section 2, we have that $N_P(x) = N_\Lambda(0)$, where

$$\Lambda := \bigcup_{i \in \mathbb{I}(x)} \Lambda_i, \quad \Lambda_i := \{h \in \mathbb{R}^n | h_j \geq 0 \quad \forall j \in I^i(x)\}$$

Moreover, (2) translates to our setting as

$$N_\Lambda(0) \subseteq \bigcap_{i \in \mathbb{I}(x)} \Lambda_i^* \cup \bigcup_{i \in \mathbb{I}(x)} \operatorname{bd} \Lambda_i^* \tag{8}$$

(recall that the number $N$ of cones considered in (2) has already been chosen to coincide with the cardinality of the set of active indices $\mathbb{I}(x)$ relating to the polyhedra $P_i$ in Section 2; thus it may be smaller than the original number $N$ of polyhedra).

As already observed in the proof of Lemma 3.6, the assumption of general position implies that

$$I^i(x) \cap I^k(x) = \varnothing \quad \forall i, k \in \mathbb{I}(x), i \neq k. \tag{9}$$

On the other hand, since $x \in \operatorname{bd} P$, one also has that $I^i(x) \neq \varnothing$ for all $i \in \mathbb{I}(x)$. Putting together these two arguments, the assumption $|\mathbb{I}(x)| \geq 2$ leads to $|I^i(x)| < n$ for all $i \in \mathbb{I}(x)$. Since

$$\Lambda_i^* = \operatorname{con}\{-e^j | j \in I^i(x)\}, \tag{10}$$

it follows that $\Lambda_i^*$ is a closed set which is contained in a linear space of dimension strictly less than $n$. We infer that $\Lambda_i^* = \operatorname{bd} \Lambda_i^*$ for all $i \in \mathbb{I}(x)$. Now, given an arbitrary $i \in \mathbb{I}(x)$, Lemma 3.6 provides us with a sequence $\{y^{(\nu)}\} \in P$ such that $y^{(\nu)} \rightarrow_\nu x$, $\mathbb{I}(y^{(\nu)}) = \{i\}$ and $I^i(y^{(\nu)}) = I^i(x)$. Therefore, locally around each $y^{(\nu)}$, $P$ coincides with the translated orthant $P_i$, and so

$$\hat{N}_P(y^{(\nu)}) = \hat{N}_{P_i}(y^{(\nu)}) = \operatorname{con}\{-e^j | j \in I^i(y^{(\nu)})\} = \operatorname{con}\{-e^j | j \in I^i(x)\} = \Lambda_i^* = \operatorname{bd} \Lambda_i^*. \tag{11}$$

Consequently, as $i \in \mathbb{I}(x)$ was chosen arbitrarily, it follows that

$$\operatorname{bd} \Lambda_i^* \subseteq \operatorname*{Limsup}_{y \to x, y \in P} \hat{N}_P(y) = N_P(x) \quad \forall i \in \mathbb{I}(x). \tag{12}$$

Finally, combining (9) and (10), one gets that

$$\Lambda_i^* \cap \Lambda_k^* = \{0\} \quad \forall i, k \in \mathbb{I}(x), i \neq k.$$

Since we assumed that $|\mathbb{I}(x)| \geq 2$, this entails the identity $\cap_{i=1}^N \Lambda_i^* = \{0\}$, whence, by (8), (12) and (11) the asserted identity

$$N_P(x) = N_\Lambda(0) = \bigcup_{i \in \mathbb{I}(x)} \operatorname{bd} \Lambda_i^* = \bigcup_{i \in \mathbb{I}(x)} \operatorname{con}\{-e^j | j \in I^i(x)\}.$$

■

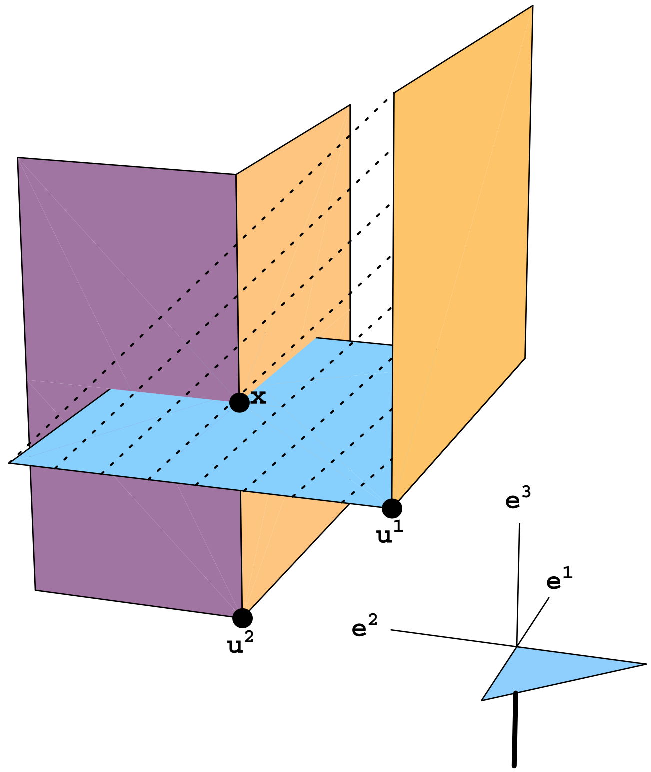
Figure 2 illustrates Proposition 3.7. In the example of the figure, one has that

$$\mathbb{I}(x) = \{1, 2\}; \quad I^1(x) = \{3\}; \quad I^2(x) = \{1, 2\}.$$

Hence, Proposition 3.7 yields that $N_P(x) = \operatorname{con}\{-e^3\} \cup \operatorname{con}\{-e^1, -e^2\}$.

Summarizing the previous sections, both special cases, the half-spaces and the translated orthants provide representations of the normal cone, where the upper estimation in (2) is realized as an equality (see the last lines in the proofs of Prop. 3.2 and Prop. 3.7, respectively). In general, the normal cone may be strictly smaller than the upper estimate due to overlapping parts in the union of polyhedra. To deal with this general situation, we provide in the following sections a precise formula for calculating the normal cone.

Figure 2: Illustration of Prop. 3.7. The point $x$ belongs to the union of the two orthants $u^1 + \mathbb{R}^3_+$, $u^2 + \mathbb{R}^3_+$. The normal cone to this union at $x$ is illustrated as an attachment to the coordinate system. See discussion below.




# 4 The Case of Two Polyhedral Cones

In this section, we start to analyze the normal cone to the union of polyhedral cones without further assumptions on their structure as in the special cases of the previous section. However, since a formula for the limiting cone is difficult to obtain at once for an arbitrary finite union, we focus first on the case of two components and consider the general case in section 6.

So, let $\Lambda := \Lambda_1 \cup \Lambda_2$, where $\Lambda_1, \Lambda_2$ are polyhedral cones. Using the decomposition

$$\Lambda \backslash \{0\} = (\Lambda_1 \backslash \Lambda_2) \cup (\Lambda_2 \backslash \Lambda_1) \cup ((\Lambda_1 \cap \Lambda_2) \backslash \{0\}),$$

we may invoke (1) to obtain the identity

$$N_\Lambda(0) = [\Lambda_1^* \cap \Lambda_2^*] \cup \bigcup_{x \in \Lambda_1 \backslash \Lambda_2} \hat{N}_{\Lambda_1}(x) \cup \bigcup_{x \in \Lambda_2 \backslash \Lambda_1} \hat{N}_{\Lambda_2}(x) \cup \bigcup_{x \in (\Lambda_1 \cap \Lambda_2) \backslash \{0\}} \hat{N}_\Lambda(x). \quad (13)$$

Here, we exploited the fact that, for $x \in \Lambda_1 \backslash \Lambda_2$, the union $\Lambda$ coincides locally around $x$ with $\Lambda_1$, so $\hat{N}_\Lambda(x) = \hat{N}_{\Lambda_1}(x)$, and similarly with $x \in \Lambda_2 \backslash \Lambda_1$. The following observation allows to omit the last contribution in (13):

**Lemma 4.1** *$\hat{N}_\Lambda(x) \subseteq \Lambda_1^* \cap \Lambda_2^*$ for all $x \in (\Lambda_1 \cap \Lambda_2)\backslash\{0\}$.*

**Proof.** Since $x \in \Lambda_1 \cap \Lambda_2$, we may apply the calculus rules for Fréchet normal cones:

$$\hat{N}_\Lambda(x) = \hat{N}_{\Lambda_1 \cup \Lambda_2}(x) = \hat{N}_{\Lambda_1}(x) \cap \hat{N}_{\Lambda_2}(x) \subseteq \Lambda_1^* \cap \Lambda_2^*.$$

■

The lemma allows to reduce (13) to

$$N_\Lambda(0) = [\Lambda_1^* \cap \Lambda_2^*] \cup \bigcup_{x \in \Lambda_1 \backslash \Lambda_2} \hat{N}_{\Lambda_1}(x) \cup \bigcup_{x \in \Lambda_2 \backslash \Lambda_1} \hat{N}_{\Lambda_2}(x). \tag{14}$$

Our aim is to represent $N_\Lambda(0)$ by a formula which merely falls back on the data of the original cones $\Lambda_1$ and $\Lambda_2$. Everything is obvious for the first part in (14), which is just the intersection of the polar cones and, which we would like to refer to as the solid part of $N_\Lambda(0)$. The reason is that, in contrast with the remaining contributions in (13), it is typically of full dimension (see Figure 1 for an illustration though in the context of nonpolyhedral cones). Since the second and third terms in (14) are symmetric, we focus our analysis now on calculating $\hat{N}_{\Lambda_1}(x)$ for $x \in \Lambda_1 \backslash \Lambda_2$. In order to do so, it is convenient to assume an explicit description of $\Lambda_1$ and $\Lambda_2$. Accordingly, let

$$\begin{aligned} \Lambda_1 &= \{x \in \mathbb{R}^n |\, \langle c_i, x \rangle \leq 0 \quad (i = 1, \dots, p)\} \\ \Lambda_2 &= \{x \in \mathbb{R}^n |\, \langle d_j, x \rangle \leq 0 \quad (j = 1, \dots, q)\}. \end{aligned} \tag{15}$$

Then,

$$\Lambda_1 \backslash \Lambda_2 = \bigcup_{j=1,\dots,q} P^j, \tag{16}$$

where

$$P^j := \{x \in \mathbb{R}^n |\, \langle c_i, x \rangle \leq 0 \quad (i = 1, \dots, p), \quad \langle -d_j, x \rangle < 0\}.$$

In the following, we fix an arbitrary index $j \in \{1, \dots, q\}$ and calculate the partial contribution

$$\bigcup_{x \in P^j} \hat{N}_{\Lambda_1}(x) \tag{17}$$

of $P^j$ to the second term in (14). With each $x \in P^j$, we associate the active index set

$$I(x) := \{i \in \{1, \dots, p\} |\, \langle c_i, x \rangle = 0\}.$$

Moreover, we introduce the following two families of index sets:

$$\begin{aligned}\mathcal{I}_1 &:= \{I(x) | x \in P^j\} \\ \mathcal{I}_2 &:= \{I \subseteq \{1, \dots, p\} | d_j \notin \text{span}\,\{c_i | i \in I\} + \text{con}\,\{c_i | i \in I^c\}\}.\end{aligned}$$

Here, $I^c := \{1, \dots, p\} \backslash I$ and, as before, 'con' denotes the conic convex hull whereas 'span' refers to the linear hull.

**Lemma 4.2** *$\mathcal{I}_1 \subseteq \mathcal{I}_2$ and for any $I \in \mathcal{I}_2$, there exists some $x \in P^j$ such that $I \subseteq I(x)$.*

**Proof.** By Motzkin's Theorem, the condition

$$d_j \notin \text{span}\,\{c_i | i \in I\} + \text{con}\,\{c_i | i \in I^c\}$$

is equivalent with the existence of some $x$ such that

$$\langle c_i, x \rangle = 0 \quad (i \in I), \quad \langle c_i, x \rangle \leq 0 \quad (i \in I^c), \quad \langle -d_j, x \rangle < 0.$$

Now, the two assertions of the Lemma follow immediately from the respective definitions of $\mathcal{I}_1$ and $P^j$. ■

**Proposition 4.3** *The partial contribution (17) computes as*

$$\bigcup_{x \in P^j} \hat{N}_{\Lambda_1}(x) = \bigcup_{I \in \mathcal{I}_2} \text{con}\ \{c_i | i \in I\}.$$

**Proof.** First, we use the well-known identity

$$\hat{N}_{\Lambda_1}(x) = \text{con}\ \{c_i | i \in I(x)\}, \tag{18}$$

which holds true for all $x \in \Lambda_1$ and so for all $x \in P^j$. Thus,

$$\bigcup_{x \in P^j} \hat{N}_{\Lambda_1}(x) = \bigcup_{x \in P^j} \text{con}\ \{c_i | i \in I(x)\} = \bigcup_{I \in \mathcal{I}_1} \text{con}\ \{c_i | i \in I\} \subseteq \bigcup_{I \in \mathcal{I}_2} \text{con}\ \{c_i | i \in I\},$$

where the last inclusion follows from the first statement of Lemma 4.2. On the other hand, let

$$h \in \bigcup_{I \in \mathcal{I}_2} \text{con}\ \{c_i | i \in I\}$$

be arbitrary, so $h \in \text{con}\ \{c_i | i \in I\}$ for some $I \in \mathcal{I}_2$. From the second statement of Lemma 4.2, we derive the existence of some $x \in P^j$ such that $I \subseteq I(x)$. Then, (18) implies that

$$\text{con}\ \{c_i | i \in I\} \subseteq \text{con}\ \{c_i | i \in I(x)\} = \hat{N}_{\Lambda_1}(x),$$

whence $h \in \hat{N}_{\Lambda_1}(x)$. This establishes the reverse inclusion

$$\bigcup_{I \in \mathcal{I}_2} \text{con}\ \{c_i | i \in I\} \subseteq \bigcup_{x \in P^j} \hat{N}_{\Lambda_1}(x).$$

■

**Theorem 4.4** *The limiting normal cone to the union $\Lambda$ of two polyhedral cones $\Lambda_1$ and $\Lambda_2$ may be represented by the formula*

$$N_\Lambda(0) = [\Lambda_1^* \cap \Lambda_2^*] \cup \bigcup_{j=1}^{q} \bigcup_{I \in \mathcal{A}_j} \text{con }\{c_i | i \in I\} \cup \bigcup_{i=1}^{p} \bigcup_{J \in \mathcal{B}_i} \text{con }\{d_j | j \in J\},$$

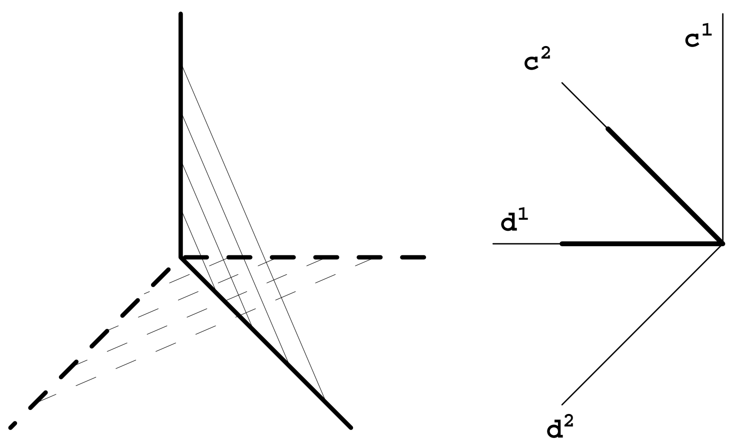
*where*

$$\begin{aligned} \mathcal{A}_j \ &: \ = \{I \subseteq \{1, \dots, p\} | d_j \notin \text{span}\,\{c_i | i \in I\} + \text{con}\,\{c_i | i \in I^c\}\} \\ \mathcal{B}_i \ &: \ = \{J \subseteq \{1, \dots, q\} | c_i \notin \text{span}\,\{d_j | i \in J\} + \text{con}\,\{d_j | i \in J^c\}\}. \end{aligned}$$

**Proof.** The result follows directly from Proposition 4.3 upon aggregating the contributions of all $P^j$ via (16), exchanging the role of $\Lambda_1$ and $\Lambda_2$ in (16) and applying (14).

■

Figure 3: Illustration of Theorem 4.4.




The following example illustrates Theorem 4.4:

**Example 4.5** *Consider the union $\Lambda := \Lambda_1 \cup \Lambda_2 \subseteq \mathbb{R}^2$, where $\Lambda_1$ and $\Lambda_2$ are described according to (15) by*

$$c_1 = (0,1), \quad c_2 = (-1,1), \quad d_1 = (-1,0), \quad d_2 = (-1,-1)$$

*(see Figure 3). Then,*

$$\Lambda_1^* = \text{con}\,\{c_1, c_2\}, \quad \Lambda_2^* = \text{con}\,\{d_1, d_2\}.$$

*The upper estimate (2) provides the cone*

$$[\Lambda_1^* \cap \Lambda_2^*] \cup \text{bd}\,\Lambda_1^* \cup \text{bd}\,\Lambda_2^* = \mathbb{R}_+(\{c_1\} \cup \{c_2\} \cup \{d_1\} \cup \{d_2\})$$

*which is the union of the thin half-rays in the right part of Figure 3. On the other hand, one immediately identifies the normal cone from the figure as*

$$N_\Lambda(0) = \mathbb{R}_+(\{\{c_2\} \cup \{d_1\})$$

*which is the union of the thick half-rays in the right part of Figure 3. This example is an instance for a situation where the normal cone is strictly smaller than the upper estimate (2). None of the special cases from Section 3 applies in this situation and actually cannot apply due to the normal cones coinciding with the upper estimate in all of these cases. On the other hand, one may use Theorem 4.4 here. For instance, from the relations*

$$\begin{array}{rcl} d_1, d_2 & \in & \operatorname{span}\{c_1\} + \operatorname{con}\{c_2\}; \quad d_1, d_2 \notin \operatorname{span}\{c_2\} + \operatorname{con}\{c_1\}; \\ d_1, d_2 & \in & \operatorname{span}\{c_1\} + \operatorname{span}\{c_2\}; \quad d_1, d_2 \notin \operatorname{con}\{c_1\} + \operatorname{con}\{c_2\} \end{array}$$

*one derives that* $\mathcal{A}_1 = \mathcal{A}_2 = \{\{2\}, \varnothing\}$*. Similarly,* $\mathcal{B}_1 = \mathcal{B}_2 = \{\{1\}, \varnothing\}$*. Since* $\Lambda_1^* \cap \Lambda_2^* = \{0\}$ *(i.e., the solid part vanishes, see Figure 3), Theorem 4.4 provides*

$$N_\Lambda(0) = \operatorname{con}\{c_2\} \cup \operatorname{con}\{d_1\} = \mathbb{R}_+(\{\{c_2\} \cup \{d_1\}).$$

# 5 An Algorithm for the Numerical Calculation

The formula provided by Theorem 4.4 may be used for a numerical calculation. Leaving aside the solid part, all one has to do is to determine the index sets $\mathcal{A}_j$ and $\mathcal{B}_i$. For the $\mathcal{A}_j$, this amounts to checking the relations

$$d_j \notin \operatorname{span}\{c_i | i \in I\} + \operatorname{con}\{c_i | i \in I^c\}$$

for all $j \in \{1, \dots, q\}$ and all subsets $I \subseteq \{1, \dots, p\}$ (analogously for the $\mathcal{B}_i$). This can be done numerically, for instance, via solving the linear program

$$\max\{\langle d_j, x\rangle \,|\, \langle c_i, x\rangle = 0 \quad (i \in I), \quad \langle c_i, x\rangle \le 0 \quad (i \in I^c), \quad x \in [-1, 1]^n\}$$

(see the equivalence in the proof of Lemma 4.2). The relation above will be satisfied whenever the optimal value of this program is strictly positive, otherwise it will be violated. As an illustration, we determine the normal cone to the union of the two cones $\Lambda_1 = \{x \in \mathbb{R}^5 | Cx \le 0\}$ and $\Lambda_2 = \{x \in \mathbb{R}^5 | Dx \le 0\}$, where $C$ and $D$ were randomly generated as:

$$C = \begin{pmatrix} 0.11 & 0.77 & -0.74 & 0.06 & -0.51 \\ -0.11 & -0.51 & 0.82 & -0.42 & -0.10 \\ -0.45 & 0.99 & 0.33 & 0.48 & 0.61 \\ 0.36 & -0.72 & -0.47 & 0.95 & -0.87 \\ -0.05 & -0.68 & 0.19 & 0.12 & 0.84 \\ -0.46 & -0.06 & -0.94 & 0.35 & 0.65 \end{pmatrix} \quad D = \begin{pmatrix} -0.55 & -0.76 & -0.22 & -0.24 & 0.90 \\ -0.75 & 0.44 & 0.28 & -0.71 & -0.11 \\ 0.17 & -0.25 & -0.66 & -0.24 & -0.79 \\ -0.57 & 0.15 & 0.64 & -0.63 & 0.89 \\ -0.79 & 0.58 & 0.02 & -0.76 & 0.77 \\ 0.35 & -0.76 & 0.49 & 0.87 & 0.10 \end{pmatrix}.$$

Applying the procedure described above and leaving aside the obvious solid part, we get the following two contributions to the normal cone:

Contribution by $\Lambda_1 \setminus \Lambda_2$ :
$\text{con}\,\{c2, c3, c4, c5\} \cup \text{con}\,\{c1, c3, c4, c5\} \cup \text{con}\,\{c1, c2, c4, c5\}$
$\cup\text{con}\,\{c1, c2, c3, c5\} \cup \text{con}\,\{c1, c2, c3, c4\}$

Contribution by $\Lambda_2 \setminus \Lambda_1$ :
$\text{con}\,\{d2, d3, d4, d6\} \cup \text{con}\,\{d1, d3, d4, d6\} \cup \text{con}\,\{d2, d4, d5, d6\}$
$\cup\text{con}\,\{d2, d3, d5, d6\} \cup \text{con}\,\{d2, d3, d4, d5\} \cup \text{con}\,\{d1, d4, d5, d6\}$
$\cup\text{con}\,\{d1, d3, d5, d6\} \cup \text{con}\,\{d1, d3, d4, d5\}$

It has to be noted that the formula in Theorem 4.4 does not generate minimal representations of the normal cone because certain parts may be contained in others and some part may appear in copies. The representation given in the example above is minimal.

Although, the example demonstrates that the numerical calculation of the normal cone is possible in principle, it becomes quickly inefficient when the number of inequalities describing the two cones exceeds ten or so. The reason is that the index sets $\mathcal{A}_j$ and $\mathcal{B}_i$ are determined by checking all subsets of $\{1, \ldots, p\}$ and $\{1, \ldots, q\}$, respectively. In the following we derive a more efficient procedure, where only possibly small subsets have to be checked. As in the previous section, by symmetry, we may restrict our considerations to the contribution (16).

We introduce the following selection operator $\sigma : [\mathbb{R}\backslash\{0\}]^{m+1} \rightrightarrows [\mathbb{R}\backslash\{0\}]^m$ for $m+1$ nonzero vectors by

$$\sigma\left(\{v_1, \ldots, v_m, w\}\right) = \{v_i | \mathbb{R}_+ v_i \text{ is an extremal ray of } \text{con}\,\{v_1, \ldots, v_m, w\}\}.$$

Fix an arbitrary $j \in \{1, \ldots, q\}$. The selection operator $\sigma$ provides a partition of the total index set $\{1, \ldots, p\}$ into

$$I_1 := \{i \in \{1, \ldots, p\} | c_i \in \sigma\left(\{c_1, \ldots, c_p, -d_j\}\right)\}, \quad I_2 := \{1, \ldots, p\}\backslash I_1. \tag{19}$$

Note that, by definition of $\sigma$ and $I_1$, one always has that

$$c_k \in \text{con}\,\left[\{c_i | i \in I_1\} \cup \{-d_j\}\right] \quad \forall k \in \{1, \ldots, p\}. \tag{20}$$

Finally, as a refinement to the argumentation in the previous section, we introduce the index set

$$\mathcal{I}_3 := \{I \subseteq I_1 | d_j \notin \text{span}\,\{c_i | i \in I\} + \text{con}\,\{c_i | i \in I_1\backslash I\}\}.$$

Clearly, $\mathcal{I}_3 \subseteq \mathcal{I}_2$ (compare definition of $\mathcal{I}_2$ in the previous section). Now, generalizing Lemma 4.2, we get the following result:

**Lemma 5.1** *For any $I \in \mathcal{I}_3$, there exists some $x \in P^j$ such that $I \subseteq I(x)$. Moreover, if the set $\{c_1, \dots, c_p\}$ is positive linearly independent and is a minimal set describing $\Lambda_1$, then, $\mathcal{I}_1 \subseteq \mathcal{I}_3$.*

**Proof.** The first statement follows immediately from Lemma 4.2 along with the inclusion $\mathcal{I}_3 \subseteq \mathcal{I}_2$. Concerning the second statement, let $I \in \mathcal{I}_1$ be arbitrary. By definition, there is some $x \in P^j$ such that $I = I(x)$. We show first that $I_2 \cap I = \varnothing$ for $I_2$ introduced in (19). Choose an arbitrary $i^* \in I_2$. From (20), one gets that

$$c_{i^*} \in \text{con}\, [\{c_i | i \in I_1\} \cup \{-d_j\}] .$$

From Gordan's Lemma, it follows that the strict inequality system

$$\langle c_i, u \rangle < 0 \quad (i \in I_1), \quad \langle -d_j, u \rangle < 0, \quad \langle c_{i^*}, u \rangle > 0 \tag{21}$$

has no solution. On the other hand, from $x \in P^j$, we know that

$$\langle c_i, x \rangle \leq 0 \quad (i \in I_1), \quad \langle -d_j, x \rangle < 0, \quad \langle c_{i^*}, x \rangle \leq 0. \tag{22}$$

We claim that the equality

$$0 = \sum_{i \in I_1} \lambda_i c_i - \lambda_0 c_{i^*}. \tag{23}$$

does not hold for coefficients $\lambda_i \geq 0$ $(i \in I_1 \cup \{0\})$ other than the trivial one. Indeed, if $\lambda_0 = 0$, then

$$\sum_{i \in I_1} \lambda_i c_i = 0,$$

where not all of the $\lambda_i$'s vanish. This would mean that the set $\{c_i | i \in I_1\}$ is positive linearly independent, and much more this holds true for the larger set $\{c_1, \dots, c_p\}$, whence a contradiction with the first assumption of our Proposition. On the other hand, if $\lambda_0 \neq 0$, then $c_{i^*} \in \text{con}\, \{c_i | i \in I_1\}$ which contradicts the second assumption of our Proposition (the inequality $\langle c_{i^*}, y \rangle \leq 0$ would follow then from the inequalities $\langle c_i, y \rangle \leq 0$ for $i \in I_1$, and, since $i^* \notin I_1$, this would allow to delete $c_{i^*}$ from the set of vectors $\{c_1, \dots, c_p\}$ describing $\Lambda_1$). The stated nonexistence of a relation (23) with nonnegative coefficients, not all of them being zero, allows to apply Gordan's Lemma once more and to derive the existence of some $\xi$ satisfying the strict inequality system

$$\langle c_i, \xi \rangle < 0 \quad (i \in I_1), \quad \langle c_{i^*}, \xi \rangle > 0. \tag{24}$$

We put

$$x^t := t\xi + (1-t)x \quad (t \geq 0)$$

and obtain from (22) along with our assumption $\langle c_{i^*}, x \rangle = 0$, that, for sufficiently small $t > 0$,

$$\langle c_i, x^t \rangle < 0 \quad (i \in I_1), \quad \langle -d_j, x^t \rangle < 0, \quad \langle c_{i^*}, x^t \rangle > (1-t) \langle c_{i^*}, x \rangle .$$

Now, if $i^* \in I$, then $\langle c_{i^*}, x\rangle = 0$ and $\langle c_{i^*}, x^t\rangle > 0$. This, however, contradicts our observation that (21) has no solution. Therefore, the assumption $i^* \in I$ must be wrong, so $I_2 \cap I = \varnothing$ and, thus, $I \subseteq I_1$. Moreover, observe that $x \in P^j$ and $I = I(x)$ imply that the system

$$\langle c_i, x\rangle = 0 \quad (i \in I), \quad \langle c_i, x\rangle \leq 0 \quad (i \in I_1 \backslash I), \quad \langle -d_j, x\rangle < 0$$

has a solution. By Motzkin's Theorem, it follows that

$$d_j \notin \text{span}\,\{c_i | i \in I\} + \text{con}\,\{c_i | i \in I_1 \backslash I\}.$$

Summarizing, $I \in \mathcal{I}_3$. ■

Now, using the stronger statement of Lemma 5.1 rather than that of Lemma 4.2 in the proof of Proposition 4.3, we may replace the index set $\mathcal{I}_2$ there by the smaller index set $\mathcal{I}_3$. From here, we derive a refined formula for the normal cone as compared to Theorem 4.4. In the statement of the result, we have to take care about the fact that the index set $I_1$ actually depends on $j$ (which has been arbitrarily fixed before).

**Theorem 5.2** *If the sets $\{c_1, \ldots, c_p\}$ and $\{d_1, \ldots, d_q\}$ in (15) are positive linearly independent and are minimal sets describing $\Lambda_1$ and $\Lambda_2$, respectively, then the limiting normal cone to the union $\Lambda = \Lambda_1 \cup \Lambda_2$ may be represented by the formula*

$$N_\Lambda(0) = [\Lambda_1^* \cap \Lambda_2^*] \cup \bigcup_{j=1}^{q} \bigcup_{I \in \mathcal{A}_j^*} \text{con}\ \{c_i | i \in I\} \cup \bigcup_{i=1}^{p} \bigcup_{J \in \mathcal{B}_i^*} \text{con}\ \{d_j | j \in J\},$$

*where*

$$\begin{aligned}
\mathcal{A}_j^* \ &: \ = \{I \subseteq I_1(j) | d_j \notin \text{span}\,\{c_i | i \in I\} + \text{con}\,\{c_i | i \in I_1(j) \backslash I\}\} \\
\mathcal{B}_i^* \ &: \ = \{J \subseteq J_1(i) | c_i \notin \text{span}\,\{d_j | i \in J\} + \text{con}\,\{d_j | i \in J_1(i) \backslash J\}\} \\
I_1(j) \ &: \ = \{i \in \{1, \ldots, p\} | c_i \in \sigma(\{c_1, \ldots, c_p, -d_j\})\} \quad (j \in \{1, \ldots, q\}) \\
J_1(i) \ &: \ = \{i \in \{1, \ldots, p\} | d_j \in \sigma(\{d_1, \ldots, d_q, -c_i\})\} \quad (i \in \{1, \ldots, p\}).
\end{aligned}$$

The advantage of Theorem 5.1 over Theorem 4.4 is that the index sets $\mathcal{A}_j^*$ and $\mathcal{B}_i^*$ are (possibly much) smaller than the original index sets $\mathcal{A}_j$ and $\mathcal{B}_i$. This comes at the price of requiring positive linear independence for the $c_i$ and $d_j$. However, this additional assumption will be satisfied as long as $\Lambda_1$ and $\Lambda_2$ do not contain nontrivial vector subspaces. Theorem 5.1 suggests the following algorithm for calculating the normal cone:

**Algorithm 5.3** *Given positive linearly independent sets $\{c_1, \ldots, c_p\}$, $\{d_1, \ldots, d_q\}$ in (15), determine the contribution of $\Lambda_1 \setminus \Lambda_2$ to the normal cone as follows (and the contribution of $\Lambda_2 \setminus \Lambda_1$ by symmetry):*

1. *Eliminate those vectors $c_i$ from the set $\{c_1, \dots, c_p\}$, which are not extremal rays of* $\text{con}\,\{c_1, \dots, c_p\}$. *Doing so, redundant $c_i$ will be removed from the description (15). We assume now, that $\{c_1, \dots, c_p\}$ is free of redundance.*

2. *Put $j := 0$ and $M := \varnothing$.*

3. *Put $j := j + 1$. Determine the set $E$ of extremal rays in* $\text{con}\,\{c_1, \dots, c_p, -d_j\}$ *and put $I_1 := \{i \in \{1, \dots, p\} | c_i \in E\}$. Put $S := \{\varnothing\}$.*

4. *Select $I \in 2^{I_1} \backslash S$ and put $S := S \cup \{I\}$.*

5. *Solve the linear program*

$$\alpha := \max\{\langle d_j, x\rangle \,|\, \langle c_i, x\rangle = 0 \ (i \in I_1),\ \langle c_i, x\rangle \le 0 \ (i \in I_1 \backslash I),\ x \in [-1,1]^n\}.$$

6. *If $\alpha > 0$, then $M := M \cup \text{con}\,\{c_i | i \in I\}$.*

7. *If $S \neq 2^{I_1}$ then go to 4.*

8. *If $j < p$ then go to 3.*

9. *Select maximal elements in the union $M$ to obtain a union $\tilde{M}$ which is free of redundance. $\tilde{M}$ is the desired contribution of $\Lambda_1 \setminus \Lambda_2$ to the normal cone.*

The set $S$ acting in Steps 3 to 7 of the algorithm serves to select all possible subsets of $I_1$ which are then checked in Step 5 for the defining relation of the index family $\mathcal{A}_j^*$ introduced in the statement of Theorem 5.2. Evidently, all which is needed to realize this algorithm are codes to solve a linear program and to find the extremal rays in a finitely generated cone. For the latter problem, one may use, for instance, the Fukuda's code 'cdd' (cf. [3]).

# 6 The Case of $N$ Polyhedral Cones

We consider now a general finite union

$$\Lambda := \bigcup_{i=1}^{N} \Lambda_i \tag{25}$$

of polyhedral cones $\Lambda_i$. Generalizing the ideas of Section 4, we partition $\Lambda$ as

$$\Lambda = \bigcup_{\mathbb{I} \subseteq \{1,\dots,N\}, \mathbb{I} \neq \varnothing} \left[\Lambda^{\mathbb{I}} \backslash \Lambda_{\mathbb{I}^c}\right],$$

where, for arbitrary $\mathbb{I} \subseteq \{1,\dots,N\}, \mathbb{I} \neq \varnothing$, we make use of the concise notation

$$\Lambda^{\mathbb{I}} := \bigcap_{i\in\mathbb{I}} \Lambda_i, \quad \Lambda_{\mathbb{I}} := \bigcup_{i\in\mathbb{I}} \Lambda_i, \quad \mathbb{I}^c := \{1,\dots,N\}\backslash\mathbb{I}$$

and adopt the convention $\Lambda_\varnothing := \varnothing$. Now, we develop the first equality in (1) as

$$N_\Lambda(0) = \bigcup_{x\in\Lambda} \hat{N}_\Lambda(x) = \bigcup_{\varnothing\neq\mathbb{I}\subseteq\{1,\dots,N\}} \bigcup_{x\in\Lambda^{\mathbb{I}}\backslash\Lambda_{\mathbb{I}^c}} \bigcap_{k\in\mathbb{I}} \hat{N}_{\Lambda_k}(x). \tag{26}$$

Here, the second equality follows from the fact that, for $x \in \Lambda^{\mathbb{I}}\backslash\Lambda_{\mathbb{I}^c}$, $\Lambda$ coincides, locally around $x$, with $\Lambda_{\mathbb{I}}$, whence

$$\hat{N}_\Lambda(x) = \hat{N}_{\Lambda_{\mathbb{I}}}(x) = \bigcap_{k\in\mathbb{I}} \hat{N}_{\Lambda_k}(x)$$

due to $x \in \cap_{k\in\mathbb{I}}\Lambda_k$. It is convenient, to assume now an explicit description of the polyhedral cones:

$$\Lambda_i = \left\{x \in \mathbb{R}^n \,\middle|\, \left\langle c_j^{(i)}, x\right\rangle \leq 0 \quad (j = 1,\dots,n_i)\right\} \quad (i = 1,\dots,N).$$

For $\mathbb{I} \subseteq \{1,\dots,N\}$, we introduce the following cartesian product of index sets:

$$\mathcal{J}_{\mathbb{I}} := \prod_{i\in\mathbb{I}} \{1,\dots,n_i\}.$$

For any integer vector $J = (J_1,\dots,J_{|\mathbb{I}^c|}) \in \mathcal{J}_{\mathbb{I}^c}$, we put

$$P_{\mathbb{I}}^J := \left\{x \in \mathbb{R}^n \,\middle|\, \left\langle c_j^{(i)}, x\right\rangle \leq 0 \quad \forall j \in \{1,\dots,n_i\}\ \forall i \in \mathbb{I}; \quad \left\langle c_{J_i}^{(i)}, x\right\rangle > 0 \quad \forall i \in \mathbb{I}^c\right\}.$$

From the very definitions, it follows that

$$\Lambda^{\mathbb{I}}\backslash\Lambda_{\mathbb{I}^c} = \bigcup_{J\in\mathcal{J}_{\mathbb{I}^c}} P_{\mathbb{I}}^J.$$

Consequently, one may continue (26) as

$$N_\Lambda(0) = \bigcup_{\varnothing\neq\mathbb{I}\subseteq\{1,\dots,N\}} \bigcup_{J\in\mathcal{J}_{\mathbb{I}^c}} \bigcup_{x\in P_{\mathbb{I}}^J} \bigcap_{k\in\mathbb{I}} \hat{N}_{\Lambda_k}(x). \tag{27}$$

Our goal is to get rid of any dependence on $x$ in the formula for $N_\Lambda(0)$ in a way that only the describing data for the $\Lambda_i$ remain there. Obviously, for each $x \in P_{\mathbb{I}}^J$, there exist subsets $\mathcal{J}_{x,i} \subseteq \{1,\dots,n_i\}$ for $(i = 1,\dots,|\mathbb{I}|)$, such that

$$\begin{aligned} \left\langle c_j^{(i)}, x\right\rangle &= 0 \quad \forall j \in \mathcal{J}_{x,i}\, \forall i \in \mathbb{I}; \\ \left\langle c_j^{(i)}, x\right\rangle &< 0 \quad \forall j \in \{1,\dots,n_i\} \setminus \mathcal{J}_{x,i}\, \forall i \in \mathbb{I}; \\ \left\langle c_{J_i}^{(i)}, x\right\rangle &> 0 \quad \forall i \in \mathbb{I}^c. \end{aligned} \tag{28}$$

For such $x$ and a fixed $k$, one has

$$\hat{N}_{\Lambda_k}(x) = \text{con}\,\left\{c_j^{(k)}\,|\,j \in \mathcal{J}_{x,k}\right\}.$$

Note that, by convention, $\text{con}\,\varnothing = \{0\}$. For any subset $\mathcal{J} = \prod_{i\in\mathbb{I}} \mathcal{J}_i \subseteq \mathcal{J}_\mathbb{I}$, we put

$$\begin{aligned} R_\mathbb{I}^{\mathcal{J},J} \ &: = \text{con}\,\left\{\left\{c_{J_i}^{(i)}\,|\,i \in \mathbb{I}^c\right\} \cup \left\{-c_j^{(i)}\,|i \in \mathbb{I},\, j \in \{1,\dots,n_i\} \setminus \mathcal{J}_i\right\}\right\} \\ S_\mathbb{I}^{\mathcal{J}} \ &: = \text{span}\,\left\{c_j^{(i)}\,|i \in \mathbb{I},\, j \in \mathcal{J}_i\right\}. \end{aligned}$$

The solvability of (28) is equivalent, via Motzkin's Theorem, to the condition

$$R_\mathbb{I}^{\mathcal{J}_x,J} \cap S_\mathbb{I}^{\mathcal{J}_x} = \{0\},$$

where $\mathcal{J}_x = \prod_{i\in\mathbb{I}} \mathcal{J}_{x,i}$. We have shown that $x \in P_\mathbb{I}^J$ if and only if there exists some $\mathcal{J} = \prod_{i\in\mathbb{I}} \mathcal{J}_i \subseteq \mathcal{J}_\mathbb{I}$ such that $R_\mathbb{I}^{\mathcal{J},J} \cap S_\mathbb{I}^{\mathcal{J}} = \{0\}$. Therefore, given $\mathbb{I} \subseteq \{1,\dots,N\}, \mathbb{I} \neq \varnothing$ and $J = (J_1,\dots,J_{|\mathbb{I}^c|}) \in \mathcal{J}_{\mathbb{I}^c}$, we may write

$$\bigcup_{x\in P_\mathbb{I}^J}\bigcap_{k\in\mathbb{I}} \hat{N}_{\Lambda_k}(x) = \bigcup_{\mathcal{J}\in\mathcal{A}_\mathbb{I}^J}\bigcap_{k\in\mathbb{I}} \text{con}\,\left\{c_j^{(k)}\,|\,j \in \mathcal{J}_k\right\},$$

where

$$\mathcal{A}_\mathbb{I}^J = \{\mathcal{J} \subseteq \mathcal{J}_\mathbb{I} | R_\mathbb{I}^{\mathcal{J},J} \cap S_\mathbb{I}^{\mathcal{J}} = \{0\}\}.$$

Combining this with (27), we may state the desired formula for the normal cone just in terms of the data describing the polyhedral cones $\Lambda_i$ in the following

**Theorem 6.1** *The limiting normal cone to a finite union of polyhedral cones as in (25) calculates as (for notation see text above)*

$$N_\Lambda(0) = \bigcup_{\varnothing\neq\mathbb{I}\subseteq\{1,\dots,N\}}\ \bigcup_{J\in\mathcal{J}_{\mathbb{I}^c}}\ \bigcup_{\mathcal{J}\in\mathcal{A}_\mathbb{I}^J}\bigcap_{k\in\mathbb{I}} \text{con}\,\left\{c_j^{(k)}\,|\,j \in \mathcal{J}_k\right\}.$$

# 7 An Application

Consider a constraint set mapping $\Gamma : \mathbb{R}^k \rightrightarrows \mathbb{R}^m$ defined by

$$\Gamma(p) := \bigcup_{i=1}^N \{y \in \mathbb{R}^m | A^i f(p,y) \leq b^i\}, \tag{29}$$

where $f : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable, and, for $i = 1,\dots,N$, $A^i$ are matrices of order $(l_i, n)$ and $b^i \in \mathbb{R}^{l_i}$.

Let $\bar{y} \in \Gamma(\bar{p})$. We want to examine the Aubin property (see [10]) of $\Gamma$ around $(\bar{p},\bar{y})$.

**Proposition 7.1** *Assume that* $\nabla f(\bar{p},\bar{y})$ *is surjective and define*

$$P_i := \{x \in \mathbb{R}^n | A^i x \leq b^i\} \quad (i = 1, \ldots, N); \quad P := \bigcup_{i=1}^{N} P_i.$$

*Then,* $\Gamma$ *has the Aubin property around* $(\bar{p},\bar{y})$ *if and only if the implication*

$$\begin{pmatrix} p^* \\ 0 \end{pmatrix} \in \nabla f(\bar{p},\bar{y})^T N_P(f(\bar{p},\bar{y})) \Longrightarrow p^* = 0 \tag{30}$$

*holds true.*

**Proof.** Clearly, $\operatorname{gph}\Gamma = \{(p,y) \in \mathbb{R}^k \times \mathbb{R}^m | f(p,y) \in P\}$. By virtue of [10, Exercise 8.14 and Exercise 10.7],

$$N_{\operatorname{gph}\Gamma}(\bar{p},\bar{y}) = \left\{ (p^*,y^*) \in \mathbb{R}^k \times \mathbb{R}^m \,\middle|\, \begin{pmatrix} p^* \\ y^* \end{pmatrix} \in \nabla f(\bar{p},\bar{y})^T N_P(f(\bar{p},\bar{y})) \right\}.$$

It suffices now to apply the Mordukhovich criterion ([10, Theorem 9.40]) to arrive at the implication (30). ■

As an application of Proposition 7.1, we consider a linear probabilistic constraint as it arises in stochastic optimization problems:

$$\mathbb{P}(Ty \geq \xi) \geq \alpha. \tag{31}$$

Here, $\xi$ denotes an $s$-dimensional random vector and $\mathbb{P}$ is a probability measure. The meaning of (31) is that a decision vector $y$ is declared to be feasible, if the stochastic inequality system $Ty \geq \xi$ is satisfied with a probability not smaller than $\alpha \in [0,1]$. Assuming that $\xi$ has a discrete distribution, one can show (see Remark 1 in [1]) that there exists a finite number, say $N$, of points $q^i$ such that (31) can be equivalently rewritten as

$$Ty \in \bigcup_{i=1}^{N} \left(\{q^i\} + \mathbb{R}^s_+\right).$$

Assume that we want to add a deterministic safety buffer of magnitude $\bar{p} > 0$ to the inequality system $Ty \geq \xi$. Then, (31) will be replaced by

$$\mathbb{P}(Ty \geq \xi + \bar{p}\mathbf{1}) \geq \alpha,$$

where $\mathbf{1} = (1, \ldots, 1)$. The meaning of this modified constraint is that a decision $y$ is feasible if $Ty$ over-dominates the random vector $\xi$ by a value of at least $\bar{p}$ at a probability of at least $\alpha$. As the choice of an appropriate value for $\bar{p}$ may be arbitrary, it may be interesting to know, how the set of feasible decisions $y$ changes upon perturbations of a nominal value $\bar{p}$. Passing to the equivalent description of the probabilistic constraint presented above, but now adding the dependence on some variable safety buffer $p$, we might be led to investigate the Aubin property of the mapping

$$\Gamma(p) := \left\{ y \,\middle|\, Ty - p\mathbf{1} \in \bigcup_{i=1}^{N} \left(\{q^i\} + \mathbb{R}^s_+\right) \right\},$$

at a point $(\bar{p}, \bar{y})$, where $\bar{y}$ is feasible for the nominal buffer $\bar{p}$. Obviously, we are in the setting of Proposition 7.1 by putting (for $i = 1, \ldots, N$)

$$f(p,y) := Ty - p\mathbf{1}; \quad A^i := -I; \quad b^i := -q^i; \quad P_i := \{q^i\} + \mathbb{R}^s_+.$$

If $T$ is surjective, then $\nabla f(\bar{p}, \bar{y}) = (-\mathbf{1}, T)$ is surjective too and Proposition 7.1 may be invoked to derive that the Aubin property around $(\bar{p}, \bar{y})$ of the considered contraint set mapping is equivalent with the constraint qualification (30) which turns out to be always satisfied. Indeed, if

$$\begin{pmatrix} p^* \\ 0 \end{pmatrix} \in \nabla f(\bar{p}, \bar{y})^T \, N_P\left(f(\bar{p}, \bar{y})\right),$$

then there exists some $z^* \in N_P\left(f(\bar{p}, \bar{y})\right)$ with $T^T z^* = 0$ and $p^* = -\mathbf{1}^T z^*$. Surjectivity of $T$ implies that $z^* = 0$ and, thus, $p^* = 0$.

However, in many situations, $T$ may fail to have full rank (e.g., in stochastic network design problems, where the number of inequalities may be substantially larger than the dimension of the decision vector, see [8]). Then, an application of Proposition 7.1 is not possible in the formulation chosen before because $\nabla f(\bar{p}, \bar{y}) = (-\mathbf{1}, T)$ may not be surjective either. Fortunately, we can find another description for the same constraint set mapping by putting (for $i = 1, \ldots, N$)

$$f(p,y) := (p,y); \quad A^i := (\mathbf{1}, -T); \quad b^i := -q^i; \quad P_i := \{(x,t) | Tx - t\mathbf{1} \geq q^i\}.$$

As surjectivity of $\nabla f(\bar{p}, \bar{y}) = I$ is always satisfied, we may invoke once more Proposition 7.1 to derive that the Aubin property around $(\bar{p}, \bar{y})$ of the considered contraint set mapping is equivalent with the constraint qualification

$$\begin{pmatrix} p^* \\ 0 \end{pmatrix} \in N_P(\bar{p}, \bar{y}) \Longrightarrow p^* = 0.$$

In order to check this relation, one has to be able to calculate the normal cone to the finite union $P$ of the polyhedra $P_i$.

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