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Two-scale homogenization for evolutionary variational inequalities via the energetic formulation

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Abstract

This paper is devoted to the two-scale homogenization for a class of rate-independent systems described by the energetic formulation or equivalently by an evolutionary variational inequality. In particular, we treat the classical model of linearized elastoplasticity with hardening. The associated nonlinear partial differential inclusion has periodically oscillating coefficients, and the aim is to find a limit problem for the case that the period tends to 0.

Our approach is based on the notion of *energetic solutions* which is phrased in terms of a stability condition and an energy balance of an energy-storage functional and a dissipation functional. Using the recently developed method of *weak and strong two-scale convergence* via *periodic unfolding*, we show that these two functionals have a suitable two-scale limit, but now involving the macroscopic variable in the physical domain as well as the microscopic variable in the periodicity cell. Moreover, relying on an abstract theory of Γ-convergence for the energetic formulation using so-called *joint recovery sequences* it is possible to show that the solutions of the problem with periodicity converge to the energetic solution associated with the limit functionals.

1 Introduction

Our aim is to provide homogenization results for evolutionary variational inequalities of the type:

$$\forall\, v \in \mathcal{Q}: \ \langle \mathcal{A}q - \ell(t), v - \dot{q} \rangle + \mathcal{R}(v) - \mathcal{R}(\dot{q}) \geq 0. \tag{1.1}$$

Here $\mathcal{Q}$ is a Hilbert space with dual $\mathcal{Q}^*$, the continuous linear operator $\mathcal{A} : \mathcal{Q} \to \mathcal{Q}^*$ is symmetric and positive definite on the cone on which $\mathcal{R}$ is finite. The forcing ℓ lies in $\mathrm{C}^1([0,T], \mathcal{Q}^*)$, and the dissipation functional $\mathcal{R} : \mathcal{Q} \to [0, \infty)$ is convex, lower semicontinuous and positively homogeneous of degree 1, i.e., $\mathcal{R}(\gamma q) = \gamma \mathcal{R}(q)$ for all $\gamma \geq 0$ and $q \in \mathcal{Q}$. The latter property of $\mathcal{R}$ leads to rate independence.

Problem (1.1) has many different equivalent formulations. For our purposes the *energetic formulation* for rate-independent hysteresis problems is especially appropriate, cf. [MT99, MT04, Mie05]. This formulation is solely based on the energy-storage functional $\mathcal{E} : [0,T] \times \mathcal{Q} \to \mathbb{R}$ defined via $\mathcal{E}(t,q) = \frac{1}{2}\langle \mathcal{A}q, q\rangle - \langle \ell(t), q\rangle$ and the dissipation functional $\mathcal{R}$. Thus, homogenization of an evolutionary problem can be reduced to some extent to the homogenization of functionals. A function $q : [0,T] \to \mathcal{Q}$ is called an *energetic solution* associated with the functionals $\mathcal{E}$ and $\mathcal{R}$, if for all $t \in [0,T]$ it satisfies the *global stability condition* (S) and the *energy balance* (E):

$$\text{(S)} \quad \forall\, q \in \mathcal{Q}: \ \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, q) + \mathcal{R}(q - q(t));$$

$$\text{(E)} \quad \mathcal{E}(t, q(t)) + \textstyle\int_t^0 \mathcal{R}(\dot{q}(s))\, \mathrm{d}t = \mathcal{E}(0, q(0)) - \int_0^t \langle \dot{\ell}(s), q(s)\rangle\, \mathrm{d}s.$$

We also say that q solves the *energetic formulation* (S) & (E) associated with $\mathcal{E}$ and $\mathcal{R}$.

The purpose of this paper is to consider a family of energy functionals $(\mathcal{E}_\varepsilon)_\varepsilon$ and of dissipation functionals $(\mathcal{R}_\varepsilon)_\varepsilon$ which are defined as integrals over a domain $\Omega \subset \mathbb{R}^d$ and where the densities depend periodically on x with a period proportional to ε. More precisely, for a periodicity lattice Λ we denote by $\mathcal{Y} = \mathbb{R}^d/_{\Lambda}$ the periodicity torus. For a tensor-valued mapping $\mathbb{A} : \mathcal{Y} \to \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^m)$ and a function $\rho : \mathcal{Y}\times\mathbb{R}^m \to \mathbb{R}$ we define the functionals

$$\mathcal{E}_\varepsilon(t,u,z) = \int_\Omega \tfrac{1}{2}\big\langle\!\big\langle \mathbb{A}(\tfrac{x}{\varepsilon})\binom{e(u)}{z}, \binom{e(u)}{z}\big\rangle\!\big\rangle \,\mathrm{d}x - \langle \ell(t), u\rangle \quad \text{and} \quad \mathcal{R}_\varepsilon(z) = \int_\Omega \rho(\tfrac{x}{\varepsilon}, z(x))\,\mathrm{d}x$$

on the space $\mathcal{Q} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega)^m$.

The task is now to describe the limiting behavior of the associated energetic solutions. Because of the nonsmoothness and the hysteretic behavior of the evolution of the systems it will not be possible to find a homogenized limit equation in the classical sense. This would mean to find limiting functionals defined on Ω again. Instead we will need the so-called two-scale homogenization that decomposes solutions into macroscopic and microscopic behavior.

The classical notion of two-scale convergence has been introduced by Nguetseng in 1989 ([Ngu89]) and further developed by Allaire in 1992 ([All92]). It was aimed at a better description of sequences of oscillating functions and thus at the derivation of a new homogenization method. In [LNW02], an overview of the main homogenization problems which have been studied by this technique is given. This concept is now applied in a variety of quite different applications in continuum mechanics, see, e.g., [HJM94, Vis96, BLM96, Vis97, Alb00, EKK02, MS02]. Moreover, even in engineering this method is used extensively for numerical simulations. There the unit periodicity cell is usually called a "representative unit cell".

To explain our results in some detail we introduce a few new notions. The two-scale method relies on a micro-macro-decomposition of points $x \in \mathbb{R}^d$ via

$$x = \mathcal{N}_\varepsilon(x) + \varepsilon\mathcal{R}_\varepsilon(x) \quad \text{with } \mathcal{N}_\varepsilon(x) = \varepsilon \left[\tfrac{x}{\varepsilon}\right]_\Lambda \text{ and } \mathcal{R}_\varepsilon(x) = \left\{\tfrac{x}{\varepsilon}\right\}_Y,$$

where $[\widetilde{x}]_\Lambda$ is the closest lattice point to $\widetilde{x}$ and $\{\widetilde{x}\}_Y$ is the remainder, see Section 2.1 for the exact details. The decomposition of functions is then done by the so-called *periodic unfolding* introduced in [CDG02, CDD04, CDD06]:

$$(\mathcal{T}_\varepsilon u)(x,y) = u_{\mathrm{ex}}(\mathcal{N}_\varepsilon(x)+\varepsilon y),$$

where u_{ex} is the extention of $u : \Omega \to \mathbb{R}$ by 0 to all of $\mathbb{R}^d$. Thus, functions in $\mathrm{L}^p(\Omega)$ are mapped to functions $U = \mathcal{T}_\varepsilon u \in \mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$.

In Section 2.2 we discuss this periodic unfolding operator together with a newly introduced *folding operator* $\mathcal{F}_\varepsilon : \mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y}) \to \mathrm{L}^p(\Omega)$, which is a kind of pseudo inverse as well as the adjoint operator (when taking the dual p). In particular, we give special care to the complications arising from the mismatch of Ω and a finite union of small cells of the type $\varepsilon(\lambda+Y)$.

In Section 2.3 we introduce our notion of *weak and strong two-scale convergence*:

$$u_\varepsilon \stackrel{\mathrm{w2}}{\rightharpoonup} U \quad \Longleftrightarrow \quad \mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup U_{\mathrm{ex}} \text{ in } \mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y}),$$

$$u_\varepsilon \stackrel{\mathrm{s2}}{\to} U \quad \Longleftrightarrow \quad \mathcal{T}_\varepsilon u_\varepsilon \to U_{\mathrm{ex}} \text{ in } \mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y}).$$

This definition is an adaptation of the definitions in [Vis04] to the case that Ω has a boundary. Nevertheless, the convergences on the right-hand side are asked to occur in $\mathrm{L}^p(\mathbb{R}^d{\times}\mathcal{Y})$, since the support of $\mathcal{T}_\varepsilon u$ is in general not contained in $\Omega{\times}\mathcal{Y}$. We relate our definitions to the ones which are used in [Ngu89, All92, CD99, LNW02] and show that our strengthening makes many relations more natural. For instance, it is easy to show that the scalar product of a weakly two-scale convergent family and of a strongly convergent family converges to the scalar product of the two limits.

In Section 2.4 we recall the classical results on the two-scale limits of sequences of gradients and explicitly construct a *gradient folding operator* $\mathcal{G}_\varepsilon : \mathrm{H}^1_0(\Omega){\times}\mathrm{L}^2(\Omega,\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})) \to \mathrm{H}^1_0(\Omega)$ such that for all (u_0,U_1) we have $\nabla\mathcal{G}_\varepsilon(u_0,U_1) \xrightarrow{\mathrm{s2}} \nabla_x u_0 + \nabla_y U_1$ and $\mathcal{G}_\varepsilon(u_0,U_1) \rightharpoonup u_0$ in $\mathrm{H}^1_0(\Omega)$. Based on these results we provide the relevant two-scale Γ-limit results for the functionals $\mathcal{E}_\varepsilon(t,\cdot)$ and $\mathcal{R}_\varepsilon$. Under simple additional assumptions, the two-scale limits are

$$\boldsymbol{E}(t,u_0,U_1,Z) = \int_{\Omega\times\mathcal{Y}} \tfrac{1}{2}\big\langle\!\!\big\langle \mathbb{A}(y)\big(\begin{smallmatrix} \boldsymbol{e}_x(u_0)+\boldsymbol{e}_y(U_1) \\ Z\end{smallmatrix}\big), \big(\begin{smallmatrix} \boldsymbol{e}_x(u_0)+\boldsymbol{e}_y(U_1) \\ Z\end{smallmatrix}\big)\big\rangle\!\!\big\rangle \,\mathrm{d}y\,\mathrm{d}x - \langle \ell(t), u_0\rangle$$

and

$$\boldsymbol{R}(Z) = \int_{\Omega\times\mathcal{Y}} \rho(y, Z(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

The convergence of $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ to $\boldsymbol{E}$ and $\boldsymbol{R}$ can be seen as a type of two-scale Mosco convergence, i.e., Γ-convergence in the weak and in the strong topology, see [MRS06]. Recovery sequences (also called *realizing sequences* in [JKO94]) in the strong two-scale convergence sense are obtained via our explicit operators $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon$.

In Section 3 we formulate our rate-independent evolution systems and we provide existence and uniqueness theorems for energetic formulations associated with $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ on the one hand and with $\boldsymbol{E}$ and $\boldsymbol{R}$ on the other hand. The importance is that we obtain uniform a priori Lipschitz bounds for the energetic solutions $q_\varepsilon = (u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}$. The solutions $Q = (u_0,U_1,Z) : [0,T] \to \boldsymbol{Q}$ are defined on the space $\boldsymbol{Q} = \boldsymbol{H}{\times}\boldsymbol{Z}$ with

$$\boldsymbol{H} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d{\times}\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d, \quad \boldsymbol{Z} = \mathrm{L}^2(\Omega;\mathrm{L}^2(\mathcal{Y}))^m = \mathrm{L}^2(\Omega{\times}\mathcal{Y})^m,$$

with $\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}) = \{\, U \in \mathrm{H}^1(\mathcal{Y}) \mid \int_{\mathcal{Y}} U(y)\,\mathrm{d}y = 0\,\}$.

The final Section 4 establishes the relation between the solutions q_ε and Q. The main result is Theorem 4.3 and it states that if the initial data $q_\varepsilon(0)$ strongly two-scale cross-converge to Q^0, written as $q_\varepsilon(0) \xrightarrow{\mathrm{s2c}} Q^0$ and defined as

$$u_\varepsilon \rightharpoonup u_0 \text{ in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d, \quad \nabla u_\varepsilon \xrightarrow{\mathrm{s2}} \nabla_x u_0{+}\nabla_y U_1 \text{ in } \mathrm{L}^2(\Omega,\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})), \quad z_\varepsilon \xrightarrow{\mathrm{s2}} Z \text{ in } \mathrm{L}^2(\Omega{\times}\mathcal{Y}),$$

then for all $t \in [0,T]$ we have $q_\varepsilon(t) \xrightarrow{\mathrm{s2c}} Q(t)$ where Q is the unique energetic solution associated with $\boldsymbol{E}$ and $\boldsymbol{R}$ with the initial value $Q(0) = Q^0$. In terms of evolutionary variational inequalities this means that the solutions $q_\varepsilon = (u_\varepsilon, z_\varepsilon)$ of

$$\langle \mathrm{D}\mathcal{E}_\varepsilon(t,q_\varepsilon), v{-}\dot q_\varepsilon\rangle + \mathcal{R}_\varepsilon(v) - \mathcal{R}_\varepsilon(\dot q_\varepsilon) \geq 0 \quad \text{for all } v \in \mathcal{Q}$$

strongly two-scale cross-converge to the solution $Q = (u_0,U_1,Z)$ of

$$\langle \mathrm{D}\boldsymbol{E}(t,Q), V{-}\dot Q\rangle + \boldsymbol{R}(V) - \boldsymbol{R}(\dot Q) \geq 0 \quad \text{for all } V \in \boldsymbol{Q},$$

if the initial conditions satisfy $q_\varepsilon(0) \xrightarrow{\mathrm{s2c}} Q(0)$ for $\varepsilon \to 0$.

The crucial tool for proving this convergence is the abstract Γ-convergence theory developed in [MRS06]. The main difficulty in the theory is to show that weak (two-scale) limits of stable states are again stable. In [MRS06, Eqn. (2.16)] a sufficient condition is provided that asks for the existence of a *joint recovery sequence* $(\widehat{q}_\varepsilon)_\varepsilon$ such that

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t,\widetilde{q}_\varepsilon)+\mathcal{R}_\varepsilon(\widetilde{z}_\varepsilon-z_\varepsilon)-\mathcal{E}_\varepsilon(t,q_\varepsilon) \leq \boldsymbol{E}(t,\widetilde{Q})+\boldsymbol{R}(\widetilde{Z}-Z)-\boldsymbol{E}(t,Q) \quad \text{and} \quad \widetilde{q}_\varepsilon \overset{\text{w2c}}{\rightharpoonup} \widetilde{Q},$$

where q_ε is a given family of stable states with $q_\varepsilon \overset{\text{w2c}}{\rightharpoonup} Q$ and $\widetilde{Q}$ is an arbitrary test state, cf. Prop. 4.5. In our situation this condition can be fulfilled by exploiting the quadratic nature of the energies, which leads to some cancellation of differences of the energies, namely $\mathcal{E}_\varepsilon(t,\widetilde{q}_\varepsilon)-\mathcal{E}_\varepsilon(t,q_\varepsilon)$ converges to $\boldsymbol{E}(t,\widetilde{Q})-\boldsymbol{E}(t,Q)$, if $q_\varepsilon \overset{\text{w2c}}{\rightharpoonup} Q$ and $\widetilde{q}_\varepsilon-q_\varepsilon \overset{\text{s2c}}{\longrightarrow} \widetilde{Q}-Q$ strong. Here it is important that our notion of weak and strong convergence allows us to conclude convergence of scalar products, see Prop. 2.4(d).

As far as we know, this is the first homogenization work for a nonlinear and nonsmooth evolutionary problems except for [Nes06]. The latter work treats more general evolution laws and is not restricted to the rate-independent setting. However, it is more restrictive in the constitutive laws and proves the convergence only in an averaged sense over microscopic phase shifts of the cells. Similar variational inequalities are treated in [CPS04, Yos01], but with different constraints and without time dependence.

We hope that our methods simplify and clarify the theory of two-scale convergence and thus provide ideas and tools for solving more general problems.

2 Two-scale convergence

We recall here the definition of the two-scale convergence and several important results concerning this notion (see [Ngu89, All92, CD99, LNW02]). In particular, the presented results are based on [CDG02, Vis04], where the notions of *periodic unfolding* (also called 'two-scale decomposition' in the latter work) and *periodic folding*, which is called 'averaging operator' in [CDG02, Sect. 5]. In the following subsections we take special care of the problems that are associated with the fact that we want to work on a bounded domain Ω and that this is only approximately compatible with microscopic periodicity. This gives rise to a certain notational complication but allows us a very precise and efficient definition of weak and strong two-scale convergence in Section 2.3. Note also Example 2.7 that shows that this special care is necessary to avoid problems at the boundary.

2.1 Basic definitions of the two-scale variables

Let $d \in \mathbb{N}$ be the space dimension. The periodicity in $\mathbb{R}^d$ is expressed by a d-dimensional periodicity lattice

$$\Lambda = \{\, \lambda = \textstyle\sum_{j=1}^d k_j b_j \mid k=(k_1,k_2,...,k_d) \in \mathbb{Z}^d \,\},$$

where $\{b_1,...,b_d\}$ is an arbitrary basis in $\mathbb{R}^d$. The associated unit cell is $Y = \{\, x = \sum_1^d \gamma_j b_j \mid \gamma_j \in [-1/2,1/2) \,\} \subset \mathbb{R}^d$, such that $\mathbb{R}^d$ is the disjoint union of the translated cells $\lambda{+}Y$, if λ ranges all of Λ. Following [Vis04], we distinguish the unit cell from the periodicity cell $\mathcal{Y}$, which is obtained by identifying the opposite faces of $\overline{Y}$, or we may set $\mathcal{Y} = \mathbb{R}^d/_\Lambda$. Thus, $\mathcal{Y}$ has the structure of a torus. For most applications one may assume that $\Lambda = \mathbb{Z}^d$, $Y = [-1/2, 1/2)^d$, and $\mathcal{Y} = \mathbb{R}^d/_{Z^d} = \mathbb{T}^d$, the d-dimensional standard torus. However, our theory covers the general case. Yet, we will be slightly inconsistent and use y to denote elements of Y and $\mathcal{Y}$ simultaneously by relying on the natural identification between $y{+}\Lambda \in \mathcal{Y}$ and $y \in Y$.

On $\mathbb{R}^d$ we define the mappings $[\,\cdot\,]_\Lambda$ and $\{\cdot\}_Y$ such that

$$[\,\cdot\,]_\Lambda : \mathbb{R}^d \to \Lambda, \quad \{\cdot\}_Y : \mathbb{R}^d \to Y, \quad x = [x]_\Lambda + \{x\}_Y \text{ for all } x \in \mathbb{R}^d.$$

We also use the notation $\{\,\cdot\,\}_\mathcal{Y}$ such that $\{x\}_\mathcal{Y} = x \operatorname{mod} \Lambda \in \mathcal{Y}$. Obviously a function f defined on R^d is Λ-periodic if $f(x) = f(\{x\}_Y)$ for $x \in \mathbb{R}^d$ and we may identify f with a function $\widetilde{f}$ defined on $\mathcal{Y}$. Note that $\mathrm{L}^p(Y)$ and $\mathrm{L}^p(\mathcal{Y})$ may be identified in contrast to $\mathrm{C}^k(Y)$ and $\mathrm{C}^k(\mathcal{Y}) = \mathrm{C}^k_{\mathrm{per}}(\overline{Y})$. Similarly, we use $\mathrm{H}^1(\mathcal{Y}) = \mathrm{H}^1_{\mathrm{per}}(\overline{Y})$, which is different from $\mathrm{H}^1(Y)$. A non-standard space, which we will need in the sequel, is

$$\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}) := \{\, f \in \mathrm{H}^1(\mathcal{Y}) \mid \textstyle\int_\mathcal{Y} f(y)\,\mathrm{d}y = 0\,\}. \tag{2.1}$$

We now introduce a small length-scale parameter $\varepsilon > 0$ and want to study functions which have fast periodic oscillations on the microscopic periodicity cell εY. We *decompose* the points $x \in \Omega \subset \mathbb{R}^d$ such that

$$x = \mathcal{N}_\varepsilon(x) + \varepsilon\mathcal{R}_\varepsilon(x) \quad \text{with } \mathcal{N}_\varepsilon(x) = \varepsilon\left[\tfrac{x}{\varepsilon}\right]_\Lambda \text{ and } \mathcal{R}_\varepsilon(x) = \left\{\tfrac{x}{\varepsilon}\right\}_Y.$$

Thus, $\mathcal{N}_\varepsilon \in \varepsilon\Lambda$ denotes the macroscopic center of the small cell $\mathcal{N}_\varepsilon(x){+}\varepsilon Y$ that contains x and $\mathcal{R}_\varepsilon$ denotes the fine-scale part of x. With this we define a *decomposition map* $\mathcal{D}_\varepsilon$ and a *composition map* $\mathcal{S}_\varepsilon$ (cf. [Vis04]) as follows

$$\mathcal{D}_\varepsilon : \begin{cases} \mathbb{R}^d & \to & \mathbb{R}^d{\times}\mathcal{Y} \\ x & \mapsto & (\mathcal{N}_\varepsilon(x), \mathcal{R}_\varepsilon(x)) \end{cases} \qquad \mathcal{S}_\varepsilon : \begin{cases} \mathbb{R}^d{\times}\mathcal{Y} & \to & \mathbb{R}^d, \\ (x, y) & \mapsto & \mathcal{N}_\varepsilon(x){+}\varepsilon y, \end{cases}$$

where in the last sum some $y \in \mathcal{Y}$ is identified with $y \in Y \subset \mathbb{R}^d$. For the construction of periodic unfolding operator and folding operator in the next subsection, the following simple properties of $\mathcal{D}_\varepsilon$ and $\mathcal{S}_\varepsilon$ are essential:

$$\mathcal{D}_\varepsilon(\mathcal{S}_\varepsilon(x, y)) = (\mathcal{N}_\varepsilon(x), y) \quad \text{and} \quad \mathcal{S}_\varepsilon(\mathcal{D}_\varepsilon(x)) = x \qquad \text{for all } (x, y) \in \mathbb{R}^d{\times}\mathcal{Y}. \tag{2.2}$$

If Ω does not coincide with $\mathbb{R}^d$ then certain technicalities arise from the fact that the image of $\mathcal{D}_\varepsilon$ is not contained in $\Omega{\times}\mathcal{Y}$. Similarly, we note that $\mathcal{S}_\varepsilon(\Omega{\times}\mathcal{Y})$ is not contained in Ω. To handle this, we introduce, for a fixed open domain Ω, the following subsets of Λ:

$$\Lambda^-_\varepsilon = \{\, \lambda \in \Lambda \mid \varepsilon(\lambda{+}Y) \subset \overline{\Omega}\,\} \text{ and } \Lambda^+_\varepsilon = \{\, \lambda \in \Lambda \mid \varepsilon(\lambda{+}Y) \cap \Omega \neq \emptyset\,\}.$$

Using this, we define the domains Ω^-_ε and Ω^+_ε via $\Omega^\pm_\varepsilon = \operatorname{int}\Big(\cup_{\lambda\in\Lambda^\pm_\varepsilon} \varepsilon(\lambda{+}Y)\Big)$. Clearly, we have $\Omega^-_\varepsilon \subset \Omega \subset \Omega^+_\varepsilon$. Moreover, we have $[\Omega^\pm_\varepsilon]^\pm_\varepsilon = \Omega^\pm_\varepsilon$, $\Omega \subset N_{\varepsilon\,\mathrm{diam}(Y)}(\Omega^-_\varepsilon)$ and $\Omega^+_\varepsilon \subset N_{\varepsilon\,\mathrm{diam}(Y)}(\Omega)$, where $\mathrm{diam}(Y)$ is the diameter of Y and $N_\delta(A)$ denotes the δ-neighborhood of the set A.

Moreover, we set $[\Omega\times\mathcal{Y}]_\varepsilon = \mathcal{S}_\varepsilon^{-1}(\Omega) = \{\,(x,y)\,|\,\mathcal{S}_\varepsilon(x,y)\in\Omega\,\}$ and note the relations

$$\Omega_\varepsilon^-\times\mathcal{Y} \subset [\Omega\times\mathcal{Y}]_\varepsilon \subset \overline{\Omega_\varepsilon^+}\times\mathcal{Y}, \tag{2.3}$$

which will significantly be used later on. From now on we will assume that Ω satisfies

$$\Omega \text{ is open and bounded} \quad \text{and} \quad |\partial\Omega| = 0. \tag{2.4}$$

This guarantees that $|\Omega\backslash\Omega_\varepsilon^-|+|\Omega_\varepsilon^+\backslash\Omega| \to 0$ for $\varepsilon\to 0$ which will be used later. To see this, denote by ϕ_ε the characteristic function of the set $N_{\varepsilon\,\mathrm{diam}(Y)}(\partial\Omega)$, then $\Omega\backslash\Omega_\varepsilon^- \cup \Omega_\varepsilon^+\backslash\Omega \subset N_{\varepsilon\,\mathrm{diam}(Y)}(\partial\Omega)$ and for all $x\notin\partial\Omega$ we have $\phi_\varepsilon(x)\to 0$ for $\varepsilon\to 0$. Hence, we conclude $|\Omega\backslash\Omega_\varepsilon^-|+|\Omega_\varepsilon^+\backslash\Omega| \le |N_{\varepsilon\,\mathrm{diam}(Y)}(\partial\Omega)| = \int_{\mathbb{R}^d}\phi_\varepsilon\,\mathrm{d}x \to 0$ for $\varepsilon\to 0$. The second condition in (2.4) is certainly satisfied, if Ω has a Lipschitz boundary.

2.2 Folding and periodic unfolding operators

The notion of two-scale convergence is intrinsically linked with a suitable "two-scale embedding" of the function space $\mathrm{L}^p(\Omega)$ into the two-scale space $\mathrm{L}^p(\Omega\times\mathcal{Y})$. Such a mapping will be called a *periodic unfolding operator*. Moreover, for a two-scale function U defined on $\Omega\times\mathcal{Y}$ it is desirable to find a function u_ε defined on Ω that has the corresponding microscopic behavior. A mapping from the two-scale space into the original function space $\mathrm{L}^p(\Omega)$ will be called a *folding operator*.

The natural candidate for the periodic unfolding operator was introduced in [CDG02] and reads

$$\mathcal{T}_\varepsilon : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y});\ v\mapsto v_{\mathrm{ex}}\circ\mathcal{S}_\varepsilon, \tag{2.5}$$

where $v_{\mathrm{ex}}\in\mathrm{L}^p(\mathbb{R}^d)$ is obtained from v by extending it by 0 outside of Ω. By definition, we immediately have the product rule:

$$\tfrac{1}{p}+\tfrac{1}{q}=\tfrac{1}{r}\le 1,\ u\in\mathrm{L}^p(\Omega),\ v\in\mathrm{L}^q(\Omega) \implies \mathcal{T}_\varepsilon(uv) = (\mathcal{T}_\varepsilon u)\,(\mathcal{T}_\varepsilon v)\in\mathrm{L}^r(\Omega\times\mathcal{Y}). \tag{2.6}$$

In general, the support of $\mathcal{T}_\varepsilon v$ is $\overline{[\Omega\times\mathcal{Y}]_\varepsilon}$ which is not contained in $\Omega\times\mathcal{Y}$. This discrepancy in support is the main reason why we repeat the definitions of the operators and the different versions of two-scale convergence in detail. Most previous work either deals with $\Omega=\mathbb{R}^d$ or is not very precise about the supports. However, as was noted in [LNW02], see also our Examples 2.3 and 2.7, we need to be careful here.

A variant of $\mathcal{T}_\varepsilon$ that maps continuous functions u into continuous ones can be found in [Vis04].

As candidates for folding operators simple choices are given in the form

$$\widehat{F}_\varepsilon : \mathrm{F}(\Omega\times\mathcal{Y}) \to \mathrm{F}(\mathbb{R}^d);\ U\mapsto U\circ\mathcal{D}_\varepsilon, \quad \text{and} \quad F_\varepsilon : \mathrm{F}(\Omega\times\mathcal{Y}) \to \mathrm{F}(\mathbb{R}^d);\ U\mapsto U\circ D_\varepsilon, \tag{2.7}$$

where D_ε is the simple decomposition $D_\varepsilon : x\mapsto (x,\{\frac{x}{\varepsilon}\}_{\mathcal{Y}})$. Both of these choices are not suitable, if for the function space "F" we choose L^p since the image of Ω under $\mathcal{D}_\varepsilon$ and D_ε, respectively, is a set of measure 0 in $\mathbb{R}^d\times\mathcal{Y}$. However, the folding operator F_ε is well-defined as a mapping from $\mathrm{C}^k(\mathbb{R}^d\times\mathcal{Y})$ into $\mathrm{C}^k(\mathbb{R}^d)$ and has the big advantage that the image of $\Omega\times\mathcal{Y}$ under D_ε is equal to Ω. In fact, this is the basis of the classical definition of two-scale convergence, see (2.9).

The main point in this subsection is that we use a very particular folding operator $\mathcal{F}_\varepsilon$ that is well adapted to the classical L^p-spaces, namely

$$\mathrm{L}^p(\Omega\times\mathcal{Y}) = \mathrm{L}^p(\Omega;\mathrm{L}^p(\mathcal{Y})) = \mathrm{L}^p(\mathcal{Y};\mathrm{L}^p(\Omega)) \text{ for } p\in[1,\infty).$$

These are the relevant ones for elliptic partial differential equations and our aim is to avoid spaces involving continuous functions like $\mathrm{L}^p(\Omega,\mathrm{C}(\mathcal{Y}))$ (on which $\widehat{F}_\varepsilon$ is well-defined). Our folding operator is a variant of the averaging operator $\mathcal{U}_\varepsilon$ defined in [CDG02, Sect. 5], since we take special care on the domain Ω.

On $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$ we first define the classical projector to piecewise constant functions on each $\varepsilon(\lambda+Y)$ via

$$(\mathcal{P}_\varepsilon U)(x,y) = ⨍_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} U(\xi,y)\,\mathrm{d}\xi,$$

where $⨍_A$ denotes the average over A, i.e., $⨍_A g(a)\,\mathrm{d}a = \frac{1}{|A|}\int_A g(a)\,\mathrm{d}a$. Clearly $(\mathcal{P}_\varepsilon)^2=\mathcal{P}_\varepsilon$, $\|\mathcal{P}_\varepsilon U\|_p \le \|U\|_p$, and $\mathcal{P}_\varepsilon U \to U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ for all $U\in\mathrm{L}^p(\Omega\times\mathcal{Y})$.

Our folding operator $\mathcal{F}_\varepsilon$ is now defined as follows:

$$\mathcal{F}_\varepsilon : \mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y}) \to \mathrm{L}^p(\Omega);\ U \mapsto \mathcal{P}_\varepsilon(\chi_\varepsilon U)\circ\mathcal{D}_\varepsilon)|_\Omega \quad \text{with } \chi_\varepsilon = \chi_{[\Omega\times\mathcal{Y}]_\varepsilon}. \tag{2.8}$$

Note that the folding operator is defined for functions on the full space $\mathbb{R}^d\times\mathcal{Y}$ and takes values in the functions on Ω. The construction with the characteristic function $\chi_\varepsilon : \mathbb{R}^d\times\mathcal{Y}\to\{0,1\}$ guarantees that satisfies $\chi_\varepsilon = \mathcal{P}_\varepsilon\chi_\varepsilon$ and $\mathrm{sppt}(\chi_\varepsilon\circ\mathcal{D}_\varepsilon)=\overline{\Omega}$, which follows from the definition of $[\Omega\times\mathcal{Y}]_\varepsilon$ and from (2.2).

The following proposition summarizes the properties of the folding operator and the periodic unfolding operator. We restrict ourselves to the case $p\in(1,\infty)$, and leave the obvious generalizations for $p=1$ and $p=\infty$ to the reader. In fact, in our application we will only use $p=p'=2$, which is especially nice.

Proposition 2.1 *Let $p\in(1,\infty)$ and $p'=p/(p-1)$. Then, the folding operator $\mathcal{F}_\varepsilon : \mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})\to\mathrm{L}^p(\Omega)$ and the periodic unfolding operators $\mathcal{T}_\varepsilon : \mathrm{L}^p(\Omega)\to\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$ and $\widetilde{\mathcal{T}}_\varepsilon : \mathrm{L}^{p'}(\Omega)\to\mathrm{L}^{p'}(\mathbb{R}^d\times\mathcal{Y})$ satisfy*

(a) $\|\mathcal{T}_\varepsilon u\|_{\mathrm{L}^{p'}(\mathbb{R}^d\times\mathcal{Y})} = \|u\|_{\mathrm{L}^{p'}(\Omega)}$ and $\mathrm{sppt}(\mathcal{T}_\varepsilon u)\subset\overline{[\Omega\times\mathcal{Y}]_\varepsilon}$ for all $u\in\mathrm{L}^{p'}(\Omega)$;

(b) $\|\mathcal{F}_\varepsilon U\|_{\mathrm{L}^p(\Omega)} \le \|U\|_{\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})}$ for all $U\in\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$;

(c) $\mathcal{F}_\varepsilon$ is the adjoint of $\widehat{\mathcal{T}}_\varepsilon$, i.e., $\mathcal{F}_\varepsilon = \big(\widehat{\mathcal{T}}_\varepsilon\big)'$;

(d) $\mathcal{F}_\varepsilon\circ\mathcal{T}_\varepsilon = \mathrm{id}_{\mathrm{L}^p(\Omega)}$ and $(\mathcal{T}_\varepsilon\circ\mathcal{F}_\varepsilon)^2 = \mathcal{T}_\varepsilon\circ\mathcal{F}_\varepsilon = \chi_\varepsilon\mathcal{P}_\varepsilon$.

All these identities can be obtained by elementary calculations via decomposing $\mathbb{R}^d$ into $\cup_{\lambda\in\Lambda}\varepsilon(\lambda+Y)$.

2.3 Weak and strong two-scale convergence

Following [Ngu89, All92, CD99, LNW02] a family $(u_\varepsilon)_\varepsilon$ in $\mathrm{L}^p(\Omega)$ is called *two-scale convergent* to a function $U\in\mathrm{L}^p(\Omega\times\mathcal{Y})$ and write $u_\varepsilon \overset{2}{\rightharpoonup} U$, if for all test functions $\psi:\Omega\times\mathcal{Y}\to\mathbb{R}$ we have

$$\lim_{\varepsilon\to0}\int_\Omega u_\varepsilon(x)\psi(x,\{\tfrac{x}{\varepsilon}\}_\mathcal{Y})\,\mathrm{d}x = \int_\Omega\int_\mathcal{Y} U(x,y)\psi(x,y)\,\mathrm{d}y\,\mathrm{d}x \quad \text{for all } \psi\in\Psi. \tag{2.9}$$

The choice of the set of test functions Ψ is important here, cf. [LNW02]. The weakest notion occurs if we take $\Psi = \mathrm{C}_\mathrm{c}^\infty(\Omega\times\mathcal{Y})$, which corresponds to a kind of distributional convergence. If $p' = p/(p-1)$ denotes the dual exponent to $p \in (1,\infty)$, the choice $\Psi = \mathrm{L}^{p'}(\Omega, \mathrm{C}(\mathcal{Y}))$ is advocated in [LNW02], since it guarantees weak convergence of $(u_\varepsilon)_\varepsilon$ to $\int_\mathcal{Y} U(\cdot,y)\,\mathrm{d}y$ in $\mathrm{L}^p(\Omega)$. Note that two-scale convergence can also be defined using the folding operator F_ε defined in (2.7)

$$u_\varepsilon \overset{2}{\rightharpoonup} U \quad \iff \quad \langle u_\varepsilon, F_\varepsilon\psi\rangle_\Omega = \langle u_\varepsilon, \psi\circ D_\varepsilon\rangle_\Omega \to \langle U,\psi\rangle_{\Omega\times\mathcal{Y}}.$$

Here we follow the notions from [Vis04], but modify them to fit the case $\Omega \subsetneq \mathbb{R}^d$, for defining weak and strong two-scale convergence via the periodic unfolding operators $\mathcal{T}_\varepsilon$.

Definition 2.2 *Let $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)}$ be a family in $\mathrm{L}^p(\Omega)$ with $p \in (1,\infty)$.*

(a) We say that u_ε weakly two-scale converges *to $U \in \mathrm{L}^p(\Omega\times\mathcal{Y})$ and write "$u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$", if $\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup U_\mathrm{ex}$ in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$.*

(b) We say that u_ε strongly two-scale converges *to $U \in \mathrm{L}^p(\Omega\times\mathcal{Y})$ and write "$u_\varepsilon \overset{\mathrm{s2}}{\to} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$", if $\mathcal{T}_\varepsilon u_\varepsilon \to U_\mathrm{ex}$ (strongly) in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$.*

As the supports of $\mathcal{T}_\varepsilon u_\varepsilon$ are contained in $\overline{[\Omega\times\mathcal{Y}]_\varepsilon} \subset \overline{\Omega_\varepsilon^+}\times\mathcal{Y}$, it is clear that any possible accumulation point U of $(\mathcal{T}_\varepsilon)_\varepsilon$ has its support in $\overline{\Omega}\times\mathcal{Y}$. Because of $|\partial\Omega| = 0$ we have $\mathrm{L}^p(\Omega\times\mathcal{Y}) = \mathrm{L}^p(\overline{\Omega}\times\mathcal{Y})$ and hence accumulation points of $(\mathcal{T}_\varepsilon u_\varepsilon)_\varepsilon$ can be uniquely described by elements in $\mathrm{L}^p(\Omega\times\mathcal{Y})$. Nevertheless, it is important that our definition involves a convergence statement in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$, i.e., we need to consider functions outside of $\Omega\times\mathcal{Y}$. If the convergence was only asked for the restrictions on $\Omega\times\mathcal{Y}$, then different notions would occur.

Example 2.3 *We choose $\Omega = (0,1)$ and $Y = [0,1)$. Along the sequence $\varepsilon_k = (k^3-1)/k^4 \to 0$ we consider the functions*

$$u_{\varepsilon_k}(x) = a_k \text{ for } x \in (1-1/k^2, 1) \text{ and } 0 \text{ otherwise,}$$

which satisfy $\|u_{\varepsilon_k}\|_{\mathrm{L}^2(\Omega)} = |a_k|/k$. The periodic unfolding $U_k = \mathcal{T}_{\varepsilon_k}u_{\varepsilon_k} \in \mathrm{L}^2(\mathbb{R}\times\mathcal{Y})$ reads

$$U_k(x,y) = a_k \text{ if } \big(x \in (1-1/k^2, 1+(k-1)/k^2) \text{ and } y \in (0,1/k)\big) \quad \text{ and } 0 \text{ else.}$$

The support of U_k only has a small part in $\Omega\times\mathcal{Y}$ while the most part is in $(\Omega_{\varepsilon_k}^+ \setminus \Omega)\times\mathcal{Y}$. Hence, $U_k\big|_{\Omega\times Y}$ has a much smaller norm, namely $\|U_k\big|_{\Omega\times\mathcal{Y}}\|_{\mathrm{L}^2(\Omega\times\mathcal{Y})} = |a_k|/k^{3/2}$. Thus, for $a_k = o(k^{3/2})$ we have $U_k\big|_{\Omega\times Y} \to 0$ strongly in $\mathrm{L}^2(\Omega\times\mathcal{Y})$ which implies $u_k \overset{2}{\rightharpoonup} 0$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})$.

However, $u_{\varepsilon_k} \overset{\mathrm{w2}}{\rightharpoonup} U$ holds if and only if $a_k = O(k)$ and then $U \equiv 0$. Moreover, $u_{\varepsilon_k} \overset{\mathrm{s2}}{\to} U$ if and only if $a_k = o(k)$ and $U \equiv 0$ then.

Using the fact that the folding operator is the adjoint of the periodic unfolding operator, we may equivalently define weak two-scale convergence in a way similar to the classical definition (2.9), namely

$$u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} U \text{ in } \mathrm{L}^p(\Omega\times\mathcal{Y}) \iff \forall\, V \in \mathrm{L}^{p'}(\Omega\times\mathcal{Y}) : \int_\Omega u_\varepsilon\,\mathcal{F}_\varepsilon V\,\mathrm{d}x \to \int_\Omega\int_\mathcal{Y} U\,V\,\mathrm{d}y\,\mathrm{d}x. \quad (2.10)$$

Note that we have simply replaced the folding operator $F_\varepsilon : U \mapsto U \circ D_\varepsilon$ by the more sophisticated version $\mathcal{F}_\varepsilon$ that allows us to take general L^p functions. Moreover, the test functions V are allowed to have a support bigger than $\overline{\Omega}\times\mathcal{Y}$. As we are interested in $\varepsilon \to 0$, it suffices to consider $V \in \mathrm{L}^{p'}(N_\delta(\Omega)\times\mathcal{Y})$ for any $\delta > 0$, whereas $\delta = 0$ will lead to a strictly weaker notion of convergence.

The definitions of weak and strong two-scale convergence are obtained by transferring convergence to the classical weak and strong convergences in the classical space $\mathrm{L}^p(\Omega\times\mathcal{Y})$.

Proposition 2.4 *Let $p \in (1,\infty)$ and $p' = p/(p-1)$ and assume that Ω satisfies* (2.4)*.*

(a) If $u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, then $\|u_\varepsilon\|_{\mathrm{L}^p(\Omega)}$ is bounded for $\varepsilon \to 0$.

(b) If $u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, then $u_\varepsilon \overset{2}{\rightharpoonup} U$. (The reverse implication is in general not true).

(c) If $u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} U$ and $\|u_\varepsilon\|_{\mathrm{L}^p(\Omega)} \to \|U\|_{\mathrm{L}^p(\Omega\times\mathcal{Y})}$, then $u_\varepsilon \overset{\mathrm{s2}}{\longrightarrow} U$.

(d) If $u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ and $v_\varepsilon \overset{\mathrm{s2}}{\longrightarrow} V$ in $\mathrm{L}^{p'}(\Omega\times\mathcal{Y})$, then $\langle u_\varepsilon, v_\varepsilon\rangle_\Omega \to \langle U, V\rangle_{\Omega\times\mathcal{Y}}$.

(e) For each $U \in \mathrm{L}^p(\Omega\times\mathcal{Y})$ there exists a family $(u_\varepsilon)_\varepsilon$ such that $u_\varepsilon \overset{\mathrm{s2}}{\longrightarrow} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ (simply take $u_\varepsilon = \mathcal{F}_\varepsilon U_{\mathrm{ex}}$).

(f) For each $w \in \mathrm{L}^p(\Omega)$ we have $\mathcal{T}_\varepsilon w \overset{\mathrm{s2}}{\longrightarrow} Ew$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, where $E : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\Omega\times\mathcal{Y})$ is defined via $Ev(x,y) = v(x)$.

(g) For $p \in (1,\infty)$, $q \in (1,\infty]$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, let $u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ and $v_\varepsilon \overset{\mathrm{s2}}{\longrightarrow} V$ in $\mathrm{L}^q(\Omega\times\mathcal{Y})$, then $u_\varepsilon v_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} UV$ in $\mathrm{L}^r(\Omega\times\mathcal{Y})$. If additionally $u_\varepsilon \overset{\mathrm{s2}}{\longrightarrow} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, then $u_\varepsilon v_\varepsilon \overset{\mathrm{s2}}{\longrightarrow} UV$ in $\mathrm{L}^r(\Omega\times\mathcal{Y})$.

Proof: Parts (a), (c), (d), and (g) are immediate consequences of the corresponding results of weak and strong convergence in $\mathrm{L}^p(\Omega\times\mathcal{Y})$.

Property (b) will be a consequence of Prop. 2.5 below.

Property (e) follows as the projector $\mathcal{P}_\varepsilon$ on $\mathrm{L}^p(\Omega\times\mathcal{Y})$ satisfies $\mathcal{P}_\varepsilon U \to U$ and the characteristic function χ_ε (cf. (2.8)) converges pointwise a.e. to $\chi_{\Omega\times\mathcal{Y}}$.

For property (f) we use the fact that the unfolding operators $\mathcal{T}_\varepsilon$ have norm 1 and that for $w \in \mathrm{C}^1(\overline{\Omega})$ some calculation gives $\|\mathcal{T}_\varepsilon w - Ew\|_{\mathrm{L}^p(\Omega\times\mathcal{Y})} \leq 2\,\mathrm{diam}Y\,\varepsilon\,|\Omega|^{1/p}\|\nabla w\|_{\mathrm{L}^\infty}$. However, because of (2.4) the smooth functions are dense and the assertion follows. ∎

In fact, the difference between $\overset{2}{\rightharpoonup}$ and $\overset{\mathrm{w2}}{\rightharpoonup}$ disappears, if we a priori impose boundedness of the sequence.

Proposition 2.5 *Let $(u_\varepsilon)_\varepsilon$ be a bounded family in $\mathrm{L}^p(\Omega)$ with $p \in (1,\infty)$. Then, the following statements are equivalent:*

$$\text{(i) } u_\varepsilon \overset{2}{\rightharpoonup} U \textit{ in } \mathrm{L}^p(\Omega\times\mathcal{Y}), \quad \text{(ii) } \mathcal{T}_\varepsilon u_\varepsilon\big|_{\Omega\times\mathcal{Y}} \rightharpoonup U \textit{ in } \mathrm{L}^p(\Omega\times\mathcal{Y}), \quad \text{(iii) } u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} U \textit{ in } \mathrm{L}^p(\Omega\times\mathcal{Y}).$$

Proof: For the equivalence between (i) and (ii) see [LNW02, CDD06]. The definition of $\overset{\mathrm{w2}}{\rightharpoonup}$ shows that (iii) implies (ii). Moreover, using (2.10) and the boundedness of $(u_\varepsilon)_\varepsilon$ it is sufficient to show $\int_\Omega u_\varepsilon \mathcal{F}_\varepsilon V\,\mathrm{d}x \to \int_\Omega\int_\mathcal{Y} UV\,\mathrm{d}y\,\mathrm{d}x$ on the dense subset $\Psi = \mathrm{C}^\infty_\mathrm{c}(\Omega\times\mathcal{Y})$. However, on Ψ we have $\|\mathcal{F}_\varepsilon\psi - F_\varepsilon\psi\|_{\mathrm{L}^p(\Omega)} = O(\varepsilon)$ and thus (i) implies (iii). ∎

The next result provides an improvement of part (g) in Prop. 2.4.

Proposition 2.6 *Let $p \in [1,\infty)$ and let $(u_\varepsilon)_\varepsilon \xrightarrow{\mathrm{s2}} U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$. Moreover, consider a bounded sequence $(m_\varepsilon)_\varepsilon$ in $\mathrm{L}^\infty(\Omega)$ such that $\mathcal{T}_\varepsilon m_\varepsilon(x,y) \to M(x,y)$ for a.e. $x \in \Omega\times\mathcal{Y}$. Then, $m_\varepsilon u_\varepsilon \xrightarrow{\mathrm{s2}} MU$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$.*

Proof: By the assumption, $U_\varepsilon = \mathcal{T}_\varepsilon u_\varepsilon$ is bounded in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ and hence there is a subsequence and a majorant $g \in \mathrm{L}^p(\Omega\times\mathcal{Y})$ such that $|U_{\varepsilon_k}(x,y)| \leq g(x,y)$ and $U_{\varepsilon_k}(x,y) \to U(x,y)$ a.e. in $\Omega\times\mathcal{Y}$. Because of the assumptions on m_ε we find that $\mathcal{T}_{\varepsilon_k}(m_{\varepsilon_k}U_{\varepsilon_k}) = \mathcal{T}_{\varepsilon_k}m_{\varepsilon_k}\mathcal{T}_{\varepsilon_k}U_{\varepsilon_k}$ also has a joint majorant and converges pointwise a.e. From this we conclude $\mathcal{T}_{\varepsilon_k}m_{\varepsilon_k}U_{\varepsilon_k} \rightharpoonup MU$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$. Since the limit of all subsequences is the same the usual contradiction argument provides the convergence of the whole family. ∎

The following example shows that the statement in Prop. 2.4(d) is not true if we do not insist on the convergence of $\mathcal{T}_\varepsilon u_\varepsilon$ and $\mathcal{T}_\varepsilon v_\varepsilon$ in $\mathrm{L}^p(\mathbb{R}^d\times\mathcal{Y})$. In [LNW02, Thm. 11] a related result to (c) is proved, namely $\int_\Omega \tau u_\varepsilon v_\varepsilon \,\mathrm{d}x \to \int_\Omega \tau \int_{\mathcal{Y}} UV \,\mathrm{d}y\,\mathrm{d}x$ for all $\tau \in \mathrm{C}_\mathrm{c}^\infty(\Omega)$, where the cut-off function τ that is 0 near the boundary $\partial\Omega$ is needed to compensate for the usage of the weaker notion of two-scale convergence $\xrightharpoonup{2}$ defined in (2.9). In [LNW02, Thm. 11] strong two-scale convergence is implicitly defined by two-scale convergence $\xrightharpoonup{2}$ and additional norm convergence, see Prop. 2.4(c).

Example 2.7 *We take $\Omega = (0,1)$, $Y = [0,1)$, ε_k, and u_{ε_k} as in Example 2.3. Moreover, we let $a_k = k$ and $v_{\varepsilon_k} = u_{\varepsilon_k}$. Obviously, we have $\int u_{\varepsilon_k}v_{\varepsilon_k}\,\mathrm{d}x = \|u_{\varepsilon_k}\|^2_{\mathrm{L}^2(\Omega)} = 1$. However, as shown above we have $\mathcal{T}_{\varepsilon_k}u_{\varepsilon_k}\big|_{\Omega\times\mathcal{Y}} \to U_\Omega \equiv 0$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})$. Hence, Prop. 2.4(d) does not hold for the limits U_Ω and V_Ω defined in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ only.*

2.4 Two-scale convergence of gradients

We now deal with bounded sequences in $\mathrm{W}^{1,p}(\Omega)$. The two-scale convergence for the associated gradients provides an additional structure. To formulate the result we define

$$\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}) = \{\, w \in \mathrm{W}^{1,p}(\mathcal{Y}) \mid \int_{\mathcal{Y}} w(y)\,\mathrm{d}y = 0 \,\}$$

and note that $\mathrm{L}^p(\Omega;\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$ is the set of functions V in $\mathrm{L}^p(\Omega\times\mathcal{Y}) = \mathrm{L}^p(\Omega;\mathrm{L}^p(\Omega))$ such that $\int_{\mathcal{Y}} V(x,y)\,\mathrm{d}y = 0$ for a.a. $x \in \Omega$ and that $\nabla_y V$ (in the sense of distributions) lies again in $\mathrm{L}^p(\Omega\times\mathcal{Y})$.

Theorem 2.8 *Let $(v_\varepsilon)_\varepsilon$ be a sequence in $\mathrm{W}^{1,p}(\Omega)$ such that $v_\varepsilon \rightharpoonup v_0$ weakly in $\mathrm{W}^{1,p}(\Omega)$, where $p \in (1,\infty)$. Then $v_\varepsilon \xrightarrow{\mathrm{s2}} Ev_0$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$, and there exist a subsequence $(v_{\varepsilon'})_{\varepsilon'}$ and a function $V_1 \in \mathrm{L}^p(\Omega;\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$ such that*

$$\nabla v_{\varepsilon'} \xrightharpoonup{\mathrm{w2}} E\nabla_x v_0 + \nabla_y V_1.$$

Proof: Since $v_\varepsilon \rightharpoonup v_0$ weakly in $\mathrm{W}^{1,p}(\Omega)$ implies by the compact embedding that $v_\varepsilon \to v_0$ (strongly) in $\mathrm{L}^p(\Omega)$. Now using Propositions 2.1(a) and 2.4 we have $\|\mathcal{T}_\varepsilon v_\varepsilon - Ev_0\|_p \leq \|\mathcal{T}_\varepsilon(v_\varepsilon - v_0)\|_p + \|\mathcal{T}_\varepsilon v_0 - Ev_0\|_p \to 0$. Thus, $v_\varepsilon \xrightarrow{\text{s2}} Ev_0$ is established.

The weak two-scale convergence of the gradients along a subsequence can be deduced by exploiting the corresponding result from the classical two-scale convergence, see [Ngu89, All92]. Since weak convergence in $\mathrm{W}^{1,p}(\Omega)$ implies boundedness of the gradients, the desired result follows using Prop. 2.5. ■

Like for the strong two-scale convergence for functions we also need a density result for gradients converging in the two-scale sense. These results will be used to construct recovery sequences for the Γ limits below. We first provide an explicit construction that is based on a smoothing procedure using the heat kernels for $\mathbb{R}^d$ and $\mathcal{Y}$. After that we provide a second construction which is based in ideas in [Vis04] and involves the solutions of elliptic problems.

Proposition 2.9 *Let $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}^d$ as above. Then, for every function $(u_0, U_1) \in \mathrm{W}^{1,p}(\Omega) \times \mathrm{L}^p(\Omega; \mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$ there exists a family $(u_\varepsilon)_\varepsilon$ in $\mathrm{W}^{1,p}(\Omega)$ such that $u_\varepsilon \rightharpoonup u_0$ in $\mathrm{W}^{1,p}(\Omega)$ and that $\nabla u_\varepsilon \xrightarrow{\text{s2}} E\nabla u_0 + \nabla_y U_1$.*

Proof: It is sufficient to prove the result for $u_0 \equiv 0$, since we may shift any sequence by u_0. Note that by Prop. 2.4(f) we have $\mathcal{T}_\varepsilon \nabla u_0 \xrightarrow{\text{s2}} E\nabla u_0$.

Hence it suffices to find for each $V_1 \in \mathrm{L}^p(\Omega; \mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))$ a family $(v_\varepsilon)_\varepsilon$ such that

$$v_\varepsilon \rightharpoonup 0 \text{ in } \mathrm{W}^{1,p}(\Omega) \quad \textit{and} \quad \nabla v_\varepsilon \xrightarrow{\text{s2}} \nabla_y V_1 \text{ in } \mathrm{L}^p(\Omega \times \mathcal{Y}).$$

For this we use the heat kernels $H_{\mathbb{R}^d}$ and $H_\mathcal{Y}$ defined via

$$H_{\mathbb{R}^d}(t,\xi) = \tfrac{1}{(4\pi t)^{d/2}} \exp\big(|\xi|^2/(4t)\big) \quad \text{and} \quad H_\mathcal{Y}(t,\eta) = \textstyle\sum_{\lambda\in\Lambda} H_{\mathbb{R}^d}(t, \eta{+}\lambda).$$

For $t > 0$ we now define the functions

$$V(t,x,y) = \textstyle\int_{\mathbb{R}^d} \int_\mathcal{Y} H_{\mathbb{R}^d}(t, x{-}\xi) H_\mathcal{Y}(t, y{-}\eta)(V_1)_{\mathrm{ex}}(\xi,\eta)\, \mathrm{d}\eta\, \mathrm{d}\xi. \tag{2.11}$$

The classical semigroup theory for the parabolic equation $\partial_t V = \Delta_{\mathbb{R}^d} V + \Delta_\mathcal{Y} V$ implies $V(t,\cdot) \in \mathrm{C}^\infty(\mathbb{R}^d \times \mathcal{Y})$ for $t > 0$ and

$$\begin{aligned} &\forall\, \alpha, \beta \in \mathrm{N}_0^d\ \exists\, C_{\alpha,\beta} > 0\ \forall\, t > 0: \quad \|\mathrm{D}_x^\alpha \mathrm{D}_y^\beta V(t,\cdot)\|_{\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})} \leq C \,/\, t^{(|\alpha|+|\beta|)/2}, \\ &\delta(t) = \|\nabla_y V(t,\cdot) - \nabla_y V_1\|_{\mathrm{L}^p(\mathbb{R}^d \times \mathcal{Y})} \to 0 \text{ for } t \searrow 0. \end{aligned}$$

We now define the two-scale function $v(\varepsilon,t,\cdot) \in \mathrm{W}^{1,p}(\Omega)$ via $v(\varepsilon,t,x) = \varepsilon\, V(t,x,\{\frac{x}{\varepsilon}\}_\mathcal{Y})$. We will choose $t = t_\varepsilon$ suitably to define $v_\varepsilon = v(\varepsilon, t_\varepsilon, \cdot)$. As a first result we obtain

$$\|v_\varepsilon\|_{\mathrm{L}^p(\Omega)} \leq \varepsilon |\Omega|^{1/p} \|V(t_\varepsilon,\cdot)\|_{\mathrm{C}^0(\Omega\times\mathcal{Y})} \leq \varepsilon C_{\mathrm{Sob}} \|V(t_\varepsilon,\cdot)\|_{\mathrm{W}^{k,p}(\Omega\times\mathcal{Y})} \leq C\varepsilon t_\varepsilon^{-k/2},$$

where $k > (d{+}d)/p$ and C_{Sob} is the corresponding embedding constant for $\mathrm{W}^{k,p}(\Omega\times\mathcal{Y})$ into $\mathrm{C}^0(\Omega\times\mathcal{Y})$. Below we will choose t_ε such that $\varepsilon\, t_\varepsilon^{-k/2} \to 0$ for $\varepsilon \to 0$ and thus we conclude $v_\varepsilon \to 0$ in $\mathrm{L}^p(\Omega)$.

For the gradients we obtain $\nabla v_\varepsilon(\varepsilon,x) = \varepsilon \nabla_x V(t_\varepsilon, x, \{\frac{x}{\varepsilon}\}_{\mathcal{Y}}) + \nabla_y V(t_\varepsilon, x, \{\frac{x}{\varepsilon}\}_{\mathcal{Y}})$. Using $\|\mathcal{T}_\varepsilon \nabla v_\varepsilon - \nabla_y V_1\|_{\mathrm{L}^p(\Omega\times\mathcal{Y})} \leq \|\mathcal{T}_\varepsilon v_\varepsilon - \nabla_y V(t_\varepsilon,\cdot)\|_{\mathrm{L}^p(\Omega\times\mathcal{Y})} + \delta(t_\varepsilon)$ with $\delta(t_\varepsilon)\to 0$ and recalling $\mathcal{T}_\varepsilon u(x,y) = (u\circ S_\varepsilon)(x,y) = u(\mathcal{N}_\varepsilon(x){+}\varepsilon y)$ it suffices to estimate

$$
\begin{aligned}
&|(\mathcal{T}_\varepsilon \nabla v_\varepsilon)(x,y) - V(t_\varepsilon,x,y)| \\
&\leq \varepsilon |\nabla_x V(t_\varepsilon, \mathcal{N}_\varepsilon(x), y)| + |\nabla_y V(t_\varepsilon, \mathcal{N}_\varepsilon(x), y) - \nabla_y V(t_\varepsilon, x, y)| \\
&\leq \varepsilon \|\nabla_x V(t_\varepsilon,\cdot)\|_{\mathrm{C}^0(\Omega\times\mathcal{Y})} + \varepsilon \mathrm{diam}(Y) \|\nabla_x \nabla_y V(t_\varepsilon,\cdot)\|_{\mathrm{C}^0(\Omega\times\mathcal{Y})} \\
&\leq C_1 \varepsilon C_{\mathrm{Sob}} \|V(t_\varepsilon,\cdot)\|_{\mathrm{W}^{k+2,p}(\Omega\times\mathcal{Y})} \;\leq\; C_2 \varepsilon\, t_\varepsilon^{-(k+2)/2}.
\end{aligned}
$$

Letting $t_\varepsilon = \varepsilon^\gamma$ with $\gamma \in (0, 2/(2{+}k))$ we obtain $\mathcal{T}_\varepsilon v_\varepsilon \rightharpoonup V_1$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ and the result is proved. ■

The second construction is more direct and allows us to do unfolding and folding as well. It is based on [Vis04, Thm. 6.1] but we take care of the problems with the boundary $\partial\Omega$. For simplicity, we restrict to the case $p = 2$ and assume Dirichlet boundary conditions. We define the intermediate space $\mathcal{L} = \mathrm{L}^2(\Omega)\times \mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})^d$, the two-scale Hilbert space $\mathcal{H} = \mathrm{H}^1_0(\Omega)\times \mathrm{L}^2(\mathbb{R}^d, \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$, and the two norm-preserving linear operators

$$
\mathbb{T}_\varepsilon : \left\{ \begin{array}{ccc} \mathrm{H}^1_0(\Omega) & \to & \mathcal{L}, \\ u & \mapsto & (u, \mathcal{T}_\varepsilon \nabla u), \end{array} \right. \qquad
\mathbb{F}_\varepsilon : \left\{ \begin{array}{ccc} \mathcal{H} & \to & \mathcal{L}, \\ (u_0, U_1) & \mapsto & (u_0, (E\nabla_x u_0{+}\nabla_y U_1)_{\mathrm{ex}}), \end{array} \right.
$$

For norm-preservation of $\mathbb{F}_\varepsilon$ we equip $\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})$ with the norm $\|U_1\|^2_{\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})} = \|\nabla_y U_1\|_{\mathrm{L}^2(\mathcal{Y})}$.

In particular the images $\mathcal{X}^\varepsilon_{\mathbb{T}} := \mathbb{T}_\varepsilon \mathrm{H}^1(\Omega)$ and $\mathcal{X}^\varepsilon_{\mathbb{F}} = \mathbb{F}_\varepsilon \mathcal{H}$ are closed subspaces of $\mathrm{L}^2_{\mathrm{av}}(\mathcal{Y})$. We let $\mathbb{Q}^\varepsilon_{\mathbb{T}}$ and and $\mathbb{Q}^\varepsilon_{\mathbb{F}}$ be the orthogonal projections onto $\mathcal{X}^\varepsilon_{\mathbb{T}}$ and $\mathcal{X}^\varepsilon_{\mathbb{F}}$, respectively. Then, we are able to define a *gradient unfolding operator* $\mathcal{T}^{(1)}_\varepsilon = \mathbb{F}^{-1}_\varepsilon \mathbb{Q}^\varepsilon_{\mathbb{F}} \mathbb{T}_\varepsilon : \mathrm{H}^1_0(\Omega) \to \mathcal{H}$ and a *gradient folding operator* $\mathcal{G}_\varepsilon$ via

$$
\mathcal{G}_\varepsilon : \left\{ \begin{array}{ccc} \mathcal{H} & \to & \mathrm{H}^1_0(\Omega), \\ (u_0, U_1) & \mapsto & \mathbb{T}^{-1}_\varepsilon\Big(\mathbb{Q}^\varepsilon_{\mathbb{T}}\big(\mathbb{F}_\varepsilon(u_0,U_1)\big)\Big). \end{array} \right. \tag{2.12}
$$

As the operators $\mathcal{T}^{(1)}_\varepsilon$ and $\mathcal{G}_\varepsilon$ are compositions of norm-preserving operators and orthogonal projections they have a norm not exceeding 1. The following result shows that the definition of $\mathcal{G}_\varepsilon$ is such that it relates to solving an auxiliary elliptic problem and that it provides a recovery sequence with strongly two-scale convergent gradients.

Proposition 2.10 *For given $(u_0, U_1) \in \mathcal{H}$ the function $\mathcal{G}_\varepsilon(u_0, U_1)$ is uniquely characterized as the solution $v \in \mathrm{H}^1_0(\Omega)$ of the weak elliptic problem*

$$
\int_\Omega (v - u_0)\, w + \big(\nabla v - \mathcal{F}_\varepsilon(E\nabla_x u_0{+}\nabla_y U_1)\big)\cdot \nabla w \,\mathrm{d}x = 0 \textit{ for all } w \in \mathrm{H}^1_0(\Omega). \tag{2.13}
$$

Moreover, for $\varepsilon \to 0$, we have the convergences

$$
\mathcal{G}_\varepsilon(u_0,U_1) \rightharpoonup u_0 \textit{ in } \mathrm{H}^1_0(\Omega) \quad \textit{and} \quad \nabla \mathcal{G}_\varepsilon(u_0,U_1) \xrightarrow{\mathrm{s2}} E\nabla_x u_0 + \nabla_y U_1 \textit{ in } \mathrm{L}^2(\Omega\times\mathcal{Y}). \tag{2.14}
$$

Proof: At first, we fix ε and let $v = \mathcal{G}_\varepsilon(u_0, U_1)$ is such that $\mathbb{T}_\varepsilon v$ is the orthogonal projection of $\mathbb{F}_\varepsilon(u_0, U_1)$ onto $\mathcal{X}^\varepsilon_\mathbb{T} = \mathbb{T}_\varepsilon \mathrm{H}^1(\Omega)$. Denoting by $\langle\cdot,\cdot\rangle_\mathcal{L}$ the scalar product in $\mathcal{L}$ this means that for all $w \in \mathrm{H}^1_0(\Omega)$ we have

$$\begin{aligned} 0 &= \langle\, \mathbb{T}_\varepsilon v - \mathbb{F}_\varepsilon(u_0, U_1)\,, \mathbb{T}_\varepsilon w\rangle_\mathcal{L} \\ &= \textstyle\int_\Omega (v-u_0)\, w \,\mathrm{d}x + \int_{\mathbb{R}^d\times\mathcal{Y}} (\mathcal{T}_\varepsilon(\nabla v) - \nabla_x u_0 - \nabla_y U_1)\cdot \mathcal{T}_\varepsilon(\nabla w)\,\mathrm{d}y\,\mathrm{d}x \\ &= \textstyle\int_\Omega (v-u_0)\, w\,\mathrm{d}x + \int_\Omega (\nabla v)\cdot(\nabla w)\,\mathrm{d}x - \int_\Omega \mathcal{F}_\varepsilon(\nabla_x u_0 + \nabla_y U_1)\cdot \nabla w\,\mathrm{d}x. \end{aligned}$$

Here we use the definitions of $\mathbb{T}_\varepsilon$ and $\mathbb{F}_\varepsilon$ as well as the properties of $\mathcal{T}_\varepsilon$ in Prop. 2.1(a) and (c). Clearly the last line give (2.13).

To show the desired convergence we recall that the operators $\mathcal{G}_\varepsilon : \mathcal{H} \to \mathrm{H}^1(\Omega)$ have a norm bounded by 1. Hence, it suffices to proof the desired convergence on a dense subset, namely $\mathcal{C} = \mathrm{C}^2_\mathrm{c}(\Omega)\times\mathrm{C}^2_\mathrm{c}(\Omega\times\mathcal{Y})$. For $(u_0, U_1)\in\mathcal{C}$ we write $u_\varepsilon = (\mathcal{G}_\varepsilon(u_0, U_1))$ in the form

$$u_\varepsilon(x) = v_\varepsilon(x) + g_\varepsilon(x) \quad \text{with } v_\varepsilon(x) = u_0(x) + \varepsilon\, U_1(x, \{\tfrac{x}{\varepsilon}\}_\mathcal{Y}),$$

where g_ε is the solution of the weak elliptic problem

$$\begin{aligned} &\textstyle\int_\Omega g_\varepsilon w + \nabla g_\varepsilon\cdot\nabla w\,\mathrm{d}x = \ell_\varepsilon(w) \quad \text{for all } w\in \mathrm{H}^1_0(\Omega), \\ &\text{where } \ell_\varepsilon(w) = \textstyle\int_\Omega (u_0 - v_\varepsilon)w + \big(\mathcal{F}_\varepsilon(E\nabla_x u_0 + \nabla_y U_1) - \nabla v_\varepsilon\big)\cdot\nabla w\,\mathrm{d}x. \end{aligned} \tag{2.15}$$

Clearly, the family $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $\mathrm{H}^1_0(\Omega)$. Moreover, we have $\|u_0 - v_\varepsilon\|_{\mathrm{L}^\infty} \leq C_1\varepsilon$ which implies $v_\varepsilon \rightharpoonup u_0$ in $\mathrm{H}^1_0(\Omega)$. Using $\nabla v_\varepsilon(x) = \nabla u_0(x) + \nabla_y U_1(x, \{\frac{x}{\varepsilon}\}_\mathcal{Y}) + \varepsilon\nabla_x U_1(x, \{\frac{x}{\varepsilon}\}_\mathcal{Y})$ and $(u_0, U_1)\in\mathcal{C}$ we have $\|\mathcal{T}_\varepsilon\nabla v_\varepsilon - (E\nabla_x u_0 - \nabla_y U_1)_\mathrm{ex}\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})} \leq C_2\varepsilon$, i.e., $\nabla v_\varepsilon \xrightarrow{\mathrm{s2}} E\nabla_x u_0 - \nabla_y U_1$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})$.

Hence, it suffices to show $\|g_\varepsilon\|_{\mathrm{H}^1(\Omega)} \to 0$, as this implies $\nabla g_\varepsilon \xrightarrow{\mathrm{s2}} 0$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})$. From (2.15) we have

$$\begin{aligned} \|g_\varepsilon\|^2_{\mathrm{H}^1(\Omega)} &\leq \|(u_0 - v_\varepsilon, \mathcal{F}_\varepsilon(E\nabla_x u_0 + \nabla_y U_1) - \nabla v_\varepsilon)\|^2_\mathcal{L} \\ &= \|u_0 - v_\varepsilon\|^2_{\mathrm{L}^2(\Omega)} + \|E\nabla_x u_0 + \nabla_y U_1) - \mathcal{T}_\varepsilon\nabla\|^2_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})} \leq C_3\varepsilon^2. \end{aligned}$$

This finishes the proof of the convergence result (2.14). ■

Finally, let us note that we may extend the construction to functions $u, u_0 \in \mathrm{H}^1(\Omega)$, namely without Dirichlet boundary conditions. In fact, for $u_0\in\mathrm{H}^1(\Omega)$ we obtain a recovery sequence $u_\varepsilon = u_0 + \mathcal{G}_\varepsilon(0, U_1)$ by simply employing the above result and Prop. 2.4(f).

2.5 Two-scale Γ-limits

We now discuss the question how functionals behave under two-scale convergence. This relates strongly to the question of homogenization. The two-scale convergence results we present here are well-known in the literature, but often they are not easily accessible. Thus, we repeat here some simple versions which can be easily deduced by our theory and which are sufficient for our application in the next section. For more advanced results we refer to [All92, CD99, CDD06].

Let $W : \mathcal{Y}\times\mathbb{R}^m \to \mathbb{R}_\infty := \mathbb{R}\cup\{\infty\}$ be a normal integrand, which means that for each $u \in \mathbb{R}^m$ the function $y \mapsto W(y,u)$ is measurable and that for a.e. $y \in \mathcal{Y}$ the function $u \mapsto W(y,u)$ is lower semi-continuous. Recalling our definitions of $\mathcal{T}_\varepsilon$, $\mathcal{F}_\varepsilon$, and of $[\Omega\times\mathcal{Y}]_\varepsilon$ (cf. the line above (2.3)) we obtain the following central formulas

$$\int_\Omega W\left(\{\tfrac{x}{\varepsilon}\}_{\mathcal{Y}}, u(x)\right) \mathrm{d}x = \int_{[\Omega\times\mathcal{Y}]_\varepsilon} W(y, \mathcal{T}_\varepsilon u\,(x,y))\,\mathrm{d}y\,\mathrm{d}x \text{ for all } u \in \mathrm{L}^p(\Omega). \tag{2.16}$$

This identity follows by a simple decomposition of Ω_ε^+ into small cells $\mathcal{N}_\varepsilon(\xi)+\varepsilon Y$ and using the definition of $\mathcal{T}_\varepsilon$.

The next two lemmas are the basis of the two-scale Γ-convergence for the functionals

$$\mathcal{W}_\varepsilon\colon \begin{cases} \mathrm{L}^p(\Omega) \to & \mathbb{R}_\infty, \\ \quad u \quad \mapsto & \int_\Omega W(\{\tfrac{x}{\varepsilon}\}_{\mathcal{Y}}, u(x))\,\mathrm{d}x \end{cases} \quad \text{and} \quad \boldsymbol{W}\colon \begin{cases} \mathrm{L}^p(\Omega\times\mathcal{Y}) \to & \mathbb{R}_\infty, \\ \quad U \quad \mapsto & \int_{\Omega\times\mathcal{Y}} W(y, U(x,y))\,\mathrm{d}y\,\mathrm{d}x. \end{cases}$$

Lemma 2.11 *Assume that $p \in (1,\infty)$, that Ω is as above, and that $W : \mathcal{Y}\times\mathbb{R}^m \to \mathbb{R}_\infty$ is a convex normal integrand, i.e., $W(y,\cdot) : \mathbb{R}^m \to \mathbb{R}_\infty$ is convex for a.e. $y \in \mathcal{Y}$. Moreover, let W be bounded from below by $W(y,u) \geq -h(y)$ for a.e. $y \in \mathcal{Y}$ with $h \in \mathrm{L}^1(\Omega)$. Then,*

$$u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} U \text{ in } \mathrm{L}^p(\Omega\times\mathcal{Y}) \quad \Longrightarrow \quad \boldsymbol{W}(U) \;\leq\; \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon).$$

Proof: We choose an increasing sequence A_k, $k \in \mathbb{N}$ of open subsets of Ω such that

$$A_k \subset A_{k+1} \Subset \Omega \quad \text{ and } |\Omega\backslash A_k| \to 0 \text{ for } k \to \infty.$$

Then, for each k there exists ε_0 such that $A_k\times\mathcal{Y} \subset \Omega_\varepsilon^-\times\mathcal{Y} \subset [\Omega\times\mathcal{Y}]_\varepsilon$ for $\varepsilon \in (0,\varepsilon_0)$.

Now consider a family with $u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} U$. Using (2.16) and $W \geq 0$ we find

$$\mathcal{W}_\varepsilon(u_\varepsilon) = \int_{[\Omega\times\mathcal{Y}]_\varepsilon} W(y, \mathcal{T}_\varepsilon u_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x \geq \int_{A_k\times\mathcal{Y}} W(y, \mathcal{T}_\varepsilon u_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x \;-\; \int_{\Omega\backslash A_k} h(y)\,\mathrm{d}y.$$

In the right-hand side we may pass to the limit inferior for $\varepsilon \to 0$, as $\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup U$ in $\mathrm{L}^p(\Omega\times\mathcal{Y})$ and as W is a convex normal integrand. We obtain

$$\liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon) \geq \int_{A_k\times\mathcal{Y}} W(y, U(x,y))\,\mathrm{d}y\,\mathrm{d}x \;-\; \int_{\Omega\backslash A_k} h(y)\,\mathrm{d}y.$$

Since k was arbitrary, we may consider now the limit $k \to \infty$. The second term tends to 0 as $|\Omega\backslash A_k| \to 0$ whereas the first term converges to $\boldsymbol{W}(U)$. ■

Lemma 2.12 *Assume that $p \in (1,\infty)$, and that Ω is as above.*
(a) Let $W : \mathcal{Y}\times\mathbb{R}^m \to \mathbb{R}$ be a Caratheodory function, i.e., $W(y,\cdot)$ is continuous for a.e. $y \in \mathcal{Y}$ and $W(\cdot,u)$ is measurable for each $u \in \mathbb{R}^d$. Moreover, assume that there is a function $h \in \mathrm{L}^1(\mathcal{Y})$ and a constant $C > 0$ such that $|W(y,u)| \leq h(y) + C(1+|u|)^p$ for all $u \in \mathbb{R}^m$ and a.e. $y \in \mathcal{Y}$. Then,

$$u_\varepsilon \overset{\mathrm{s2}}{\longrightarrow} U \text{ in } \mathrm{L}^p(\Omega\times\mathcal{Y}) \quad \Longrightarrow \quad \boldsymbol{W}(U) = \lim_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon).$$

In particular, this implies that $\mathcal{W}_\varepsilon(\mathcal{F}_\varepsilon U_{\rm ex}) \to \boldsymbol{W}(U)$.
(b) Let $W : \mathcal{Y}{\times}\mathbb{R}^m \to \mathbb{R}_\infty$ *be a normal integrand such that for a.e.* $y \in \mathcal{Y}$ *the function* $W(y,\cdot)$ *is convex and that* $|W(y,0)| \le h(y)$ *for some* $h \in \mathrm{L}^1(\mathcal{Y})$*. Then,*

$$\boldsymbol{W}(U) = \lim_{\varepsilon\to 0} \mathcal{W}_\varepsilon(\mathcal{F}_\varepsilon U_{\rm ex}) \quad \textit{for all } U \in \mathrm{L}^p(\Omega{\times}\mathcal{Y}).$$

Proof: ad (a). We let $U_\varepsilon = \mathcal{T}_\varepsilon u_\varepsilon$, then formula (2.16) gives

$$\begin{aligned}
&\mathcal{W}_\varepsilon(u_\varepsilon) = \textstyle\int_{[\Omega\times\mathcal{Y}]_\varepsilon} W(y, U_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x = \boldsymbol{W}(U) + I_1^\varepsilon + I_\varepsilon^2 \\
&\text{with } I_1^\varepsilon = \textstyle\int_{\Omega\times\mathcal{Y}} \big[W(y,U_\varepsilon(x,y)) - W(y,U(x,y))\big]\,\mathrm{d}y\,\mathrm{d}x = \boldsymbol{W}(U_\varepsilon) - \boldsymbol{W}(U), \\
&\text{and } \ I_2^\varepsilon = \textstyle\int_{[\Omega\times\mathcal{Y}]_\varepsilon} W(y,U_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x - \int_{\Omega\times\mathcal{Y}} W(y,U_\varepsilon(x,y))\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

We have $I_1^\varepsilon \to 0$ because of $U_\varepsilon \to U_{\rm ex}$ in $\mathrm{L}^p(\mathbb{R}^d{\times}\mathcal{Y})$ and the strong continuity of the functional $\boldsymbol{R}$. For the later property we use the continuity of $W(y,\cdot)$ and the growth restrictions, cf. [Dac89, Val88].

For $I_2^\varepsilon \to 0$ we note that both integrals have the same integrand. Moreover, the difference of the domains $\Omega{\times}\mathcal{Y}$ and $[\Omega{\times}\mathcal{Y}]_\varepsilon$ is contained in $B_\varepsilon = \big(\Omega_\varepsilon^+ \backslash \Omega_\varepsilon^-\big){\times}\mathcal{Y}$. By condition (2.4) the Lebesgue measure of this set tends to 0, whence $I_2^\varepsilon \to 0$ and we conclude

$$|I_2^\varepsilon| \le \textstyle\int_{B_\varepsilon} h(y) + C(1{+}|U_\varepsilon(x,y)|)^p\,\mathrm{d}y\,\mathrm{d}x \to 0,$$

where again $U_\varepsilon \to U$ is used to obtain the equi-integrability of $|U_\varepsilon|^p$.

ad (b). We again use (2.16) for $u = \mathcal{F}_\varepsilon U_{\rm ex}$ and note that $\mathcal{T}_\varepsilon \mathcal{F}_\varepsilon U_{\rm ex} = \chi_\varepsilon \mathcal{P}_\varepsilon U_{\rm ex}$ by Prop. 2.1(d). With this we find

$$\begin{aligned}
&\mathcal{W}_\varepsilon(\mathcal{F}_\varepsilon U_{\rm ex}) = \int_{[\Omega\times\mathcal{Y}]_\varepsilon} W(y, \mathcal{P}_\varepsilon U_{\rm ex}(x,y))\,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbb{R}^d\times\mathcal{Y}} \chi_\varepsilon(x,y) W(y, \mathcal{P}_\varepsilon U_{\rm ex}(x,y))\,\mathrm{d}y\,\mathrm{d}x \\
&\le_{(1)} \int_{\mathbb{R}^d\times\mathcal{Y}} \chi_\varepsilon(x,y) \fint_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} W(y, U_{\rm ex}(y,\xi))\,\mathrm{d}\xi\;\mathrm{d}y\,\mathrm{d}x \;=_{(2)} \int_{\mathbb{R}^d\times\mathcal{Y}} \chi_\varepsilon(\xi,y) W(y, U_{\rm ex}(y,\xi))\,\mathrm{d}y\,\mathrm{d}\xi \\
&\le_{(3)} \boldsymbol{W}(U) + \int_{(\Omega_\varepsilon^+\backslash\Omega)\times\mathcal{Y}} h(y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

For $\le_{(1)}$ we have used convexity of $W(y,\cdot)$ and Jensen's inequality. The equality $=_{(2)}$ uses the fact that the integrand is piecewise constant in x on each $\mathcal{N}_\varepsilon(x){+}\varepsilon Y$. For $\le_{(3)}$ we use $\chi_\varepsilon \le \chi_{\Omega\times\mathcal{Y}} + \chi_{(\Omega_\varepsilon^+\backslash\Omega)\times\mathcal{Y}}$ and $U_{\rm ex} = 0$ outside of $\Omega{\times}\mathcal{Y}$. Using $h \in \mathrm{L}^1(\mathcal{Y})$ and (2.4) we find $\limsup_\varepsilon \mathcal{W}_\varepsilon(\mathcal{F}_\varepsilon U_{\rm ex}) \le \boldsymbol{W}(U)$. The opposite inequality $\liminf_\varepsilon \mathcal{W}_\varepsilon(\mathcal{F}_\varepsilon U_{\rm ex}) \ge \boldsymbol{W}(U)$ was established in Lemma 2.11. ∎

The following result states that the two-scale functional $\boldsymbol{W}$ can be considered as the two-scale Γ-limit of the functionals $\mathcal{W}_\varepsilon$ in the sense of Mosco, i.e., it is the two-scale Γ-limit in the weak as well as in the strong topology.

Corollary 2.13 *Let* $p \in (1,\infty)$ *and* Ω *be as above. Moreover, let* $W : \mathcal{Y}{\times}\mathbb{R}^m \to \mathbb{R}$ *be a convex, normal integrand satisfying the bounds* $W(y,u) \ge -h(y)$ *and* $W(y,0) \le h(y)$ *for all* $u \in \mathbb{R}^m$ *and a.a.* $y \in \mathcal{Y}$ *with* $h \in \mathrm{L}^1(\mathcal{Y})$*. Then, we have*
(i) Lower estimate: $\quad u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} U$ *in* $\mathrm{L}^p(\Omega{\times}\mathcal{Y}) \implies \boldsymbol{W}(U) \le \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon)$.
(ii) Recovery sequence: $\forall\, U \in \mathrm{L}^p(\Omega{\times}\mathcal{Y})\ \exists\,(u_\varepsilon)_\varepsilon$: $u_\varepsilon \overset{\mathrm{s2}}{\to} U$ *and* $\boldsymbol{W}(U) = \lim_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon)$.

Remark 2.14 *It is possible to generalize the above results to the case that the density W also depends on the macroscopic variable $x \in \Omega$. The central identity (2.16) is easily generalized to*

$$\int_\Omega W_\varepsilon(x, u(x))\,\mathrm{d}x = \int_{[\Omega\times\mathcal{Y}]_\varepsilon} W_\varepsilon(\mathcal{S}_\varepsilon(x,y), \mathcal{T}_\varepsilon u(x,y))\,\mathrm{d}y\,\mathrm{d}x \quad \text{for all } u \in \mathrm{L}^p(\Omega).$$

Thus, if we want to realize a general Caratheodory functions $W : \Omega\times\mathcal{Y}\times\mathbb{R}^m \to \mathbb{R}_\infty$ in the two-scale limit functional $\boldsymbol{W}$, we define $\mathcal{W}_\varepsilon$ via the approximate energy density

$$W_\varepsilon(x,u) = \widehat{W}_\varepsilon(x, \left\{\tfrac{x}{\varepsilon}\right\}_\mathcal{Y}, u) \quad \text{with } \widehat{W}_\varepsilon(x,y,u) = ⨍_{\mathcal{N}_\varepsilon(x)+\varepsilon Y} W(\xi, y, u)\,\mathrm{d}\xi,$$

instead of the traditionally used $W(x, \left\{\frac{x}{\varepsilon}\right\}_\mathcal{Y}, u)$. Note that W_ε satisfies $W_\varepsilon(\mathcal{S}_\varepsilon(x,y), u) = \widehat{W}_\varepsilon(x,y,u) \to W(x,y,u)$ a.e. for $\varepsilon \to 0$.

Under some mild additional conditions it is then possible to pass to the limit as in Lemmas 2.11 and 2.12, see also Prop. 2.6. This also resolves the difficulties addressed in [CDG02, Thm. 2]. This will be subject of future research.

2.6 Two-scale cross-convergence

Finally we present a result concerning functional involving gradients. For families $((u_\varepsilon, z_\varepsilon))_\varepsilon$ in $\mathrm{W}^{1,p}(\Omega)\times\mathrm{L}^p(\Omega)$ we define the notions of *weak* and *strong two-scale cross-convergence* as follows:

$$(u_\varepsilon, z_\varepsilon) \overset{\text{w2c}}{\rightharpoonup} (u_0, U_1, Z) \text{ in } \boldsymbol{X}_p \quad \Longleftrightarrow \quad \begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{in } \mathrm{W}^{1,p}(\Omega), \\ \nabla u_\varepsilon \overset{\text{w2}}{\rightharpoonup} E\nabla u_0 + \nabla_y U_1 & \text{in } \mathrm{L}^p(\Omega\times\mathcal{Y}), \\ z_\varepsilon \overset{\text{w2}}{\rightharpoonup} Z & \text{in } \mathrm{L}^p(\Omega\times\mathcal{Y}), \end{cases}$$

$$(u_\varepsilon, z_\varepsilon) \overset{\text{s2c}}{\longrightarrow} (u_0, U_1, Z) \text{ in } \boldsymbol{X}_p \quad \Longleftrightarrow \quad \begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{in } \mathrm{W}^{1,p}(\Omega), \\ \nabla u_\varepsilon \overset{\text{s2}}{\longrightarrow} E\nabla u_0 + \nabla_y U_1 & \text{in } \mathrm{L}^p(\Omega\times\mathcal{Y}), \\ z_\varepsilon \overset{\text{s2}}{\longrightarrow} Z & \text{in } \mathrm{L}^p(\Omega\times\mathcal{Y}), \end{cases}$$

where $\boldsymbol{X}_p = \mathrm{W}^{1,p}(\Omega)\times\mathrm{L}^p(\Omega;\mathrm{W}^{1,p}_{\mathrm{av}}(\mathcal{Y}))\times\mathrm{L}^p(\Omega\times\mathcal{Y})$. The final result on two-scale Γ-convergence now provides relations between the functionals

$$\begin{aligned} \Phi_\varepsilon(u,z) &= \int_\Omega \phi(\left\{\tfrac{x}{\varepsilon}\right\}_\mathcal{Y}, u(x), \nabla u(x), z(x))\,\mathrm{d}x \qquad \text{and} \\ \boldsymbol{\Phi}_\varepsilon(u_0, U_1, Z) &= \int_{\Omega\times\mathcal{Y}} \phi(y, u_0(x), \nabla u_0(x){+}\nabla_y U_1(x,y), Z(x,y))\,\mathrm{d}x. \end{aligned}$$

Proposition 2.15 *Let $p \in (1,\infty)$ and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Assume that $\phi : \mathcal{Y}\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\times\mathbb{R}^m \to \mathbb{R}$ is a Caratheodory function (measurable in $y \in \mathcal{Y}$ and continuous in $(u,F,z) \in \mathbb{R}^k\times\mathbb{R}^{k\times d}\times\mathbb{R}^m \to \mathbb{R}$) satisfying the bound $|\phi(y,u,A,z)| \leq h(y) + C(1{+}|u|{+}|A|{+}|z|)^p$ for $h \in \mathrm{L}^1(\mathcal{Y})$. Then, we have*

$$(u_\varepsilon, z_\varepsilon) \overset{\text{s2c}}{\longrightarrow} (u_0, U_1, Z) \text{ in } \boldsymbol{X}_p \quad \Longrightarrow \quad \Phi_\varepsilon(u_\varepsilon, z_\varepsilon) \to \boldsymbol{\Phi}(u_0, U_1, Z).$$

Moreover, if $\phi(y,\cdot)$ is convex for a.a. $y \in \mathcal{Y}$, we also have

$$(u_\varepsilon, z_\varepsilon) \overset{\text{w2c}}{\rightharpoonup} (u_0, U_1, Z) \text{ in } \boldsymbol{X}_p \quad \Longrightarrow \quad \boldsymbol{\Phi}(u_0, U_1, Z) \leq \liminf_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon, z_\varepsilon).$$

The proof is a direct consequence of combining Lemmas 2.11 and 2.12(a).

3 Evolutionary variational inequality

3.1 Abstract result

For the convenience of the reader we recall the standard existence and uniqueness results for evolutionary variational inequalities, see, e.g., [BS96, Vis94, Mie05]. We start with a Hilbert space $\mathcal{Q}$ with dual $\mathcal{Q}^*$ and dual pairing $\langle \cdot, \cdot \rangle : \mathcal{Q}^* \times \mathcal{Q} \to \mathbb{R}$ and a positive semidefinite operator $\mathcal{A} \in \mathrm{Lin}(\mathcal{Q}, \mathcal{Q}^*)$, i.e., $\mathcal{A} = \mathcal{A}^*$ and $\langle \mathcal{A}q, q \rangle \geq 0$ for all $q \in \mathcal{Q}$. For a function $\ell \in \mathrm{C}^1([0,T], \mathcal{Q}^*)$ we define the energy functional

$$\mathcal{E}(t,q) = \frac{1}{2}\langle \mathcal{A}q, q \rangle - \langle \ell(t), q \rangle.$$

Moreover, let a dissipation functional $\mathcal{R} : \mathcal{Q} \to [0, \infty]$ be given that is convex, lower semi-continuous and positively homogeneous of degree 1, viz.,

$$\mathcal{R}(\gamma q) = \gamma \mathcal{R}(q) \quad \text{for all } \gamma \geq 0 \text{ and } q \in \mathcal{Q}.$$

The *energetic formulation* (S) & (E) of the rate-independent hysteresis problem associated with $\mathcal{E}$ and $\mathcal{R}$ is based on the *global stability condition* (S) and the *energy balance* (E):

$$\begin{aligned} &(\mathrm{S}): \quad \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \widetilde{q}) + \mathcal{R}(\widetilde{q} - q(t)) \quad \text{for every } \widetilde{q} \in \mathcal{Q}, \\ &(\mathrm{E}): \quad \mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{R}}(q; [0,t]) = \mathcal{E}(0, q(0)) + \textstyle\int_0^t \partial_s \mathcal{E}(s, q(s))\,\mathrm{d}s, \end{aligned}$$

where $\mathrm{Diss}_{\mathcal{R}}(q; [r,s]) = \int_r^s \mathcal{R}(\dot{q}(t))\,\mathrm{d}t$ and $\partial_s \mathcal{E}(s, q(s)) = -\langle \dot{\ell}(s), q(s) \rangle$. We call $q : [0,T] \to \mathcal{Q}$ satisfying (S) and (E) for all $t \in [0,T]$ an *energetic solution* associated with $(\mathcal{E}, \mathcal{R})$.

The stability condition can be formulated in terms of the *sets of stable states*

$$\mathcal{S}(t) = \{\, q \in \mathcal{Q} \mid \mathcal{E}(t,q) \leq \mathcal{E}(t, \widehat{q}) + \mathcal{R}(\widehat{q} - q) \text{ for every } \widehat{q} \in \mathcal{Q} \,\}.$$

Now, (S) just means $q(t) \in \mathcal{S}(t)$.

There are several equivalent formulation for (S) & (E), for instance the subdifferential inclusion $0 \in \partial\mathcal{R}(\dot{q}(t)) + \mathrm{D}_q\mathcal{E}(t, q(t))$ or the variational inequality

$$\langle \mathcal{A}q(t) - \ell(t), v - \dot{q}(t) \rangle + \mathcal{R}(v) - \mathcal{R}(\dot{q}(t)) \geq 0 \quad \text{for every } v \in \mathcal{Q}. \tag{3.1}$$

For these equivalences, we refer to [MT04, Mie05], where also a proof of the following existence and uniqueness result can be found.

Theorem 3.1 *Let $\ell \in \mathrm{C}^1([0,T], \mathcal{Q}^*)$ and $q_0 \in \mathcal{S}(0)$. Moreover, assume that the following coercivity condition holds:*

$$\exists\, \alpha > 0\ \forall\, v \in \mathcal{Q} \text{ with } \mathcal{R}(v) < \infty: \quad \langle Av, v \rangle \geq \alpha \|v\|^2. \tag{3.2}$$

Then, the energetic problem (S) & (E) *has a unique solution $q \in \mathrm{C}^{\mathrm{Lip}}([0,T], \mathcal{Q})$ with*

$$\|q(t) - q(s)\|_{\mathcal{Q}} \leq \frac{\mathrm{Lip}_{\mathcal{Q}^*}(\ell)}{\alpha} |t - s| \quad \text{for all } s,t \in [0,T].$$

For the reader's convenience we repeat the main argument for the a priori estimate. Assume that for t the derivative $\dot q(t)$ exists. Using (3.1) with $v = 0$ we find $\langle \mathcal{A}q(t) - \ell(t), -\dot q(t)\rangle - \mathcal{R}(\dot q(t)) \le 0$. For a sequence $t_n \to t$ where (3.1) holds we test with $v = \mu\dot(t)$, divide by μ and consider the limit $\mu \to \infty$. Using 1-homogeneity of $\mathcal{R}$ we obtain $\langle \mathcal{A}q(t_n) - \ell(t_n), \dot q(t)\rangle + \mathcal{R}(\dot q(t)) \le 0$. Adding this to the above estimate gives

$$\langle \big(\mathcal{A}q(t_n) - \ell(t_n)\big) - \big(\mathcal{A}q(t) - \ell(t)\big), \dot q(t)\rangle \le 0.$$

Assuming $t_n > t$ we may divide the above inequality and pass to the limit to find $\langle \mathcal{A}\dot q(t) - \dot\ell(t), \dot q(t)\rangle \le 0$. For $t_n < t$ we find the opposite inequality. Since we may approach t by sequences from both sides, this implies $\langle \mathcal{A}\dot q(t), \dot q(t)\rangle = \langle \dot\ell(t), \dot q(t)\rangle$. Now, (3.2) leads to the desired result $\alpha\|\dot q(t)\| \le \|\ell(t)\|_*$.

3.2 Elastoplasticity with periodic coefficients

In this section we formulate the continuum mechanics that describes the rate-independent evolution of an elastoplastic body under prescribed loading. This model is the classical one introduced by Moreau and is still used in many engineering applications, cf. [Mor76, HR99].

The body occupies a domain $\Omega \subset \mathbb{R}^d$, which is assumed to be a nonempty connected bounded open set with Lipschitz boundary $\partial\Omega$. As above we have a length scale parameter ε and a periodicity lattice Λ with unit cell $Y \subset \mathbb{R}^d$. With $u : \Omega \to \mathbb{R}^d$ we denote the displacement of the body and $z : \Omega \to \mathbb{R}^m$ denotes a vector of internal variables which will account for inelastic effects due to plastic strains and plastic hardening.

The material properties are assumed to be periodic with respect to the microscopic lattice $\varepsilon\Lambda$, which leads to the dependence on $\big\{\frac{x}{\varepsilon}\big\}_{\mathcal{Y}}$. The energy functional $\mathcal{E}_\varepsilon$ is based on a stored-energy density $W : \mathcal{Y}\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^m \to \mathbb{R}; (y, \boldsymbol{e}, z) \mapsto W(y, \boldsymbol{e}, z)$, where $\mathbb{R}^{d\times d}_{\mathrm{sym}} = \{\, A \in \mathbb{R}^{d\times d} \,|\, A = A^{\mathsf{T}}\,\}$ and $\boldsymbol{e} = \boldsymbol{e}(u) = \frac12(\nabla u + \nabla u^{\mathsf{T}}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is the linearized strain tensor. With this, $\mathcal{E}_\varepsilon$ takes the form

$$\mathcal{E}_\varepsilon(t, u, z) = \int_\Omega W(\{\tfrac{x}{\varepsilon}\}_{\mathcal{Y}}, \boldsymbol{e}(u)(x), z(x))\,\mathrm{d}x - \langle \ell(t), u\rangle$$
$$\text{with } \langle \ell(t), u\rangle = \textstyle\int_\Omega u(x)\cdot f_{\mathrm{ap}}(t,x)\,\mathrm{d}x + \int_{\partial\Omega} u(\xi)\cdot g_{\mathrm{ap}}(t,\xi)\,\mathrm{d}\xi,$$

where f_{ap} and g_{ap} are the applied, time-dependent loading in the volume and on the surface, respectively. We assume that they satisfy $f_{\mathrm{ap}} \in \mathrm{C}^1([0,T], \mathrm{L}^2(\Omega;\mathbb{R}^d))$ and $g_{\mathrm{ap}} \in \mathrm{C}^1([0,T], \mathrm{L}^2(\partial\Omega;\mathbb{R}^d))$, such that $\ell \in \mathrm{C}^1([0,T], \mathrm{H}^1(\Omega;\mathbb{R}^d)^*)$.

For the stored energy W we assume that it is a quadratic form in $(\boldsymbol{e}, z)$, namely

$$W(y, \boldsymbol{e}, z) = \frac12 \big\langle\!\big\langle \mathbb{A}(y)\tbinom{\boldsymbol{e}}{z}, \tbinom{\boldsymbol{e}}{z} \big\rangle\!\big\rangle,$$

where $\mathbb{A}(y) : \mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^m \to \mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^m$ is a positive semidefinite linear operator and $\big\langle\!\big\langle \tbinom{\boldsymbol{e}}{z}, \tbinom{\widetilde{\boldsymbol{e}}}{\widetilde z} \big\rangle\!\big\rangle = \sum_{i,j=1}^d \boldsymbol{e}_{ij}\widetilde{\boldsymbol{e}}_{ij} + \sum_{k=1}^m z_k\widetilde z_k$ is the scalar product on $\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^m$.

The dissipation potential $\mathcal{R}_\varepsilon$ is defined via a dissipation density $\rho : \mathcal{Y}\times\mathbb{R}^m \to [0,\infty]$ in the form $\mathcal{R}_\varepsilon(\dot z) = \int_\Omega \rho(\big(\big\{\frac{x}{\varepsilon}\big\}_{\mathcal{Y}}\big), \dot z(x))\,\mathrm{d}x$. Rate-independence is imposed by assuming that

$\rho(y,\cdot)$ is positively homogeneous of degree 1 (for short: 1-homogeneous). Note that ρ is not assumed to be symmetric (i.e., $\rho(y,-\dot z)\neq\rho(y,\dot z)$ is allowed), since this freedom is necessary to model hardening.

Our precise assumptions on the material data $\mathbb{A}$ and ρ are

$$\mathbb{A}\in \mathrm{L}^\infty(\mathcal{Y},\mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^m)) \text{ with } \mathbb{A}(y)=\mathbb{A}(y)^{\mathsf{T}}\geq 0, \tag{3.3a}$$

$$\rho:\mathcal{Y}\to[0,\infty] \text{ is a convex, normal integrand and } \rho(y,\cdot) \text{ is 1-homogeneous,} \tag{3.3b}$$

$$\begin{aligned}&\exists\,\widehat\alpha>0\ \forall_{a.a.} y\in\mathcal{Y}\ \forall\binom{e}{z}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}^m \text{ with } \rho(y,z)<\infty:\\ &\qquad\qquad\left\langle\!\left\langle \mathbb{A}(y)\tbinom{e}{z},\tbinom{e}{z}\right\rangle\!\right\rangle\geq\widehat\alpha\left|\tbinom{e}{z}\right|^2.\end{aligned} \tag{3.3c}$$

Remark 3.2 *Here we describe the exact setting for the linearized theory of elastoplasticity which is the motivation of this work. However, in the sequel of the paper we do not rely on the further specifications given here.*

The basis of linearized elastoplasticity is the additive split of the strain into an elastic part $\boldsymbol{e}_{\mathrm{el}}=\boldsymbol{e}(u)-\boldsymbol{p}$ and an plastic part $\boldsymbol{p}=\mathbb{B}(y)z$, where $\mathbb{B}(y):\mathbb{R}^m\to\mathbb{R}^{d\times d}_{\mathrm{sym}}$ is a linear mapping. Then, W is taken in the form

$$W(y,\boldsymbol{e},z)=\big\langle\mathbb{C}(y)(\boldsymbol{e}-\mathbb{B}(y)z),\boldsymbol{e}-\mathbb{B}(y)z\big\rangle_{d\times d}+\langle\mathbb{H}(y)z,z\rangle_m,$$

where $\mathbb{C}(y):\mathbb{R}^{d\times d}_{\mathrm{sym}}\to\mathbb{R}^{d\times d}_{\mathrm{sym}}$ is the symmetric (fourth order) elasticity tensor and $\mathbb{H}(y)$ denotes the hardening tensor. This means that $\mathbb{A}$ has the block structure $\left(\begin{smallmatrix}\mathbb{A} & \mathbb{C}\mathbb{B}\\ -\mathbb{B}^*\mathbb{C} & \mathbb{H}+\mathbb{B}^*\mathbb{C}\mathbb{B}\end{smallmatrix}\right)$. The typical case of isotropic hardening may be written in the way that $z=(\boldsymbol{p},h)$, where $\boldsymbol{p}\in(\mathbb{R}^{d\times d}_{\mathrm{sym}})_0=\{\,A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}\,|\,\mathrm{tr}\,A=0\,\}$ is the (deviatoric) plastic strain (i.e., $\mathbb{B}(y)(\boldsymbol{p},h)=\boldsymbol{p}$) and $h\in\mathbb{R}$ is the isotropic hardening parameter and $\mathbb{H}(y)$ is taken as $\kappa(y)>0$. Moreover, ρ is assumed to have the form

$$\rho(y,(\dot{\boldsymbol{p}},\dot h))=\begin{cases} r(y)\dot h & \text{for } \dot h\geq 0 \text{ and } \dot{\boldsymbol{p}}\in\dot h\,\Sigma(y),\\ \infty & \text{otherwise,}\end{cases}$$

where $r(y)>0$ and $\Sigma(y)\subset(\mathbb{R}^{d\times d}_{\mathrm{sym}})^*_0$ is the compact and convex elastic domain (with $\partial\Sigma(y)$ being the yield surface) at the point $y\in\mathcal{Y}$ for the the initial hardening state $h=1$.

The coercivity assumption (3.3c) then follows if we assume that there exist positive constants c and C such that for a.a. $y\in\mathcal{Y}$ we have the estimates

$$\kappa(y)\geq c,\ \langle\mathbb{C}(y)\boldsymbol{e},\boldsymbol{e}\rangle\geq c|\boldsymbol{e}|^2 \text{ for all } \boldsymbol{e},\ \ |\boldsymbol{\sigma}|\leq C \text{ for all } \boldsymbol{\sigma}\in\Sigma(y).$$

Note that the restriction $\rho(y,(\boldsymbol{p},h))<\infty$ implies $|\boldsymbol{p}|\leq Ch$.

Finally, we fix the function spaces by prescribing Dirichlet boundary conditions $u=0$ along the part Γ_{Dir} of $\partial\Gamma$. This defines the underlying Hilbert space

$$\mathcal{Q}=\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d\times\mathrm{L}^2(\Omega)^m \qquad \text{with } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)=\{\,u\in\mathrm{H}^1(\Omega)\,|\,u_{\Gamma_{\mathrm{Dir}}}=0\,\}.$$

The domain Ω and the Dirichlet boundary part Γ_{Dir} are specified further in the next result to guarantee coercivity of the energy $\mathcal{E}_\varepsilon$.

Proposition 3.3 (Korn's inequality) *Let $\Omega \subset \mathbb{R}^d$ be a connected, open, bounded set with Lipschitz boundary Γ. Moreover, let Γ_{Dir} be a measurable subset of Γ, such that $\int_{\Gamma_{\mathrm{Dir}}} 1 \,\mathrm{d}a > 0$. Then there exists a constant $C_{\mathrm{Korn}} > 0$, such that*

$$\int_\Omega |\boldsymbol{e}(u)|^2 \,\mathrm{d}x \geq C_{\mathrm{Korn}} \|u\|^2_{\mathrm{H}^1(\Omega)} \quad \textit{for all} \;\; u \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d. \tag{3.4}$$

Clearly, we may write $\mathcal{E}_\varepsilon(t, \boldsymbol{e}, z) = \frac{1}{2}\langle\!\langle \mathcal{A}_\varepsilon \binom{u}{z}, \binom{u}{z} \rangle\!\rangle - \langle \widetilde{\ell}(t), \binom{\boldsymbol{e}}{z} \rangle$, where $\mathcal{A}_\varepsilon : \mathcal{Q} \to \mathcal{Q}^*$ is symmetric and positive semi-definite. Moreover, combining assumption (3.3c) and Korn's inequality, we find for all $\binom{\boldsymbol{e}}{z} \in \mathcal{Q}$ with $\mathcal{R}_\varepsilon(z) < \infty$ the coercivity estimate

$$\langle\!\langle \mathcal{A}_\varepsilon \tbinom{u}{z}, \tbinom{u}{z} \rangle\!\rangle \geq \widehat{\alpha} \| \tbinom{\boldsymbol{e}(u)}{z} \|^2_{\mathrm{L}^2(\Omega)} \geq \alpha \| \tbinom{u}{z} \|^2_{\mathcal{Q}} \quad \text{with } \alpha = \widehat{\alpha} \min\{1, C_{\mathrm{Korn}}\}. \tag{3.5}$$

We call $q_\varepsilon = (u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}$ an ***energetic solution*** associated with $(\mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, if for all $t \in [0,T]$ the *stability condition* (S^ε) and the *energy balance* (E^ε) hold:

$$\begin{aligned} &(\mathrm{S}^\varepsilon) \;\; \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) \leq \mathcal{E}_\varepsilon(t, \widetilde{u}, \widetilde{z}) + \mathcal{R}_\varepsilon(\widetilde{z} - z_\varepsilon(t)) \quad \text{for every } (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}, \\ &(\mathrm{E}^\varepsilon) \;\; \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \textstyle\int_0^t \mathcal{R}_\varepsilon(\dot{z}_\varepsilon(s)) \,\mathrm{d}s = \mathcal{E}_\varepsilon(0, u_\varepsilon(0), z_\varepsilon(0)) - \textstyle\int_0^t \langle \ell(s), u(s) \rangle \,\mathrm{d}s. \end{aligned} \tag{3.6}$$

Applying the abstract Theorem 3.1 we immediately obtain the following existence and uniqueness result which contains an a priori Lipschitz bound that is independent of $\varepsilon > 0$.

Proposition 3.4 *Let $\ell \in \mathrm{C}^{\mathrm{Lip}}([0,T], (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$. Then for all $\varepsilon > 0$ and all stable $(u^0_\varepsilon, z^0_\varepsilon) \in \mathcal{Q}$ there exists a unique solution $(u_\varepsilon, z_\varepsilon) \in \mathrm{C}^{\mathrm{Lip}}([0,T], \mathcal{Q})$ of (S^ε) & (E^ε) with $(u_\varepsilon(0), z_\varepsilon(0)) = (u^0_\varepsilon, z^0_\varepsilon)$. Moreover, all these solutions satisfy*

$$\|(u_\varepsilon(t), z_\varepsilon(t)) - (u_\varepsilon(s), z_\varepsilon(s))\|_{\mathcal{Q}} \leq \tfrac{\mathrm{Lip}_{\mathcal{Q}^*}((\ell,0))}{\alpha} |t-s| \quad \textit{for all } t, s \in [0,T], \tag{3.7}$$

where α is defined in (3.5) *and is independent of ε.*

3.3 The two-scale homogenized problem

Instead of the functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ we may consider their two-scale limits. As the energy storage functional depends on the gradient of u, we use the notion of two-scale cross-convergence introduced in Section 2.6 on the space

$$\boldsymbol{Q} = \boldsymbol{H} \times \boldsymbol{Z} \text{ with } \boldsymbol{H} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega, \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))^d \text{ and } \boldsymbol{Z} = \mathrm{L}^2(\Omega \times \mathcal{Y})^m.$$

We use $U = (u_0, U_1)$ for the elements in $\boldsymbol{H}$ and Z for the internal elements lying in $\boldsymbol{Z}$. The functionals $\boldsymbol{E}$ and $\boldsymbol{R}$ are defined via

$$\begin{aligned} &\boldsymbol{E}(t, U, Z) = \textstyle\int_{\Omega \times \mathcal{Y}} \frac{1}{2} \langle\!\langle \mathbb{A}(y) \binom{\widehat{\boldsymbol{e}}(U)}{Z}, \binom{\widehat{\boldsymbol{e}}(U)}{Z} \rangle\!\rangle - \langle \ell(t), u_0 \rangle, \\ &\quad \text{where } \widehat{\boldsymbol{e}}(U) = \boldsymbol{e}_x(u_0) + \boldsymbol{e}_y(U_1) = \tfrac{1}{2}(\nabla_x u_0 + (\nabla_x u_0)^\mathsf{T}) + \tfrac{1}{2}(\nabla_y U_1 + (\nabla_y U_1)^\mathsf{T}), \\ &\boldsymbol{R}(Z) = \textstyle\int_{\Omega \times \mathcal{Y}} \rho(y, Z(x,y)) \,\mathrm{d}y \,\mathrm{d}x. \end{aligned}$$

Again we define the energetic formulation for $\boldsymbol{E}$ and $\boldsymbol{R}$ on $\boldsymbol{Q}$ via the global stability condition (**S**) and the energy balance (**E**). As above, a mapping $(U, Z) : [0,T] \to \boldsymbol{H} \times \boldsymbol{Z} = \boldsymbol{Q}$ is called an energetic solution associated with $\boldsymbol{E}$ and $\boldsymbol{R}$ if for all $t \in [0,T]$ we have

$$\begin{aligned} &(\mathbf{S}) \quad \boldsymbol{E}(t, U(t), Z(t)) \leq \boldsymbol{E}(t, \widetilde{U}, \widetilde{Z}) + \boldsymbol{R}(\widetilde{Z} - Z(t)) \quad \text{for all } (\widetilde{U}, \widetilde{Z}) \in \boldsymbol{H} \times \boldsymbol{Z}, \\ &(\mathbf{E}) \quad \boldsymbol{E}(t, U(t), Z(t)) + \textstyle\int_0^t \boldsymbol{R}(\dot{Z}(s)) \,\mathrm{d}s = \boldsymbol{E}(0, U(0), Z(0)) - \textstyle\int_0^t \langle \ell(s), u_0(s) \rangle \,\mathrm{d}s. \end{aligned} \tag{3.8}$$

Using the abstract existence Theorem 3.1 we again obtain the following result as soon as we have established the coercivity assumption (3.2) for the energy $\boldsymbol{E}$.

Proposition 3.5 *Let $\ell \in \mathrm{C}^{\mathrm{Lip}}([0,T],(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$. Then for all stable $Q^0=(U^0,Z^0)\in \boldsymbol{Q}$, $(\mathbf{S})\,\&\,(\mathbf{E})$ has a unique solution $Q=(U,Z)\in \mathrm{C}^{\mathrm{Lip}}([0,T],\boldsymbol{Q})$ with $Q(0)=Q^0$.*

Proof: It remains to prove that $\boldsymbol{A}:\boldsymbol{Q}\to\boldsymbol{Q}^*$, which is defined via $\boldsymbol{E}(t,U,Z)=\frac{1}{2}\big\langle \boldsymbol{A}\binom{U}{Z},\binom{U}{Z}\big\rangle_{\boldsymbol{Q}}-\langle \ell(t),u_0\rangle_{\mathrm{H}^1}$, satisfies (3.2),

$$\exists\,\alpha>0\;\forall\,(U,Z)\in\boldsymbol{Q}\text{ with }\boldsymbol{R}(Z)<\infty:\quad \big\langle \boldsymbol{A}\tbinom{U}{Z},\tbinom{U}{Z}\big\rangle_{\boldsymbol{Q}}\geq \alpha\|(U,Z)\|^2_{\boldsymbol{Q}}. \tag{3.9}$$

By our Assumption (3.3c), we immediately obtain the lower estimate

$$\big\langle \boldsymbol{A}\tbinom{U}{Z},\tbinom{U}{Z}\big\rangle_{\boldsymbol{Q}}\geq \widehat{\alpha}\|(\widehat{\boldsymbol{e}}(U),Z)\|^2_{\mathrm{L}^2(\Omega\times\mathcal{Y})}\text{ for all }(U,Z)\in\boldsymbol{Q}. \tag{3.10}$$

Next, we use an orthogonality condition for the two-scale limit of gradients. If $\nabla u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} E\nabla_x u_0+\nabla_y U_1$ in $\mathrm{L}^2(\Omega\times\mathcal{Y})$, then

$$\textstyle\int_{\Omega\times\mathcal{Y}}|\nabla_x u_0(x){+}\nabla_y U_1(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x=\int_\Omega|\nabla u_0(x)|^2\,\mathrm{d}x+\int_{\Omega\times\mathcal{Y}}|\nabla_y U_1(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x.$$

The mixed terms drop out, since $E\nabla u_0(x,\cdot)$ is constant on $\mathcal{Y}$, while $\nabla_y U_1(x,\cdot)$ has average 0 as it is a derivative of a periodic function. For the symmetric strains we similarly obtain

$$\|\widehat{\boldsymbol{e}}((u_0,U_1))\|^2_{\mathrm{L}^2(\Omega\times\mathcal{Y})}=\|\boldsymbol{e}(u_0)\|^2_{\mathrm{L}^2(\Omega)}+\|\boldsymbol{e}_y(U_1)\|^2_{\mathrm{L}^2(\Omega\times\mathcal{Y})}.$$

With $K_{\mathcal{Y}}=2\pi^2\min\{\,|\lambda|^2\mid 0\neq\lambda\in\Lambda\,\}$ we have the Korn-Poincaré type inequalities:

$$\forall\,V\in\mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}):\quad \|\boldsymbol{e}_y(V)\|^2_{\mathrm{L}^2(\mathcal{Y})}\geq K_{\mathcal{Y}}\|V\|^2_{\mathrm{L}^2(\mathcal{Y})}\quad\text{and}\quad \|\boldsymbol{e}_y(V)\|^2_{\mathrm{L}^2(\mathcal{Y})}\geq\tfrac{1}{2}\|\nabla_y V\|^2_{\mathrm{L}^2(\mathcal{Y})}.$$

This follows easily by writing $V(y)=\sum_\Lambda V_\lambda \mathrm{e}^{2\mathrm{i}\pi\lambda\cdot y}$ and using Plancherel's identity. Inserting these estimates into (3.10) and employing Korn's inequality for u_0 we obtain

$$\big\langle \boldsymbol{A}\tbinom{U}{Z},\tbinom{U}{Z}\big\rangle_{\boldsymbol{Q}}\geq\widehat{\alpha}\Big(C_{\mathrm{Korn}}\|u_0\|^2_{\mathrm{H}^1(\Omega)}+\tfrac{K_{\mathcal{Y}}}{1+2K_{\mathcal{Y}}}\textstyle\int_\Omega\|U_1(x,\cdot)\|^2_{\mathrm{H}^1(\mathcal{Y})}\,\mathrm{d}x+\|Z\|^2_{\mathrm{L}^2(\Omega\times\mathcal{Y})}\Big),$$

which provides the desired estimate (3.9). ■

4 Convergence results

This final section addresses the question under which conditions the solutions $(u_\varepsilon,z_\varepsilon)$ of $(\mathrm{S}^\varepsilon)\,\&\,(\mathrm{E}^\varepsilon)$ have a two-scale limit (U,Z) which is a solution of $(\mathbf{S})\,\&\,(\mathbf{E})$. The convergence is taken in the sense of two-scale cross-convergence and we can build on our theory in Section 4.3.

In particular, the results of Section 2.5 state that $\boldsymbol{E}$ and $\boldsymbol{R}$ are the Γ-limits of the families $(\mathcal{E}_\varepsilon)_\varepsilon$ and $(\mathcal{R}_\varepsilon)_\varepsilon$, respectively, in the Mosco sense.

Proposition 4.1 *Let $\Omega \subset \mathbb{R}^d$ be bounded with Lipschitz boundary. Moreover, let $\mathcal{E}_\varepsilon$, $\mathcal{R}_\varepsilon$, $\boldsymbol{E}$, and $\boldsymbol{R}$ be defined as above such that* (3.3) *and* $\ell \in \mathrm{C}^0([0,T], (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$ *hold. Then, for each $t \in [0,T]$ we have the following convergences*

$$(u_\varepsilon, z_\varepsilon) \overset{\text{w2c}}{\rightharpoonup} (u_0, U_1, Z) \in \boldsymbol{Q} \quad \Longrightarrow \quad \left\{ \begin{array}{c} \boldsymbol{E}(t, u_0, U_1, Z) \leq \liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon), \\ \boldsymbol{R}(Z) \leq \liminf_{\varepsilon \to 0} \mathcal{R}_\varepsilon(z_\varepsilon); \end{array} \right. \tag{4.1a}$$

$$\begin{array}{l} \forall\, (u_0, U_1, Z) \in \boldsymbol{Q}\ \exists\, ((u_\varepsilon, z_\varepsilon))_\varepsilon : \\ (u_\varepsilon, z_\varepsilon) \overset{\text{s2c}}{\longrightarrow} (u_0, U_1, Z) \text{ in } \boldsymbol{Q} \text{ and } \end{array} \left\{ \begin{array}{c} \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) \to \boldsymbol{E}(t, u_0, U_1, Z), \\ \mathcal{R}_\varepsilon(z_\varepsilon) \to \boldsymbol{R}(Z), \end{array} \right. \tag{4.1b}$$

where for the recovery sequence in (4.1b) *we may take* $(u_\varepsilon, z_\varepsilon) = (u_0 + \mathcal{G}_\varepsilon(0, U_1), \mathcal{F}_\varepsilon Z)$ *with $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon$ as defined in* (2.8) *and* (2.12)*, respectively.*

Here it is important that $\mathcal{G}_\varepsilon$ maps into $\mathrm{H}^1_0(\Omega)$, such that $u_0 + \mathcal{G}_\varepsilon(0, U_1) \in \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$.

Our convergence result for the solutions $(u_\varepsilon, z_\varepsilon) \in \mathrm{C}^{\mathrm{Lip}}([0,T], \mathcal{Q})$ of (S^ε) & (E^ε) to a solution $(U, Z) \in \mathrm{C}^{\mathrm{Lip}}([0,T], \boldsymbol{Q})$ will be an adapted and simplified variant of the two abstract Theorems 3.1 and 3.3 in [MRS06]. The abstract theory is formulated on one single space $\widehat{\mathcal{Q}}$ but in fact, the results there are easily generalized to the setting needed here. The following remark gives the alternative way of embedding everything into one big function space $\widehat{\mathcal{Q}}$.

Remark 4.2 *To show that our situation is included exactly in this setting we choose*

$$\widehat{\mathcal{Q}} = \widehat{\mathcal{H}} \times \widehat{\mathcal{Z}} \quad \text{with } \widehat{\mathcal{H}} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\mathbb{R}^d; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y})) \text{ and } \widehat{\mathcal{Z}} = \mathrm{L}^2(\mathbb{R}^d \times \mathcal{Y})$$

and define an ε-dependent embedding $(u, z) \mapsto (\mathcal{Q}_\varepsilon u, \mathcal{U}_\varepsilon u, \mathcal{T}_\varepsilon z)$, where the $\mathcal{Q}_\varepsilon : \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \to \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ and $\mathcal{U} : \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \to \mathrm{L}^2(\mathbb{R}^d; \mathrm{H}^1_{\mathrm{av}}(\mathcal{Y}))$ can be defined as indicated in [CDG02]. Define H_ε as the subspace of $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$ containing the functions u such that $\fint_{\varepsilon(\lambda+Y)} u(x)\,\mathrm{d}x = 0$ for all $\lambda \in \Lambda^-_\varepsilon$, see Section 2.1. Then, let $\mathcal{Q}_\varepsilon$ be the orthogonal projection to the orthogonal complement of H_ε and set $\mathcal{U}_\varepsilon u = \frac{1}{\varepsilon}(\mathrm{id} - \mathcal{Q}_\varepsilon)u$. Finally, we define the functionals in $\widehat{\mathcal{Q}}$ via

$$\widehat{\mathcal{E}}_\varepsilon(t, u_0, \widehat{U}_1, \widehat{Z}) = \left\{ \begin{array}{ll} \mathcal{E}_\varepsilon(t, u, z) & \text{if } (u_0, \widehat{U}_1, \widehat{Z}) = (u, \mathcal{Q}_\varepsilon u, \mathcal{T}_\varepsilon z), \\ \infty & \text{else}, \end{array} \right.$$

$$\widehat{\mathcal{E}}_0(t, u_0, \widehat{U}_1, \widehat{Z}) = \left\{ \begin{array}{ll} \boldsymbol{E}(t, u_0, U_1, Z) & \text{if } \mathrm{sppt}(\widehat{U}_1, \widehat{Z}) \subset \overline{\Omega} \times \mathcal{Y}, \\ \infty & \text{else}, \end{array} \right.$$

$$\widehat{\mathcal{R}}_\varepsilon(\widehat{Z}) = \left\{ \begin{array}{ll} \mathcal{R}_\varepsilon(z) & \text{if } \widehat{Z} = \mathcal{T}_\varepsilon z, \\ \infty & \text{else}, \end{array} \right. \qquad \widehat{\mathcal{R}}_0(Z) = \left\{ \begin{array}{ll} \boldsymbol{R}(Z) & \text{if } \mathrm{sppt}(Z) \subset \overline{\Omega} \times \mathcal{Y}, \\ \infty & \text{else}. \end{array} \right.$$

Hence, under the additional assumption that for all considered functions the corresponding functionals have finite values, we have concluded that weak and strong convergence in $\widehat{\mathcal{Q}}$ is equivalent to weak or strong two-scale convergence of families $(u_\varepsilon, z_\varepsilon)_\varepsilon$ in $\mathcal{Q}$ towards a limit $(u_0, U_1, Z) \in \boldsymbol{Q}$.

Now we are able to formulate the main result of this paper. It states that the solutions $(u_\varepsilon, z_\varepsilon)_\varepsilon$ of the ε-periodic problem (S^ε) & (E^ε) strongly two-scale cross-converge to a solution (U, Z) of the two-scale homogenized problem $(\mathbf{S})$ & $(\mathbf{E})$ under the sole assumption that the initial conditions strongly two-scale cross-converge.

Theorem 4.3 *Let $(u_\varepsilon, z_\varepsilon) : [0,T] \to \mathcal{Q}$ be the solution for (S^ε) & (E^ε) as obtained in Prop. 3.4. Assume that the initial data satisfy*

$$(u_\varepsilon(0), z_\varepsilon(0)) \overset{\text{s2c}}{\longrightarrow} Q^0 = (u^0, U^0, Z^0) \text{ in } \boldsymbol{Q}.$$

Then Q^0 is stable (i.e., $Q^0 \in \boldsymbol{S}(0)$) and

$$\forall\, t \in [0,T] : \quad (u_\varepsilon(t), z_\varepsilon(t)) \overset{\text{s2c}}{\longrightarrow} Q(t) = (u_0(t), U_1(t), Z(t)) \text{ in } \boldsymbol{Q},$$

where $Q : [0,T] \to \boldsymbol{Q}$ is the unique solution of $(\mathbf{S})$ & $(\mathbf{E})$ with initial condition $Q(0) = Q^0$ as provided in Prop. 3.5.

Recall the definition of the stable sets

$$\begin{aligned} \mathcal{S}_\varepsilon(t) &= \{\, (u,z) \in \mathcal{Q} \mid \forall (\widetilde{u}, \widetilde{z}) \in \mathcal{Q}\colon \mathcal{E}_\varepsilon(t, \widetilde{u}, \widetilde{z}) \leq \mathcal{E}_\varepsilon(0, \widetilde{u}, \widetilde{z}) - \mathcal{R}_\varepsilon(\widetilde{z} - z) \,\}, \\ \boldsymbol{S}(t) &= \{\, (U,Z) \in \boldsymbol{Q} \mid \forall (\widetilde{U}, \widetilde{Z}) \in \boldsymbol{Q}\colon \boldsymbol{E}(t, \widetilde{U}, \widetilde{Z}) \leq \boldsymbol{E}(0, \widetilde{U}, \widetilde{Z}) - \boldsymbol{R}(\widetilde{Z} - Z) \,\}. \end{aligned}$$

Remark 4.4 *In [MRS06] the convergence of the initial condition and of the solutions is formulated in terms of the underlying topology, which in the present setting means weak two-scale cross-convergence. However, the abstract theory assumes convergence of the initial energies and proves convergence of the energies $\mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) \to \boldsymbol{E}(t, U(t), Z(t))$. Because of uniform convexity (cf. (3.9)) we see that weak convergence and energy convergence implies strong convergence. The details of this argument are worked out at the end of the proof of Theorem 4.3. See also [Vis84] for general arguments of this type.*

The main difficulty in the proof of the desired result is to prove that the weak limit of stable states is again stable. In [MRS06] this property is reduced to a property which postulates the existence of suitable *joint recovery sequences* for a combination of $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$. In our setting this reads as follows.

Proposition 4.5 *For $t \in [0,T]$ assume $(u_\varepsilon, z_\varepsilon) \in \mathcal{S}_\varepsilon(t)$ and $(u_\varepsilon, z_\varepsilon) \overset{\text{w2c}}{\rightharpoonup} (u_0, U_1, Z)$ in $\boldsymbol{Q}$.*
(a) Then, for each $(\widetilde{u}_0, \widetilde{U}_1, \widetilde{Z}) \in \boldsymbol{Q}$ there exists a joint recovery family $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)_\varepsilon$ with $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) \overset{\text{w2c}}{\rightharpoonup} (\widetilde{u}_0, \widetilde{U}_1, \widetilde{Z})$ in $\boldsymbol{Q}$ such that

$$\limsup_{\varepsilon \to 0} \big[\mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) + \mathcal{R}_\varepsilon(\widetilde{z}_\varepsilon - z_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) \big] \leq \boldsymbol{E}(t, \widetilde{U}, \widetilde{Z}) + \boldsymbol{R}(\widetilde{Z} - Z) - \boldsymbol{E}(t, U, Z). \tag{4.2}$$

(b) As a consequence $(u_0, U_1, Z) \in \boldsymbol{S}(t)$.

Proof: ad (a). We give the joint recovery sequence explicitly in the form

$$(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) = (u_\varepsilon, z_\varepsilon) + (\widetilde{u}_0 - u_0 + \mathcal{G}_\varepsilon(0, \widetilde{U}_1 - U_1)\,,\ \mathcal{F}_\varepsilon(\widetilde{Z} - Z)).$$

Note that the arguments for $\mathcal{G}_\varepsilon$ and $\mathcal{F}_\varepsilon$ do not depend on ε. Hence, by Prop. 2.10 and Prop. 2.4 we obtain the important relation

$$(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) - (u_\varepsilon, z_\varepsilon) = (\widetilde{u}_0 - u_0 + \mathcal{G}_\varepsilon(0, \widetilde{U}_1 - U_1)\,,\ \mathcal{F}_\varepsilon(\widetilde{Z} - Z)) \xrightarrow{\text{s2c}} (\widetilde{u}_0 - u_0, \widetilde{U}_1 - U_1, \widetilde{Z} - Z). \quad (4.3)$$

In turn, this implies the obvious convergence $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) \overset{\text{w2c}}{\rightharpoonup} (\widetilde{u}_0, \widetilde{U}_1, \widetilde{Z})$.

From (4.3) and Lemma 2.12(b) we obtain $\mathcal{R}_\varepsilon(\widetilde{z}_\varepsilon - z_\varepsilon) \to \boldsymbol{R}(\widetilde{Z} - Z)$.

For the energies we use the quadratic nature and obtain

$$\mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) = \tfrac{1}{2} \int_\Omega \left\langle\!\left\langle \mathbb{A}(\{\tfrac{x}{\varepsilon}\}_\mathcal{Y}) \binom{e(\widetilde{u}_\varepsilon - u_\varepsilon)}{\widetilde{z}_\varepsilon + z_\varepsilon}, \binom{e(\widetilde{u}_\varepsilon - u_\varepsilon)}{\widetilde{z}_\varepsilon + z_\varepsilon} \right\rangle\!\right\rangle \mathrm{d}x - \langle \ell(t), \widetilde{u}_\varepsilon - u_\varepsilon \rangle.$$

The last term obviously converges to $\langle \ell(t), \widetilde{u}_0 - u_0 \rangle$ by the usual weak convergence in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$. Under the integral we have a quadratic form, where the right factor weakly two-scale converges to $\binom{\widetilde{\mathbf{e}}(\widetilde{U} + U)}{\widetilde{Z} + Z}$ in $\mathrm{L}^2(\Omega \times \mathcal{Y})$. The left-hand factor is a product of the multiplicator $m_\varepsilon = \mathbb{A}(\{\frac{\cdot}{\varepsilon}\}_\mathcal{Y})$ and a strongly two-scale convergent sequence with limit $\binom{\widetilde{\mathbf{e}}(\widetilde{U} - U)}{\widetilde{Z} - Z}$ in $\mathrm{L}^2(\Omega \times \mathcal{Y})$. As $\mathcal{T}_\varepsilon m_\varepsilon(x, y) = \mathbb{A}(y)$ Prop. 2.6 implies

$$\mathbb{A}(\{\tfrac{\cdot}{\varepsilon}\}_\mathcal{Y}) \binom{e(\widetilde{u}_\varepsilon - u_\varepsilon)}{\widetilde{z}_\varepsilon + z_\varepsilon} \xrightarrow{\text{s2}} \mathbb{A} \binom{\widetilde{\mathbf{e}}(\widetilde{U} - U)}{\widetilde{Z} - Z} \text{ in } \mathrm{L}^2(\Omega \times \mathcal{Y}).$$

Since a scalar product of a weakly and a strongly converging sequence converges (see Prop. 2.4(d)), we conclude

$$\mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon) \to \boldsymbol{E}(t, \widetilde{U}, \widetilde{Z}) - \boldsymbol{E}(t, U, Z).$$

Thus, we have established (4.2) in the stronger version that the limsup is a limit and the "$\leq$" is "$=$".

ad (b). This is a direct consequence of part (a). Let (U, Z) be the limit of stable states and take any test state $(\widetilde{U}, \widetilde{Z}) \in \boldsymbol{Q}$. Now take the joint recovery sequence obtained in part (a) and insert $(\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)$ into the stability condition for $(u_\varepsilon, z_\varepsilon)$, namely

$$0 \leq \mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon) + \mathcal{R}_\varepsilon(\widetilde{z}_\varepsilon - z_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon, z_\varepsilon).$$

As the right-hand side converges we conclude $0 \leq \boldsymbol{E}(t, \widetilde{U}, \widetilde{Z}) + \boldsymbol{R}(\widetilde{Z} - Z) - \boldsymbol{E}(t, U, Z)$ and stability is established as $(\widetilde{U}, \widetilde{Z})$ was arbitrary. ∎

Proof: [of Theorem 4.3]
By Prop. 3.4 we know that the family $(u_\varepsilon, z_\varepsilon)_\varepsilon$ is uniformly bounded in $\mathrm{C}^{\mathrm{Lip}}([0, T], \mathcal{Q})$. As closed balls in $\mathcal{Q}$ are weakly compact and have a metrizable topology, the Arzela-Ascoli theorem can be applied in $\mathrm{C}^0([0, T], \mathcal{Q}_{\mathrm{weak}})$ and we find a subsequence $(\varepsilon_k)_{k \in \mathbb{N}}$ with $0 < \varepsilon_k \to 0$ such that

$$\forall\, t \in [0, T] : (u_{\varepsilon_k}(t), z_{\varepsilon_k}(t)) \overset{\text{w2c}}{\rightharpoonup} (U(t), Z(t)) \text{ in } \boldsymbol{Q}.$$

By the lower semi-continuity of the norm, we have $(U, Z) \in \mathrm{C}^{\mathrm{Lip}}([0, T], \boldsymbol{Q})$ and it remains to show that (U, Z) is a solution of $(\mathbf{S})$ & $(\mathbf{E})$. As the initial condition (U^0, Z^0) is known the solution is unique and we even conclude that the whole family converges (by the standard argument via contradiction).

By Prop. 4.5 we know that $(U(t), Z(t))$ is stable for all $t \in [0, T]$, hence (**S**) is satisfied and we have to establish the energy balance (**E**) in (3.8). For this, we pass to the limit $\varepsilon \to 0$ in (E^ε), cf. (3.6). The first term on the right-hand side converges, as the energy $\mathcal{E}_\varepsilon(0, u_\varepsilon(0), z_\varepsilon(0))$ converges applying the strong two-scale cross-convergence and Prop. 2.15. The second term converges by Lebesgue's dominated convergence theorem as the integrands are uniformly bounded and converge pointwise.

To treat the left-hand side of (E^ε) we let $e_\varepsilon(t) = \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t))$ and $d_\varepsilon(t) = \int_0^t \mathcal{R}_\varepsilon(z_\varepsilon(s))\,\mathrm{d}s$. By the above, we know that $r_\varepsilon(t) = e_\varepsilon(t) + d_\varepsilon(t)$ converges to $r_0(t)$, which is the limit of the right-hand side. We let $e^*(t) = \limsup_{\varepsilon\to 0} e_\varepsilon(t)$ and $d_*(t) = \liminf_{\varepsilon\to 0} d_\varepsilon(t)$ and conclude $e^*(t) + d_*(t) = r_0(t)$. Now we use the lower estimates for the functionals. For the stored energy we use (4.1a) to obtain

$$\boldsymbol{E}(t, U(t), Z(t)) \le \liminf_{\varepsilon\to 0} e_\varepsilon(t) \le \limsup_{\varepsilon\to 0} e_\varepsilon(t) = e^*(t).$$

For the dissipation integral we use $\int_0^t \boldsymbol{R}(\dot{Z}(s))\,\mathrm{d}s = \sup \sum_{j=1}^N \boldsymbol{R}(Z(t_j) - Z(t_{j-1}))$, where the supremum is taken over all finite partitions of $[0, t]$. Again by (4.1a) we find

$$\begin{aligned}&\textstyle\sum_{j=1}^N \boldsymbol{R}(Z(t_j) - Z(t_{j-1})) \le \liminf_{\varepsilon\to 0} \sum_{j=1}^N \mathcal{R}_\varepsilon(z_\varepsilon(t_j) - z_\varepsilon(t_{j-1})) \\ &\le \liminf_{\varepsilon\to 0} \textstyle\int_0^t \mathcal{R}_\varepsilon(\dot{z}_\varepsilon(s))\,\mathrm{d}s = d_*(t).\end{aligned} \tag{4.4}$$

Thus, recalling $e^* + d_* = r_0$ we proved the lower energy estimate

$$\boldsymbol{E}(t, U(t), Z(t)) + \textstyle\int_0^T \boldsymbol{R}(\dot{Z}(s))\,\mathrm{d}s \le e^*(t) + d_*(t) = \boldsymbol{E}(0, U(0), Z(0)) - \int_0^t \langle \ell(s), u_0(s)\rangle\,\mathrm{d}s.$$

The upper energy estimate (just replace "$\le$" by "$\ge$") follows from the already established stability of (U, Z), see [MTL02, Thm. 2.5] or [MM05, Thm. 4.4]. Thus, (**E**) holds and, moreover, we also conclude that the inequality in (4.4) must be an equality. This in turn implies that $\boldsymbol{E}(t, U(t), Z(t)) = e^*(t) = \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t))$.

As the value of $t \in [0, T]$ is kept from now on, we omit it in the rest of the proof. From the above and using the weak two-scale convergence $q_\varepsilon = (u_\varepsilon, z_\varepsilon) \stackrel{\mathrm{w2c}}{\rightharpoonup} Q = (u_0, U_1, Z)$ we want to conclude $q_\varepsilon \xrightarrow{\mathrm{s2c}} Q$.

For this, we define $\widehat{q}_\varepsilon = (u_0 + \mathcal{G}_\varepsilon(0, U_1), \mathcal{F}_\varepsilon Z) \in \mathcal{Q}$, which satisfies $\widehat{q}_\varepsilon \xrightarrow{\mathrm{s2c}} Q$. Moreover, we have

$$\begin{aligned}&\tfrac{\alpha}{2}\|\widehat{q}_\varepsilon - q_\varepsilon\|_{\mathcal{Q}}^2 \le \tfrac{1}{2}\langle\!\langle \mathcal{A}_\varepsilon(\widehat{q}_\varepsilon - q_\varepsilon), (\widehat{q}_\varepsilon - q_\varepsilon)\rangle\!\rangle \\ &= \mathcal{E}_\varepsilon(t, q_\varepsilon) - \mathcal{E}_\varepsilon(t, \widehat{q}_\varepsilon) + \langle\!\langle \mathcal{A}_\varepsilon \widehat{q}_\varepsilon - \ell, q_\varepsilon - \widehat{q}_\varepsilon\rangle\!\rangle \\ &\to e^* - \boldsymbol{E}(t, Q) + 0 = 0.\end{aligned}$$

For the convergence note that the first term was treated above, that the second term converges because of "$\xrightarrow{\mathrm{s2c}}$" and Prop. 2.15, and that the third term converges as a scalar product, since the left-hand term is strongly convergent and while the right-hand term weakly converges to 0, see Prop. 2.4(d). Finally, we conclude by noting that

$$\begin{aligned}&\|(\mathcal{T}_\varepsilon(\nabla u_\varepsilon), z_\varepsilon) - (E\nabla_x u_0 + \nabla_y U_1, Z)\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})} \le \\ &\|(\mathcal{T}_\varepsilon(\nabla u_\varepsilon - \nabla\widehat{u}_\varepsilon), z_\varepsilon - \widehat{z}_\varepsilon)\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})} + \delta_\varepsilon \le \|(u_\varepsilon, z_\varepsilon) - (\widehat{u}_\varepsilon, \widehat{z}_\varepsilon)\|_{\mathcal{Q}} + \delta_\varepsilon \to 0\end{aligned}$$

with $\delta_\varepsilon = \|(\mathcal{T}_\varepsilon(\nabla\widehat{u}_\varepsilon), \widehat{z}_\varepsilon) - (E\nabla_x u_0 + \nabla_y U_1, Z)\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathcal{Y})} \to 0$ because of $\widehat{q}_\varepsilon \xrightarrow{\mathrm{s2c}} Q$. This establishes $q_\varepsilon \xrightarrow{\mathrm{s2c}} Q$ and we are done. ■

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