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Feedback stabilization of magnetohydrodynamic equations

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Abstract

We prove the local exponential stabilizability for the MHD system, with internally distributed feedback controllers. These controllers take values in a finite dimensional space which is the unstable manifold of the elliptic part of the linearized operator. The stabilization of the linear system is derived using a unique continuation property for systems of parabolic and elliptic equations, as well as the equivalence between controllability and feedback stabilizability in the case of finite dimensional systems. The feedback that stabilizes the linearized system is also stabilizing the nonlinear system in the domain of a fractional power of the elliptic operator.

1 Introduction

This paper is concerned with the study of the local exponential stabilization for the magnetohydrodynamic (MHD) equations, with feedback controllers, localized in a subdomain and taking values in a finite dimensional space.

The idea is to linearize the system around a stationary state and then construct a feedback controller stabilizing the linear system. The last step is to show that the same controller stabilizes, locally in a specified space, the nonlinear system. In order to stabilize the linear system one needs to project the system on the stable and unstable subspaces corresponding to a spectral decomposition of the elliptic part. The unstable subspace is finite dimensional and the projected system on it is exactly controllable, as a consequence of the approximate controllability of the original linearized system; one may thus construct a feedback stabilizing this finite dimensional linear system. The projected system on the stable subspace is, of course, asymptotically stable and, in fact, the feedback for the finite dimensional system is stabilizing the initial linearized equations. The approximate controllability of the linearized system is a consequence of the unique continuation property for the adjoint system. We prove this by adapting the Carleman inequality obtained by O.Yu. Imanuvilov (see [10]) in order to establish exact controllability of parabolic equations and coupling it (as in [9] when deriving an observability inequality for linearized Navier-Stokes equations) with a refined Carleman inequality for elliptic equations obtained by O.Yu.Imanuvilov and J.-P.Puel in [14],[15]. The fact that the feedback controller constructed in the linear case is also stabilizing the nonlinear system is proved by using the solution of a Lyapunov equation.

The corresponding problem for Navier-Stokes equations was first studied by V.Barbu in [2], where the method of spectral decomposition was applied. There, the exact controllability of the finite dimensional projection was derived as a consequence of the exact null controllability of the linearized Navier-Stokes equations (for the controllability of Navier-Stokes equations see the papers of O.Yu.Imanuvilov [12],[13]). Later, V.Barbu and R.Triggiani [7] proved the feedback stabilizability of Navier-Stokes with distributed controllers taking values in a finite dimensional space. Feedback stabilization of Navier-Stokes equations with boundary tangential controllers was studied by V.Barbu, I.Lasiecka and R. Triggiani in [5] and [6]. The key in obtaining the exact controllability for the finite dimensional system is here a Kalman type condition, derived as a consequence of the unique continuation property for a stationary Stokes type system. In both situations the feedback stabilizing the initial nonlinear Navier-Stokes system is the solution of an algebraic Ricatti equation coming from a quadratic optimal control problem with infinite horizon. Nevertheless, in [5] it was also emphasized the fact that the feedback stabilizing the linearized system is also stabilizing the nonlinear one.

We also mention here the result of exact controllability for the magnetohydrodynamic equations obtained by V.Barbu, T.Havarneanu, C.Popa and S.S.Sritharan in [3], [4] (see also [11]). We did not choose to use this exact controllability result in order to derive controllability for the finite dimensional projection since exact controllability is a more involved result and, moreover, supplementary regularity for the stationary solution is needed. Also, a unique continuation result does not depend on the boundary conditions, as in the case of observability inequalities of Carleman type, and may thus be applied to the stabilization of other systems.

2 Preliminaries

Let $\Omega \subset \mathbf{R}^3$ be a bounded connected set with C^2 boundary $\partial\Omega$. Let $Q = \Omega \times (0, \infty)$, $\Sigma = \partial\Omega \times (0, \infty)$, $\mathbf{n}$ is the unit exterior normal to $\partial\Omega$. We consider in the paper the following MHD controlled system:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \nu\Delta y + (y \cdot \nabla)y - (B \cdot \nabla)B + \nabla(\dfrac{1}{2}B^2 + p) = f + \chi_\omega u & \text{in } Q, \\ \dfrac{\partial B}{\partial t} + \eta \text{ curl curl } B + (y \cdot \nabla)B - (B \cdot \nabla)y = P(\chi_\omega v) & \text{in } Q, \\ \nabla \cdot y = 0, \ \nabla \cdot B = 0 & \text{in } Q, \\ y = 0, \ B \cdot \mathbf{n} = 0, (\text{curl } B) \times \ \mathbf{n} = 0 & \text{on } \Sigma \\ y(\cdot, 0) = y_0, \ B(\cdot, 0) = B_0 & \text{in } \Omega. \end{cases} \tag{1}$$

The functions that appear in the system are $y = (y_1, y_2, y_3) : \Omega \times (0, T) \to \mathbf{R}^3$ is the velocity field, $p : \Omega \times (0, T) \to \mathbf{R}$ is the pressure, $B = (B_1, B_2, B_3) : \Omega \times (0, T) \to \mathbf{R}^3$ is the magnetic field. The functions $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) : \omega \times (0, T) \to \mathbf{R}^3$

are the controllers and $\chi_\omega : L^2(\omega) \to L^2(\Omega)$ is the operator extending the functions in $L^2(\omega)$ with 0 to the whole Ω. We will suppose that $u, v \in \mathcal{U} := L^2(0,T;(L^2(\omega))^3)$. The coefficients ν, η are the positive kinematic viscosity and the magnetic resistivity coefficients.

Denote by

$$H = \{z \in (L^2(\Omega)^3) : \nabla \cdot z = 0, z \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

endowed with the L^2 norm and

$$V_1 = H \cap (H_0^1)^3$$
$$V_2 = H \cap (H^1)^3$$

endowed with the H^1 norm. We denote by $|\cdot|$ and $(\cdot,\cdot)$ the L^2 norm respectively the L^2 scalar product.

We also recall here the standard estimate on the trilinear term appearing in the Navier-Stokes equations and, consequently in the MHD system (see [18]). Let for $m \geq 0, V^m := H \cap (H^m(\Omega))^3$ with norm $\|\cdot\|_m$. Then the trilinear form

$$b(u,v,w) := \int_\Omega [(u \cdot \nabla)v] \cdot w dx = \int_\Omega \sum_{i,j=1}^{3} u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

is well defined on $(V^1)^3$ and extends to $V^{m_1} \times V^{m_2+1} \times V^{m_3}$ when $m_1 + m_2 + m_3 \geq \frac{3}{2}$ and $m_i \neq \frac{3}{2}$ or when at leas one of $m_i = \frac{3}{2}$ and $m_1 + m_2 + m_3 > \frac{3}{2}$. In these situations we have:

$$|b(u,v,w)| \leq C\|u\|_{m_1}\|v\|_{m_2+1}\|w\|_{m_3}. \tag{2}$$

Also, it is antisymmetric in the last two variables: $b(u,v,w) = -b(u,w,v)$.

Consider for a given $f \in (H^{-1}(\Omega))^3$ a steady state variational solution $(\bar{y}, \bar{B}, \bar{p}) \in V_1 \times V_2 \times L^2(\Omega)$ of (1):

$$\begin{cases} -\nu\Delta\bar{y} + (\bar{y}\cdot\nabla)\bar{y} + \nabla(\frac{1}{2}\bar{B}^2) - (\bar{B}\cdot\nabla)\bar{B} + \nabla\bar{p} = f & \text{in } \Omega, \\ \eta \text{ curl curl } \bar{B} + (\bar{y}\cdot\nabla)\bar{B} - (\bar{B}\cdot\nabla)\bar{y} = 0 & \text{in } \Omega, \\ \nabla\cdot\bar{y} = 0,\ \nabla\cdot\bar{B} = 0 & \text{in } \Omega, \\ \bar{y} = 0,\ \bar{B}\cdot\mathbf{n} = 0, (\text{curl } \bar{B}) \times \mathbf{n} = 0 & \text{on } \Sigma. \end{cases} \tag{3}$$

We will assume the following hypothesis on the regularity of the stationary solution:

$$\textbf{(H)} \qquad \bar{y}, \bar{B} \in W^{1,3}(\Omega) \cap L^\infty(\Omega). \tag{4}$$

In order to write (1) in an abstract form we define the following two operators (P is the Leray projection):

$$A_1 y = -P\Delta y \text{ for } y \in D(A_1)$$

$$A_2 B = \text{curl (curl } B) \text{ for } B \in D(A_2)$$

where

$$D(A_1) := (H^2(\Omega))^3 \cap V_1,$$
$$D(A_2) := \{B \in (H^2(\Omega))^3 \cap V_2 \mid (\text{curl } B) \times \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

With no loss of generality we will suppose that $\nu = \eta = 1$ and system (1) may thus be written as

$$\begin{cases} y' + A_1 y + P(y \cdot \nabla y) - P(B \cdot \nabla B) = Pf + P(\chi_\omega u), \\ B' + A_2 B + y \cdot \nabla B - B \cdot \nabla y = P(\chi_\omega v), \\ y(0) = y_0, \ B(0) = B_0. \end{cases} \tag{5}$$

The main question we address in this paper is to find a feedback control $(u, v) = K(y, B)$ such that, if (y_0, B_0) is in a neighborhood of $(\bar{y}, \bar{B})$ (in a topology to be specified) then system (5) admits a global weak solution that satisfies an estimate of the form:

$$\|(y, B) - (\bar{y}, \bar{B})\| \leq Ce^{-\gamma t} \|(y_0 - \bar{y}, B_0 - \bar{B})\|$$

with C, γ positive constants and the norm is in a space that a will be specified. Moreover, the feedback control we will construct will take values in a finite dimensional space.

In order to do this, we need to study the difference between the solution of (1) and the stationary solution satisfying (3). After renaming by y, B, p, y_0, B_0 the quantities $y - \bar{y}$, $B - \bar{B}$, $p - \bar{p}$, $y_0 - \bar{y}$ and respectively $B_0 - \bar{B}$, we obtain the following system that we have now to stabilize in 0:

$$\begin{cases} y' + A_1 y & +P(y \cdot \nabla \bar{y} + \bar{y} \cdot \nabla y - B \cdot \nabla \bar{B} + \bar{B} \cdot \nabla B) + \\ & +P(y \cdot \nabla y - B \cdot \nabla B) = P(\chi_\omega u), \\ B' + A_2 B & +y \cdot \nabla \bar{B} + \bar{y} \nabla B - B \cdot \nabla \bar{y} - \bar{B} \cdot \nabla y + \\ & +(y \cdot \nabla B - B \cdot \nabla y) = P(\chi_\omega v), \\ & y(0) = y_0, \ B(0) = B_0. \end{cases} \tag{6}$$

The first thing we are doing is to find a feedback that stabilizes the linearized system:

$$\begin{cases} y' + A_1 y + P(y \cdot \nabla \bar{y} + \bar{y} \cdot \nabla y - B \cdot \nabla \bar{B} + \bar{B} \cdot \nabla B) = P(\chi_\omega u), \\ B' + A_2 B + P(y \cdot \nabla \bar{B} + \bar{y} \nabla B - B \cdot \nabla \bar{y} - \bar{B} \cdot \nabla y) = P(\chi_\omega v), \\ y(0) = y_0, \ B(0) = B_0. \end{cases} \tag{7}$$

We observe (see [4]) that in the second equation one has to introduce a supplementary Leray projection since otherwise we could not obtain a solution (y, B) of (7) with a divergence free B. Denote by $\mathcal{A}$ the following operator:

$$\mathcal{A}\left(\begin{array}{c} y \\ B \end{array}\right) = \left(\begin{array}{c} A_1 y + P(y\cdot\nabla\bar{y} + \bar{y}\cdot\nabla y - B\cdot\nabla\bar{B} + \bar{B}\cdot\nabla B) \\ A_2 B + P(y\cdot\nabla\bar{B} + \bar{y}\nabla B - B\cdot\nabla\bar{y} - \bar{B}\cdot\nabla y) \end{array}\right), \tag{8}$$

with $D(\mathcal{A}) = D(A_1)\times D(A_2) \subset H\times H$ and by $\mathcal{B} : (L^2(\omega))^3 \times (L^2(\omega))^3 \to H\times H$

$$\mathcal{B}\left(\begin{array}{c} u \\ v \end{array}\right) = \left(\begin{array}{c} P(\chi_\omega u) \\ P(\chi_\omega v) \end{array}\right).$$

Then the linear controlled system (7) is written in the abstract form

$$\left\{\begin{array}{l} z' + \mathcal{A}z = \mathcal{B}w \\ z(0) = z_0 \end{array}\right., \tag{9}$$

where we denoted by $z = (y, B)^T$, $w = (u, v)^T$ and the solution corresponding to the control w will be denoted as z^w.

The stabilization result concerning the linearized MHD system, that will be proved in Section 3, is the following:

Theorem 1 ***i)*** *The operator $-\mathcal{A}$ generates an analytic semigroup in $H\times H$, with compact resolvent.*

ii) *The linear system (7) is approximately controllable in any time T.*

iii) *There exist a finite dimensional subspace $U \subset (L^2(\omega))^3 \times (L^2(\omega))^3$ and a linear continuous operator $K : H\times H \to U$ such that the operator $-\mathcal{A}+\mathcal{B}K$ generates an analytic semigroup of negative type i.e. a semigroup satisfying an estimate of the form:*

$$\|e^{-t(\mathcal{A}-\mathcal{B}K)}\| \leq Ce^{-\delta t},\ t>0. \tag{10}$$

where C, δ are positive constants. Moreover, for any positive δ there exists such a feedback K with a corresponding change of the constant $C = C(\delta)$ and of the finite dimensional space U.

The main result of this paper, that will be proved in Section 4, concerns the null stabilization of the nonlinear system (6), and consequently of system (1) around the stationary solution satisfying (5):

Theorem 2 *There exist $\delta > 0$, $C > 0$, a neighborhood $\mathcal{V}_\rho$ of 0 in $D(\mathcal{A}^{\frac{1}{4}})$, a finite dimensional subspace $U \subset (L^2(\omega))^3 \times (L^2(\omega))^3$ and a continuous linear feedback operator $K : H\times H \to U$ such that system (6) with $y_0 \in \mathcal{V}_\rho$ admits a global weak solution that satisfies:*

$$|\mathcal{A}^{\frac{1}{4}}(y(t), B(t))| \leq C|\mathcal{A}^{\frac{1}{4}}(y_0, B_0)|e^{-\delta t},\ t>0. \tag{11}$$

3 Feedback stabilization of the linearized MHD system. Proof of Theorem 1

i) The operator $\mathcal{A}$ admits the representation $\mathcal{A} = A + A_0$ where

$$A\begin{pmatrix} y \\ B \end{pmatrix} = \begin{pmatrix} A_1 y \\ A_2 B \end{pmatrix},$$

$$A_0\begin{pmatrix} y \\ B \end{pmatrix} = \begin{pmatrix} P(y\cdot\nabla\bar{y} + \bar{y}\cdot\nabla y - B\cdot\nabla\bar{B} + \bar{B}\cdot\nabla B) \\ P(y\cdot\nabla\bar{B} + \bar{y}\nabla B - B\cdot\nabla\bar{y} - \bar{B}\cdot\nabla y) \end{pmatrix}$$

with $D(A) = D(\mathcal{A}) \subset H \times H$ and $D(A_0) = V_1 \times V_2 \subset H \times H$. Remark that, since $\bar{y}, \bar{B} \in W^{1,3}(\Omega) \cap L^\infty$, for $y \in V_1, B \in V_2$ the products of the type $y \cdot \nabla\bar{B}, \bar{B}\cdot\nabla y$ appearing in the definition of A_0 are in L^2 and it is easy to see that A_0 is closed, A is semi-positive self-adjoint operator and $D(A_0) \subset D(A)$. Moreover, an estimate of the type

$$|A_0 y| \le \varepsilon |Ay| + C(\varepsilon)|y|$$

is standard to prove (see e.g. [16]) and it implies that $-\mathcal{A}$ is the generator of an analytic semigroup. Compactness of the resolvent is, finally, a consequence of the Rellich theorem on the compact embedding for Sobolev spaces on bounded domains (i.e. $D(\mathcal{A})$ is compactly embedded in $H \times H$).

The fact that $-\mathcal{A}$ has compact resolvent and generates an analytic semigroup implies that its spectrum $\sigma(\mathcal{A})$ is discrete, with no finite accumulation points and is contained in an angular domain $V_{\alpha,\theta} := \{z \in \mathbf{C} : \arg(z-\alpha) \in (-\theta,\theta)\}$ with some $\alpha \in \mathbf{R}$, $\theta \in (0, \frac{\pi}{2})$.

ii) Approximate controllability in time T for problem (9) is equivalent to the unique continuation property for the dual equation, i.e. if ξ is a solution of the dual equation

$$-\xi' + \mathcal{A}^*\xi = 0 \quad t \in (0,T) \tag{12}$$

and

$$\mathcal{B}^*\xi \equiv 0,\ t \in (0,T),$$

then $\xi \equiv 0$.

Let $\xi = (\zeta, C)^T$. Then, the dual equation (12) may be rewritten as (note that curl curl $B = -\Delta B + \nabla(\mathrm{div} B)$):

$$\begin{cases} -\zeta_t - \Delta\zeta + (\nabla C)\bar{B} - (\nabla\zeta)\bar{y} + (\nabla^T\bar{B})C + (\nabla^T\bar{y})\zeta + \nabla\pi = 0 & \text{in } Q, \\ -C_t - \Delta C + (\nabla\zeta)\bar{B} - (\nabla C)\bar{y} - (\nabla^T\bar{B})\zeta - (\nabla^T\bar{y})C + \nabla\rho = 0 & \text{in } Q, \\ \nabla\cdot\zeta = 0,\ \nabla\cdot C = 0 & \text{in } Q \\ \zeta = 0,\ C\cdot\mathbf{n} = 0,\ (\mathrm{curl}\ C)\times\mathbf{n} = 0 & \text{on } \Sigma. \end{cases} \tag{13}$$

For a vectorial function $\phi : \Omega \to \mathbf{R}^3$ ($\mathbf{R}^3 \sim M_{3\times 1}(\mathbf{R})$) we denote by:

$$D^s\phi = \nabla\phi + (\nabla\phi)^T, \ D^a\phi = \nabla\phi - (\nabla\phi)^T.$$

When computing the adjoint equation, if we make a further integration (actually use the antisymmetry of the trilinear term b, see (2)) the dual equation (12) takes the form:

$$\begin{cases} -\zeta_t - \Delta\zeta - (D^s\zeta)\bar{y} + (D^aC)\bar{B} + \nabla\pi = 0 & \text{in } Q, \\ -C_t - \Delta C + (D^s\zeta)\bar{B} - (D^aC)\bar{y} + \nabla\rho = 0 & \text{in } Q, \\ \nabla\cdot\zeta = 0, \ \nabla\cdot C = 0 & \text{in } Q \\ \zeta = 0, \ C\cdot\mathbf{n} = 0, \ (\text{curl } C)\times\mathbf{n} = 0 & \text{on } \Sigma. \end{cases} \tag{14}$$

Actually, the difference between the two forms of the dual equation is hidden in the pressure terms, but we will see that it is more convenient to work with the latter.

The unique continuation property that has to be proved reads:

$$\zeta = 0, \ C = 0 \text{ in } \omega\times(0,T) \Longrightarrow \zeta \equiv 0, C \equiv 0 \text{ in } Q \tag{15}$$

This assertion will be proved in Section 5.

iii) We separate now the spectrum of $\mathcal{A}$ in a stable part and an unstable one. Let $\delta > 0$ be such that $\sigma(\mathcal{A}) \cap \{\lambda | \text{ Re } \lambda = \delta\} = \phi$. Let $\sigma_1 = \sigma(\mathcal{A}) \cap \{\lambda | \text{ Re } \lambda < \delta\}$, $\sigma_2 = \sigma(\mathcal{A}) \cap \{\lambda | \text{ Re } \lambda > \delta\}$. It is clear that σ_1 is a finite set and $\sigma_2 \subset V_{\delta,\theta'}$ or some $\theta' \in (0, \frac{\pi}{2})$. Correspondingly, the complexified space $(H\times H)^c$ is decomposed as a direct sum of two closed subspaces $H_1 \bigoplus H_2$, subspaces which are invariant for $\mathcal{A}$ (we denoted also by $\mathcal{A}$ the complexified operator) and $\sigma(\mathcal{A}|H_i) = \sigma_i, i = 1, 2$. Of course H_1 is finite dimensional and let $N = \dim H_1$. Denote by P_N the projection onto H_1 given by the direct sum $H_1 \bigoplus H_2$ and by $Q_N = I - P_N$. Then, with $z_1 = P_N z$ and $z_2 = Q_N z$, equation (9) projects in two equations:

$$z_1' + \mathcal{A}z_1 = P_N\mathcal{B}w \tag{16}$$

$$z_2' + \mathcal{A}z_2 = Q_N\mathcal{B}w \tag{17}$$

The operator $-\mathcal{A}_2 = -Q_N\mathcal{A}$ generates on H_2 a stable analytic semigroup that satisfies:

$$|e^{-t\mathcal{A}_2}z_2^0| \le Ce^{-\delta t}|z_2^0| \tag{18}$$

Equation (16) is a finite dimensional linear equation in the space H_1. Moreover, equation (16) is exactly controllable in any time T. Indeed, we proved that equation (9) is approximately controllable in any time T, so the set $\{z^w(T) : w \in L^2(0,T;(L^2(\omega))^2\}$ is dense in $H\times H$. So the projection of this set, through P_N, on H_1, which is finite dimensional, is the whole space that is $\{z_1^w(T) : w \in L^2(0,T;(L^2(\omega))^2)\} = H_1$. Moreover, if we choose as $U \subset (L^2(\omega))^2$ an N dimensional subspace such that Im $P_N\mathcal{B} = P_N\mathcal{B}(U)$ the pair $(\mathcal{A}_1, P_N\mathcal{B})$ remains exactly controllable in any time T and thus is completely stabilizable (see [20]), i.e. for any $\delta_1 > 0$ there exits a linear operator $K_1 : H_1 \to U$ and a constant $C = C(\delta_1)$ such that

$$\|e^{-t(\mathcal{A}_1 - P_N \mathcal{B} K_1)}\| \leq C e^{-\delta_1 t}. \tag{19}$$

The feedback K, that we will prove to stabilize the linear system (9), is defined as

$$K = \text{ Re } \tilde{K}, \ \tilde{K} = K_1 \circ P_N.$$

We denote by $z^{\tilde{K}}, z_1^{\tilde{K}}, z_2^{\tilde{K}}$ the corresponding solutions of (9),(16) respectively (17). The only estimate to put in evidence is on the corresponding solution of (17) because we have by the complete stabilization of (16) that

$$|z_1^{\tilde{K}}(t)| \leq C e^{-\delta_1 t} |z_1^0|. \tag{20}$$

Variations of constants formula gives

$$z_2^{\tilde{K}}(t) = e^{-t\mathcal{A}_2} z_2^0 + \int_0^t e^{-(t-s)\mathcal{A}_2} Q_N \mathcal{B} K_1 z_1(s) ds$$

Passing to the norm and using the estimates (18) and (20) we obtain

$$|z_2^{\tilde{K}}(t)| \leq C e^{-\delta t} |z_2^0| + \int_0^t C e^{-\delta(t-s)} e^{-\delta_1 s} |z_1^0| ds,$$

from where, for a $\delta_1 > \delta$ and a constant $C = C(\delta, \delta_1)$,

$$|z_2^{\tilde{K}}(t)| \leq C e^{-\delta t} |z_0|.$$

This, together with (20), give (10) and we conclude the proof of the theorem. ■

4 Local stabilization of the MHD system. Proof of Theorem 2

Lemma 4.1 *Let $\tilde{H}$ be a Hilbert space with norm $|\cdot|$ and scalar product $(\cdot,\cdot)$ and let $-\tilde{\mathcal{A}}$ be the generator of an analytic semigroup of negative type satisfying an estimate of the type* (10) *and such that $D(\tilde{\mathcal{A}}) = D(\tilde{\mathcal{A}}^*)$. Then, the quadratic functional*

$$h(z) = \int_0^\infty |\tilde{\mathcal{A}}^{\frac{1}{2}} e^{-t\tilde{\mathcal{A}}} z|^2 dt$$

is finite for all $z \in \tilde{H}$ and defines an equivalent norm in $\tilde{H}$.

Proof From the theory of interpolation spaces (see [19] or [8]) we know that for such operators and for $\theta \in (0,1)$, the interpolation space

$$[D(\tilde{\mathcal{A}}), \tilde{H}]_\theta = \{z \in \tilde{H} : t^{\theta - \frac{1}{2}} \tilde{\mathcal{A}} e^{-t\tilde{\mathcal{A}}} z \in L^2(0,\infty;\tilde{H})\} = D(\tilde{\mathcal{A}}^{1-\theta}).$$

An equivalent norm ($0 \notin \sigma(\tilde{\mathcal{A}})$) is given by

$$\|t^{\theta-\frac{1}{2}}\tilde{\mathcal{A}}e^{-t\tilde{\mathcal{A}}}z\|_{L^2(0,\infty;\tilde{H})}$$

We prove first that there exists a positive constant C such that for $z \in D(\tilde{\mathcal{A}}^{\frac{1}{2}})$,

$$h(z) \leq C\|z\|^2_{D(\tilde{\mathcal{A}}^{\frac{1}{2}})}.$$

Indeed, integrating by parts in the integral of h we obtain

$$h(z) = \int_0^\infty t(\tilde{\mathcal{A}}^{\frac{1}{2}}e^{-t\tilde{\mathcal{A}}}z, \tilde{\mathcal{A}}^{\frac{3}{2}}e^{-t\tilde{\mathcal{A}}}z)dt =$$

$$= \int_0^\infty t((\tilde{\mathcal{A}}^{\frac{1}{2}})^*\tilde{\mathcal{A}}^{-\frac{1}{2}}\tilde{\mathcal{A}}e^{-t\tilde{\mathcal{A}}}z, \tilde{\mathcal{A}}e^{-t\tilde{\mathcal{A}}}z)dt \leq C\int_0^\infty t|\tilde{\mathcal{A}}e^{-t\tilde{\mathcal{A}}}z|^2dt \leq C|z|^2,$$

where we have also used the fact that $(\tilde{\mathcal{A}}^{\frac{1}{2}})^*\tilde{\mathcal{A}}^{-\frac{1}{2}}$ is an isomorphism of $\tilde{H}$.

For the reverse inequality we use the fact that for positive θ

$$|\tilde{\mathcal{A}}^\theta e^{-t\tilde{\mathcal{A}}}z| \leq \frac{C(\theta)}{t^\theta}|z|.$$

So, if in the previous inequality we take $\theta = \frac{1}{2}$ and instead of z, $\tilde{\mathcal{A}}^{\frac{1}{2}}e^{-t\tilde{\mathcal{A}}}z$, we find that

$$t|\tilde{\mathcal{A}}e^{-2t\tilde{\mathcal{A}}}z|^2 \leq C|\mathcal{A}^{\frac{1}{2}}e^{-t\tilde{\mathcal{A}}}z|^2$$

and the reverse inequality

$$c\|z\|^2_{D(\tilde{\mathcal{A}}^{\frac{1}{2}})} \leq h(z)$$

is immediate. Of course, the two inequalities extend to the whole $\tilde{H}$ and the proof of the lemma is complete. ■

We turn now to the proof of Theorem 2. Let $\tilde{H} = H \times H$ and $\tilde{\mathcal{A}} = \mathcal{A} - BK$, where K is the feedback constructed in Theorem 1. Consider the functional

$$h(z) = \int_0^\infty |\tilde{\mathcal{A}}^{\frac{3}{4}}e^{-t\tilde{\mathcal{A}}}z|^2dt.$$

$\tilde{\mathcal{A}}$ also defines an analytic semigroup of negative type in $D(\tilde{\mathcal{A}})$, $D(\tilde{\mathcal{A}}) = D(\tilde{\mathcal{A}}^*) = D(A)$, from Lemma 4.1 we deduce that

$$h(z) \sim |\tilde{\mathcal{A}}^{\frac{1}{4}}z|^2.$$

We denote also by h the bilinear form giving $h(z) = h(z,z)$. Given $\zeta \in \tilde{H}$ it defines, via the scalar product in $\tilde{H}$, a linear continuous functional on $D(\tilde{\mathcal{A}}^{\frac{1}{4}})$, so there exists an unique $S\zeta \in D(\tilde{\mathcal{A}}^{\frac{1}{4}})$ such that $(\zeta, z) = h(S\zeta, z)$ for all $z \in D(\tilde{\mathcal{A}}^{\frac{1}{4}})$. It is clear that S is self-adjoint in $\tilde{H}$. Denote by $R = S^{-1}$ which is an unbounded self-adjoint operator with dense domain $D(R) = \text{Im } S^{-1}$ with $D(R) \subset D(\tilde{\mathcal{A}}^{\frac{1}{4}})$.

We prove that $D(\tilde{\mathcal{A}}^{\frac{1}{2}}) \subset D(R)$ with continuous imbedding. To see this we have to prove that if $\zeta \in D(\tilde{\mathcal{A}}^{\frac{1}{2}})$ then the linear functional $z \to h(\zeta, z)$ extends as a linear continuous functional on $\tilde{H}$. Let $\zeta = \tilde{\mathcal{A}}^{-\frac{1}{2}}\eta$, with $\eta \in \tilde{H}$. We have

$$h(\zeta, z) = \int_0^\infty (\tilde{\mathcal{A}}^{\frac{1}{2}} e^{-t\tilde{\mathcal{A}}}\eta, (\tilde{\mathcal{A}}^{-\frac{1}{4}})^* \tilde{\mathcal{A}}^{\frac{1}{4}} \tilde{\mathcal{A}}^{\frac{1}{2}} e^{-t\tilde{\mathcal{A}}} z) dt,$$

So using Lemma 4.1 and the fact that $(\tilde{\mathcal{A}}^{-\frac{1}{4}})^* \tilde{\mathcal{A}}^{\frac{1}{4}}$ is bounded in $\tilde{H}$ we have

$$h(\zeta, z) \leq C \left(\int_0^\infty |\mathcal{A}^{\frac{1}{2}} e^{-t\tilde{\mathcal{A}}}\eta|^2 \right)^{\frac{1}{2}} \left(\int_0^\infty |\mathcal{A}^{\frac{1}{2}} e^{-t\tilde{\mathcal{A}}} z|^2 \right)^{\frac{1}{2}} \leq C|\tilde{\mathcal{A}}^{\frac{1}{2}}\zeta||z|.$$

So we have that $\zeta \in D(R)$,

$$(R\zeta, z) = h(\zeta, z),$$

and

$$|R\zeta| \leq C|\tilde{\mathcal{A}}^{\frac{1}{2}}\zeta|. \tag{21}$$

As usual, one may easily derive the Lyapunov equation satisfied by R:

$$(Rz, \tilde{\mathcal{A}}z) = \frac{1}{2}|\tilde{\mathcal{A}}^{\frac{3}{4}} z|^2. \tag{22}$$

We are now in a position to stabilize the nonlinear system (6). We introduce the feedback K and system (6) may be rewritten as

$$\begin{cases} z' + \tilde{\mathcal{A}}z = g(z) \\ \\ z(0) = z_0 \end{cases} \tag{23}$$

where the nonlinearity g is

$$g \begin{pmatrix} y \\ B \end{pmatrix} = \begin{pmatrix} P(B \cdot \nabla B - y \cdot \nabla y) \\ P(B \cdot \nabla y - y \cdot \nabla B) \end{pmatrix}$$

The idea, the same as in [2] or [7], is to multiply equation (23), scalarly in $\tilde{H}$, with Ry and integrate. The problem is that we do not know if (23) has a global strong solution (difficulty of the same nature as in the case of Navier-Stokes equations). So, one has to consider an approximate equation, to show that this is stable and that its solution is converging to a weak solution of our problem, which conserves the exponential decay at infinity.

For $\zeta \in H^1(\Omega, \mathbf{R}^3)$ and $\varepsilon > 0$ we define the truncation $T_\varepsilon(\zeta) = \zeta$ if $\|\zeta\| \leq \dfrac{1}{\varepsilon}$ and $T_\varepsilon(\zeta) = \dfrac{\zeta}{\varepsilon\|\zeta\|}$ if $\|\zeta\| > \dfrac{1}{\varepsilon}$. For $z = (y, B)^T \in V$ we denote by $T_\varepsilon(z) = (T_\varepsilon(y), T_\varepsilon(B))^T$ and denote by $g_\varepsilon(z) = g(T_\varepsilon(z))$. The approximate equation is:

$$\begin{cases} z' + \tilde{\mathcal{A}}z = g_\varepsilon(z) \\ \\ z(0) = z_0. \end{cases} \tag{24}$$

For this equation one may prove (as in [2]) that it has a strong solution $z_\varepsilon \in C([0,T];H) \cap L^2(0,T;V) \cap L^2_{loc}((0,T],D(\tilde{\mathcal{A}}))$. Moreover, one has for $t \in [0,T]$ the bound

$$|z_\varepsilon(t)|^2 + \int_0^t \left(\|z_\varepsilon\|_V^2 + \|\frac{\partial z_\varepsilon}{\partial t}\|_{V'}^{\frac{4}{3}} \right) \le C_T |z_0|^2$$

and $z_\varepsilon \to z$ strongly in $L^2(0,T;H)$ and weakly in $L^2(0,T;V)$. We also have, by the Aubin compactness theorem that $z_\varepsilon \to z$ strongly in $L^2(0,T,D(\tilde{\mathcal{A}}^{\frac{1}{4}}))$. The limit z is a weak solution of (23).

We multiply now equation (24) by Rz_ε and obtain

$$\frac{d}{dt}(Rz_\varepsilon, z_\varepsilon) + (\tilde{\mathcal{A}} z_\varepsilon, Rz_\varepsilon) = (g_\varepsilon(z_\varepsilon), Rz_\varepsilon). \tag{25}$$

Standard estimates using the inequality (2) for the trilinear term b show that, for $z \in D(\tilde{\mathcal{A}}^{\frac{3}{4}})) \subset H^{\frac{3}{2}}, \zeta \in V$, one has

$$|(g_\varepsilon(z), \zeta)| \le |(g(z), \zeta)| \le C\|z\|_1 \|z\|_{\frac{3}{2}} |\zeta|. \tag{26}$$

Using the interpolation inequality

$$\|z\|_1^2 = |\tilde{\mathcal{A}}^{\frac{1}{2}} z|^2 \le |\tilde{\mathcal{A}}^{\frac{1}{4}} z||\tilde{\mathcal{A}}^{\frac{3}{4}} z| \le C(Rz,z)^{\frac{1}{2}} |\tilde{\mathcal{A}}^{\frac{3}{4}} z|$$

and (26) with $\zeta = Rz_\varepsilon$, we obtain from (25), (21) and (22) that

$$\frac{d}{dt}(Rz_\varepsilon, z_\varepsilon) + |\tilde{\mathcal{A}}^{\frac{3}{4}} z_\varepsilon|^2 \le C(Rz_\varepsilon, z_\varepsilon)^{\frac{1}{2}} |\tilde{\mathcal{A}}^{\frac{3}{4}} z_\varepsilon|^2 \tag{27}$$

With $\rho = \dfrac{1}{4C^2}$ it is easy to see that the set $\mathcal{V}_\rho = \{z_0 : (Rz_0, z_0) < \rho\}$ is invariant under the flow generated by (24). Moreover, for $z_0 \in \mathcal{V}_\rho$ one has

$$\frac{d}{dt}(Rz_\varepsilon, z_\varepsilon) + \frac{1}{2}|\tilde{\mathcal{A}}^{\frac{3}{4}} z_\varepsilon|^2 \le 0$$

and thus, since $c(Rz_\varepsilon, z_\varepsilon) \le \|z_\varepsilon\|_{\frac{1}{2}}^2 \le C|\tilde{\mathcal{A}}^{\frac{3}{4}} z_\varepsilon|^2$, with some positive constant δ,

$$\frac{d}{dt}(Rz_\varepsilon, z_\varepsilon) + \delta(Rz_\varepsilon, z_\varepsilon) \le 0.$$

Integrating the last inequality one finds that

$$|(Rz_\varepsilon, z_\varepsilon)| \le |(Rz_0, z_0)| e^{-\delta t}.$$

Now, because $z_\varepsilon \to z$ strongly in $L^2(0,T,D(\tilde{\mathcal{A}}^{\frac{1}{4}}))$ then for t *a.e.* $|(Rz_\varepsilon, z_\varepsilon)| \to |(Rz,z)|$ and thus

$$|(Rz, z)| \le |(Rz_0, z_0)| e^{-\delta t}.$$

and the inequality (11) follows. The proof is complete. ■

5 Unique continuation for systems of mixed parabolic-elliptic equations

The purpose of this section is to study the unique continuation property for systems of parabolic-elliptic equations in order to apply these results to the unique continuation of system (14).

In a first step, we will obtain estimates for parabolic and elliptic equations that will have as consequence the unique continuation property established by J.-C. Saut and B. Scheurer in [17]. The systems for which such a unique property holds are systems of linear parabolic and elliptic equations coupled in the terms of order 0 and 1. In this form this result is applicable to the dual equation (12) in the form (13). This is however requiring higher regularity for the stationary solution $(\bar{y}, \bar{B})$.

The result in [17] is not applicable to the dual system in form (14) because, when applying the divergence to the equations in (14), the elliptic equations for π and ρ contain second order derivatives in the other unknowns ζ and C. That is the reason for which we will make use of the more refined estimates for elliptic equations, obtained by O.Yu. Imanuvilov and J.-P- Puel in [14], [15], and couple them with the estimates for the parabolic part of the system.

The heat equation

To make computations as simple and transparent as possible, we consider first the simplest case of (backward) heat equation:

$$\frac{\partial y}{\partial t} + \Delta y = f \quad \text{in} \quad \Omega \times (0, T) \tag{28}$$

Let $\omega \subset \Omega$. We intend to derive estimates that will have as a consequence the fact that if $f \equiv 0$ and $y = 0$ on $\omega \times (0, T)$ then $y \equiv 0$ on $Q^T := \Omega \times (0, T)$. Also, for a given function z we will denote by $z_t, z_{,i}, z_{,ti}$ etc. the partial derivatives $\dfrac{\partial z}{\partial t}, \dfrac{\partial z}{\partial x_i}, \dfrac{\partial^2 z}{\partial t \partial x_i}$ and we use the the convention for summation when indices are repeated.

Let $0 < r < R$, with $B_R \subset\subset \Omega$. It is enough to prove that if $y = 0$ in $Q_r := B_r \times (0, T)$ then $y = 0$ on $Q_R := B_R \times (0, T)$. If this is not the case, with no loss of generality (by modifying R and replacing the time interval $(0, T)$ with a smaller interval (t_1, t_2)), we will assume that

$$m := \inf_{t \in (0,T)} \int_{\partial B_R} |y(x,t)|^2 d\sigma_x > 0. \tag{29}$$

We choose an auxiliary function $\psi \in C^2(\overline{B_R})$ with the following properties:

$$\psi|_{\partial B_R} = 0,\ \psi|_{B_R} > 0,\ \{x | \nabla \psi(x) = 0\} \subset B_{\frac{r}{2}}, \frac{\partial \psi}{\partial \mathbf{n}} = -1 \text{ on} \partial B_R. \tag{30}$$

Obviously, it is easy to construct such a function and it may be chosen radially symmetric with an unique critical point, namely in the center of the ball. Let for

$\lambda > 0$

$$\alpha(x,t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_{C(\bar{B}_R)}}}{t^2(T-t)^2}, \ \varphi(x,t) = \frac{e^{\lambda\psi(x)}}{t^2(T-t)^2}$$

and $\bar{\alpha}(t) = \alpha|\partial B_R, \bar{\varphi}(t) = \varphi|\partial B_R$. Denote by $z := e^{s\alpha}y$ for some $s, \lambda > 0$. Using (28), the equations satisfied by z is:

$$z_t + \left[z_{,ii} - 2s\alpha_{,i}z_{,i} + (s^2\alpha_{,i}^2 - s\alpha_{,ii} - s\alpha_t)z\right] = fe^{s\alpha} \tag{31}$$

We reorder the terms in equation (31) and, denoting by

$$X(x,t) = \left[z_{,ii} + (s^2\alpha_{,i}^2 + s\alpha_{,ii} - s\alpha_t)z\right],$$

$$F(x,t) = -2s(\alpha_{,i}z_{,i} + s\alpha_{,ii}z)$$

we rewrite (31) as

$$z_t + X(x,t) + F(x,t) = fe^{s\alpha} \tag{32}$$

We multiply (32) with $X(x,t)$, we integrate on Q_R and, by Cauchy inequality we obtain:

$$\int_{Q_R} z_t X(x,t)dxdt + \int_{Q_R} X(x,t)F(x,t)dxdt \leq \frac{1}{2}\int_{Q_R} f^2 e^{2s\alpha} \tag{33}$$

We proceed as in [10] (see also [1]) to evaluate the integrals in (33). The idea is to put in evidence the dominant terms in s, λ, φ as well as their signs. We will see that the dominant term concerning z will be $s^3\lambda^4\varphi^3z^2$ and the other terms, dominated by it for s, λ big enough, like $s^3\lambda^3\varphi^3$, will be generically denoted as $l.o.t(z^2)$. The same for $|\nabla z|^2$ where the dominant term will be found to be $s\lambda^2\varphi|\nabla z|^2$ and the lower order terms will be denoted as $l.o.t.(|\nabla z|^2)$. When talking about y the dominant term will be found to be $s^3\lambda^4\varphi^3e^{2s\alpha}z^2$ and lower order terms like $s^2\lambda^4\varphi^2e^{2s\alpha}z^2$ will be denoted shortly as $l.o.t.(y^2)$. Similar notations will be used for the terms containing $|\nabla y|^2$. Finally, the dominant terms, with positive sign, will be found in the left part of the final equality.

I. $\int_{Q_R} z_t X(x,t)dxdt$:

$$\int_{Q_R} z_t X(x,t)dxdt = \int_{Q_R} z_t z_{,ii} + \left(\frac{z^2}{2}\right)_t \left(s^2\alpha_{,i}^2 + s\alpha_{,ii} - s\alpha_t\right)dxdt =$$

$$= \int_{\Sigma_R} z_t z_{,i}\mathbf{n}_i d\sigma dt - \int_{Q_R} z_{,ti}z_{,i} + \frac{z^2}{2}\left(s^2\alpha_{,i}^2 + s\alpha_{,ii} - s\alpha_t\right)_t dxdt =$$

$$= \int_{\Sigma_R} z_t z_{,i}\mathbf{n}_i d\sigma dt + \int_{Q_R} l.o.t.(z^2)dxdt \tag{34}$$

II. $\int_{Q_R} X(x,t)F(x,t)dxdt$:

$$\textbf{a)} - 2\int_{Q_R} z_{,ii}s\alpha_{,j}z_{,j}dxdt =$$

$$= 2\int_{Q_R} z_{,i}s\alpha_{,ij}z_{,j} + z_{,i}z_{,ij}s\alpha_{,j}dxdt - 2\int_{\Sigma_R} z_{,i}\mathbf{n}_i s\alpha_{,j}z_{,j}dxdt =$$

$$= \int_{Q_R} 2z_{,i}z_{,j}s\alpha_{,ij} - z_{,i}^2 s\alpha_{,jj}dxdt + \int_{\Sigma_R} z_i^2 s\alpha_{,j}\mathbf{n}_j - 2z_{,i}\mathbf{n}_i s\alpha_{,j}z_{,j}\sigma dt. \tag{35}$$

b) $$-2\int_{Q_R} z_{,ii}s\alpha_{,jj}z = 2\int_{Q_R} z_{,i}zs\alpha_{,ijj} + z_{,i}^2 s\alpha_{,jj}dxdt - 2\int_{\Sigma_R} z_{,i}\mathbf{n}_i s\alpha_{,jj}zd\sigma dt \tag{36}$$

c) $$-2\int_{Q_R} s^3\alpha_{,i}^2\alpha_{,j}zz_{,j} = \int_{Q_R} s(\alpha_{,i}^2\alpha_{,j})_{,j}z^2dxdt - \int_{\Sigma_R} s^3\alpha_{,i}^2\alpha_{,j}\mathbf{n}_j z^2 d\sigma dt \tag{37}$$

d.The other terms in $\int_{Q_R} X(x,t)F(x,t)dxdt$ are of lower order in s, λ and may be dominated by using Cauchy's inequality.

First estimates, for all solutions:

$$\int_{Q_R\setminus Q_r} s^3\lambda^4\varphi^3 e^{2s\alpha}y^2dxdt + \int_{Q_R\setminus Q_r} s\lambda^2\varphi e^{2s\alpha}|\nabla y|^2 + \int_{\Sigma_R} ... \leq$$

$$\leq C\left[\int_{Q_r}((s^3\lambda^3\varphi^3 + s^4\lambda^4\varphi^2)y^2 + s\lambda\varphi|\nabla y|^2)e^{2s\alpha}dxdt\right] + \int_{Q_R} f^2 e^{2s\alpha}dxdt \tag{38}$$

where the constant C does not depend on the solution y and $s \geq s_0(\lambda), \lambda \geq \lambda_0$.

We estimate now the integrals on Σ_R denoted in the above formula, for short, $\int_{\Sigma_R} ...$. Remember that $\frac{\partial\psi}{\partial\mathbf{n}} \equiv -1$ on ∂B_R.

(i) $$-\int_{\Sigma_R} s^3\alpha_{,i}^2\alpha_{,j}\mathbf{n}_j z^2dxdt = s^3\lambda^3\int_{\Sigma_R}\bar{\varphi}^3 e^{2s\bar{\alpha}}|y|^2d\sigma dt \tag{39}$$

Since $z = e^{s\alpha}y$, $z_{,i} = (s\alpha_{,i}y + y_{,i})e^{s\alpha}$ and we have:

(ii) $$-2\int_{\Sigma_R} z_{,i}\mathbf{n}_i s\alpha_{,j}z_{,j}d\sigma dt = -2\int_{\Sigma_R}(s\alpha_{,i}y + y_{,i})\mathbf{n}_i s\alpha_{,j}(s\alpha_{,j}y + y_j)e^{2s\alpha}d\sigma dt =$$

$$= -2\int_{\Sigma_R}\left(s^3\alpha_{,i}\alpha_{,j}^2\mathbf{n}_i|y|^2 - sy_{,i}\mathbf{n}_i\alpha_{,j}y_{,j} - s^2\alpha_{,i}\mathbf{n}_i\alpha_{,j}yy_{,j} - s^2y_{,i}\mathbf{n}_i\alpha_{,j}^2 y\right)e^{2s\alpha}d\sigma dt =$$

$$= 2s^3\lambda^3\int_{\Sigma_R}\bar{\varphi}^3 e^{2s\bar{\alpha}}|y|^2d\sigma dt + \int_{\Sigma_R} l.o.t \tag{40}$$

(iii) $$\int_{\Sigma_R} z_{,i}^2 s\alpha_{,j}\mathbf{n}_j d\sigma dt = \int_{\Sigma_R}(s\alpha_{,i}y + y_{,i})^2 s\alpha_{,j}\mathbf{n}_j e^{2s\alpha} =$$

$$= \int_{\Sigma_R} s^3\alpha_{,i}^2\alpha_{,j}\mathbf{n}_j e^{2s\alpha}y^2 + y_{,i}^2 s\alpha_{,j}\mathbf{n}_j e^{2s\alpha} + 2s\alpha_{,i}yy_{,i}s\alpha_{,j}\mathbf{n}_j e^{2s\alpha}d\sigma dt =$$

$$= -s^3\lambda^3\int_{\Sigma_R}\bar{\varphi}^3 e^{2s\bar{\alpha}}y^2d\sigma dt + \int_{\Sigma_R} l.o.t.d\sigma dt \tag{41}$$

$$\textbf{(iv)}\ -2\int_{\Sigma_R} z_{,i}\mathbf{n}_i s\alpha_{,jj} z d\sigma dt = -2\int_{\Sigma_R} (s\alpha_{,i}y + y_i)\mathbf{n}_i s\alpha_{,jj} y e^{2s\alpha} d\sigma dt =$$

$$= -2\int_{\Sigma_R} s^2\alpha_{,i}\mathbf{n}_i\alpha_{,jj}y^2 e^{2s\alpha} d\sigma dt - 2\int_{\Sigma_R} s\alpha_{,jj}\frac{\partial y}{\partial \mathbf{n}} y e^{2s\alpha} d\sigma dt \tag{42}$$

$$\textbf{(v)}\ \int_{\Sigma_R} z_t z_{,i}\mathbf{n}_i d\sigma dt = \int_{\Sigma_R} (s\alpha_t y + y_t)(s\alpha_{,i}y + y_{,i})\mathbf{n}_i e^{2s\alpha} d\sigma dt$$

$$= s^2\int_{\Sigma_R} (\alpha_t\alpha_{,i}\mathbf{n}_i y^2 + l.o.t) e^{2s\alpha} d\sigma dt \tag{43}$$

If we look now to the formulas (39)-(43) we find that the dominant term for the integral on Σ_R is

$$2s^3\lambda^3\int_{\Sigma}\varphi^3 y^2 e^{2s\alpha} d\sigma dt = 2k^3 s^3\lambda^3\int_0^T e^{2s\bar{\alpha}}\bar{\varphi}^3\int_{\partial\Omega} y^2 d\sigma dt$$

In this situation, for $s, \lambda > 0$ big enough, the integral on Σ_R becomes positive. So, we finally may write the final estimate:

$$\int_{Q_R\setminus Q_r} s^3\lambda^4\varphi^3 e^{2s\alpha}|y|^2 + s\lambda^2\varphi e^{2s\alpha}|\nabla y|^2 dxdt + \int_{\Sigma_R} s^3\lambda^3|y|^2 e^{2s\bar{\alpha}}\bar{\varphi}^3 d\sigma dt \leq \tag{44}$$

$$\leq C\Big(\int_{Q_r} s^3\lambda^4\varphi^3 e^{2s\alpha}|y|^2 + s\lambda^2\varphi e^{2s\alpha}|\nabla y|^2 dxdt\Big) + \int_{Q_R} f^2 e^{2s\alpha} dxdt$$

This implies immediately that if $f \equiv 0$ and $y = 0$ on $B_r \times (0,T)$ then $y = 0$ on $B_R \times (0,T)$.

The case of general parabolic operators

Consider now an elliptic operator with the form

$$Ly = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 y}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j\frac{\partial y}{\partial x_j} + cy \tag{45}$$

where

i) $a_{ij} \in C^1(Q^T)$, $a_{ij} = a_{ji}$ and define a uniformly positive definite matrix: $\sum_{i,j=1}^n a_{ij}(x)\xi^i\xi^j \geq \beta|\xi|^2$ for some positive constant β and all $(x,t) \in Q$.

ii) $b_i \in L^\infty(Q^T)$

iii) $c \in L^\infty(0,T;L^n(\Omega))$ if $n \geq 3$ and $c \in L^p(\Omega)$ for some $p > 1$ if $n = 2$.

Then, if $y \in L^2(0,T;H^2_{loc}(\Omega))$ satisfies

$$y_t + Ly = 0 \quad \text{in } Q^T$$

and $y \equiv 0$ in $\omega \times (0,T)$, then $y \equiv 0$ in $\Omega \times (0,T)$.

This is the content of Theorem 1.1 in [17] and we show how to derive this result by means of an inequality of type (44). Consider the nonhomogeneous equation

$$y_t + Ly = f \quad \text{ in } Q^T.$$

The key in obtaining in this case estimates of the type (44) is to move the first order and zero order terms in the right hand side, incorporate them in the free term f, and at the end of the computations, which in this case work almost identically as in the model case, to apply the Hölder inequality and dominate them with the terms in the left hand side in inequality (44)(see also the inequality (38)).

More precisely, denote by $\tilde{f} = f - \sum_{i=1}^{n} b_i \dfrac{\partial y}{\partial x_i} - cy$. The term $\int_{Q_R} \tilde{f}^2 e^{2s\alpha} dxdt$ is estimated by $\int_{Q_R} f^2 e^{2s\alpha} dxdt$ and the following terms (multiplied by some independent constant):

$$\int_{Q_R} b_i^2 y_{,i}^2 e^{2s\alpha} dxdt \leq C \int_{Q_R} |\nabla y|^2 e^{2s\alpha} dxdt = \int_{Q_R} l.o.t.(|\nabla y|^2) dxdt$$

and

$$\int_{Q_R} c^2 y^2 e^{2s\alpha} dxdt \leq \left(\int_{Q_R} c^n e^{ns\alpha} \right)^{\frac{2}{n}} \left(\int_{Q_R} (ye^{s\alpha})^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}.$$

But

$$\left(\int_{Q_R} c^n e^{ns\alpha} \right)^{\frac{2}{n}} \leq e^{2s\bar{\alpha}} \|c\|_{L^n}^2 = o(s) \|c\|_{L^n}^2$$

and by the Sobolev embedding theorem

$$\left(\int_{Q_R} (ye^{s\alpha})^{\frac{2n}{n-2}} dxdt \right)^{\frac{n-2}{n}} \leq \int_{Q_R} |\nabla (ye^{s\alpha})|^2 dxdt \leq$$

$$\leq 2 \int_{Q_R} \left(|\nabla y|^2 + s^2 \lambda^2 \varphi^2 y^2 \right) e^{2s\alpha} dxdt = \int_{Q_R} l.o.t(y^2) + l.o.t(|\nabla y|^2) dxdt.$$

The computations follow now the same lines as in the model case.

We observe here that, with no essential changes in the proof, the same estimates hold for elliptic equations of the form $Ly = f$. ■

Systems of parabolic - elliptic equations

Consider now systems of the type

$$\begin{cases} \dfrac{\partial y_i}{\partial t} + L_i y_i + \displaystyle\sum_{k=1}^{m} \sum_{l=1}^{n} \gamma_{ikl}(x,t) \frac{\partial y_k}{\partial x_l} + \sum_{k=1}^{m} \varsigma_{ik}(x,t) y_k = 0, \ 1 \leq i \leq p \\ \\ L_i y_i + \displaystyle\sum_{k=1}^{m} \sum_{l=1}^{n} \gamma_{ikl}(x,t) \frac{\partial y_k}{\partial x_l} + \sum_{k=1}^{m} \varsigma_{ik}(x,t) y_k = 0, \quad p \leq i \leq m \end{cases} \tag{46}$$

where $\gamma_{ikl} \in L^\infty_{loc}(Q^T)$, $\varsigma_{ik} \in L^\infty(0,T;L^n_{loc}(\Omega))$.

The unique continuation result in this case says that if $y_i = 0$ on $\omega \times (0,T), i = \overline{1,m}$, then $y_i \equiv 0$ on $\Omega \times (0,T), i = \overline{1,m}$. This is essentially the content of Theorem 2.1 in [17]. The proof is quite similar to the case of a single parabolic equation we described above and we just sketch it. So, for $0 < r < R$ we suppose that $y_i = 0$, on $Q_r, i = \overline{1,m}$ and we want to prove that $y_i = 0$, on $Q_R, i = \overline{1,m}$. If it is not true, we may suppose, as in the case of a single parabolic equation, that there exists an $i_0 \in \{1,\dots m\}$ such that condition (29) holds for y_{i_0}. For each $i \in \{1,\dots m\}$, for the equation $y_{i,t} + L_i y_i = f_i$ we write an estimate of type (44) and add these estimates. We keep in mind that among the terms on the boundary the one which is dominant is $\int_{\Sigma_R} s^3\lambda^3 e^{2s\bar\alpha}\bar\varphi^3 y_{i_0}^2 d\sigma dt$. We obtain, for $\lambda > \lambda_0$ and $s > s_0(\lambda)$, that

$$\sum_{i=1}^m \left(\int_{Q_R\setminus Q_r} s^3\lambda^4\varphi^3 e^{2s\alpha}|y_i|^2 + s\lambda^2\varphi e^{2s\alpha}|\nabla y_i|^2 dxdt \right) + \tag{47}$$

$$+ \int_{\Sigma_R} s^3\lambda^3 e^{2s\bar\alpha}\bar\varphi^3 |y_{i_0}|^2 d\sigma dt \le C\sum_{i=1}^m \left(\int_{Q_r} (s^3\lambda^4\varphi^3 y_i^2 + s\lambda^2\varphi|\nabla y_i|^2) e^{2s\alpha} dxdt + \right.$$

$$\left. + \int_{Q_R} \left| \sum_{k=1}^m \sum_{l=1}^n \gamma_{ikl}(x,t)\frac{\partial y_k}{\partial x_l} + \sum_{k=1}^m \varsigma_{ik}(x,t) y_k \right|^2 e^{2s\alpha} dxdt \right).$$

Using now the Hölder inequality for the right side of the inequality, as in the case of a general parabolic equation, we absorb the integrals on $Q_R \setminus Q_r$ from the right side, into the corresponding integrals in the left side, for s, λ big enough. We find then that necessarily $y_i \equiv 0$ also on $Q_R \setminus Q_r, i = \overline{1,m}$. ∎

Remark 5.1 *We may apply now this result for the unique continuation property for* (12)*. The regularity needed for the parabolic part is $\nabla\bar y, \nabla\bar B \in L^3$, $\bar y, \bar B \in L^\infty$, so $W^{1,3} \cap L^\infty$ (the same as in the hypotheses for the moment). Now, if we apply the divergence to the equations in* (13)*, we find two elliptic equations for π, ρ, containing terms of the type $\Delta\bar y \cdot \zeta$ and $\bar B_{i,j} C_{j,i}$. At this point the previous result works if we ask for the supplementary regularity $\Delta\bar y, \Delta\bar B \in L^3$ and $\nabla\bar B \in L^\infty$, that is the stationary solution should belong to $W^{2,3} \cap W^{1,\infty}$.*

Elliptic equations

We mention here the following theorem of O.Yu.Imanuvilov and J.P.Puel proved in [15](see also [14]):

Theorem 3 *Let $D \subset \mathbf{R}^n$ be bounded, open, with C^2 boundary. Let $y \in H^1(D)$*

solution of the following boundary value problem:

$$\begin{cases} Ly := & \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 y}{\partial x_i x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial y}{\partial x_j} + \\ & +\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(d_i(x)y) + c(x)y = f + \sum_{j=1}^{n}\frac{\partial f_j}{\partial x_j} \quad \text{in } D \\ y = g & \qquad\qquad \text{on } \partial D \end{cases} \tag{48}$$

where

i) $a_{ij} \in C^2(\overline{D}), b_j, c, d_i \in L^\infty(D), i,j = \overline{1,n}$, a_{ij} *verify the uniform ellipticity condition*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi^i\xi^j \geq \beta|\xi|^2, \forall \xi \in \mathbf{R}^n, x \in D$$

for some $\beta > 0$.

ii) $f, f_j \in L^2(D), j = \overline{1,n}$, $g \in H^{\frac{1}{2}}(\partial D)$.

Consider a function $\psi \in C^2(\overline{D})$ *with the following properties, analogous to* (30)*:*

$$\psi|_{\partial D} = 0,\ \psi|_D > 0,\ \{x|\nabla\psi(x) = 0\} \subset \omega_1 \subset\subset D, \frac{\partial \psi}{\partial \mathbf{n}} < 0 \text{ on } \partial D.$$

With $\eta(x) = e^{\lambda\psi(x)}$, *there exist* $\hat{\lambda}, \hat{\tau}$ *and a constant* $C > 0$ *such that for all* $\lambda \geq \hat{\lambda}, \tau \geq \hat{\tau}$ *the following inequality holds:*

$$\begin{aligned} &\int_\Omega e^{2\tau\eta}|\nabla y|^2 dx + \tau^2\lambda^2\int_\Omega \eta^2 e^{2\tau\eta}|y|^2 dx \leq \\ &\leq C\left[\tau^{\frac{1}{2}} e^{2\tau}\|g\|^2_{H^{\frac{1}{2}}(\partial D)} + \frac{1}{\tau\lambda^2}\int_\Omega \frac{|f|^2}{\eta} e^{2\tau\eta}dx + \right. \\ &\left. + \sum_{j=1}^{n}\tau\int_\Omega |f_j|^2 \eta e^{2\tau\eta}dx + \int_\omega (|\nabla y|^2 + \tau^2\lambda^2\eta^2|y|^2)e^{2\tau\eta}dx\right]. \end{aligned} \tag{49}$$

Unique continuation for system (14)

We prove now the unique continuation result for system (14), that is property (15).

For $0 < r < R$ we suppose that $\zeta = 0, C = 0$ on Q_r and we want to prove that $\zeta \equiv 0, C \equiv 0$ on Q_R. If it is not true, we may suppose, with no loss of generality, that the following analogue of condition (29) holds:

$$m := \inf_{t\in(0,T)} \int_{\partial B_R} |\zeta(x,t)|^2 + |C(x,t)|^2 d\sigma_x > 0. \tag{50}$$

From (14) we find that also $\nabla\pi = \nabla\rho = 0$ in Q_r so, since for given t $\pi(\cdot,t), \rho(\cdot,t)$ are defined up to an additive constant, we suppose with no loss of generality that

$\pi = \rho = 0$ in Q_r. By treating $\nabla\pi, \nabla\rho$ as nonhomogeneous terms in (14) we obtain first the following estimate of type (44) that is, for $\lambda > \lambda_0, s > s_0(\lambda)$:

$$\int_{Q_R\setminus Q_r} s^3\lambda^4\varphi^3 e^{2s\alpha}(|\zeta|^2+|C|^2) + s\lambda^2\varphi e^{2s\alpha}(|\nabla\zeta|^2+|\nabla C|^2)dxdt+$$
$$+\int_{\Sigma_R} s^3\lambda^3(|\zeta|^2+|C|^2)e^{2s\bar\alpha}\bar\varphi^3 d\sigma dt \le \int_{Q_R}(|\nabla\pi|^2+|\nabla\rho|^2)e^{2s\alpha}dxdt+ \quad (51)$$
$$+C\int_{Q_r}[s^3\lambda^4\varphi^3(|\zeta|^2+|C|^2)+s\lambda^2\varphi(|\nabla\zeta|^2+|\nabla C|^2)]e^{2s\alpha}dxdt.$$

We need now to estimate the pressure terms π, ρ. We apply the divergence to the equations in the dual system (14) and we obtain the following elliptic equations:

$$\Delta\pi = \text{div } ((D^s\zeta)\bar y - (D^a C)\bar B) \quad \text{in } Q^T \quad (52)$$

$$\Delta\rho = \text{div } (((D^a C)\bar B - D^s\zeta)\bar y) \quad \text{in } Q^T \quad (53)$$

We remark here that the result of J.-C. Saut and B. Scheurer on unique continuation for systems of mixed parabolic-elliptic equations does not apply since the equations for π and ρ contain second order derivatives of ζ and C. This is the reason for which the stronger estimate of J.-P. Puel and O.Yu.Imanuvilov is needed. So, applying inequality (49) to π and ρ on the ball B_R, for fixed t and taking into account that $\bar y, \bar B \in L^\infty(\Omega)$ and on Q_ω $\pi = \rho = 0$, we obtain, for all $\tau > \hat\tau, \lambda > \hat\lambda$, the inequality:

$$\int_{B_R} e^{2\tau\eta}(|\nabla\pi|^2+|\nabla\rho|^2)dx \le C\Big[\tau^{\frac12}e^{2\tau}(\|\pi\|^2_{H^{\frac12}(\partial B_R)}+\|\rho\|^2_{H^{\frac12}(\partial B_R)})+$$
$$+\tau\int_{B_R}(|(D^s\zeta)|^2+|(D^aC)|^2)\eta e^{2\tau\eta}dx\Big]. \quad (54)$$

We choose in this inequality $\tau = \dfrac{s}{t^2(T-t)^2}$, then we multiply (54) by $e^{-\frac{2s}{t^2(T-t)^2}e^{2\lambda\|\psi\|_{C(\bar B_R)}}}$ and integrate from 0 to T. We obtain:

$$\int_{Q_R} e^{2s\alpha}(|\nabla\pi|^2+|\nabla\rho|^2)dx \le$$
$$\le C\Big[\int_0^T s^{\frac12}\bar\varphi^{\frac12}e^{2s\bar\alpha}(\|\pi(t)\|^2_{H^{\frac12}(\partial B_R)}+\|\rho(t)\|^2_{H^{\frac12}(\partial B_R)})dt+ \quad (55)$$
$$+\int_{Q_R} s\varphi(|\nabla\zeta|^2+|\nabla C|^2)e^{2s\alpha}dx\Big].$$

We plug now inequality (55) into (51). We see that the integrals on Σ_R in the right hand side of (54) are of lower order with respect to $s, \lambda, \bar\varphi$ than the integrals on Σ_R in the left side of (47). The same happens with the integrals concerning $|\nabla\zeta|, |\nabla C|$

on $Q_R \setminus Q_r$. Finally, after absorbing in the left side of the inequality the lower order terms in the right side, we obtain for $\lambda > \lambda_0, s > s_0(\lambda)$:

$$\int_{Q_R \setminus Q_r} s^3\lambda^4\varphi^3 e^{2s\alpha}(|\zeta|^2 + |C|^2) + s\lambda^2\varphi e^{2s\alpha}(|\nabla\zeta|^2 + |\nabla C|^2)dxdt +$$

$$+ \int_{\Sigma_R} s^3\lambda^3(|\zeta|^2 + |C|^2)e^{2s\bar{\alpha}}\bar{\varphi}^3 d\sigma dt \le \tag{56}$$

$$\le C\int_{Q_r} \left[s^3\lambda^4\varphi^3(|\zeta|^2 + |C|^2) + s\lambda^2\varphi(|\nabla\zeta|^2 + |\nabla C|^2)\right] e^{2s\alpha}dxdt.$$

It follows that,necessarily, $\zeta = 0, C = 0$ in Q_R. ■

Remark 5.2 *For the unique continuation property of the dual equation* (14) *only the L^∞ regularity for the stationary data was needed.*

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