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An Iterative Algorithm for Multiple Stopping: Convergence and Stability

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Abstract

We present a new iterative procedure for solving the discrete multiple stopping problem and discuss the stability of the algorithm. The algorithm produces monotonically increasing approximations of the Snell envelope, which coincide with the Snell envelope after finitely many steps. Contrary to backward dynamic programming, the algorithm allows to calculate approximative solutions with only a few nestings of conditionals expectations and is, therefore, tailor-made for a plain Monte-Carlo implementation.

1 Introduction

Financial derivatives with several early exercise rights play an important role in several markets. For example, electricity markets (e.g. swing options) and interest rate markets (e.g. chooser flexible caps). The pricing problem for such instruments is equivalent to a multiple stopping problem which is solved in practice by trinomial forests usually, see Jaillet et al. (2004) and the references therein. However, this pricing procedure is restricted to models for low-dimensional underlying processes, since trees tend to explode with increasing dimension of the underlying process.

Obviously, multiple callable instruments with respect to a high dimensional interest rate model such as the very popular Libor market model, and also multiple callable options on a basket of several assets, do not meet this restriction. So new pricing methods for financial instruments with early exercise opportunities, based on high-dimensional underlying processes, are called for.

The problem of exploding computational cost, when the dimension of the underlying processes increases, is known as 'curse of dimensionality'. Even in the case of a single exercise right (i.e. the pricing problem of an American option or, equivalently, the optimal stopping problem), the classical approaches such as tree methods, initialized by Cox et al. (1979), or PDE techniques (Bensoussan and Lion, 1982; Van Moerbecke, 1976) are affected by the curse of dimensionality. Only in recent years several approaches have been proposed to overcome this problem for American style derivatives, hence the case of a single exercise right. These methods basically rely on Monte-Carlo simulation and can be roughly divided into three groups. The first group directly employs a recursive scheme for solving the stopping problem, known as backward dynamic programming. Different techniques are applied to approximate the nested conditional expectations. The stochastic mesh method by Broadie et al. (2000) and the least square regression method of Longstaff and Schwartz (2001) are among the most popular approaches in this group. An alternative to backward dynamic programming is to approximate the exercise boundary by simulation, see e.g. Andersen (1999), Ibáñez and Zapatero (2004), and Milstein et al. (2004). The third group relies on a dual approach developed in Rogers (2002), Haugh and Kogan (2004), and in a multiplicative setting by Jamshidian (2003). For a numerical treatment of this approach, see Kolodko and Schoenmakers (2003). By duality, tight price upper bounds may be constructed from given approximative processes.

The methods in these three categories can be transferred from one to several exercise opportunities because the multiple stopping problem is equivalent to a system of nested single stopping problems. Meinshausen and Hambly (2004) suggest an extension of the Longstaff and Schwartz (2001)-algorithm to several exercise rights along these lines. Their main contribution is, however, a derivation of the dual formulation under several exercise rights. Ibáñez (2004) presents a generalization of Ibáñez and Zapatero (2004) for multiple exercise opportunities.

The aim of the present paper is twofold: Firstly, we suggest an algorithm for the multiple stopping problem, which generalizes a procedure recently introduced by Kolodko and Schoenmakers (2004) for the single stopping problem. Secondly, we analyze stability of the algorithm under one as well as under several exercise rights.

The policy-improvement algorithm proposed in Kolodko and Schoenmakers (2004) is mending one of the main drawbacks of the backward dynamic programming scheme: Suppose exercise can take place at one out of k time instances. Then, in order to obtain the value of the optimal stopping problem via backward dynamic programming, one has to calculate nested conditional expectations of order k. No approximation of the time 0 value is available prior to the evaluation of the kth nested conditional expectations. This prevents the use of plain Monte-Carlo simulations for approximating the conditional expectations and requires more complicated approximation procedures for these quantities. For instance, to employ the procedure of Longstaff and Schwartz (2001), one has to choose the number of basis functions and the basis functions themselves. Moreover, the error analysis of the Longstaff and Schwartz (2001)-algorithm in Egloff (2004) suggests that the error propagation backward in time increases exponentially in the number of time steps. Contrary, the algorithm of Kolodko and Schoenmakers (2004) yields approximations of the time 0 value of the value function for every iteration step, which monotonically increase to the Snell envelope. This allows for a plain Monte-Carlo simulation of the conditional expectations. Indeed, the simulations in Kolodko and Schoenmakers (2004) show that good approximations can be obtained with a quadratic simulation (i.e. two iteration steps), even for very high($d = 40!$)-dimensional problems.

In fact, the main advantage of the algorithm in Kolodko and Schoenmakers (2004) were lost, if a multi-exercise version would be straightforwardly defined as a nesting of one-exercise versions. This would cause nested conditional expectations in each iteration step and, thus, again prevent the use of a plain Monte Carlo implementation. Instead we present a multiple exercise version of the policy-improvement algorithm in a way that the order of nestings does not depend on the number of exercise rights. It is therefore tailored for plain Monte-Carlo simulation of the conditional expectations. We also prove that the algorithm coincides with the Snell envelope under L exercise rights after the same number of iterations as needed for the nested dynamic programming algorithm proposed in Carmona and Touzi (2003). This shows that our algorithm is theoretically as good as backward dynamic programming, but superior from a practical point of view.

The second contribution of our paper is a stability analysis for the policy-improvement algorithm of Kolodko and Schoenmakers (2004) and its multi-exercise extension. In the case of a single exercise right the stability result can be put in words as follows (recall, one can think of the stopping problem as an investor trying to maximize his expected gain): The shortfall of the investor's expected gain corresponding to m steps of the perturbed algorithm below the expected gain corresponding to m steps of the theoretical algorithm converges to zero. Surprisingly, it can happen that the perturbed algorithm performs better than the theoretical one (as is shown in example 4.1). Put differently, in comparison with the theoretical algorithm, better approximations of the Snell envelope may be achieved due to simulation errors! A little weaker result is obtained in the multi-exercise case.

The paper is organized as follows: In Section 2 we pose the multiple stopping problem and explain its connection to the single stopping problem. Then in Section 3 we state the multiple exercise algorithm and prove its convergence. In particular, in Section 3.2 and 3.3 we put a main emphasis on the analysis of the building blocks of the algorithm, called one-step improvements. The results of Sections 3.2-3.3 are crucial for the discussion of stability in Section 4. Section 5 concludes.

2 On the Multiple Stopping Problem

Suppose $(Z(i)\colon\ i = 0, 1, \ldots, k)$ is a nonnegative stochastic process in discrete time on a probability space $(\Omega, \mathcal{F}, P)$ adapted to some filtration $(\mathcal{F}_i : 0 \le i \le k)$ which satisfies

$$\sum_{i=1}^{k} E|Z(i)| < \infty.$$

We may think of the process Z as a cash-flow, which an investor may exercise L times. The investors' problem is to maximize his expected gain by exercising optimally. He is subjected to the additional constraint that he has to wait a minimal time $\delta \in \mathbb{N}$ between exercising two rights. The introduction of δ avoids mathematical trivialities, as otherwise the investor would exercise all rights at the same time. To emphasize that the introduction of δ is not a mathematical oddity, we will refer to δ as the *refracting period* following the terminology from swing options.

We now formalize the multiple stopping problem. For notational convenience we trivially extend the cash-flow process by $Z(i) = 0$ and $\mathcal{F}_i = \mathcal{F}_k$ for $i > k$. Let us define $\mathcal{S}_i(L, \delta)$ as the set of $\mathcal{F}_i$ stopping vectors $(\tau_1(i), \ldots, \tau_L(i))$ such that $i \le \tau_1(i)$ and, for all $2 \le j \le L$, $\tau_{j-1}(i) + \delta \le \tau_j(i)$. The multiple stopping problem may then be stated as follows: Find a family of stopping vectors $\tau^*(i) \in \mathcal{S}_i(L, \delta)$ such that for $0 \le i \le k$

$$E^{\mathcal{F}_i}\left[\sum_{j=1}^{L} Z(\tau_j^*(i))\right] = \operatorname{esssup}_{\tau \in \mathcal{S}_i(L,\delta)} E^{\mathcal{F}_i}\left[\sum_{j=1}^{L} Z(\tau_j)\right].$$

The process on the right hand side is called the *Snell envelope* of Z under L exercise rights and we denote it by $Y_L^*(i)$. We sometimes write $Y^*(i) = Y_1^*(i)$.

The case of one exercise right $L = 1$ is very well studied. We collect some facts, which can be found in Neveu (1975).

1. The Snell envelope Y^* of Z under one exercise rights is the smallest supermartingale that dominates Z.

2. A family of optimal stopping times for the stopping problem with one exercise rights is given by
$$\tau^*(i) = \inf\{i \le j : \; Z(j) \ge Y^*(j)\}, \quad 0 \le i \le k.$$
If several optimal stopping families exist, then the above family is the family of first optimal stopping times.

The multiple stopping problem can be reduced to L nested stopping problems with one exercise right (see also Carmona and Touzi (2003) and Carmona and Dayanik (2004) for the more demanding continuous time setting). We briefly explain the reduction.

Define a sequence of processes $(X_0, \ldots, X_L, \ldots)$ as follows. $X_0 := 0$, $X_1 := Y_1^*$ is the Snell envelope of Z. X_L, $L \ge 2$, is the Snell envelope of the cash-flow $Z(i) + E^{\mathcal{F}_i} X_{L-1}(i + \delta)$ under one exercise right. We also define for $L = 1, 2, \ldots$,

$$\sigma_L^*(i) = \inf\{i \le j : \; Z(j) + E^{\mathcal{F}_j} X_{L-1}(j + \delta) \ge X_L(j)\}, \qquad i \ge 0,$$

i.e. the first optimal stopping families for the sequence of single stopping problems. It is straightforward to show by induction over L, that

$$Y_L^*(i) = X_L(i), \qquad 1 \le i \le k, \tag{1}$$

and a family of optimal stopping vectors for the multiple stopping problem with L exercise rights and cash-flow Z is given by

$$\begin{aligned} \tau_{1,L}^*(i) &= \sigma_L^*(i) \\ \tau_{d+1,L}^*(i) &= \tau_{d,L-1}^*(\sigma_L^*(i) + \delta) \quad 1 \le d \le L - 1. \end{aligned} \tag{2}$$

Note that, due to the convention $Z(i) = 0$ for $i > k$, we have $\tau_{1,L}^*(i) = \sigma_L^*(i) = i$ for $i \ge k$.

The reduction (1), (2) is intuitively clear: It basically says, that the investor has to choose the first stopping time of the stopping vector in the following way: Decide, at time j, whether it is better to take the cash-flow $Z(j)$ and enter a new contract with $L-1$ exercise rights starting at $j+\delta$, or to keep the L exercise rights. Then, after entering the stopping problem with $L-1$ exercise rights, he proceeds to behave optimally.

By the above reduction, any algorithm for single optimal stopping problems can, in principle, be applied iteratively to the multiple stopping problem. For example, Carmona and Touzi (2003) suggested to apply backward dynamic programming iteratively to the L stopping problems. This obviously leads to even higher nestings of conditional expectations than the dynamic programming approach yields for the single stopping problems, and, as a consequence, to tremendous simulation costs in a plain Monte Carlo approach.

Contrary, we are going to present an algorithm which simultaneously improves the Snell envelope under $L = 1, \ldots, D$ exercise rights with the order of nested conditional expectations for a given number of iterations independent of L.

3 An Algorithm for Multiple Stopping

3.1 The Algorithm

We now explain our new algorithm for the multiple stopping problem. In the case of a single exercise right it coincides with the procedure suggested in Kolodko and Schoenmakers (2004). The building block of the algorithm is, as in the case of one exercise right, a policy improvement. More precisely, suppose we are given the families of stopping times

$$\sigma_L(i), \quad 0 \le i \le k, \quad 1 \le L \le D,$$

trivially extended with $\sigma_L(i) = i$ for $i > k$. Recall that k is the time horizon of the real cash-flow process. We are interested in the Snell envelope with L exercise rights for all $1 \le L \le D$ and refracting period δ. We interpret $\sigma_L(i)$ as the time, when the investor exercises (possibly in a suboptimal way) the first of his L rights, given that he has not exercised prior to time i. This interpretation requires that the stopping families σ_L under consideration are consistent in the sense of the following definition:

Definition 3.1 *A family of integer-valued stopping times $(\tau(i) : 0 \le i \le k)$ is said to be* consistent, *if for $0 \le i < k$,*

$$\begin{aligned} i \le \tau(i) \le k, \ \ \tau(k) \equiv k, \\ \tau(i) > i \Rightarrow \tau(i) = \tau(i+1). \end{aligned} \tag{3}$$

Indeed, suppose $\sigma_L(i) > i$, i.e. according to our interpretation the investor has not exercised the first right prior to time $i+1$. Then he has not exercised the first right prior to time i, either. This means he will exercise the first right at times $\sigma_L(i)$ and $\sigma_L(i+1)$, which requires $\sigma_L(i) = \sigma_L(i+1)$. Note: A trivial example of a non-consistent stopping family is $\tau(i) = \min(i+1, k)$.

Given consistent stopping families σ_L, $L = 1, 2, \ldots$, we define associated stopping families $\tau_{d+1,L}$ via,

$$\begin{aligned} \tau_{1,L}(i) &= \sigma_L(i) \\ \tau_{d+1,L}(i) &= \tau_{d,L-1}\left(\sigma_L(i) + \delta\right) \quad 1 \leq d \leq L-1. \end{aligned} \tag{4}$$

$\tau_{d,L}(i)$ can be interpreted as the time, when the investor exercises the dth of his L exercise rights, provided he has not exercised his first right prior to time i.

An approximation of the Snell envelope with L exercise rights is now given by

$$Y_L(i; \sigma_1, \ldots \sigma_L) := E^{\mathcal{F}_i}\left[\sum_{d=1}^{L} Z(\tau_{d,L}(i))\right]. \tag{5}$$

Note, $Y_L(i; \sigma_1, \ldots \sigma_L)$ has a simple interpretation as the expected gain (conditional on $\mathcal{F}_i$) the investor obtains when he employs the stopping families $\sigma_1, \ldots, \sigma_L$ for exercising the cash-flows.

We then introduce intermediate processes

$$\widehat{Y}_L(i; \sigma_1, \ldots, \sigma_L) := \max_{i+1 \leq p \leq k} E^{\mathcal{F}_i}\left[\sum_{d=1}^{L} Z(\tau_{d,L}(p))\right] \tag{6}$$

on which a next exercise criterion is built,

$$\widetilde{\sigma}_L(i) := \inf\left\{ j \geq i; \;\; Z(j) + E^{\mathcal{F}_j} Y_{L-1}(j+\delta; \sigma_1, \ldots, \sigma_{L-1}) \geq \widehat{Y}_L(j; \sigma_1, \ldots, \sigma_L) \right\}, \tag{7}$$

with $Y_0(i) := 0$. Note that $\widetilde{\sigma}_L(k) = k$ since $\max \emptyset = -\infty$, and, obviously, the stopping families $\tilde{\sigma}_L$ are consistent for $1 \leq L \leq D$.

Given consistent starting families of stopping times $\sigma_L^{(0)}$, $1 \leq L \leq D$, we define iteratively,

$$\begin{aligned} \sigma_L^{(m)}(i) &:= \widetilde{\sigma}_L^{(m-1)}(i), \\ Y_L^{(m)}(i) &:= Y_L(i; \sigma_1^{(m)}, \ldots, \sigma_L^{(m)}). \end{aligned} \tag{8}$$

Canonical consistent starting families are given, for instance, by $\sigma_L^{(0)}(i) = i$, $L = 1, 2, \ldots$

Theorem 3.2 *Suppose the stopping families $\sigma_L^{(0)}(i)$ are consistent for all $1 \leq L \leq D$. Then, for all $m \in \mathbb{N}$, $1 \leq L \leq D$, and $0 \leq i \leq k$,*

$$Y_L^{(m+1)}(i) \geq Y_L^{(m)}(i).$$

Moreover, for $m \geq Lk - i$,

$$Y_L^{(m)}(i) = Y_L^{*}(i),$$

where Y_L^* denotes the Snell envelope of Z under L exercise rights.

Remark 3.3 *The algorithm provides an iteration scheme of increasing lower bounds for the Snell envelope under L exercise rights. By a dual method, developed by Rogers (2002) and Haugh and Kogan (2004), and extended to the case of several exercise rights as in Meinshausen and Hambly (2004), one can construct a family of convergent upper bounds given this family of lower bounds.*

Remark 3.4 *The reader might suggest to consider the following intuitively better algorithm: Given consistent stopping families* $\sigma_1, \ldots, \sigma_D$*, define* $\underline{\sigma}_1(i) := \widetilde{\sigma}_L(i)$*, and then, recursively in* L*,*

$$\begin{aligned} \underline{\sigma}_L(i) &= \inf\{j \geq i; \;\; Z(j) + E^{\mathcal{F}_j} Y_{L-1}(j+\delta; \underline{\sigma}_1, \ldots, \underline{\sigma}_{L-1}) \\ &\qquad \geq \widehat{Y}_L(j; \underline{\sigma}_1, \ldots, \underline{\sigma}_{L-1}, \sigma_L)\}. \end{aligned}$$

The very intuition of this modification is to use the already improved stopping times $\underline{\sigma}_1(i), \ldots, \underline{\sigma}_{L-1}(i)$ *for improving* σ_L*.*

For given consistent starting families of stopping times $\sigma_L^{(0)}$*,* $1 \leq L \leq D$ *we may then define,*

$$\begin{aligned} \sigma_L^{[m]}(i) &:= \underline{\sigma}_L^{[m-1]}(i), \\ Y_L^{[m]}(i) &:= Y_L(i; \sigma_1^{[m]}, \ldots, \sigma_L^{[m]}). \end{aligned}$$

It can be shown, that all assertions of Theorem 3.2 also hold for $Y_L^{[m]}$ *instead of* $Y_L^{(m)}$*. However, the modified algorithm* $Y_L^{[m]}$ *requires calculation of nested conditional expectations within each improvement step. Therefore it requires much higher computational costs, when the conditional expectations are approximated by Monte Carlo simulation. Indeed, the main advantage of the algorithm (8) based on (7) is that the order of nested conditional expectations for a given number of iterations does not depend on the number of exercise rights.*

Before we prove Theorem 3.2 in Section 3.4, we investigate in the next two subsections the building blocks, which we will refer to as *one-step improvements* in more detail. We first consider the case of one exercise right and generalize results of Kolodko and Schoenmakers (2004). These generalizations will be of crucial importance for investigating the stability of the proposed algorithm in Section 4.

3.2 A Generalization of the One-Step Improvement in the Case of One Exercise Right

Suppose a consistent stopping family $(\tau(i) : 1 \leq i \leq k)$ is given. We then define the process

$$Y(i; \tau) := E^{\mathcal{F}_i}\left[Z(\tau(i))\right]. \tag{9}$$

Based on the sequence $(\tau(i) : 1 \leq i \leq k)$ Kolodko and Schoenmakers (2004) construct a new family $(\widetilde{\tau}(i) : 1 \leq i \leq k)$ in the following way: Introduce an intermediate process

$$\widetilde{Y}(i;\tau) := \max_{p:\, i\leq p\leq k} E^{\mathcal{F}_i}\left[Z(\tau(p))\right], \tag{10}$$

which serves as a new exercise criterion, i.e.

$$\begin{aligned}\widetilde{\tau}(i) &:= \inf\{j : i \leq j \leq k,\ \widetilde{Y}(j,\tau) \leq Z(j)\} \\ &= \inf\{j : i \leq j \leq k,\ \max_{p:\, j\leq p\leq k} E^{\mathcal{F}_j}\left[Z(\tau(p))\right] \leq Z(j)\}, \quad 0 \leq i \leq k.\end{aligned} \tag{11}$$

Kolodko and Schoenmakers (2004), Theorem 3.1, show that $\widetilde{\tau}$ is an improvement of τ in the sense that the new strategy promises a higher expected gain for the investor than the old one, i.e.

$$Y(i;\widetilde{\tau}) \geq \widetilde{Y}(i;\tau) \geq Y(i;\tau).$$

Our first aim is to find a wider class of stopping families $\bar{\tau}$ such that

$$Y(i;\bar{\tau}) \geq \widetilde{Y}(i;\tau) \geq Y(i;\tau).$$

To this end we first compare the intermediate processes $\widetilde{Y}(i,\tau)$ and

$$\widehat{Y}(i;\tau) := \max_{p:\, i+1\leq p\leq k} E^{\mathcal{F}_i}\left[Z(\tau(p))\right]. \tag{12}$$

Lemma 3.5 *Suppose the stopping family τ is consistent. Then, for $0 \leq i \leq k$,*

$$\widetilde{Y}(i;\tau) = \mathbf{1}_{\{\tau(i)>i\}}\widehat{Y}(i;\tau) + \mathbf{1}_{\{\tau(i)=i\}} \max\left\{\widehat{Y}(i;\tau),\ Z(i)\right\}. \tag{13}$$

In particular,

$$Z(i) \geq \widetilde{Y}(i;\tau) \iff Z(i) \geq \widehat{Y}(i;\tau), \tag{14}$$

and

$$\widetilde{\tau}(i) = \inf\{j : i \leq j \leq k,\ \widehat{Y}(j) \leq Z(j)\}. \tag{15}$$

Proof. By property (3), we have,

$$\begin{aligned}E^{\mathcal{F}_i}\left[Z(\tau(i))\right] &= E^{\mathcal{F}_i}\left[\mathbf{1}_{\{\tau(i)=i\}}Z(i)\right] + E^{\mathcal{F}_i}\left[\mathbf{1}_{\{\tau(i)>i\}}Z(\tau(i+1))\right] \\ &= \mathbf{1}_{\{\tau(i)=i\}}Z(i) + \mathbf{1}_{\{\tau(i)>i\}}E^{\mathcal{F}_i}\left[Z(\tau(i+1))\right].\end{aligned}$$

Since

$$\widetilde{Y}(i;\tau) = \max\left\{\widehat{Y}(i;\tau),\ E^{\mathcal{F}_i}\left[Z(\tau(i))\right]\right\},$$

(13) follows with (14) and (15) as immediate consequences. ∎

We next define another stopping family, namely,

$$\widehat{\tau}(i) := \inf\{j : i \leq j \leq k,\ \widehat{Y}(j) < Z(j)\}. \tag{16}$$

By (15),

$$\widehat{\tau}(i) \geq \widetilde{\tau}(i). \tag{17}$$

We are now ready to state a generalization of Theorem 3.1 in Kolodko and Schoenmakers (2004), which provides the basis of our stability analysis.

Theorem 3.6 *Let $(\tau(i),\ 1 \leq i \leq k)$ be a consistent stopping family. Suppose $(\bar{\tau}(i),\ 1 \leq i \leq k)$ is also consistent and satisfies*

$$\widetilde{\tau}(i) \leq \bar{\tau}(i) \leq \widehat{\tau}(i) \quad 0 \leq i \leq k. \tag{18}$$

Then,

$$Y(i;\bar{\tau}) \geq \widetilde{Y}(i;\tau) \geq Y(i;\tau), \quad 0 \leq i \leq k.$$

Remark 3.7 *Obviously, the choices $\bar{\tau} = \widetilde{\tau}$ and $\bar{\tau} = \widehat{\tau}$ are examples of a family $\bar{\tau}$ satisfying (3) and (18).*

Proof. The second inequality is trivial. We prove the first inequality by backward induction over i. For $i = k$, note that

$$Y(k;\bar{\tau}) = Z(k) = \widetilde{Y}(k;\tau).$$

Now suppose $0 \leq i \leq k-1$, and that the assertion is already proved for $i+1$. It holds $\{\bar{\tau}(i) = i\} \subset \{\widetilde{\tau}(i) = i\}$ by (18). Hence, we obtain on the set $\{\bar{\tau}(i) = i\}$,

$$Y(i;\bar{\tau}) = Z(i) \geq \widetilde{Y}(i;\tau).$$

However, on $\{\bar{\tau}(i) > i\}$ the induction hypothesis yields,

$$\begin{aligned} Y(i;\bar{\tau}) &= E^{\mathcal{F}_i}\left[Z(\bar{\tau}(i+1))\right] = E^{\mathcal{F}_i}\left[Y(i+1;\bar{\tau})\right] \geq E^{\mathcal{F}_i}\left[\widetilde{Y}(i+1;\tau)\right] \\ &= E^{\mathcal{F}_i}\left[\max_{i+1\leq p\leq k} E^{\mathcal{F}_{i+1}}\left[Z(\tau(p))\right]\right] \geq \max_{i+1\leq p\leq k} E^{\mathcal{F}_i}\left[Z(\tau(p))\right] \\ &= \widehat{Y}(i,\tau). \end{aligned}$$

Property (18) implies $\{\bar{\tau}(i) > i\} \subset \{\widehat{\tau}(i) > i\}$. Thus, on $\{\bar{\tau}(i) > i\}$,

$$\widehat{Y}(i,\tau) \geq Z(i)$$

and, by (13),

$$\widehat{Y}(i,\tau) = \widetilde{Y}(i,\tau) \quad \text{on } \{\bar{\tau}(i) > i\}.$$

This completes the proof. ∎

Motivated by the previous theorem we introduce the notion of an *improver*:

Definition 3.8 *Suppose τ is a consistent stopping family. A stopping family $\bar{\tau}$ is called an* improver *of τ, if it satisfies (3) and (18) for $0 \le i \le k$.*

The next theorem provides another justification for the name 'improver'.

Theorem 3.9 *Suppose τ is a consistent stopping family and $\bar{\tau}$ is an improver of τ. Then*

$$Y(i,\tau) = Y^*(i) \quad \text{for } i \ge j+1$$

implies

$$Y(i,\bar{\tau}) = Y^*(i) \quad \text{for } i \ge j.$$

Proof. We will exploit the fact that the Snell envelope is the smallest supermartingale dominating Z.

By Theorem 3.6 we have, for $0 \le i \le k-1$,

$$Y(i,\bar{\tau}) \ge \widetilde{Y}(i;\tau) \ge E^{\mathcal{F}_i}\left[Z(\tau(i+1))\right] = E^{\mathcal{F}_i}\left[Y(i+1;\tau)\right].$$

Therefore, for $j \le i \le k-1$,

$$Y(i,\bar{\tau}) \ge E^{\mathcal{F}_i}\left[Y^*(i+1)\right] \ge E^{\mathcal{F}_i}\left[Y(i+1;\bar{\tau})\right].$$

This means $(Y(i,\bar{\tau}),\ j \le i \le k)$ is a supermartingale. We may also deduce from Theorem 3.6 that for $0 \le i \le k$,

$$Y(i,\bar{\tau}) \ge \mathbf{1}_{\{\bar{\tau}(i)=i\}} Z(i) + \mathbf{1}_{\{\bar{\tau}(i)>i\}} \widetilde{Y}(i;\tau).$$

However, as in the proof of Theorem 3.6, we obtain

$$\mathbf{1}_{\{\bar{\tau}(i)>i\}} \widetilde{Y}(i;\tau) \ge \mathbf{1}_{\{\bar{\tau}(i)>i\}} \widehat{Y}(i;\tau) \ge \mathbf{1}_{\{\bar{\tau}(i)>i\}} Z(i).$$

Thus, $Y(\cdot,\bar{\tau})$ dominates Z. We thus have shown that $(Y(i,\bar{\tau}),\ j \le i \le k)$ is a supermartingale dominating Z. Therefore,

$$Y(i,\bar{\tau}) \ge Y^*(i) \quad \text{for } i \ge j.$$

The reverse inequality is trivial. ∎

Remark 3.10 *The proof of the previous theorem shows, that for any improver $\bar{\tau}$,*

$$Y(i,\bar{\tau}) \ge Z(i), \quad 0 \le i \le k. \tag{19}$$

We end this section with a comparison between different improvers.

Proposition 3.11 *Suppose τ is consistent and $\bar{\tau}$ is an improver of τ. Then, for all $0 \leq i \leq k$,*

$$Y(i, \widehat{\tau}) \geq Y(i, \bar{\tau}) \geq Y(i, \widetilde{\tau}).$$

Proof. We prove the second inequality. The proof of the first one is similar. For $i = k$ even equality holds. Suppose $0 \leq i \leq k-1$ and the inequality is proved for $i+1$. Then, on $\{\bar{\tau}(i) > i\} \cap \{\widetilde{\tau}(i) > i\}$,

$$Y(i, \bar{\tau}) = E^{\mathcal{F}_i}\left[Y(i+1, \bar{\tau})\right] \geq E^{\mathcal{F}_i}\left[Y(i+1, \widetilde{\tau})\right] = Y(i, \widetilde{\tau})$$

by the induction hypothesis. On $\{\bar{\tau}(i) > i\} \cap \{\widetilde{\tau}(i) = i\}$ we have

$$Y(i, \bar{\tau}) \geq Z(i) = Y(i, \widetilde{\tau})$$

by (19). Finally, the set $\{\bar{\tau}(i) = i\} \cap \{\widetilde{\tau}(i) > i\}$ is evanescent by the definition of an improver. ■

3.3 The One-Step Improvement in the Case of Several Exercise Rights

We now investigate the one-step improvement under several exercise rights. To this end, suppose consistent stopping families $\sigma_1, \ldots, \sigma_D$ are given. Recall that $\sigma_L(i)$, $1 \leq L \leq D$, is interpreted as the time the investor exercises his first of L rights given that he has not exercised the cash-flow prior to time i. The stopping time $\tau_{d,L}(i)$, which indicates the time he exercises the dth of L rights provided he has not exercised the first of L rights prior to time i, is defined as in (4). The corresponding approximation $Y_L(i; \sigma_1, \ldots, \sigma_L)$ of the Snell envelope under L exercise rights is given by (5). Finally, the new exercise criterion is based on the process $\widehat{Y}_L(i; \sigma_1, \ldots, \sigma_L)$ defined in (6).

We will now derive representations of $Y_L(i; \sigma_1, \ldots, \sigma_L)$ and $\widehat{Y}_L(i; \sigma_1, \ldots, \sigma_L)$, which allow to extend Theorem 3.6 to the case of several exercise rights.

Lemma 3.12 *Define for $2 \leq L \leq D$ and $0 \leq i \leq k$,*

$$Z_L(i; \sigma_1, \ldots \sigma_{L-1}) = Z(i) + E^{\mathcal{F}_i}\left[Y_{L-1}(i+\delta; \sigma_1, \ldots, \sigma_{L-1})\right]. \tag{20}$$

Then,

$$\begin{aligned} Y_L(i; \sigma_1, \ldots, \sigma_L) &= E^{\mathcal{F}_i}\left[Z_L(\sigma_L(i); \sigma_1, \ldots, \sigma_{L-1})\right], \\ \widehat{Y}_L(i; \sigma_1, \ldots, \sigma_L) &= \max_{i+1 \leq p \leq k} E^{\mathcal{F}_i}\left[Z_L(\sigma_L(p); \sigma_1, \ldots, \sigma_{L-1})\right]. \end{aligned}$$

Proof. Fix $0 \leq i \leq p \leq k$. Then by (4),

$$\begin{aligned} & E^{\mathcal{F}_i}\left[Z_L(\sigma_L(p); \sigma_1, \ldots, \sigma_{L-1})\right] \\ = \; & E^{\mathcal{F}_i}\left[Z(\sigma_L(p)) + \sum_{d=1}^{L-1} Z(\tau_{d,L-1}(\sigma_L(p)+\delta))\right] \\ = \; & E^{\mathcal{F}_i}\left[Z(\tau_{1,L}(p)) + \sum_{d=1}^{L-1} Z(\tau_{d+1,L}(p))\right] = E^{\mathcal{F}_i}\left[\sum_{d=1}^{L} Z(\tau_{d,L}(p))\right]. \end{aligned}$$

■ By the previous lemma we may rewrite $\widetilde{\sigma}_L$ defined in (7) as

$$\widetilde{\sigma}_L(i) = \inf\left\{j \geq i;\ \ Z_L(j;\sigma_1,\ldots,\sigma_{L-1}) \geq \max_{p\geq j+1} E^{\mathcal{F}_j}\, Z_L(\sigma_L(p);\sigma_1,\ldots,\sigma_{L-1})\right\}. \quad (21)$$

Consequently, the step from σ_L to $\widetilde{\sigma}_L$ is a one-step improvement with one exercise right and cash-flow $Z_L(\cdot;\sigma_1,\ldots,\sigma_{L-1})$.

As in the case of one exercise right we also consider the stopping family

$$\widehat{\sigma}_L(i) = \inf\left\{j \geq i;\ \ Z_L(j;\sigma_1,\ldots,\sigma_{L-1}) > \max_{p\geq j+1} E^{\mathcal{F}_j}\, Z_L(\sigma_L(p);\sigma_1,\ldots,\sigma_{L-1})\right\}. \quad (22)$$

Definition 3.13 *A stopping family $\bar{\sigma}_L$ is said to be an L-*improver *of σ_L* with respect to *$(\sigma_1,\ldots,\sigma_{L-1})$, if $\bar{\sigma}_L$ is consistent and*

$$\widetilde{\sigma}_L(i) \leq \bar{\sigma}_L(i) \leq \widehat{\sigma}_L(i). \quad (23)$$

In abuse of terminology we will simply speak of an improver, *when L and $(\sigma_1,\ldots,\sigma_{L-1})$ are evident from the context.*

We now state a generalization of Theorem 3.6, which justifies the name 'improver'.

Theorem 3.14 *Suppose consistent stopping families $\sigma_1,\ldots,\sigma_D$ are given with respective improvers $\bar{\sigma}_1,\ldots,\bar{\sigma}_D$. Then, for $1 \leq L \leq D$ the following chain of inequalities holds,*

$$\begin{aligned} & Y_L(i;\bar{\sigma}_1,\ldots,\bar{\sigma}_L) \geq Y_L(i;\sigma_1,\ldots,\sigma_{L-1},\bar{\sigma}_L) \geq Y_L(i;\sigma_1,\ldots,\sigma_{L-1},\widetilde{\sigma}_L) \\ \geq\ & \max\left\{Y_L(i;\sigma_1,\ldots,\sigma_L),\ \widehat{Y}_L(i;\sigma_1,\ldots,\sigma_L)\right\}. \end{aligned}$$

Proof. By the previous considerations $\bar{\sigma}_L$ is also a 1-improver of σ_L with respect to the cash-flow $Z_L(\cdot;\sigma_1,\cdots,\sigma_{L-1})$ (with the convention $Z_1 = Z$). In view of Lemma 3.12 the second inequality follows from Proposition 3.11 and the third one from Theorem 3.6. We will prove the first inequality by induction over L. Note that the inequality is trivial for $L = 1$. The step from $L-1$ to L can be shown as follows. By Lemma 3.12,

$$\begin{aligned} & Y_L(i;\bar{\sigma}_1,\ldots,\bar{\sigma}_L) - Y_L(i;\sigma_1,\ldots,\sigma_{L-1},\bar{\sigma}_L) \\ =\ & E^{\mathcal{F}_i}\left[Z(\bar{\sigma}_L(i)) + Y_{L-1}(\bar{\sigma}_L(i)+\delta;\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1})\right] \\ & -E^{\mathcal{F}_i}\left[Z(\bar{\sigma}_L(i)) + Y_{L-1}(\bar{\sigma}_L(i)+\delta;\sigma_1,\ldots,\sigma_{L-1})\right] \\ =\ & E^{\mathcal{F}_i}\left[Y_{L-1}(\bar{\sigma}_L(i)+\delta;\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1}) - Y_{L-1}(\bar{\sigma}_L(i)+\delta;\sigma_1,\ldots,\sigma_{L-1})\right]. \end{aligned}$$

As the second and the third inequality are already proved, the induction hypothesis implies,

$$Y_{L-1}(\bar{\sigma}_L(i)+\delta;\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1}) \geq Y_{L-1}(\bar{\sigma}_L(i)+\delta;\sigma_1,\ldots,\sigma_{L-1}).$$

Thus,

$$Y_L(i;\bar{\sigma}_1,\ldots,\bar{\sigma}_L) - Y_L(i;\sigma_1,\ldots,\sigma_{L-1},\bar{\sigma}_L) \geq 0.$$

■

We are now ready to give the proof of Theorem 3.2

3.4 Proof of Theorem 3.2

The monotonicity assertion is a direct consequence of Theorem 3.14 since, by definition,

$$\begin{aligned} Y_L^{(m)}(i) &= Y(i;\sigma_1^{(m)},\dots\sigma_L^{(m)}) \\ \sigma_d^{(m+1)} &= \widetilde{\sigma}_d^{(m+1)}, \quad 1 \le d \le L. \end{aligned}$$

Recall that the $\bar{}$ in Theorem 3.14, can always be replaced by $\widetilde{}$ by the definition of an improver.

We prove the second assertion by induction over L. For $L = 1$, it follows by backward induction over i and making use of Theorem 3.9.

Suppose $2 \le L \le D$ and that the assertion is already proved for $L - 1$. We fix $0 \le i_0 \le k$ and $m_0 \ge Lk - i_0$. By the induction hypothesis,

$$Y_{L-1}^{(m)}(i) = Y_{L-1}^*(i),$$

for all $m \ge (L-1)k - i$. In particular,

$$Z_L^{(m+1)}(i) = Z(i) + E^{\mathcal{F}_i} Y_{L-1}^{(m)}(i+\delta) = Z(i) + E^{\mathcal{F}_i} Y_{L-1}^*(i+\delta),$$

for all $0 \le i \le k$ and $m \ge (L-1)k$. This means that from step $(L-1)k$ on we have an iteration procedure as in the case of a single exercise right, but with the cash-flow $Z(i)$ replaced by $Z(i) + E^{\mathcal{F}_i} Y_{L-1}^*(i+\delta)$. Thus, due to Theorem 3.9 the time i value of this iteration does not change anymore after $k - i$ new improvements but coincides with the Snell envelope. Hence, for $m_0 \ge Lk - i_0 = (L-1)k + k - i_0$, $Y_L^{(m_0)}(i_0)$ coincides with the time i_0 value of the Snell envelope of $Z(i) + E^{\mathcal{F}_i} Y_{L-1}^*(i+\delta)$ with one exercise right, which in turn equals $Y_L^*(i_0)$ by (1).

Remark 3.15 *The proof shows that after any $m \ge Lk - i$ improvements, not only the $\widetilde{}$-improvement, the corresponding approximation coincides with the Snell envelope under L exercise rights up from time i on.*

3.5 A Modification of the Algorithm

We now present a slight modification of the algorithm which may appear less natural but sometimes yields better approximations of the Snell envelope. We emphasize that this modification does not affect the construction of the improved stopping family, say $\widetilde{\sigma}_L$ starting with σ_L, but, is a suggestion to replace Y_L.

The modification is motivated by the well-known dynamic programming approach for constructing the Snell envelope. Under one exercise right one has

$$\begin{aligned} Y^*(k) &= k \\ Y^*(i) &= \max\left\{Z(i),\ E^{\mathcal{F}_i}[Y^*(i+1)]\right\}. \end{aligned}$$

The dynamic programming scheme suggests to define

$$y_L(i;\sigma_1,\ldots,\sigma_L) := \max\left\{Z_L(i;\sigma_1,\ldots,\sigma_{L-1}),\ E^{\mathcal{F}_i}[Y_L(i+1;\sigma_1,\ldots,\sigma_L)]\right\}, \tag{24}$$

$1 \le L \le D$, $0 \le i \le k$, given consistent stopping families $\sigma_1,\ldots,\sigma_D$. In fact, y_L has not such an intuitive interpretation as Y_L but we have, however,

$$y_L(i;\sigma_1,\ldots,\sigma_L) \ge Y_L(i;\sigma_1,\ldots,\sigma_L), \tag{25}$$

since by Lemma 3.12

$$\begin{aligned} Y_L(i;\sigma_1,\ldots,\sigma_L) &= \mathbf{1}_{\{\sigma_L(i)=i\}} Z_L(i;\sigma_1,\ldots,\sigma_{L-1}) \\ &\quad +\mathbf{1}_{\{\sigma_L(i)>i\}} E^{\mathcal{F}_i}[Y_L(i+1;\sigma_1,\ldots,\sigma_L)]. \end{aligned}$$

The following variant of Theorem 3.14 for y_L is a direct consequence of Theorem 3.14 and the definition of y_L.

Corollary 3.16 *Suppose consistent stopping families $\sigma_1,\ldots,\sigma_D$ are given with respective improvers $\bar\sigma_1,\ldots,\bar\sigma_D$. Then the following chain of inequalities holds for $1 \le L \le D$,*

$$\begin{aligned} & y_L(i;\bar\sigma_1,\ldots,\bar\sigma_L) \ge y_L(i;\sigma_1,\ldots,\sigma_{L-1},\bar\sigma_L) \ge y_L(i;\sigma_1,\ldots,\sigma_{L-1},\widetilde\sigma_L) \\ \ge\ & y_L(i;\sigma_1,\ldots,\sigma_L). \end{aligned}$$

We may thus replace $Y_L^{(m)}$ in Theorem 3.2 by

$$y_L^{(m)}(i) := y_L(i;\sigma_1^{(m)},\ldots,\sigma_L^{(m)}). \tag{26}$$

Theorem 3.17 *All assertions of theorem 3.2 remain valid, when $Y_L^{(m)}$ is replaced by $y_L^{(m)}$.*

Remark 3.18 *(i) The reader may easily verify that $y_L(i;\sigma_1,\ldots,\sigma_L)$ and $Y_L(i;\sigma_1,\ldots,\sigma_L)$ coincide, when*

$$Z_L(i;\sigma_1,\ldots,\sigma_{L-1}) \le E^{\mathcal{F}_i}[Y_L(i+1;\sigma_1,\ldots,\sigma_L)] \implies \sigma_L(i) > i,$$

and,

$$Z_L(i;\sigma_1,\ldots,\sigma_{L-1}) \le Y_L(i;\sigma_1,\ldots,\sigma_L).$$

Example 4.1-(ii) in Section 4 exhibits an example where these conditions are violated and, (under one exercise right), $Y(0;\widetilde\tau)$ is strictly smaller than $y(0;\widetilde\tau)$ for some consistent stopping family $\widetilde\tau$.
(ii) Note that

$$\begin{aligned} y_L(i;\sigma_1,\ldots,\sigma_L) &= \max\{Z_L(i;\sigma_1,\ldots,\sigma_{L-1}), \\ &\qquad E^{\mathcal{F}_i}[Z_L(\sigma_L(i+1);\sigma_1,\ldots,\sigma_{L-1})]\}. \end{aligned}$$

Thus, a Monte Carlo simulation based approximation of y_L requires the same computational cost as for Y_L. In contrast, a definition involving the maximum of Z_L and $\widehat{Y}_L$ would

cause higher costs.
(iii) Note that $y_L(0;\sigma_1,\ldots,\sigma_L)$ *can be computed without knowledge of* $\sigma_1(0),\ldots,\sigma_L(0)$. *This turns out to be a significant advantage of the algorithm for* $y_L^{(m)}$ *over the algorithm for* $Y_L^{(m)}$, *when considering stability under several exercise rights. Indeed, the introduction of* $y_L^{(m)}$ *is mainly motivated by this stability issue and inspired by the study of the Longstaff-Schwartz algorithm (Longstaff and Schwartz, 2001) in Clément et al. (2002).*

4 Stability

In this section we discuss the stability of the algorithm for multiple stopping, starting with a study of the one-step improvement under one exercise right. We will focus on the stability of Y_L rather than y_L (in (24)), since all stability results for Y_L can be simply transferred to y_L. Some details of this transfer will be given in the context of several exercise rights.

4.1 Stability of the One-Step Improvement $(L = 1)$

Suppose a consistent stopping family τ is given. As we cannot expect to know the conditional expectations analytically in general, but, may only calculate approximations, we consider instead of $\widetilde{\tau}(i)$ a sequence of stopping families

$$\widetilde{\tau}^{(N)}(i) := \inf\{j : i \le j \le k,\ \widehat{Y}(j;\tau) + \epsilon^{(N)}(j) \le Z(j)\},$$

where $N \in \mathbb{N}$, and $\epsilon^{(N)}(i)$ is a sequence of $\mathcal{F}_i$-adapted processes.

We will first show by some simple examples that we must neither expect

$$\widetilde{\tau}^{(N)}(i) \to \widetilde{\tau}(i) \quad \text{in probability,}$$

nor

$$Y(0;\widetilde{\tau}^{(N)}) \to Y(0;\widetilde{\tau}), \tag{27}$$

when

$$\lim_{N\to\infty} \epsilon^{(N)}(i) = 0, \quad P-a.s.$$

Example 4.1 *(i) Suppose* $(\xi_N)_{N\in\mathbb{N}}$ *is a sequence of independent binary trials with* $P(\xi_N = 1) = P(\xi_N = 0) = 1/2$. *We define the process* $(Z(i):\ i = 0,1)$ *by* $Z(0) = Z(1) \equiv 1$. *The* σ*-field* $\mathcal{F}_0 = \mathcal{F}_1$ *is the one generated by the sequence of trials. Moreover, the sequence of perturbations is defined by* $\epsilon^{(N)}(0) = \xi_N/N$ *and* $\epsilon^{(N)}(1) = 0$. *Then, starting with any consisting stopping family* τ, *we get*

$$\widetilde{\tau}^{(N)}(0) = \xi_N.$$

In particular, no subsequence of $\widetilde{\tau}^{(N)}(0)$ *converges in probability.*

(ii) Let $\Omega = \{\omega_0, \omega_1\}$, $\mathcal{F}$ *the power set of* Ω, *and* $P(\{\omega_1\}) = 1/4 = 1 - P(\{\omega_0\})$. *We define the process* $(Z(i) : i = 0, 1, 2)$ *by* $Z(0) = Z(2) = 2$, *and* $Z(1, \omega_0) = 1$, $Z(1, \omega_1) = 3$. $\mathcal{F}_i$ *is the filtration generated by* Z. *We start with the stopping family* $\tau(i) = i$. *As* $E[Z(1)] = 3/2$, *we have*

$$Z(0) = 2 \geq \max\{3/2,\ 2\} = \max\{E[Z(1)],\ E[Z(2)]\} = \widehat{Y}(0, \tau).$$

Therefore,

$$\widetilde{\tau}(0) = 0$$

and

$$Y(0; \widetilde{\tau}) = 2.$$

The perturbation sequence $\epsilon^{(N)}$ *is defined to be* $\epsilon^{(N)}(1) = \epsilon^{(N)}(2) \equiv 0$ *and* $\epsilon^{(N)}(0) = 1/N$. *A straightforward calculation shows that for* $N \geq 2$,

$$\widetilde{\tau}^{(N)}(0, \omega_0) = 2, \quad \widetilde{\tau}^{(N)}(0, \omega_1) = 1.$$

Thus,

$$Y(0; \widetilde{\tau}^{(N)}) = 9/4 > 2 = Y(0; \widetilde{\tau}),$$

which violates (27).

We briefly note that in this example,

$$\widetilde{\tau}(1, \omega_0) = 2, \quad \widetilde{\tau}(1, \omega_1) = 1$$

and thus

$$y(0; \widetilde{\tau}) = 9/4 > 2 = Y(0; \widetilde{\tau}),$$

i.e. the modified improvement y *performs better than* Y. *However, we emphasize, that the replacement of* Y *by* y *does not generally mend the stability problem explained in this example. Indeed, a change of time* $i \to i + 1$ *and introduction of new time 0 values, say* $Z(0) = 0$ *and* $\epsilon^{(N)}(0) = 0$, *transfers the same stability problem to* y.

At first glance, Example 4.1 paints a rather sceptical picture of the stability properties of the one-step-improvement. Indeed, the best we can now hope for, is

(ia) there is a sequence $\bar{\tau}^{(N)}$ of improvers of τ such that

$$|\widetilde{\tau}^{(N)}(i) - \bar{\tau}^{(N)}(i)| \to 0 \quad P - a.s.$$

(iia) The shortfall of $Y(i; \widetilde{\tau}^{(N)})$ below $Y(i; \widetilde{\tau})$ converges to zero P-a.s.

Note, however, that convergence of the shortfall as in (iia) is the relevant question, not convergence of the distance as in (27), since the shortfall corresponds to a change for the worse of $\widetilde{\tau}^{(N)}$ compared to $\widetilde{\tau}$. As we are interested in an improvement it suffices to guarantee that such a change for the worse converges to zero. An additional improvement of $\widetilde{\tau}^{(N)}$ compared to $\widetilde{\tau}$ due to the error processes $\epsilon^{(N)}$ may be seen as a welcome side effect!

We now prove assertions (ia) and (iia). We first introduce a new sequence of stopping families which turns out to consist of improvers. Let us define

$$\bar{\tau}^{(N)}(k) = k,$$

and for $i < k$,

$$\begin{aligned}
\bar{\tau}^{(N)}(i) = i \quad &\Longleftrightarrow \quad (\widetilde{\tau}^{(M)}(i) > i \text{ for only finitely many } M) \\
&\qquad \vee (\widetilde{\tau}^{(M)}(i) = i \text{ for infinitely many } M \text{ and } \widetilde{\tau}^{(N)}(i) = i), \\
\bar{\tau}^{(N)}(i) \neq i \quad &\Longrightarrow \quad \bar{\tau}^{(N)}(i) = \bar{\tau}^{(N)}(i+1).
\end{aligned}$$

We then have the following result:

Theorem 4.2 *Suppose*

$$\lim_{N\to\infty} \epsilon^{(N)}(i) = 0 \quad P-a.s.,$$

for all $0 \leq i \leq k$*. Then* $\bar{\tau}^{(N)}$ *is an improver of* τ *for every* $N \in \mathbb{N}$*.*

Proof. The consistent property (3) is satisfied by definition. We show (18) by backward induction over i. The case $i = k$ is immediate. Suppose now $0 \leq i \leq k-1$ and (18) is already shown for $i+1$. On $\{\widetilde{\tau}^{(M)}(i) = i$ for infinitely many $M\}$ we have, for infinitely many M (depending on ω),

$$Z(i) \geq \widehat{Y}(i,\tau) + \epsilon^{(M)}(i).$$

This means,

$$Z(i) \geq \widehat{Y}(i,\tau) \quad \text{on } \{\widetilde{\tau}^{(M)}(i) = i \text{ for infinitely many } M\},$$

as $\epsilon^{(M)}(i)$ tends to zero almost surely. However,

$$\{\bar{\tau}^{(N)}(i) = i\} \subset \{\widetilde{\tau}^{(M)}(i) = i \text{ for infinitely many } M\}.$$

Thus,

$$Z(i) \geq \widehat{Y}(i,\tau) \quad \text{on } \{\bar{\tau}^{(N)}(i) = i\}.$$

But this implies $\widetilde{\tau}(i) = i$ on $\{\bar{\tau}^{(N)}(i) = i\}$. Consequently, (18) holds on $\{\bar{\tau}^{(N)}(i) = i\}$.

On the other hand, $\{\bar{\tau}^{(N)}(i) > i\} \subset \{\widetilde{\tau}^{(M)}(i) > i$ for infinitely many $M\}$, and an analogous argument yields

$$Z(i) \leq \widehat{Y}(i,\tau) \quad \text{on } \{\bar{\tau}^{(N)}(i) > i\}.$$

Consequently, $\widehat{\tau}(i) > i$ and thus, by the induction hypothesis,

$$\bar{\tau}^{(N)}(i) = \bar{\tau}^{(N)}(i+1) \leq \widehat{\tau}(i+1) = \widehat{\tau}(i) \quad \text{on } \{\bar{\tau}^{(N)}(i) > i\}.$$

The induction hypothesis can be applied in the same way to show

$$\bar{\tau}^{(N)}(i) \geq \widetilde{\tau}(i) \quad \text{on } \{\bar{\tau}^{(N)}(i) > i\} \cap \{\widetilde{\tau}(i) > i\},$$

whereas this inequality is trivially satisfied on $\{\bar{\tau}^{(N)}(i) > i\} \cap \{\widetilde{\tau}(i) = i\}$. This completes the proof of (18). ∎

The next theorem completes the proof of assertion (ia).

Theorem 4.3

$$|\widetilde{\tau}^{(N)}(i) - \bar{\tau}^{(N)}(i)| \to 0 \quad P - a.s.,$$

or equivalently,

$$P\left(\bigcap_{N\in\mathbb{N}} \bigcup_{M=N}^{\infty} \left\{ \widetilde{\tau}^{(M)}(i) \neq \bar{\tau}^{(M)}(i) \right\} \right) = 0.$$

Proof. The statement is obvious for $i = k$. Suppose now $0 \leq i \leq k-1$ and that the statement is proved for $i+1$. Define,

$$A(N,i) = \bigcup_{M=N}^{\infty} \left\{ \widetilde{\tau}^{(M)}(i) \neq \bar{\tau}^{(M)}(i) \right\}. \tag{28}$$

Clearly,

$$A(N,i) = B(N,i) \cup C(N,i) \cup D(N,i),$$

where

$$\begin{aligned}
B(N,i) &= \bigcup_{M=N}^{\infty} \left\{ \widetilde{\tau}^{(M)}(i) = i \right\} \cap \left\{ \bar{\tau}^{(M)}(i) > i \right\}, \\
C(N,i) &= \bigcup_{M=N}^{\infty} \left\{ \widetilde{\tau}^{(M)}(i) > i \right\} \cap \left\{ \bar{\tau}^{(M)}(i) = i \right\}, \\
D(N,i) &= \bigcup_{M=N}^{\infty} \left\{ \widetilde{\tau}^{(M)}(i) > i \right\} \cap \left\{ \bar{\tau}^{(M)}(i) > i \right\} \cap \left\{ \widetilde{\tau}^{(M)}(i) \neq \bar{\tau}^{(M)}(i) \right\}.
\end{aligned}$$

Since the sets $B(N,i)$, $C(N,i)$, and $D(N,i)$ are decreasing in N, we have

$$\bigcap_{N\in\mathbb{N}} A(N,i) = \left(\bigcap_{N\in\mathbb{N}} B(N,i) \right) \cup \left(\bigcap_{N\in\mathbb{N}} C(N,i) \right) \cup \left(\bigcap_{N\in\mathbb{N}} D(N,i) \right).$$

We show, that the three sets on the right hand side are evanescent. Firstly, as $\bar{\tau}^{(M)}$ and $\widetilde{\tau}^{(M)}$ are consistent, it holds

$$D(N,i) \subset A(N, i+1).$$

Hence, the intersection of the $D(N,i)$'s is a null set by the induction hypothesis. By the definition of $\bar{\tau}^{(M)}$ we have,

$$C(N,i) \subset \bigcup_{M=N}^{\infty} \left\{ \widetilde{\tau}^{(M)}(i) > i \right\} \cap \left\{ \widetilde{\tau}^{(K)}(i) > i \text{ for only finitely many } K \right\}.$$

Thus, the intersection of the $C(N,i)$'s is a null set. A similar argument applies for the intersection of the $B(N,i)$'s. ■

Assertion (iia) follows from the next theorem.

Theorem 4.4 *Suppose that for all i, $0 \le i \le k$,*

$$\lim_{N\to\infty} \epsilon^{(N)}(i) = 0, \quad P-a.s.$$

Then, for all $0 \le i \le k$,

$$\lim_{N\to\infty} \left| Y(i, \widetilde{\tau}^{(N)}) - Y(i, \bar{\tau}^{(N)}) \right| = 0 \quad P-a.s.$$

and

$$\lim_{N\to\infty} \left(Y(i, \widetilde{\tau}^{(N)}) - Y(i, \widetilde{\tau}) \right)_{-} = 0, \quad P-a.s.$$

Remark 4.5 *By the dominated convergence theorem the above convergences also hold in $L^1(P)$.*

Proof. With $A(N,i)$ defined in (28) we obtain,

$$\begin{aligned} & \left| E^{\mathcal{F}_i}\left[Z(\widetilde{\tau}^{(N)}(i)) \right] - E^{\mathcal{F}_i}\left[Z(\bar{\tau}^{(N)}(i)) \right] \right| \\ \le & \left| E^{\mathcal{F}_i}\left[\left(Z(\widetilde{\tau}^{(N)}(i)) - Z(\bar{\tau}^{(N)}(i)) \right) \mathbf{1}_{A(N,i)} \right] \right| \\ \le & E^{\mathcal{F}_i}\left[\mathbf{1}_{A(N,i)} \max_{0\le j\le k} Z(j) \right] \to 0, \end{aligned}$$

by the dominated convergence theorem, since

$$\lim_{N\to\infty} \mathbf{1}_{A(N,i)} = 0 \quad P-a.s.$$

by Theorem 4.3. This proves the first claim. The second claim then follows from Proposition 3.11. ■

4.2 Stability of the Algorithm: The Case $L = 1$

We are now going to explain how the stability result for the one-step improvement carries over to the algorithm in the case of one exercise right. We will make use of the following perturbed monotonicity result.

Proposition 4.6 *Suppose (τ_N) is a sequence of consistent stopping families and, for all $0 \le i \le k$,*

$$\lim_{N\to\infty} (Y(i;\tau_N) - Y(i;\tau))_{-} = 0 \quad P-a.s.$$

Then, for all $0 \le i \le k$,

$$\lim_{N\to\infty} (Y(i, \widetilde{\tau}_N) - Y(i, \widetilde{\tau}))_{-} = 0 \quad P-a.s.,$$

where

$$\widetilde{\tau}_N(i) := \inf\{j : i \le j \le k,\ \widehat{Y}(j;\tau_N) \le Z(j)\}.$$

Remark 4.7 *For a constant sequence $\tau_N = \sigma$ for all N, with σ being consistent, Proposition 4.6 states:*

$$Y(i,\sigma) \geq Y(i,\tau) \quad \Longrightarrow \quad Y(i,\widetilde{\sigma}) \geq Y(i,\widetilde{\tau}).$$

By defining a preference structure on the set of stopping families in a natural way via

$$\sigma \succeq \tau \quad :\Longleftrightarrow \quad Y(i,\sigma) \geq Y(i,\tau),$$

we see that the improvement operator $\widetilde{\ }$ preserves this preference structure.

Proof. The statement will be proved by backward induction over i. The induction base $i = k$ is obvious. Suppose the statement is proved for some $1 \leq i+1 \leq k$.

We first note that by Remark 3.10,

$$\mathbf{1}_{\{\widetilde{\tau}(i)=i\}} \left(Y(i,\widetilde{\tau}_N) - Y(i,\widetilde{\tau})\right)_- \leq \left(Y(i,\widetilde{\tau}_N) - Z(i)\right)_- = 0. \tag{29}$$

We next show that the statement is true on the set $\{\widetilde{\tau}_M(i) = i$ for infinitely many $M\}$. For this we need the following preliminary consideration. By Jensen's inequality and the dominated convergence theorem, for all $p \geq i$ it holds,

$$\left(E^{\mathcal{F}_i}[Y(p,\tau_N)] - E^{\mathcal{F}_i}[Y(p,\tau)]\right)_- \leq E^{\mathcal{F}_i}\left[(Y(p,\tau_N) - Y(p,\tau))_-\right] \to 0.$$

Thus,

$$\lim_{N\to\infty} \left(\widehat{Y}(i,\tau_N) - \widehat{Y}(i,\tau)\right)_- = 0 \quad P-a.s., \tag{30}$$

since the max-operator is continuous with respect to the metric generated by the negative part. On $\{\widetilde{\tau}_M(i) = i$ for infinitely many $M\}$ we have for infinitely many M,

$$\widehat{Y}(i,\tau_M) \leq Z(i).$$

Since

$$\left(Z(i) - \widehat{Y}(i,\tau)\right)_- \leq \left(Z(i) - \widehat{Y}(i,\tau_M)\right)_- + \left(\widehat{Y}(i,\tau_M) - \widehat{Y}(i,\tau)\right)_-,$$

we may conclude from (30), that

$$Z(i) \geq \widehat{Y}(i,\tau) \quad \text{on } \{\widetilde{\tau}_M(i) = i \text{ for infinitely many } M\}.$$

Hence,

$$\{\widetilde{\tau}_M(i) = i \text{ for infinitely many } M\} \subset \{\widetilde{\tau}(i) = i\}.$$

On the latter set the statement was proved in (29).

It remains to verify the statement on

$$E(i) = \{\widetilde{\tau}_M(i) = i \text{ for only finitely many } M\} \cap \{\widetilde{\tau}(i) > i\}.$$

Define

$$N_0(i) = \mathbf{1}_{E(i)} \max\{N;\ \widetilde{\tau}_N(i) = i\} + 1,$$

and note that the process $N_0(i)$ is $\mathcal{F}_i$-adapted. Since

$$\widetilde{\tau}_N(i) > i \quad \text{on } \{N \geq N_0(i)\} \cap E(i),$$

it follows from the induction hypothesis, Jensen's inequality, and the dominated convergence theorem, that

$$\begin{aligned} & \mathbf{1}_{\{N \geq N_0(i)\} \cap E(i)} \left(Y(i, \widetilde{\tau}_N) - Y(i, \widetilde{\tau})\right)_- \\ = \; & \mathbf{1}_{\{N \geq N_0(i)\} \cap E(i)} \left(E^{\mathcal{F}_i}[Y(i+1, \widetilde{\tau}_N)] - E^{\mathcal{F}_i}[Y(i+1, \widetilde{\tau})]\right)_- \\ \leq \; & E^{\mathcal{F}_i}\left[(Y(i+1, \widetilde{\tau}_N) - Y(i+1, \widetilde{\tau}))_-\right] \to 0. \end{aligned}$$

■

For notational convenience we state the stability result of the algorithm for two improvement steps ($m = 2$) only. It is immediate, how this extends to higher iterations. We will also skip all subscripts, which are superfluous in the case of one exercise right. For instance, we write $\tau^{(1)}$ instead of $\tau^{(1)}_{1,1}$. First note that with $\tau = \tau^{(0)}$,

$$\begin{aligned} \tau^{(1)}(i) &= \widetilde{\tau}(i), \\ \tau^{(2)}(i) &= \widetilde{\widetilde{\tau}}(i) = \inf\{j : i \leq j \leq k,\ \widehat{Y}(j; \widetilde{\tau}) \leq Z(j)\}. \end{aligned}$$

Let us suppose that for $(N_1, N_2) \in \mathbb{N} \times \mathbb{N}$, sequences $\epsilon^{(N_1)}(i)$ and $\epsilon^{(N_1,N_2)}(i)$ are given such that for $0 \leq i \leq k$,

$$\lim_{N_1 \to \infty} \epsilon^{(N_1)}(i) = 0 \quad P - a.s.,$$

and, for $0 \leq i \leq k$ and $N_1 \in \mathbb{N}$,

$$\lim_{N_2 \to \infty} \epsilon^{(N_1,N_2)}(i) = 0 \quad P - a.s.$$

We then define

$$\begin{aligned} \widetilde{\tau}^{(N_1)}(i) &:= \inf\{j : i \leq j \leq k,\ \widehat{Y}(j; \tau) + \epsilon^{(N_1)}(j) \leq Z(j)\}, \\ \widetilde{\widetilde{\tau}}^{(N_1)}(i) &:= \inf\{j : i \leq j \leq k,\ \widehat{Y}(j; \widetilde{\tau}^{(N_1)}) \leq Z(j)\}, \\ \widetilde{\widetilde{\tau}}^{(N_1,N_2)}(i) &:= \inf\{j : i \leq j \leq k,\ \widehat{Y}(j; \widetilde{\tau}^{(N_1)}) + \epsilon^{(N_1,N_2)}(j) \leq Z(j)\}. \end{aligned}$$

Theorem 4.4 now yields

$$\lim_{N_1 \to \infty} \left(Y(i, \widetilde{\tau}^{(N_1)}) - Y(i, \widetilde{\tau})\right)_- = 0 \quad P - a.s., \tag{31}$$

$$\lim_{N_2 \to \infty} \left(Y(i, \widetilde{\widetilde{\tau}}^{(N_1,N_2)}) - Y(i, \widetilde{\widetilde{\tau}}^{(N_1)})\right)_- = 0 \quad P - a.s. \tag{32}$$

In view of (31) we obtain by Proposition 4.6,

$$\lim_{N_1 \to \infty} \left(Y(i, \widetilde{\widetilde{\tau}}^{(N_1)}) - Y(i, \widetilde{\widetilde{\tau}})\right)_- = 0 \quad P - a.s. \tag{33}$$

From

$$\begin{aligned}&\left(Y(i,\widetilde{\widetilde{\tau}}^{(N_1,N_2)}) - Y^{(2)}(i)\right)_- \\ \leq\ &\left(Y(i,\widetilde{\widetilde{\tau}}^{(N_1,N_2)}) - Y(i,\widetilde{\widetilde{\tau}}^{(N_1)})\right)_- + \left(Y(i,\widetilde{\widetilde{\tau}}^{(N_1)}) - Y(i,\widetilde{\widetilde{\tau}})\right)_-,\end{aligned}$$

we then obtain,

Theorem 4.8 *For all $0 \leq i \leq k$,*

$$\lim_{N_1\to\infty}\lim_{N_2\to\infty}\left(Y(i,\widetilde{\widetilde{\tau}}^{(N_1,N_2)}) - Y^{(2)}(i)\right)_- = 0$$

P-almost surely and in $L^1(P)$.

The generalization of this result to m iteration steps may be put into words as follows:

The shortfall of the investor's expected gain corresponding to m perturbed steps of the algorithm below the expected gain corresponding to m theoretical steps converges to zero.

We emphasize again that it may happen that the perturbed algorithm performs even better than the theoretical (compare Example 4.1-(ii)).

4.3 Stability under Several Exercise Rights

The stability issue becomes more involved under several exercise rights. One reason is that we cannot expect to have the inequality

$$Y_L(i;\bar{\sigma}_1,\ldots,\bar{\sigma}_L) \geq Y_L(i;\widetilde{\sigma}_1,\ldots,\widetilde{\sigma}_L),$$

where $\bar{\sigma}_1,\ldots,\bar{\sigma}_L$ are arbitrary improvers of $\sigma_1,\ldots,\sigma_L$, but only the inequalities stated in Theorem 3.14. In other words, we cannot identify a worst improver as was possible in the case of one exercise right. Theorem 3.14 suggest that we must confine ourselves with the following stability result for the one-step improvement under several rights.

Theorem 4.9 *Suppose $\sigma_1,\ldots,\sigma_D$ are consistent stopping families. Define for $1 \leq L \leq D$,*

$$\begin{aligned}\widetilde{\sigma}_L(i) &= \inf\{j \geq i;\ \ Z(j) + E^{\mathcal{F}_j}Y_{L-1}(j+\delta;\sigma_1,\ldots,\sigma_{L-1}) \\ &\qquad \geq \widehat{Y}_L(j;\sigma_1,\ldots,\sigma_L) + \epsilon_L^{(N)}(j)\},\end{aligned}$$

where for all $1 \leq L \leq D$, $0 \leq i \leq k$,

$$\lim_{N\to\infty}\epsilon_L^{(N)}(i) = 0 \quad P-a.s.$$

Then, there are sequences of improver $\bar{\sigma}_1^{(N)},\ldots,\bar{\sigma}_D^{(N)}$ of $\sigma_1,\ldots,\sigma_D$ such that, for all $1 \leq L \leq D$,

$$\lim_{N\to\infty}|\widetilde{\sigma}_L^{(N)}(i) - \bar{\sigma}_L^{(N)}(i)| = 0.$$

Moreover,

$$\lim_{N\to\infty}\left|Y_L(i;\widetilde{\sigma}_1^{(N)},\ldots,\widetilde{\sigma}_L^{(N)})-Y_L(i;\bar{\sigma}_1^{(N)},\ldots,\bar{\sigma}_L^{(N)})\right|=0 \quad P-a.s.,$$

and

$$\lim_{N\to\infty}\left(Y_L(i;\widetilde{\sigma}_1^{(N)},\ldots,\widetilde{\sigma}_L^{(N)})-Y_L(i;\sigma_1,\ldots,\sigma_{L-1},\widetilde{\sigma}_L)\right)_-=0 \quad P-a.s.$$

Proof. In view of Lemma 3.12 and Theorem 3.14, the theorem follows by straightforward reduction to the case of one exercise right. ■

Again we demonstrate the stability of the multiple stopping algorithm only for two steps ($m=2$). Suppose we are given consistent starting families $\sigma_1,\ldots,\sigma_D$ (with suppressed superscript 0 in the notation of the algorithm). Recall that

$$\begin{aligned}\sigma_L^{(1)}(i) &:= \widetilde{\sigma}_L(i),\\ \sigma_L^{(2)}(i) &:= \widetilde{\sigma}_L^{(1)}(i)=\widetilde{\widetilde{\sigma}}_L(i).\end{aligned}$$

We next consider perturbed versions,

$$\begin{aligned}\widetilde{\sigma}_L^{(N_1)}(i) &= \inf\{j\geq i;\;\; Z(j)+E^{\mathcal{F}_j}Y_{L-1}(j+\delta;\sigma_1,\ldots,\sigma_{L-1})\\ &\quad\geq \widehat{Y}_L(j;\sigma_1,\ldots,\sigma_L)+\epsilon_L^{(N_1)}(j)\},\\ \widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}(i) &= \inf\{j\geq i;\;\; Z(j)+E^{\mathcal{F}_j}Y_{L-1}(j+\delta;\widetilde{\sigma}_1^{(N_1)},\ldots,\widetilde{\sigma}_{L-1}^{(N_1)})\\ &\quad\geq \widehat{Y}_L(j;\widetilde{\sigma}_1^{(N_1)},\ldots,\widetilde{\sigma}_L^{(N_1)})+\epsilon_L^{(N_1,N_2)}(j)\},\end{aligned}$$

with

$$\begin{aligned}\lim_{N_1\to\infty}\epsilon_L^{(N_1)}(i) &= 0 \quad P-a.s.,\\ \lim_{N_2\to\infty}\epsilon_L^{(N_1,N_2)}(i) &= 0 \quad P-a.s.\end{aligned}$$

In order to iterate the stability result from the previous theorem we will now additionally assume that, for $1\leq L\leq D$, $0\leq i\leq k-1$,

$$\bar{\sigma}_L(i)=\lim_{N_1\to\infty}\widetilde{\sigma}_L^{(N_1)}(i) \tag{34}$$

exists. Note that, by Theorem 4.9, the limit $\bar{\sigma}_L$ can be rewritten as a limit of L-improvers. By the definition of an L-improver it is straightforward that $\bar{\sigma}_L$ is an L-improver itself. We postpone a discussion of assumption (34) and continue to prove stability under this assumption.

We denote by $\widetilde{\widetilde{\sigma}}_L^{(N_1)}$ the theoretical $\widetilde{\ }$-improvement of $\widetilde{\sigma}_L^{(N_1)}$. The additional assumption (34) now ensures that we can write (by applying Lemma 3.12),

$$\begin{aligned}\widetilde{\widetilde{\sigma}}_L^{(N_1)}(i) &= \inf\{j\geq i;\;\; Z_L(j;\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1})\\ &\quad\geq \max_{p\geq j+1}E^{\mathcal{F}_j}\,Z_L(\bar{\sigma}_L(p);\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1})+\widetilde{\epsilon}_L^{(N_1)}(i)\},\end{aligned}$$

where

$$\lim_{N_1\to\infty} \widetilde{\epsilon}_L^{(N_1)}(i) = 0 \quad P-a.s.$$

We now define

$$\bar{\widetilde{\sigma}}_L^{(N_1)}(k) = k,$$

and

$$\begin{aligned}
\bar{\widetilde{\sigma}}_L^{(N_1)}(i) = i \quad &\Longleftrightarrow \quad (\widetilde{\widetilde{\sigma}}_L^{(M)}(i) > i \text{ for only finitely many } M) \\
&\qquad \vee (\widetilde{\widetilde{\sigma}}_L^{(M)}(i) = i \text{ for infinitely many } M \text{ and } \widetilde{\widetilde{\sigma}}_L^{(N_1)}(i) = i), \\
\bar{\widetilde{\sigma}}_L^{(N_1)}(i) \neq i \quad &\Longrightarrow \quad \bar{\widetilde{\sigma}}_L^{(N_1)}(i) = \bar{\widetilde{\sigma}}_L^{(N_1)}(i+1).
\end{aligned}$$

By Theorem 4.3, we have for all $1 \leq L \leq D$,

$$\lim_{N_1\to\infty} |\widetilde{\widetilde{\sigma}}_L^{(N_1)}(i) - \bar{\widetilde{\sigma}}_L^{(N_1)}(i)| = 0.$$

Thus, P-almost surely,

$$\lim_{N_1\to\infty} \left| Y_L(i; \widetilde{\sigma}_1^{(N_1)}, \ldots, \widetilde{\sigma}_{L-1}^{(N_1)}, \widetilde{\widetilde{\sigma}}_L^{(N_1)}) - Y_L(i; \bar{\sigma}_1(i), \ldots, \bar{\sigma}_{L-1}(i), \bar{\widetilde{\sigma}}_L^{(N_1)}) \right| = 0. \tag{35}$$

Moreover, by Theorem 4.2, for all $1 \leq L \leq D$, $\bar{\widetilde{\sigma}}_L^{(N_1)}$ is an improver of $\bar{\sigma}_L$ with respect to the cash-flow $Z_L(\cdot; \bar{\sigma}_1, \ldots \bar{\sigma}_{L-1})$, and thus an L-improver with respect to $(\bar{\sigma}_1, \ldots, \bar{\sigma}_{L-1})$ (Lemma 3.12). Hence, by Theorem 3.14,

$$\left(Y_L(i; \bar{\sigma}_1(i), \ldots, \bar{\sigma}_{L-1}(i), \bar{\widetilde{\sigma}}_L^{(N_1)}) - Y_L(i; \bar{\sigma}_1(i), \ldots, \bar{\sigma}_{L-1}(i), \widetilde{\bar{\sigma}}_L) \right)_- = 0. \tag{36}$$

Here, again, $\widetilde{\bar{\sigma}}_L$ is the theoretical$^{\sim}$-improvement of $\bar{\sigma}_L$. Finally, by Theorem 4.9, P-almost surely,

$$\lim_{N_2\to\infty} \left(Y_L(i; \widetilde{\widetilde{\sigma}}_1^{(N_1,N_2)}, \ldots, \widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}) - Y_L(i; \widetilde{\sigma}_1^{(N_1)}, \ldots, \widetilde{\sigma}_{L-1}^{(N_1)}, \widetilde{\widetilde{\sigma}}_L^{(N_1)}) \right)_- = 0. \tag{37}$$

Clearly, the convergence in (35) and (37) also holds in $L^1(P)$. By combining (35)–(37), we obtain,

$$\lim_{N_1\to\infty} \lim_{N_2\to\infty} \left(Y_L(i; \widetilde{\widetilde{\sigma}}_1^{(N_1,N_2)}, \ldots, \widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}) - Y_L(i; \bar{\sigma}_1, \ldots, \bar{\sigma}_{L-1}, \widetilde{\bar{\sigma}}_L) \right)_- = 0, \tag{38}$$

P-almost surely and in $L^1(P)$. Recall that $\bar{\sigma}_1, \ldots, \bar{\sigma}_L$ are some theoretical improvers of $\sigma_1, \ldots, \sigma_L$. Thus, $\widetilde{\bar{\sigma}}_L$ is a theoretical two-step improvement of σ_L.

We now discuss the additional assumption (34). (34) can be violated for $i = 0$ very easily, as the following variant of Example 4.1-(i), shows.

Example 4.10 *Suppose the initial value $Z_L(0;\sigma_1,\ldots,\sigma_{L-1})$ equals the real number $\widehat{Y}_L(0;\sigma_1,\ldots,\sigma_L)$. Note that this can always be enforced for some $1 \le L \le D$ by changing the initial value $Z(0)$ of the cash-flow appropriately. Moreover, assume $\epsilon_L^{(N_1)}(0) = \xi_{N_1}/N_1$ for a sequence (ξ_{N_1}) of independent binary trials as in Example 4.1-(i). Then again,*

$$\widetilde{\sigma}_L^{(N_1)}(0) = \xi_{N_1},$$

which does not converge almost surely when N_1 tends to infinity. It is clear that more general perturbations, which take positive and non-positive values with positive probability, yield the same effect.

The problem indicated in this example was our main motivation to introduce the modified algorithm based on y_L instead of Y_L. Suppose for the moment, that (34) is satisfied for $1 \le i \le k-1$ only. Then (38) holds for $1 \le i \le k$. By the definition of y_L and Jensen's inequality we obtain,

$$\begin{aligned}&\left(y_L(i;\widetilde{\widetilde{\sigma}}_1^{(N_1,N_2)},\ldots,\widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}) - y_L(i;\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1},\widetilde{\bar{\sigma}}_L)\right)_- \\ \le\ & E^{\mathcal{F}_i}\left(Y_L(i+1;\widetilde{\widetilde{\sigma}}_1^{(N_1,N_2)},\ldots,\widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}) - Y_L(i+1;\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1},\widetilde{\bar{\sigma}}_L)\right)_-.\end{aligned}$$

Thus, by dominated convergence, for all $0 \le i \le k$,

$$\lim_{N_1\to\infty}\lim_{N_2\to\infty}\left(y_L(i;\widetilde{\widetilde{\sigma}}_1^{(N_1,N_2)},\ldots,\widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}) - y_L(i;\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1},\widetilde{\bar{\sigma}}_L)\right)_- = 0,$$

P-almost surely and in $L^1(P)$.

We summarize the previous discussion in the following theorem:

Theorem 4.11 *Suppose that for all $1 \le L \le D$ and $1 \le i \le k-1$,*

$$\bar{\sigma}_L(i) = \lim_{N_1\to\infty}\widetilde{\sigma}_L^{(N_1)}(i)$$

exist. Then $\bar{\sigma}_1,\ldots,\bar{\sigma}_D$ are improvers of $\sigma_1,\ldots,\sigma_D$ up from time 1 (they are not defined at time 0).

Define by $\widetilde{\bar{\sigma}}_L$ $(1 \le L \le D)$ the theoretical $\widetilde{\ }$-L-improver of $\bar{\sigma}_L$ with respect to $(\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1})$. Then, for all $0 \le i \le k$, $1 \le L \le D$,

$$\lim_{N_1\to\infty}\lim_{N_2\to\infty}\left(y_L(i;\widetilde{\widetilde{\sigma}}_1^{(N_1,N_2)},\ldots,\widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}) - y_L(i;\bar{\sigma}_1,\ldots,\bar{\sigma}_{L-1},\widetilde{\bar{\sigma}}_L)\right)_- = 0$$

P-almost surely and in $L^1(P)$. The corresponding result for Y_L, i.e. (38), holds up from time i=1. It holds up from time $i = 0$, when (34) is also valid for $i = 0$.

The previous theorem still calls for sufficiency criteria for assumption (34) for $1 \le i \le k-1$.

Theorem 4.12 *Suppose, for all* $1 \leq L \leq D$, $1 \leq i \leq k-1$,

$$P\left(Z_L(i;\sigma_1,\ldots,\sigma_{L-1}) = \max_{p\geq i+1} E^{\mathcal{F}_i}\left[Z_L(\sigma_L(p);\sigma_1,\ldots,\sigma_{L-1})\right]\right) = 0. \tag{39}$$

Then, (34) is satisfied for all $1 \leq i \leq k-1$ *and* $1 \leq L \leq D$. *Moreover, for* $0 \leq i \leq k-1$ *and* $1 \leq L \leq D$,

$$\lim_{N_1\to\infty}\lim_{N_2\to\infty}\left(y_L(i;\widetilde{\widetilde{\sigma}}_1^{(N_1,N_2)},\ldots,\widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}) - y_L(i;\widetilde{\sigma}_1,\ldots,\widetilde{\sigma}_{L-1},\widetilde{\widetilde{\sigma}}_L)\right)_- = 0,$$

P-almost surely and in $L^1(P)$.

Proof. By (21) and (22) assumption (39) guarantees that, for all $1 \leq L \leq D$ and $1 \leq i \leq k$,

$$\widetilde{\sigma}_L(i) = \widehat{\sigma}_L(i) \quad P-a.s.$$

This implies that the sequence $\bar{\sigma}_L^{(N)}(i)$ of Theorem 4.9 coincides with $\widetilde{\sigma}_L(i)$ for all $N \in \mathbb{N}$. In particular, Theorem 4.9 yields that for all $1 \leq L \leq D$ and $1 \leq i \leq k$,

$$\lim_{N_1\to\infty}\widetilde{\sigma}_L^{(N_1)}(i) = \widetilde{\sigma}_L(i).$$

The theorem now follows by application of Theorem 4.11. ∎

The procedure described in this section can be iterated straightforwardly. For instance, under the additional (to (34)) assumption that

$$\bar{\bar{\sigma}}_L(i) = \lim_{N_1\to\infty}\lim_{N_2\to\infty}\widetilde{\widetilde{\sigma}}_L^{(N_1,N_2)}(i) \tag{40}$$

for $1 \leq L \leq D$ and $1 \leq i \leq k-1$ exists, we obtain for $m = 3$ (with obvious notations),

$$\left(y_L(i;\widetilde{\widetilde{\widetilde{\sigma}}}_1^{(N_1,N_2,N_3)},\ldots,\widetilde{\widetilde{\widetilde{\sigma}}}_L^{(N_1,N_2,N_3)}) - y_L(i;\bar{\bar{\sigma}}_1(i),\ldots,\bar{\bar{\sigma}}_{L-1}(i),\widetilde{\widetilde{\widetilde{\sigma}}}_L)\right)_- \to 0 \tag{41}$$

P-almost surely and in $L^1(P)$, when N_3, N_2, and N_1 tend to infinity. Here, one can verify that the limit $\bar{\bar{\sigma}}_L$ improves upon $\bar{\sigma}_L$ defined in (34) up from time $i = 1$. Thus, $\bar{\bar{\sigma}}_L$, $1 \leq L \leq D$, is a two-step improvement of σ_L (up from time $i = 1$) and $\widetilde{\widetilde{\widetilde{\sigma}}}_L$, $1 \leq L \leq D$, is a three-step improvement of σ_L (up from time $i = 1$).

If, in addition to (39), we have

$$P\left(Z_L(i;\widetilde{\sigma}_1,\ldots,\widetilde{\sigma}_{L-1}) = \max_{p\geq i+1} E^{\mathcal{F}_i}\left[Z_L(\widetilde{\sigma}_L(p);\widetilde{\sigma}_1,\ldots,\widetilde{\sigma}_{L-1})\right]\right) = 0, \tag{42}$$

for $1 \leq L \leq D$, $1 \leq i \leq k-1$, then (34) and (40) are satisfied and the limit in (40) equals $\widetilde{\widetilde{\sigma}}_L(i)$. Thus, (41) yields an analogue of Theorem 4.12 for three iterations.

Remark 4.13 *In view of Theorem 4.8 the assumptions of this section can be slightly relaxed. Indeed, for the improvement of the first stopping family* σ_1 *we may apply Theorem 4.8 directly. Therefore, it suffices to assume all additional properties for* $2 \leq L \leq D$ *instead of* $1 \leq L \leq D$. *Then, of course,* $\bar{\sigma}_1$, $\bar{\bar{\sigma}}_1$ *must be replaced by* $\widetilde{\sigma}_1$, $\widetilde{\widetilde{\sigma}}_1$.

Discussion of stability results

Under one exercise right we were able to prove that the shortfall of the perturbed algorithm under the theoretical (non-perturbed) algorithm converges to zero (Theorem 4.8). Compared to this the stability results under several rights are less satisfactory in two respects. We will now explain, why the obtained results are sufficient to call the algorithm stable, and why we think that better results are unlikely to hold.

The first shortcoming is that the stability results under several rights, (even for one step), do not allow to compare the theoretical and the perturbed improvement directly. To overcome this, one could employ the improvement strategy of Remark 3.4. But, as we explained there, this would cause much higher simulation costs, when implementation the algorithm. As one of the key issues of the paper is to provide an algorithm with few nestings of conditional expectations, we decided not to go along this way. When iterating the one-step improvement, the fact that we make use of the procedure in (7) causes the following effect: After m iterations we can only guarantee that the shortfall of the perturbed algorithm below some theoretical $(m-1)$-step improvement, not below some theoretical m-step improvement, converges to zero (actually a little more, see Theorem 4.11). Hence, $(m+1)$ nestings of conditional expectations are needed to compare with some m-step improvement. This is still much less than the nestings required to calculate m steps of the improvement type introduced in Remark 3.4. We also note that the comparison with some m-step improvement instead of the theoretical algorithm does not make too much of a difference due to Remark 3.15. Moreover, Theorem 4.12 allows to compare $(m+1)$ perturbed steps of the y_L-algorithm with m theoretical steps of this algorithm.

The second drawback, compared to stability under one exercise right, is that we had to impose additional conditions in order to iterate the one-step stability. Under one exercise right the monotonicity result in Proposition 4.6 allows to circumvent these assumptions. We believe a multi-exercise version of proposition 4.6 is unlikely to hold for the following reason: Suppose $\sigma_1, \ldots, \sigma_L$ and $\sigma'_1, \ldots, \sigma'_L$ are consistent stopping families with respective improvers $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_L$ and $\widetilde{\sigma}'_1, \ldots, \widetilde{\sigma}'_L$. Then, $\widetilde{\sigma}_L$ and $\widetilde{\sigma}'_L$ may be viewed as improvers under one exercise right with respect to the different cash-flows $Z_L(i; \sigma_1, \ldots, \sigma_{L-1})$ and $Z_L(i; \sigma'_1, \ldots, \sigma'_{L-1})$. But comparisons of the quality of improvements with respect to different cash-flows even fail, when one cash-flow dominates the other. We finally note, that an assumption similar to (39) has been made in Clément et al. (2002) in order to prove stability of the Longstaff-Schwartz algorithm for the optimal stopping problem under a single exercise right.

5 Conclusion

Motivated by the pricing problem of financial instruments with multiple early exercise opportunities we presented a new algorithm for the multiple stopping problem in discrete time and proved stability results for this algorithm. From a numerical point of view, the main feature of the algorithm is that it allows to calculate an increasing and convergent family of approximations of the Snell envelope with the order of nested conditional expectations for the mth approximation independent of the number of exercise rights. The algorithm is therefore tailor-made for a plain Monte-Carlo implementation and is thus expected to be particularly powerful when the cash-flow is a function of a high-dimensional Markov process. Under a single exercise right the strength of the algorithm is demonstrated by the simulation results in Kolodko and Schoenmakers (2004). Simulations under several exercise rights will be discussed in a forthcoming paper.

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