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Solutions to muscle fiber equations and their long time behaviour

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Abstract

We consider the nonlinear initial-boundary value problem governing the dynamical displacements of a one dimensional solid body with specific stress-strain law. This constitutive law results from the modelization of the mechanisms that rules the electrically activated mechanical behaviour of cardiac muscle fibers at the microscopic level. We prove global existence and uniqueness of solutions and we study their asymptotic behaviour in time. In particular we show that under vanishing external forcing solutions asymptotically converge to an equilibrium.

Introduction

In this paper we study the wellposedness for a system of partial differential equations governing the dynamical displacements of a fiber, a one-dimensional solid body that mimics the muscle fibers contained in the heart. The mathematical analysis of this system is intended to be a first step towards the analysis of some of the models used recently to represent the mechanical behavior of the heart (see e.g. [12, 3]). Following A.V. Hill's rheological scheme for striated muscles [9, 8] (see also [6]), the constitutive law of the fiber is chosen as follow: an *active* contractile element, in series with a linear elastic passive element, modelizes the transformation of biochemical energy into mechanical work during activation and a passive, nonlinear elastic element is added in parallel to represent the relaxed muscle.

The contractile element model considered here has been proposed in [2]. As most other models of the cardiac muscle contraction, it is based on the "sliding filament hypothesis" introduced by A.F. Huxley in [13] and extended in [10, 11]. In these models, the tension between actin and myosin filaments in the sarcomeres of striated muscles is the sum of the tensions in the cross-bridges, the chemical links between actin and myosin. It is a function of the cross-bridge deformation distribution that depends on the rates at which cross-bridges fasten and unfasten. Huxley-like models are first-order hyperbolic equations describing the evolution of this distribution. They differ by their rate functions, chosen to recover experimental sarcomere force-length relations. These rates are usually chosen as functions of the cross-bridge deformations (see e.g. [15]).

More recently, observed history-dependent force-length relations have led to consider attachment and detachment rates as functions of the cross-bridge deformations and deformation velocities [20]. The model in [2] has a similar structure coming from considering attachment and detachment rates allowing to recover the Hill force-velocity relation during isotonic contraction [7] and the force-length relation during passive relaxation [18]. They are furthermore functions of an input representing the action of the electrical potential on the fiber scale and of the intracellular calcium potential on the cell scale. Positive values of the input correspond to increasing cross-bridge density (activation) and negative values to a decreasing density (active relaxation). This model is consistent with the collective behaviour of myosin molecular motors [14]. The sarcomere tension being well approximated by a combination of the first two moments of the cross-bridge deformation distribution, the force-length relation can be reduced to a simple set of two ordinary differential equations by scaling, using the method of moments. This sarcomere constitutive law is embedded in the fiber rheological model used in [3] for whole heart simulations.

A forthcoming paper [17] will be devoted to a detailed derivation of the equation of motion for the obtained one-dimensional model, as well as to its numerical aspects and simulations. Here, we focus on its analytical properties and on the qualitative behaviour of solutions to the resulting system of equations. We prove here that a unique solution exists globally for each regular data and remains globally bounded if the external forcing is bounded. In the passive relaxation case, i. e. if no excitation is present, the solution is shown to decay to an equilibrium.

From a mathematical analysis point of view, refined mechanical models of muscle contractions on fiber or organ scales have not been paid much attention to yet. Let us nevertheless mention the works of P. Colli, V. Comincioli, and others (see for instance [4, 5] and their references) concerning the cell (sarcomere) and fiber scales. They study the wellposedness of the model of Huxley with general deformation-dependent rates and nonlinear parallel element. They consider only the isometric contraction case (the total deformation of the contractile and series elastic elements is constant). This two-scale (cell and fiber) problem leads to a nonlinear and nonlocal equation for the cross-bridge deformation distribution and the contractile element deformation. In our case, due to the scaling technique used to derive the model of the contractile element, the cross-bridge deformation distribution is no more to be determined and it is then possible to consider problems involving the fiber and organ scales where the cell deformation is no more constant but varies in time and along the fiber. Technical difficulties come from the fact that, on the organ scale, the force-length relation for the sarcomere leads to an hysteresis operator in the fiber rheological model and that the inertial effects have to be considered in the equation of motion of the whole fiber.

The paper is organized as follows. The problem formulation and the main results are stated in section 1. The properties of the constitutive mapping are given in the following section and the proof of the main theorem is detailed in section 3. Finally the asymptotic behaviour of the solutions is characterized in sections 4 and 5.

1 Statement of the problem

We consider longitudinal vibrations of a fiber of normalized length 1 and mass density ϱ, and assume that the displacement $y(x,t)$, strain $\varepsilon(x,t)$ and the elastic stress component $\sigma(x,t)$

satisfy the following system of equations

$$\varrho y_{tt} = (\mu\varepsilon_t + \sigma)_x \,, \tag{1.1}$$

$$\varepsilon = y_x \,, \tag{1.2}$$

$$\sigma = E_p\varepsilon + f(\varepsilon) + E_s(\varepsilon - \varepsilon^c)\,, \tag{1.3}$$

$$\mu_c\varepsilon^c_t = E_s(\varepsilon - \varepsilon^c) - \tau^c\,, \tag{1.4}$$

$$\tau^c_t = k^c\varepsilon^c_t - (\alpha|\varepsilon^c_t| + |u|)\tau^c + \sigma_0 u^+\,, \tag{1.5}$$

$$k^c_t = -(\alpha|\varepsilon^c_t| + |u|)k^c + k_0 u^+ \tag{1.6}$$

in the domain $(x,t) \in Q :=]0,1[\times]0,\infty[$, where the subscripts $_t$ and $_x$ denote partial derivatives, $E_p, E_s, \varrho, \mu, \mu_c, \sigma_0, k_0, \alpha$ are fixed positive constants, $f : \mathbb{R} \to \mathbb{R}$ is a given constitutive function, $u : Q \to \mathbb{R}$ is a given external forcing, and $u^+ := \max\{0, u\}$ is its positive part. System (1.1) – (1.6) is coupled with initial and boundary conditions

$$y(x,0) = y^0(x)\,,\ y_t(x,0) = y^1(x)\,,\ \varepsilon^c(x,0) = \varepsilon^{oc}(x)\,,\ \tau^c(x,0) = \tau^{oc}(x)\,,\ k^c(x,0) = k^{oc}(x)\,, \tag{1.7}$$

$$y(0,t) = 0\,,\ (\mu\varepsilon_t + \sigma + g(y_t))(1,t) = \psi(t) \tag{1.8}$$

where $g : \mathbb{R} \to \mathbb{R}$, $\psi : [0,\infty[\to \mathbb{R}$ are given functions.

Equations (1.4) – (1.6) represent the constitutive law of Hill-Maxwell type presented in [2] and corresponding to the contractile element with strain ε^c, stress τ^c, and variable stiffness k^c, driven by the electric activation $u(x,t)$. The boundary condition (1.8) says that the fiber is fixed at $x = 0$ and attached to an active valve at $x = 1$, where $\psi(t)$ is the reaction of the valve. A detailed discussion on the above system can be found in [17].

The data fulfil the following hypothesis.

Hypothesis 1.1

(i) $f, g : \mathbb{R} \to \mathbb{R}$ *are non-decreasing and locally Lipschitz continuous functions satisfying the conditions* $f(0) = g(0) = 0$ *and*

$$\lim_{|x|\to\infty} \frac{f(x)}{x^3} = 0\,, \qquad \limsup_{|x|\to\infty} \frac{g(x)}{x} < \infty\,; \tag{1.9}$$

(ii) $y^0 \in W^{2,2}(0,1)$, $y^1, \varepsilon^{oc}, \tau^{oc}, k^{oc} \in W^{1,2}(0,1)$, $y^0(0) = 0$, *and there exists a constant* $k^* > 0$ *such that* $k^{oc}(x) \ge k^*$ *a. e.;*

(iii) $\psi \in L^\infty(0,\infty)\,, \psi_t \in L^\infty_{\rm loc}(0,\infty)\,;$

(iv) $u \in L^\infty(Q)$, $u_x \in L^2(0,1;\, L^\infty_{\rm loc}(0,\infty))$.

In Hypothesis 1.1, we denote by $L^\infty_{\rm loc}(0,\infty)$ the space of locally bounded measurable functions $v : [0,\infty[\to \mathbb{R}$, in which we define the seminorms

$$|v|_{[0,t]} = \sup_{s\in[0,t]} \mathrm{ess}\, |v(s)| \qquad \text{for } t \ge 0\,. \tag{1.10}$$

In fact, with the metric $\Delta(v_1,v_2) = \sup_{T>0} |v_1 - v_2|_{[0,T]}/(1+|v_1 - v_2|_{[0,T]})$, the space $L^\infty_{\rm loc}(0,\infty)$ becomes a Fréchet space.

In the next sections, we prove the following main results.

Theorem 1.2 *Let Hypothesis 1.1 hold. Then system (1.1) – (1.8) admits a unique solution on Q such that the functions $y, \varepsilon, \varepsilon^c, \sigma, \tau^c, k^c, \tau^c/k^c, \varepsilon^c_t, \tau^c_t, k^c_t$ are continuous on $\bar{Q}$ and belong to $L^\infty(Q)$, y_t is continuous on $\bar{Q}$ and belongs to $L^\infty(0,\infty;\, L^2(0,1))$, $y_{tt}, \sigma_{xt}, \varepsilon_{xt} \in L^2_{\text{loc}}(0,\infty;\, L^2(0,1))$, $\varepsilon^c_{xt}, k^c_{xt}, \tau^c_{xt} \in L^2(0,1;\, L^\infty_{\text{loc}}(0,\infty))$, and Eqs. (1.1) – (1.6) are satisfied almost everywhere.*

Let us point out the fact that $y, \varepsilon, \varepsilon^c, \sigma, \tau^c, k^c, \tau^c/k^c, \varepsilon^c_t, \tau^c_t, k^c_t$ remain *globally bounded* on Q. In other words, the stress and strain cannot arbitrarily increase beyond a certain threshold if the command u and the boundary forcing ψ remain bounded.

This result is in agreement with the natural expectation. The situation is however specific here in the sense that the internal energy functional contains in the denominator the rigidity coefficient k^c which is not a priori bounded from below (in particular if $u < 0$). We will see in the next sections that this fact causes technical difficulties.

In fact, if we are interested merely in the *existence* of global solutions, the growth conditions (1.9) can be removed. They only play a role in the derivation of the global bounds for the solution, see Remark 3.1 at the end of Section 3.

In the *passive relaxation case* $u \equiv 0$, $\psi \equiv 0$, we say something more about the asymptotic behaviour of the solution.

Theorem 1.3 *Let Hypothesis 1.1 hold with $u \equiv 0$ and $\psi \equiv 0$. Then there exist functions $k^c_\infty, \varepsilon^c_\infty, \tau^c_\infty, \varepsilon_\infty \in L^\infty(0,1)$, $k^c_\infty(x) \geq 0$ a. e., such that for $t \to \infty$ we have*

$$k^c(x,t) - k^c_\infty(x) \;\to\; 0 \quad a.\,e., \tag{1.11}$$

$$\int_0^1 \left(|\varepsilon(x,t) - \varepsilon_\infty(x)|^2 + |\varepsilon^c(x,t) - \varepsilon^c_\infty(x)|^2 + |\tau^c(x,t) - \tau^c_\infty(x)|^2\right) dx \;\to\; 0, \tag{1.12}$$

$$\int_0^1 \left(y_t^2(x,t) + \sigma^2(x,t)\right) dx \;\to\; 0, \tag{1.13}$$

$$\int_0^1 \left(|k^c_t(x,t)|^2 + |\varepsilon^c_t(x,t)|^2 + |\tau^c_t(x,t)|^2\right) dx \;\to\; 0, \tag{1.14}$$

$$\int_0^1 \left(y_{tt}^2(x,t) + \varepsilon_t^2(x,t)\right) dx \;\to\; 0. \tag{1.15}$$

The proof of the above statements is based on an estimation technique which involves several consecutive steps. We start with some easy properties of the constitutive law.

2 The constitutive mapping

Equations (1.4) – (1.6) contain the spatial variable x merely as parameter. Assuming x to be fixed, we can consider them as a system of ODEs with given input functions $u, \varepsilon \in L^\infty_{\text{loc}}(0,\infty)$. Let us note that if u identically vanishes, then the restricted mapping $\varepsilon^c \mapsto (\tau^c, k^c)$ given by (1.5) – (1.6) is causal and rate-independent. In fact, it belongs to the family of the so-called *Duhem hysteresis operators*, and an interested reader may find in [19] a detailed discussion on this subject.

We can replace (1.5) – (1.6) by

$$\tau^c_t = \frac{k^c}{\mu_c}\left(E_s(\varepsilon-\varepsilon^c)-\tau^c\right) - \left(\frac{\alpha}{\mu_c}\left|E_s(\varepsilon-\varepsilon^c)-\tau^c\right| + |u|\right)\tau^c + \sigma_0 u^+\,, \tag{2.1}$$

$$k^c_t = -\left(\frac{\alpha}{\mu_c}\left|E_s(\varepsilon-\varepsilon^c)-\tau^c\right| + |u|\right)k^c + k_0 u^+\,, \tag{2.2}$$

so that (1.4) – (1.6) becomes an ODE system with a locally Lipschitz continuous right-hand side with respect to the three unknowns $\varepsilon^c, \tau^c, k^c$, with initial conditions of the form

$$\varepsilon^c(0) = \varepsilon^{oc}\,, \tau^c(0) = \tau^{oc}\,, k^c(0) = k^{oc} > 0\,. \tag{2.3}$$

System (1.4) – (1.6) therefore admits a unique maximal solution on an interval $[0,T[$ for some $T > 0$. We now derive some estimates which will be useful in the sequel.

Lemma 2.1 *Let $u, \varepsilon \in L^\infty_{\text{loc}}(0,\infty)$ be given, and let $(\varepsilon^c, \tau^c, k^c) : [0,T[\to \mathbb{R}^3$ be the maximal solution of (1.4) – (1.6) with initial conditions (2.3). Then $T = +\infty$ and for all $t \geq 0$ we have*

(i) $0 < k^c(t) < \max\{k^{oc}, k_0\} =: \bar{k}$;

(ii) $|\tau^c(t)| \leq \max\{|\tau^{oc}|, \bar{k}/\alpha, \sigma_0\} =: \bar{\tau}$;

(iii) $\left|\dfrac{\tau^c(t)}{k^c(t)} - \varepsilon^c(t)\right| \leq \max\left\{\left|\dfrac{\tau^{oc}}{k^{oc}} - \varepsilon^{oc}\right|, \dfrac{\sigma_0}{k_0} + |\varepsilon^c|_{[0,t]}\right\}$;

(iv) $|\varepsilon^c(t)| \leq \max\{|\varepsilon^{oc}|, |\varepsilon|_{[0,t]} + \bar{\tau}/E_s\}$;

(v) $$\varepsilon_t\,\sigma - \frac{d}{dt}\left(\frac{E_p}{2}\varepsilon^2 + F(\varepsilon) + \frac{E_s}{2}(\varepsilon-\varepsilon^c)^2 + \frac{(\tau^c)^2}{2k^c}\right)$$
$$= \mu_c(\varepsilon^c_t)^2 + (\alpha|\varepsilon^c_t| + |u|)\frac{(\tau^c)^2}{2k^c} + u^+\left(\frac{k_0}{2}\left(\frac{\tau^c}{k^c}\right)^2 - \sigma_0\frac{\tau^c}{k^c}\right),$$

where σ is given by (1.3) and $F(\varepsilon) := \int_0^\varepsilon f(z)\,dz$.

Note that the estimates (i) – (iv) are all independent of u, (i) – (ii) are moreover independent of ε. Identity (v) is in fact the energy balance, where

$$U := \frac{E_p}{2}\varepsilon^2 + F(\varepsilon) + \frac{E_s}{2}(\varepsilon-\varepsilon^c)^2 + \frac{(\tau^c)^2}{2k^c} \tag{2.4}$$

is the internal energy functional. In the absence of external excitation (this means $u \equiv 0$), the right-hand side of (v) corresponds to the dissipation rate and is positive in agreement with the Second Principle of Thermodynamics. We also see that energy can be supplied to the system only if $u > 0$ and the coefficient at $|u| = u^+$ in the energy balance equation (v) is negative, that is, $0 < \tau^c < 2\sigma_0\,k^c/(k_0 + k^c)$.

Proof of Lemma 2.1. For $t \in [0,T[$ put $B(t) = \int_0^t (\alpha|\varepsilon_t^c(s)| + |u(s)|)\, ds$. Then (1.6) yields

$$\frac{d}{dt}\left(\mathrm{e}^{B(t)}\, k^c(t)\right) \;\in\; \Big]0, k_0\, \dot{B}(t)\, \mathrm{e}^{B(t)}\Big[\,, \tag{2.5}$$

and (i) follows. Similarly, from (1.5) we obtain that

$$\frac{d}{dt}\left(\mathrm{e}^{B(t)}\, |\tau^c(t)|\right) \;\le\; \left|k^c(t)\varepsilon_t^c(t) + \sigma_0\, u^+(t)\right|\, \mathrm{e}^{B(t)} \;\le\; \max\{\bar{k}/\alpha, \sigma_0\}\, \dot{B}(t)\, \mathrm{e}^{B(t)}\,, \tag{2.6}$$

hence (ii) is verified. To prove (iii), we first notice that

$$\frac{d}{dt}\left(\frac{\tau^c(t)}{k^c(t)} - \varepsilon^c(t)\right) + k_0\, \frac{u^+(t)}{k^c(t)}\left(\frac{\tau^c(t)}{k^c(t)} - \varepsilon^c(t)\right) \;=\; k_0\, \frac{u^+(t)}{k^c(t)}\left(\frac{\sigma_0}{k_0} - \varepsilon^c(t)\right)\,, \tag{2.7}$$

so that we are again in the "Gronwall" situation and argue as in (2.6). We similarly obtain (iv) from (1.4) and (ii).

Estimates (i) – (iv) enable us to conclude that $\varepsilon^c, \tau^c, k^c$ do not blow up in finite time, hence $T = +\infty$. Identity (v) can easily be checked by direct differentiation. ■

By virtue of Lemma 2.1, we may consider the mapping which with each $u, \varepsilon \in L^\infty_{\mathrm{loc}}(0,\infty)$ and $\varepsilon^{oc}, \tau^{oc} \in \mathbb{R}$, $k^{oc} > 0$ associates the solution $(\varepsilon^c, \tau^c, k^c) \in (L^\infty_{\mathrm{loc}}(0,\infty))^3$ of (1.4) – (1.6). We now show that this mapping is locally Lipschitz continuous in suitable norms.

Lemma 2.2 *Let $(\varepsilon, u), (\tilde{\varepsilon}, \tilde{u}) \in L^\infty_{\mathrm{loc}}(0,\infty) \times L^\infty_{\mathrm{loc}}(0,\infty)$ be given, and let $(\varepsilon^c, \tau^c, k^c), (\tilde{\varepsilon}^c, \tilde{\tau}^c, \tilde{k}^c)$ be the respective solutions to Eqs. (1.4) – (1.6) with corresponding initial data $(\varepsilon^{oc}, \tau^{oc}, k^{oc})$ and $(\tilde{\varepsilon}^{oc}, \tilde{\tau}^{oc}, \tilde{k}^{oc})$. Then for every $T > 0$ and every $R > 0$ such that*

$$\max\{|u|_{[0,T]}\,, |\tilde{u}|_{[0,T]}\,, |\varepsilon|_{[0,T]}\,, |\tilde{\varepsilon}|_{[0,T]}\} \;\le\; R$$

there exists a constant $K(T,R)$ depending only on T and R such that

$$\begin{aligned} &\max\left\{|\varepsilon^c - \tilde{\varepsilon}^c|_{[0,T]}\,, |\tau^c - \tilde{\tau}^c|_{[0,T]}\,, \left|k^c - \tilde{k}^c\right|_{[0,T]}\,, |\varepsilon_t^c - \tilde{\varepsilon}_t^c|_{[0,T]}\,, |\tau_t^c - \tilde{\tau}_t^c|_{[0,T]}\,, \left|k_t^c - \tilde{k}_t^c\right|_{[0,T]}\right\} \\ &\quad \le K(T,R)\, \max\left\{|\varepsilon - \tilde{\varepsilon}|_{[0,T]}\,, |u - \tilde{u}|_{[0,T]}\,, |\varepsilon^{oc} - \tilde{\varepsilon}^{oc}|, |\tau^{oc} - \tilde{\tau}^{oc}|, |k^{oc} - \tilde{k}^{oc}|\right\} \end{aligned} \tag{2.8}$$

Proof. This is again an easy Gronwall-type exercise based on the estimates from Lemma 2.1 if Eqs. (1.5) – (1.6) are replaced by (2.1) – (2.2). We omit the details. ■

3 Proof of Theorem 1.2: Space discretization

The solution to Eqs. (1.1) – (1.8) will be constructed as a limit of space-semidiscrete approximations. We choose an integer $n \in \mathbb{N}$ and replace the continuous variable $x \in [0,1]$ by the equidistant partition $x_j = j/n$ for $j = 0, 1, \ldots, n$, with the intention to let n tend to $+\infty$.

Index j stands for the approximate value at the point x_j, and the x-derivative is replaced by a difference quotient. We thus consider the following system of ODEs for $j = 1, \ldots, n$:

$$\varrho \ddot{y}_j = n\left((\mu\dot{\varepsilon}_{j+1} + \sigma_{j+1}) - (\mu\dot{\varepsilon}_j + \sigma_j)\right), \quad j = 1, \ldots, n-1, \tag{3.1}$$

$$\mu\dot{\varepsilon}_n = -\sigma_n - g(\dot{y}_n) + \psi, \tag{3.2}$$

$$y_0 = 0, \tag{3.3}$$

$$\varepsilon_j = n(y_j - y_{j-1}), \tag{3.4}$$

$$\sigma_j = E_p\varepsilon_j + f(\varepsilon_j) + E_s(\varepsilon_j - \varepsilon_j^c), \tag{3.5}$$

$$\mu_c\dot{\varepsilon}_j^c = E_s(\varepsilon_j - \varepsilon_j^c) - \tau_j^c, \tag{3.6}$$

$$\dot{\tau}_j^c = k_j^c\dot{\varepsilon}_j^c - (\alpha|\dot{\varepsilon}_j^c| + |u_j|)\tau_j^c + \sigma_0 u_j^+, \tag{3.7}$$

$$\dot{k}_j^c = -(\alpha|\dot{\varepsilon}_j^c| + |u_j|)k_j^c + k_0 u_j^+, \tag{3.8}$$

with $u_j(t) = n\int_{x_{j-1}}^{x_j} u(x,t)\,dx$ for $j = 1, \ldots, n$, and with initial conditions

$$y_j(0) = y^0(x_j),\ \varepsilon_j^c(0) = \varepsilon^{oc}(x_j),\ \tau_j^c(0) = \tau^{oc}(x_j),\ k_j^c(0) = k^{oc}(x_j) \quad \text{for} \quad j = 1, \ldots, n, \tag{3.9}$$

$$\dot{y}_j(0) = y^1(x_j) \quad \text{for} \quad j = 1, \ldots, n-1. \tag{3.10}$$

The value of $\dot{y}_n(0)$ cannot be prescribed. However, we can rewrite (3.2) as

$$\dot{y}_n(t) = (\mu n I + g)^{-1}(\mu n \dot{y}_{n-1}(t) - \sigma_n(t) + \psi(t)), \tag{3.11}$$

where $I : \mathbb{R} \to \mathbb{R}$ is the identity mapping. For $t = 0$ we have in particular

$$\begin{aligned} &\mu n(\dot{y}_n(0) - y^1(1)) + g(\dot{y}_n(0)) - g(y^1(1)) \\ &\quad = \mu n(y^1(1 - (1/n)) - y^1(1)) - \sigma_n(0) - g(y^1(1)) + \psi(0), \end{aligned} \tag{3.12}$$

hence

$$|\dot{y}_n(0) - y^1(1)| \le \frac{1}{\sqrt{n}}\left(\int_{1-(1/n)}^1 |y_x^1(x)|^2\,dx\right)^{1/2} + \frac{1}{\mu n}(|\sigma_n(0)| + |g(y^1(1))| + |\psi(0)|). \tag{3.13}$$

System (3.1) – (3.8) is of the form $\dot{Y} = \Phi(Y, t)$ provided (3.2) is written in the form (3.11), and Eqs. (3.7) – (3.8) are transformed similarly as in (2.1) – (2.2). The mapping Φ is locally Lipschitz continuous in Y and measurable in t. There exists therefore for each $n \in \mathbb{N}$ a unique maximal solution to (3.1) – (3.10) defined on an interval $[0, T_n[$, $T_n > 0$.

We now derive a series of estimates which will enable us to pass to the limit as $n \to \infty$. We will systematically use the convention that C, c denote any suitable positive constants (C being "large" and c "small"), depending only on the data and independent of n and t.

Estimate 1. Test (3.1) by $\frac{1}{n}\dot{y}_j$ and sum over $j = 1, \ldots, n-1$. This yields for $t \in [0, T_n[$ that

$$\frac{d}{dt}\left(\frac{1}{n}\sum_{j=1}^{n-1}\frac{\varrho}{2}\dot{y}_j^2(t)\right) + \frac{1}{n}\sum_{j=1}^{n}\left(\mu\dot{\varepsilon}_j^2(t) + \sigma_j(t)\dot{\varepsilon}_j(t)\right) + \dot{y}_n(t)\,g(\dot{y}_n(t)) = \dot{y}_n(t)\,\psi(t), \tag{3.14}$$

where we have similarly as in Lemma 2.1 (v) that

$$\sigma_j\,\dot\varepsilon_j \;=\; \frac{d}{dt}\left(\frac{E_p}{2}\varepsilon_j^2 + F(\varepsilon_j) + \frac{E_s}{2}(\varepsilon_j-\varepsilon_j^c)^2 + \frac{(\tau_j^c)^2}{2k_j^c}\right) \tag{3.15}$$
$$+\,\mu_c(\dot\varepsilon_j^c)^2 + (\alpha|\dot\varepsilon_j^c| + |u_j|)\frac{(\tau_j^c)^2}{2k_j^c} + u_j^+\left(\frac{k_0}{2}\left(\frac{\tau_j^c}{k_j^c}\right)^2 - \sigma_0\,\frac{\tau_j^c}{k_j^c}\right).$$

From the inequalities $k_0z^2 - 2\sigma_0 z \ge -\sigma_0^2/k_0$ for all $z\in\mathbb{R}$ and

$$|\dot y_i(t)| \;\le\; \frac{1}{n}\sum_{j=1}^{n}|\dot\varepsilon_j(t)| \;\le\; \left(\frac{1}{n}\sum_{j=1}^{n}\dot\varepsilon_j^2(t)\right)^{1/2} \qquad \forall i=1,\dots,n\,, \tag{3.16}$$

we obtain under Hypothesis 1.1 the crucial estimate

$$\frac{d}{dt}\left(\frac{1}{n}\sum_{j=1}^{n-1}\frac{\varrho}{2}\dot y_j^2(t) + \frac{1}{n}\sum_{j=1}^{n}\left(\frac{E_p}{2}\varepsilon_j^2(t) + F(\varepsilon_j(t)) + \frac{E_s}{2}(\varepsilon_j(t)-\varepsilon_j^c(t))^2 + \frac{(\tau_j^c)^2(t)}{2k_j^c(t)}\right)\right) \tag{3.17}$$
$$+\,\dot y_n(t)\,g(\dot y_n(t)) + \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\mu}{2}\dot\varepsilon_j^2(t) + \mu_c(\dot\varepsilon_j^c)^2(t)\right) \;\le\; C\,.$$

This is enough to conclude that the solution to (3.1) – (3.10) is global, i. e. $T_n = +\infty$ for all $n\in\mathbb{N}$.

Estimate 2. Test (3.1) by $\frac{1}{n}y_j$ and sum over $j=1,\dots,n-1$. For each $t>0$ we then have

$$\frac{d}{dt}\left(\frac{1}{n}\sum_{j=1}^{n-1}\varrho\dot y_j(t)\,y_j(t) + \frac{1}{n}\sum_{j=1}^{n}\frac{\mu}{2}\varepsilon_j^2(t)\right) \;+\; y_n(t)\,g(\dot y_n(t)) + \frac{1}{n}\sum_{j=1}^{n}\sigma_j(t)\,\varepsilon_j(t) \tag{3.18}$$
$$=\; \frac{1}{n}\sum_{j=1}^{n-1}\varrho\dot y_j^2(t) + y_n(t)\,\psi(t)\,,$$

where

$$\sigma_j\,\varepsilon_j \;=\; \frac{d}{dt}\left(\frac{\mu_c}{2}(\varepsilon_j^c)^2\right) + E_p\varepsilon_j^2 + f(\varepsilon_j)\,\varepsilon_j + E_s(\varepsilon_j-\varepsilon_j^c)^2 + \tau_j^c\,\varepsilon_j^c \tag{3.19}$$
$$\ge\; \frac{d}{dt}\left(\frac{\mu_c}{2}(\varepsilon_j^c)^2\right) + \frac{E_p}{2}\varepsilon_j^2 + F(\varepsilon_j) + \frac{E_s}{2}(\varepsilon_j-\varepsilon_j^c)^2 - C$$

by virtue of Lemma 2.1, hence

$$\frac{d}{dt}\left(\frac{1}{n}\sum_{j=1}^{n-1}\varrho\dot y_j(t)\,y_j(t) + \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\mu}{2}\varepsilon_j^2(t) + \frac{\mu_c}{2}(\varepsilon_j^c)^2(t)\right)\right) \tag{3.20}$$
$$+\frac{1}{n}\sum_{j=1}^{n}\left(\frac{E_p}{2}\varepsilon_j^2(t) + F(\varepsilon_j(t)) + \frac{E_s}{2}(\varepsilon_j(t)-\varepsilon_j^c(t))^2\right)$$
$$\le\; \frac{1}{n}\sum_{j=1}^{n-1}\varrho\dot y_j^2(t) + |y_n(t)|\,\left(|g(\dot y_n(t))| + |\psi(t)|\right) + C$$

for all $t > 0$. We further estimate the right-hand side of (3.20) using the inequalities

$$\frac{1}{n}\sum_{j=1}^{n-1} \dot y_j^2(t) \;\le\; \frac{1}{n}\sum_{j=1}^{n} \dot\varepsilon_j^2(t)\,, \tag{3.21}$$

$$|y_i(t)| \;\le\; \frac{1}{n}\sum_{j=1}^{n} |\varepsilon_j(t)| \;\le\; \left(\frac{1}{n}\sum_{j=1}^{n} \varepsilon_j^2(t)\right)^{1/2} \qquad \forall i = 1,\dots,n\,, \tag{3.22}$$

$$|g(\dot y_n(t))| \;\le\; C\,\left(\dot y_n(t)\, g(\dot y_n(t))\right)^{1/2}\,, \tag{3.23}$$

where (3.23) follows from the condition (1.9). We now fix some (small) $c > 0$ and define the extended energy functionals

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{n}\sum_{j=1}^{n-1} \varrho\left(\frac{1}{2}\dot y_j^2 + c\dot y_j y_j\right)(t) \qquad (3.24)\\ &\quad + \frac{1}{n}\sum_{j=1}^{n}\left(\frac{E_p + c\mu}{2}\varepsilon_j^2 + F(\varepsilon_j) + \frac{E_s}{2}(\varepsilon_j - \varepsilon_j^c)^2 + \frac{c\mu_c}{2}(\varepsilon_j^c)^2 + \frac{(\tau_j^c)^2}{2k_j^c}\right)(t)\,,\\ \mathcal{E}_1(t) &= \frac{1}{n}\sum_{j=1}^{n}\left(\dot\varepsilon_j^2 + (\dot\varepsilon_j^c)^2 + \varepsilon_j^2 + (\varepsilon_j^c)^2 + F(\varepsilon_j)\right)(t)\,. \qquad (3.25) \end{aligned}$$

Using (3.17) and (3.20) – (3.23) we find another small $c > 0$ such that

$$\dot{\mathcal{E}}(t) + c\mathcal{E}_1(t) + \frac{1}{2}\dot y_n(t)\, g(\dot y_n(t)) \;\le\; C \qquad \forall t > 0\,. \tag{3.26}$$

We would like to replace $\mathcal{E}_1(t)$ in (3.26) by $\mathcal{E}(t)$. In other words, we have to get the term $(\tau_j^c)^2/k_j^c$ in (3.24) under control. This can be done by invoking Lemma 2.1 which yields

$$\frac{(\tau_j^c)^2(t)}{k_j^c(t)} \;\le\; |\tau_j^c(t)|\left(\left|\frac{\tau_j^c(t)}{k_j^c(t)} - \varepsilon_j^c(t)\right| + |\varepsilon_j^c(t)|\right) \;\le\; C\left(1 + \left|\varepsilon_j^c\right|_{[0,t]}\right) \qquad \text{for } \; t \ge 0\,, \tag{3.27}$$

so that by (3.26) and (3.21) – (3.22) we have

$$\dot{\mathcal{E}}(t) + c\mathcal{E}(t) + \frac{1}{2}\dot y_n(t)\, g(\dot y_n(t)) \;\le\; C\left(1 + \frac{1}{n}\sum_{j=1}^{n}\left|\varepsilon_j^c\right|_{[0,t]}\right) \qquad \forall t > 0\,. \tag{3.28}$$

Using the fact

$$\mathcal{E}(0) \;\le\; C\left(|y^1|_\infty^2 + |y_x^0|_\infty^2 + |\varepsilon^{oc}|_\infty^2 + |\tau^{oc}|_\infty^2/k^*\right) \;\le\; C\,, \tag{3.29}$$

where $|\cdot|_\infty$ denotes the norm of $L^\infty(0,1)$, we obtain from (3.28) the estimate

$$\frac{1}{n}\sum_{j=1}^{n-1}\dot y_j^2(t) + \frac{1}{n}\sum_{j=1}^{n}\left(\varepsilon_j^2(t) + (\varepsilon_j^c)^2(t)\right) \;\le\; C\,\mathcal{E}(t) \;\le\; C\left(1 + \frac{1}{n}\sum_{j=1}^{n}\left|\varepsilon_j^c\right|_{[0,t]}\right) \quad \forall t \ge 0\,. \tag{3.30}$$

Estimate 3. We use a discrete variant of the trick proposed in [1] and introduce new variables by putting

$$p_i(t) \;=\; \frac{1}{n}\sum_{j=i}^{n-1}\dot y_j(t)\,, \qquad q_i(t) \;=\; \varrho p_i(t) + \mu\varepsilon_i(t) \tag{3.31}$$

for $i = 1, \ldots, n$. According to usual conventions, this means in particular that $p_n(t) = 0$. Then (3.1) – (3.2) yield

$$\dot{q}_i(t) + \sigma_i(t) + g(\dot{y}_n(t)) \;=\; \psi(t) \quad \forall t > 0 \;\; \forall i = 1, \ldots, n\,. \tag{3.32}$$

We rewrite Eq. (3.32) as

$$\mu \dot{q}_i(t) + (E_p + E_s) q_i(t) - \mu E_s \varepsilon_i^c(t) + \mu f(\varepsilon_i(t)) \;=\; a_i(t)\,, \tag{3.33}$$

where, by (3.23),

$$|a_i(t)| \;\leq\; C \left(1 + |p_i(t)| + \sqrt{\dot{y}_n(t)\, g(\dot{y}_n(t))}\right)\,. \tag{3.34}$$

Equation (3.6) can be written in the form

$$\mu^2 \mu_c \dot{\varepsilon}_i^c(t) - \mu E_s q_i(t) + \mu^2 E_s \varepsilon_i^c(t) \;=\; b_i(t) \tag{3.35}$$

with (cf. Lemma 2.1)

$$|b_i(t)| \;\leq\; C\,(1 + |p_i(t)|)\,. \tag{3.36}$$

We now test (3.33) by $q_i(t)$ and (3.35) by $\varepsilon_i^c(t)$. By Hypothesis 1.1 (i) we have $f(\varepsilon_i(t))\, q_i(t) \geq 0$ whenever $\varepsilon_i(t)\, q_i(t) \geq 0$. Hence, putting $\bar{f}(z) = \max\{|f(\varrho z/\mu)|, |f(-\varrho z/\mu)|\}$ for $z \in \mathbb{R}$ we have

$$-f(\varepsilon_i(t))\, q_i(t) \;\leq\; \varrho \bar{f}(p_i(t))\, |p_i(t)| \qquad \forall t \geq 0\,. \tag{3.37}$$

Using the fact that the matrix $A = \begin{pmatrix} E_p + E_s & -\mu E_s \\ -\mu E_s & \mu^2 E_s \end{pmatrix}$ is symmetric and positive definite, we obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{\mu}{2} q_i^2(t) + \frac{\mu^2 \mu_c}{2} (\varepsilon_i^c)^2(t)\right) + \lambda \left(q_i^2(t) + (\varepsilon_i^c)^2(t)\right) \\ \leq\; -\mu\, f(\varepsilon_i(t))\, q_i(t) + |q_i(t)|\, |a_i(t)| + |\varepsilon_i^c(t)|\, |b_i(t)|\,, \end{aligned} \tag{3.38}$$

where $\lambda > 0$ is the smallest eigenvalue of A. Using (3.34), (3.36) we thus obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{\mu}{2} q_i^2(t) + \frac{\mu^2 \mu_c}{2} (\varepsilon_i^c)^2(t)\right) + c \left(q_i^2(t) + (\varepsilon_i^c)^2(t)\right) \\ \leq\; C\left(1 + \bar{f}(p_i(t))\, |p_i(t)| + p_i^2(t) + \dot{y}_n(t)\, g(\dot{y}_n(t))\right)\,. \end{aligned} \tag{3.39}$$

From (3.30) – (3.31) it follows that

$$p_i^2(t) \;\leq\; \left(\frac{1}{n} \sum_{j=1}^{n-1} |\dot{y}_j(t)|\right)^2 \;\leq\; \frac{1}{n} \sum_{j=1}^{n-1} \dot{y}_j^2(t) \;\leq\; C \left(1 + \frac{1}{n} \sum_{j=1}^{n} \left|\varepsilon_j^c\right|_{[0,t]}\right)\,, \tag{3.40}$$

hence, by (1.9) and (3.37), for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$\bar{f}(p_i(t))\, |p_i(t)| \;\leq\; C_\delta + \delta \left(\frac{1}{n} \sum_{j=1}^{n} \left|\varepsilon_j^c\right|_{[0,t]}\right)^2 \;\leq\; C_\delta + \frac{\delta}{n} \sum_{j=1}^{n} \left|\varepsilon_j^c\right|_{[0,t]}^2\,. \tag{3.41}$$

Putting

$$\mathcal{F}_i(t) = \mathcal{E}(t) + c \left(\frac{\mu}{2} q_i^2(t) + \frac{\mu^2 \mu_c}{2} (\varepsilon_i^c)^2(t)\right) \tag{3.42}$$

for $c > 0$ sufficiently small, we obtain from (3.28), (3.39) – (3.40) that

$$\dot{\mathcal{F}}_i(t) + c\mathcal{F}_i(t) \ \le \ C\left(1 + \frac{1}{n}\sum_{j=1}^{n}\left|\varepsilon_j^c\right|_{[0,t]} + C_\delta + \frac{\delta}{n}\sum_{j=1}^{n}\left|\varepsilon_j^c\right|_{[0,t]}^2\right) \tag{3.43}$$

with $\mathcal{F}_i(0) \le \mathcal{E}(0) + C(|y^1|_\infty^2 + |y_x^0|_\infty^2 + |\varepsilon^{oc}|_\infty^2) \le C$. This yields

$$\mathcal{F}_i(t) \ \le \ C\left(1 + \frac{1}{n}\sum_{j=1}^{n}\left|\varepsilon_j^c\right|_{[0,t]} + C_\delta + \frac{\delta}{n}\sum_{j=1}^{n}\left|\varepsilon_j^c\right|_{[0,t]}^2\right) \tag{3.44}$$

and, in particular,

$$(\varepsilon_i^c)^2(t) \ \le \ C\left(1 + \frac{1}{n}\sum_{j=1}^{n}\left|\varepsilon_j^c\right|_{[0,t]} + C_\delta + \frac{\delta}{n}\sum_{j=1}^{n}\left|\varepsilon_j^c\right|_{[0,t]}^2\right) \quad \forall i = 1,\ldots,n \ \ \forall t \ge 0\,. \tag{3.45}$$

Choosing $\delta > 0$ sufficiently small, we conclude that $\frac{1}{n}\sum_{j=1}^{n}\left|\varepsilon_j^c\right|_{[0,t]}^2 \le C$, hence

$$\frac{1}{n}\sum_{j=1}^{n-1}\dot{y}_j^2(t) + \max_{j=1,\ldots,n}\left|\varepsilon_j^c\right|_{[0,t]} + \max_{j=1,\ldots,n}\left|\varepsilon_j\right|_{[0,t]} \ \le \ C \quad \forall t \ge 0\,. \tag{3.46}$$

Finally, from (3.5) – (3.8) we obtain that

$$\max_{j=1,\ldots,n}\left|\dot{\varepsilon}_j^c\right|_{[0,t]} + \max_{j=1,\ldots,n}\left|\dot{\tau}_j^c\right|_{[0,t]} + \max_{j=1,\ldots,n}\left|\dot{k}_j^c\right|_{[0,t]} \ \le \ C \quad \forall t \ge 0\,. \tag{3.47}$$

Estimate 4. Test (3.1) by $-(\dot{\varepsilon}_{j+1} - \dot{\varepsilon}_j)$ and sum over $j = 1,\ldots,n-1$. We obtain

$$\frac{d}{dt}\left(\frac{1}{n}\sum_{j=1}^{n}\frac{\varrho}{2}\dot{\varepsilon}_j^2\right) - \varrho\ddot{y}_n\dot{\varepsilon}_n + \mu n\sum_{j=1}^{n-1}(\dot{\varepsilon}_{j+1} - \dot{\varepsilon}_j)^2 + n\sum_{j=1}^{n-1}(\sigma_{j+1} - \sigma_j)(\dot{\varepsilon}_{j+1} - \dot{\varepsilon}_j) \ = \ 0\,, \tag{3.48}$$

where

$$\begin{aligned}
&\mu(\dot{\varepsilon}_{j+1} - \dot{\varepsilon}_j)^2 + (\sigma_{j+1} - \sigma_j)(\dot{\varepsilon}_{j+1} - \dot{\varepsilon}_j) \qquad (3.49)\\
&\quad= \ \frac{d}{dt}\left(\frac{E_p}{2}(\varepsilon_{j+1} - \varepsilon_j)^2 + \frac{E_s}{2}\left((\varepsilon_{j+1} - \varepsilon_j) - (\varepsilon_{j+1}^c - \varepsilon_j^c)\right)^2\right)\\
&\qquad + \mu(\dot{\varepsilon}_{j+1} - \dot{\varepsilon}_j)^2 + \mu_c(\dot{\varepsilon}_{j+1}^c - \dot{\varepsilon}_j^c)^2 + (f(\varepsilon_{j+1}) - f(\varepsilon_j))(\dot{\varepsilon}_{j+1} - \dot{\varepsilon}_j) + (\tau_{j+1}^c - \tau_j^c)(\dot{\varepsilon}_{j+1}^c - \dot{\varepsilon}_j^c)\\
&\quad\ge \ \frac{d}{dt}\left(\frac{E_p}{2}(\varepsilon_{j+1} - \varepsilon_j)^2 + \frac{E_s}{2}\left((\varepsilon_{j+1} - \varepsilon_j) - (\varepsilon_{j+1}^c - \varepsilon_j^c)\right)^2\right)\\
&\qquad + \frac{\mu}{2}(\dot{\varepsilon}_{j+1} - \dot{\varepsilon}_j)^2 + \frac{\mu_c}{2}(\dot{\varepsilon}_{j+1}^c - \dot{\varepsilon}_j^c)^2 - C\left((\varepsilon_{j+1} - \varepsilon_j)^2 + (\tau_{j+1}^c - \tau_j^c)^2\right)\,.
\end{aligned}$$

To estimate the term $-\varrho\ddot{y}_n\dot{\varepsilon}_n$, we put $G(z) = \int_0^z g(s)\,ds$ for $z \in \mathbb{R}$, and use (3.2) to derive the identity

$$-\varrho\ddot{y}_n\dot{\varepsilon}_n \ = \ \frac{\varrho}{\mu}\frac{d}{dt}\left(G(\dot{y}_n) + (\sigma_n - \psi)\dot{y}_n\right) - \frac{\varrho}{\mu}(\dot{\sigma}_n - \dot{\psi})\dot{y}_n\,. \tag{3.50}$$

Using (3.2) once again leads to

$$\dot{\varepsilon}_n \dot{y}_n + \frac{1}{\mu} \dot{y}_n\, g(\dot{y}_n) \;=\; -\frac{1}{\mu}\,(\sigma_n - \psi)\,\dot{y}_n\,, \tag{3.51}$$

and (3.5) for $j = n$ yields

$$\begin{aligned} -\dot{\sigma}_n\,\dot{y}_n \;&=\; -(E_p + f'(\varepsilon_n) + E_s)\,\dot{\varepsilon}_n\,\dot{y}_n + E_s\,\dot{\varepsilon}^c_n\,\dot{y}_n \\ &=\; \frac{1}{\mu}\,(E_p + f'(\varepsilon_n) + E_s)\,\dot{y}_n\,g(\dot{y}_n) + \left(\frac{1}{\mu}\,(E_p + f'(\varepsilon_n) + E_s)\,(\sigma_n - \psi) + E_s\,\dot{\varepsilon}^c_n\right)\,\dot{y}_n \end{aligned} \tag{3.52}$$

Combining Eqs. (3.50) and (3.52) with (3.46) – (3.47) we finally find out that

$$-\varrho\ddot{y}_n\dot{\varepsilon}_n \;\geq\; \frac{\varrho}{\mu}\frac{d}{dt}\,(G(\dot{y}_n) + (\sigma_n - \psi)\,\dot{y}_n) + c\,\dot{y}_n\,g(\dot{y}_n) - (C + |\dot{\psi}|)\,|\dot{y}_n|\,. \tag{3.53}$$

Before integrating (3.48) from 0 to t, we need to estimate the expression

$$\frac{1}{n}\sum_{j=1}^{n}\dot{\varepsilon}_j^2(0) + n\sum_{j=1}^{n-1}\left((\varepsilon_{j+1} - \varepsilon_j)^2(0) + (\varepsilon^c_{j+1} - \varepsilon^c_j)^2(0)\right) + G(\dot{y}_n(0)) + (\sigma_n(0) - \psi(0))\,\dot{y}_n(0) \tag{3.54}$$

(up to appropriate constant coefficients) from above independently of n. We have

$$n\sum_{j=1}^{n-1}(\varepsilon_{j+1} - \varepsilon_j)^2(0) \;=\; n^3\sum_{j=1}^{n-1}\left(\int_{x_{j-1}}^{x_j}\int_{\xi}^{\xi+(1/n)} y^0_{xx}(\eta)\,d\eta\,d\xi\right)^2 \;\leq\; \int_0^1 |y^0_{xx}(\eta)|^2\,d\eta\,, \tag{3.55}$$

an estimate for $\dot{y}_n(0)$ follows from (3.17), and the other terms are straightforward. From (3.48) – (3.55) it thus follows that

$$\begin{aligned} &\frac{1}{n}\sum_{j=1}^{n}\dot{\varepsilon}_j^2(t) + n\sum_{j=1}^{n-1}\left((\varepsilon_{j+1} - \varepsilon_j)^2(t) + (\varepsilon^c_{j+1} - \varepsilon^c_j)^2(t)\right) + G(\dot{y}_n(t)) \\ &\quad + \int_0^t \left(n\sum_{j=1}^{n-1}\left((\dot{\varepsilon}_{j+1} - \dot{\varepsilon}_j)^2 + (\dot{\varepsilon}^c_{j+1} - \dot{\varepsilon}^c_j)^2\right) + \dot{y}_n\,g(\dot{y}_n)\right)(s)\,ds \\ &\quad \leq\; C\left((1 + |\dot{\psi}(t)|)|\dot{y}_n(t)| + \int_0^t\left(n\sum_{j=1}^{n-1}\left((\varepsilon_{j+1} - \varepsilon_j)^2 + (\tau^c_{j+1} - \tau^c_j)^2\right) + |\dot{y}_n|\right)(s)\,ds\right)\,. \end{aligned} \tag{3.56}$$

The right-hand side of (3.56) can be estimated using (3.16) for $i = n$ and the inequalities $G(\dot{y}_n(t)) \geq 0$, $\dot{y}_n(t)\,g(\dot{y}_n(t)) \geq 0$. Lemma 2.2 together with the standard Gronwall argument enable us to conclude that for every $T > 0$ there exists $K(T) > 0$ independent of n such that for all $t \in [0, T]$ we have

$$\begin{aligned} &\frac{1}{n}\sum_{j=1}^{n}\dot{\varepsilon}_j^2(t) + n\sum_{j=1}^{n-1}\left((\varepsilon_{j+1} - \varepsilon_j)^2(t) + (\varepsilon^c_{j+1} - \varepsilon^c_j)^2(t)\right) \\ &\quad + \int_0^t \left(n\sum_{j=1}^{n-1}\left((\dot{\varepsilon}_{j+1} - \dot{\varepsilon}_j)^2(s) + (\dot{\varepsilon}^c_{j+1} - \dot{\varepsilon}^c_j)^2(s)\right)\,ds\right) \;\leq\; K(T) \end{aligned} \tag{3.57}$$

As a consequence of (3.57) we have

$$n \sum_{j=1}^{n-1} |\varepsilon_{j+1} - \varepsilon_j|^2_{[0,T]} \ \le \ K(T)\,, \quad n \sum_{j=1}^{n-1} \int_0^T (\dot\sigma_{j+1} - \dot\sigma_j)^2(t)\, dt \ \le \ K(T)\,, \tag{3.58}$$

and Lemma 2.2 entails

$$n \sum_{j=1}^{n-1} \left(\left|\tau^c_{j+1} - \tau^c_j\right|^2_{[0,T]} + \left|k^c_{j+1} - k^c_j\right|^2_{[0,T]} \right) \ \le \ K(T)\,, \tag{3.59}$$

$$n \sum_{j=1}^{n-1} \left(\left|\dot\varepsilon^c_{j+1} - \dot\varepsilon^c_j\right|^2_{[0,T]} + \left|\dot\tau^c_{j+1} - \dot\tau^c_j\right|^2_{[0,T]} + \left|\dot k^c_{j+1} - \dot k^c_j\right|^2_{[0,T]} \right) \ \le \ K(T). \tag{3.60}$$

From Eq. (3.1) we finally obtain

$$\frac{1}{n} \sum_{j=1}^{n-1} \int_0^T \ddot y_j^2(t)\, dt \ \le \ K(T)\,. \tag{3.61}$$

Proof of Theorem 1.2. We fix $T > 0$, and for $n \in \mathbb{N}$ we construct the interpolates for $x \in [x_{j-1}, x_j[\ ,\ j = 1, \dots, n$, and $t \in [0,T]$ by the formula

$$\bar y^{(n)}(x,t) \ = \ y_{j-1}(t)\,, \tag{3.62}$$

$$\hat y^{(n)}(x,t) \ = \ y_{j-1}(t) + (x - x_j)\, \varepsilon_j(t) \tag{3.63}$$

with continuous extension to $x = 1$. For the other quantities occurring in Eqs. (1.1) – (1.6) we use the recipe

$$\bar w^{(n)}(x,t) \ = \ w_j(t)\,, \tag{3.64}$$

$$\hat w^{(n)}(x,t) \ = \ w_{j-1}(t) + n(x - x_j)\,(w_j(t) - w_{j-1}(t)) \tag{3.65}$$

with the convention $w_0 = w_1$, where w stands for any item on the list $\{\varepsilon, \sigma, \varepsilon^c, \tau^c, k^c\}$. Note that for all $(x,t) \in Q_T :=]0,1[\times]0,T[$ and for each of those "w" we have

$$\begin{aligned} |\bar w^{(n)}(x,t) - \hat w^{(n)}(x,t)|^2 \ &\le \ \max_j |w_j(t) - w_{j-1}(t)|^2 \le \sum_{j=1}^{n} |w_j(t) - w_{j-1}(t)|^2 \\ &\le \ \frac{1}{n} \int_0^1 |\hat w_x^{(n)}|^2(x,t)\, dx\,. \end{aligned} \tag{3.66}$$

The estimates (3.46) – (3.47) and (3.57) – (3.61) imply that

$\bar y^{(n)}, \bar y_t^{(n)}, \bar\varepsilon^{(n)}, \bar\varepsilon^{c(n)}, \bar\sigma^{(n)}, \bar\tau^{c(n)}, \bar k^{c(n)}, \bar\varepsilon_t^{c(n)}, \bar\tau_t^{c(n)}, \bar k_t^{c(n)}$ are uniformly bounded in $L^\infty(Q)$, (3.67)

$\bar y_{tt}^{(n)}, \hat\varepsilon_{xt}^{(n)}, \hat\sigma_{xt}^{(n)}$ are uniformly bounded in $L^2(Q_T)$, (3.68)

$\hat\varepsilon_{xt}^{c(n)}, \hat\tau_{xt}^{c(n)}, \hat k_{xt}^{c(n)}$ are uniformly bounded in $L^2(0,1;\, L^\infty(0,T))$. (3.69)

The approximations have been chosen so as to satisfy a. e. in Q the system

$$
\begin{aligned}
\varrho \bar{y}_{tt}^{(n)} &= (\mu \hat{\varepsilon}_t^{(n)} + \hat{\sigma}^{(n)})_x \,, && (3.70)\\
\bar{\varepsilon}^{(n)} &= \hat{y}_x^{(n)} \,, && (3.71)\\
\bar{\sigma}^{(n)} &= E_p \bar{\varepsilon}^{(n)} + f(\bar{\varepsilon}^{(n)}) + E_s(\bar{\varepsilon}^{(n)} - \bar{\varepsilon}^{c(n)}) \,, && (3.72)\\
\mu_c \bar{\varepsilon}_t^{c(n)} &= E_s(\bar{\varepsilon}^{(n)} - \bar{\varepsilon}^{c(n)}) - \bar{\tau}^{c(n)} \;,\,, && (3.73)\\
\bar{\tau}_t^{c(n)} &= \bar{k}^{c(n)} \bar{\varepsilon}_t^{c(n)} - (\alpha |\bar{\varepsilon}_t^{c(n)}| + |\bar{u}^{(n)}|) \bar{\tau}^{c(n)} + \sigma_0 (\bar{u}^{(n)})^+ \,, && (3.74)\\
\bar{k}_t^{c(n)} &= -(\alpha |\bar{\varepsilon}_t^{c(n)}| + |\bar{u}^{(n)}|) \bar{k}^{c(n)} + k_0 (\bar{u}^{(n)})^+ \,, && (3.75)
\end{aligned}
$$

where $\bar{u}^{(n)}(x,t) = u_j$ for $x \in [x_{j-1}, x_j[\,$, $j = 1, \ldots, n$, and $t \geq 0$, as well as the boundary conditions

$$\hat{y}^{(n)}(0,t) = 0 \,, \quad (\mu \hat{\varepsilon}_t^{(n)} + \hat{\sigma}^{(n)} + g(\hat{y}_t^{(n)}))(1,t) \;=\; \psi(t) \,. \tag{3.76}$$

By the usual compactness argument based on the estimates (3.66) – (3.69) and Sobolev's embedding theorems, we obtain the existence of a solution on Q_T with the required regularity by selecting a subsequence, if necessary, and passing to the limit as $n \to \infty$. We also note that on the fixed interval $[0,T]$, the values of $k^{c(n)}(x,t)$ are bounded from below by a constant $c(T)$, so that we may pass to the uniform limit also in the term $\tau^{c(n)}/k^{c(n)}$. The first boundary condition is preserved under uniform limit, the second one is verified when passing to the limit in the identity

$$
\begin{aligned}
&\int_0^T \int_0^1 \left(\mu \hat{\varepsilon}_t^{(n)} + \hat{\sigma}^{(n)} + g(\hat{y}_t^{(n)}) \right)(x,t)\, a'(x)\, b(t)\, dx\, dt && (3.77)\\
&\qquad = -\int_0^T \int_0^1 \left(\mu \hat{\varepsilon}_{xt}^{(n)} + \hat{\sigma}_x^{(n)} + g'(\hat{y}_t^{(n)}) \bar{\varepsilon}_t^{(n)} \right)(x,t)\, a(x)\, b(t)\, dx\, dt
\end{aligned}
$$

for every $a \in W^{1,2}(0,1)$ and $b \in L^2(0,T)$ such that $a(0) = 0$. The proof of Theorem 1.2 will be complete if we prove that the solution is unique for every $T > 0$; it can then be extended to the whole Q. Let y_1, y_2 be two solutions. We test the difference of Eqs. (1.1) written for y_1 and y_2 by $(y_1 - y_2)_t$. The uniqueness proof is again based on a straightforward use of the Gronwall lemma combined with Lemma 2.2 and we omit the details here. ■

Remark 3.1 As mentioned in Section 1, the growth conditions are not necessary for the existence of global solutions. Indeed, Estimate 1 is independent of the growth of f and g, and (3.17) yields for all $T > 0$ and $t \in [0,T]$ that

$$\frac{1}{n}\sum_{j=1}^{n-1} \dot{y}_j^2(t) + \frac{1}{n}\sum_{j=1}^{n} \left(\varepsilon_j^2 + F(\varepsilon_j) + (\varepsilon_j^c)^2\right)(t) \;+\; \int_0^T \frac{1}{n}\sum_{j=1}^{n} \left(\dot{\varepsilon}_j^2 + (\dot{\varepsilon}_j^c)^2\right)(\tau)\, d\tau \;\leq\; K(T) \,. \tag{3.78}$$

In order to get the term $g(\dot{y}_n)$ in Estimates 2 and 3 under control, we introduce auxiliary functions

$$P_n(t) \;=\; \frac{1}{n}\sum_{j=1}^{n-1} p_j(t) \,, \qquad S_n(t) \;=\; \frac{1}{n}\sum_{j=1}^{n} \sigma_j(t) \,. \tag{3.79}$$

Summing up Eqs. (3.32) over $i = 1, \ldots, n$ and dividing by n we obtain

$$\mu \dot{y}_n(t) + \varrho \dot{P}_n(t) + S_n(t) + g(\dot{y}_n(t)) \;=\; \psi(t) \,. \tag{3.80}$$

This enables us to rewrite (3.18) in the form

$$\frac{d}{dt}\left(\frac{1}{n}\sum_{j=1}^{n-1}\varrho\dot y_j(t)\,y_j(t)+\frac{1}{n}\sum_{j=1}^{n}\frac{\mu}{2}\varepsilon_j^2(t)-\frac{\mu}{2}y_n^2(t)-\varrho P_n(t)\,y_n(t)\right)+\frac{1}{n}\sum_{j=1}^{n}\sigma_j(t)\,\varepsilon_j(t) \tag{3.81}$$
$$= \frac{1}{n}\sum_{j=1}^{n-1}\varrho\dot y_j^2(t)+y_n(t)\,S_n(t)-\varrho P_n(t)\,\dot y_n(t)\,.$$

By (3.78), (3.16) and (3.22), we have for $t\in[0,T]$ that $|P_n(t)|\le K(T)$, $|y_n(t)|\le K(T)$, $\int_0^T|\dot y_n(t)|\,dt\le K(T)$, and it remains to estimate the term involving $f(\varepsilon_j)$ in $S_n(t)$ without using the growth condition (1.9). Integrating (3.81) from 0 to T and using (3.19), (3.78), we obtain

$$\int_0^T\frac{1}{n}\sum_{j=1}^{n}\varepsilon_j(t)\,f(\varepsilon_j(t))\,dt\ \le\ K^*(T)\left(1+\int_0^T\frac{1}{n}\sum_{j=1}^{n}|f(\varepsilon_j(t))|\,dt\right), \tag{3.82}$$

where $K^*(T)$ is a constant independent of n which we keep fixed from now on. For $t\in[0,T]$ set $J=\{1,\dots,n\}$, $J_n(t)=\{j\in J\,;\,|\varepsilon_j(t)|\ge 2K^*(T)\}$. We have

$$\begin{aligned}2K^*(T)\int_0^T\frac{1}{n}\sum_{j\in J_n(t)}|f(\varepsilon_j(t))|\,dt\ &\le\ \int_0^T\frac{1}{n}\sum_{j=1}^{n}\varepsilon_j(t)\,f(\varepsilon_j(t))\,dt\\ &\le\ K^*(T)\left(1+\int_0^T\frac{1}{n}\sum_{j=1}^{n}|f(\varepsilon_j(t))|\,dt\right)\\ &\le\ K(T)+K^*(T)\int_0^T\frac{1}{n}\sum_{j\in J_n(t)}|f(\varepsilon_j(t))|\,dt\,,\end{aligned} \tag{3.83}$$

hence

$$\int_0^T|S_n(t)|\,dt\ \le\ K(T)\,. \tag{3.84}$$

We now use (3.80) again to modify Estimate 3. Introducing new variables

$$r_i(t)\ =\ q_i(t)-\mu y_n(t)-\varrho P_n(t)\ =\ \mu(\varepsilon_i-y_n)+\varrho(p_i-P_n)\,, \tag{3.85}$$

we rewrite (3.32)–(3.33) in the form

$$\mu\dot r_i(t)+(E_p+E_s)r_i(t)-\mu E_s\varepsilon_i^c(t)+\mu f(\varepsilon_i(t))\ =\ \tilde a_i(t)\,, \tag{3.86}$$

where

$$\tilde a_i(t)\ =\ \mu S_n(t)+\varrho p_i(t)-(E_p+E_s)(\mu y_n(t)+\varrho P_n(t))\,. \tag{3.87}$$

Equation (3.35) now reads

$$\mu^2\mu_c\dot\varepsilon_i^c(t)-\mu E_s r_i(t)+\mu^2E_s\varepsilon_i^c(t)\ =\ \tilde b_i(t) \tag{3.88}$$

with $|\tilde b_i(t)|\ \le\ K(T)$. We test (3.86) by $r_i(t)$ and (3.88) by $\varepsilon_i^c(t)$. Arguing as in (3.37) ($r_i(t)$ and $\varepsilon_i(t)$ have again different signs only if they are both "small"), we obtain a counterpart of (3.38) in the form

$$\frac{d}{dt}\left(\frac{\mu}{2}r_i^2+\frac{\mu^2\mu_c}{2}(\varepsilon_i^c)^2\right)(t)+\lambda\left(r_i^2+(\varepsilon_i^c)^2\right)(t)\ \le\ K(T)(1+|\varepsilon_i^c(t)|+|r_i(t)|\,(1+|S_n(t)|))\,. \tag{3.89}$$

The L^1-bound for S_n in (3.84) is sufficient to conclude that $|\varepsilon_i|$, $|\varepsilon_i^c|$ are bounded in $L^\infty(0,T)$ uniformly with respect to i and n, and the assertion follows as in the proof of Theorem 1.2.

4 Proof of Theorem 1.3: Asymptotic behaviour

In this section, we prove Theorem 1.3. The fact that under vanishing external forcing $u \equiv 0$ and $\psi \equiv 0$, the solution asymptotically converges to an equilibrium, is not surprising. However, the set of possible equilibria is a continuum, and the problem has to be handled properly. For the moment, only the estimate (3.67) is independent of the time interval, and more estimates will be needed. We will proceed formally, manipulating directly with Eqs. (1.1) – (1.8), keeping in mind that for instance the time differentiation in Estimates 7 and 8 below is rigorous only for the discrete system (3.1) – (3.10), and an underlying limit passage as $n \to \infty$ is tacitly assumed.

Invoking the identity (2.6) and putting

$$\varphi^0(x) \;=\; \frac{\tau^{oc}(x)}{k^{oc}(x)} - \varepsilon^{oc}(x) \qquad \text{for} \quad x \in [0,1] \tag{4.1}$$

we may eliminate τ^c from the system and rewrite (1.1) – (1.6) in the form

$$\begin{aligned} \varrho y_{tt} &= (\mu\varepsilon_t + \sigma)_x\,, && (4.2)\\ \varepsilon &= y_x\,, && (4.3)\\ \sigma &= E_p\varepsilon + f(\varepsilon) + E_s(\varepsilon - \varepsilon^c)\,, && (4.4)\\ \mu_c\varepsilon^c_t &= E_s(\varepsilon - \varepsilon^c) - k^c(\varepsilon^c - \varphi^0)\,, && (4.5)\\ k^c_t &= -\alpha|\varepsilon^c_t|k^c\,. && (4.6) \end{aligned}$$

For all $x \in [0,1]$, the function $k^c(x,\cdot)$ is non-decreasing, hence there exists the limit

$$k^c_\infty(x) \;=\; \lim_{t\to\infty} k^c(x,t) \;\geq\; 0\,, \qquad k^c_\infty \in L^\infty(0,1)\,. \tag{4.7}$$

The expected limit values $\varepsilon_\infty(x), \varepsilon^c_\infty(x)$ for ε and ε^c, respectively, must for every $x \in [0,1]$ satisfy the system

$$\begin{aligned} E_p\varepsilon_\infty + f(\varepsilon_\infty) + E_s(\varepsilon_\infty - \varepsilon^c_\infty) &= 0\,, && (4.8)\\ E_s(\varepsilon_\infty - \varepsilon^c_\infty) - k^c_\infty(\varepsilon^c_\infty - \varphi^0) &= 0\,, && (4.9) \end{aligned}$$

which obviously admits a unique solution $\varepsilon_\infty, \varepsilon^c_\infty \in L^\infty(0,1)$. More precisely, we have

$$E_p\varepsilon_\infty + f(\varepsilon_\infty) + \frac{E_s k^c_\infty}{E_s + k^c_\infty}(\varepsilon_\infty - \varphi^0) \;=\; 0\,, \tag{4.10}$$

hence

$$|\varepsilon_\infty| \;\leq\; |\varphi^0| \quad \text{a.e.}\,, \qquad |\varepsilon_\infty - \varphi^0| \;\leq\; |\varphi^0| \quad \text{a.e.} \tag{4.11}$$

From the identity

$$\varepsilon^c_\infty - \varphi^0 \;=\; \frac{E_s}{E_s + k^c_\infty}(\varepsilon_\infty - \varphi^0) \tag{4.12}$$

we obtain also

$$|\varepsilon^c_\infty - \varphi^0| \;\leq\; |\varphi^0| \quad \text{a.e.} \tag{4.13}$$

Putting $\tilde\varepsilon(x,t)=\varepsilon(x,t)-\varepsilon_\infty(x)$, $\tilde\varepsilon^c(x,t)=\varepsilon^c(x,t)-\varepsilon^c_\infty(x)$, $\tilde\varphi^0(x)=\varphi^0(x)-\varepsilon^c_\infty(x)$, $\tilde y(x,t)=\int_0^x\tilde\varepsilon(z,t)\,dz$, we may rewrite the system (4.2) – (4.6) in the form

$$\begin{aligned}
\varrho\tilde y_{tt} &= (\mu\tilde\varepsilon_t+\sigma)_x\,, &(4.14)\\
\tilde\varepsilon &= \tilde y_x\,, &(4.15)\\
\sigma &= E_p\tilde\varepsilon+f(\varepsilon)-f(\varepsilon_\infty)+E_s(\tilde\varepsilon-\tilde\varepsilon^c)\,, &(4.16)\\
\mu_c\tilde\varepsilon^c_t &= E_s(\tilde\varepsilon-\tilde\varepsilon^c)-k^c_\infty\,\tilde\varepsilon^c-(k^c-k^c_\infty)(\tilde\varepsilon^c-\tilde\varphi^0)\,, &(4.17)\\
k^c_t &= -\alpha|\tilde\varepsilon^c_t|k^c\,, &(4.18)
\end{aligned}$$

with boundary conditions

$$\tilde y(0,t)=0\,,\ (\mu\tilde\varepsilon_t+\sigma+g(\tilde y_t))(1,t)=0, \tag{4.19}$$

and with accordingly modified initial conditions (note that the $W^{1,2}$-regularity is lost – the initial conditions for $\tilde\varepsilon$ and $\tilde\varepsilon^c$ now belong to $L^\infty(0,1)$ only).

Estimate 5. Testing (4.14) by $\tilde y_t$ we obtain similarly as in (3.14) – (3.15) that

$$\begin{aligned}
&\frac{d}{dt}\int_0^1\Big(\frac{\varrho}{2}\tilde y_t^2+\frac{E_p}{2}\tilde\varepsilon^2+F(\varepsilon)-F(\varepsilon_\infty)-\tilde\varepsilon\,f(\varepsilon_\infty)+\frac{E_s}{2}(\tilde\varepsilon-\tilde\varepsilon^c)^2+\frac{k^c_\infty}{2}(\tilde\varepsilon^c)^2 \qquad (4.20)\\
&\qquad+\frac{1}{2}(k^c-k^c_\infty)\,(\tilde\varepsilon^c-\tilde\varphi^0)^2\Big)(x,t)\,dx\\
&\qquad+\int_0^1\Big(\mu\tilde\varepsilon_t^2+\mu_c(\tilde\varepsilon^c_t)^2+\frac{\alpha}{2}k^c|\tilde\varepsilon^c_t|(\tilde\varepsilon^c-\tilde\varphi^0)^2\Big)\,(x,t)\,dx+(\tilde y_t\,g(\tilde y_t))(1,t)\ =\ 0\,.
\end{aligned}$$

Estimate 6. We now test (4.14) by $\tilde y$. This yields

$$\begin{aligned}
&\frac{d}{dt}\int_0^1\Big(\varrho\,\tilde y_t\,\tilde y+\frac{\mu}{2}\tilde\varepsilon^2+\frac{\mu_c}{2}(\tilde\varepsilon^c)^2\Big)(x,t)\,dx \qquad (4.21)\\
&\qquad+\int_0^1\Big(E_p\tilde\varepsilon^2+(f(\varepsilon)-f(\varepsilon_\infty))\,\tilde\varepsilon+E_s\,(\tilde\varepsilon-\tilde\varepsilon^c)^2+\frac{1}{2}(k^c+k^c_\infty)(\tilde\varepsilon^c)^2\\
&\qquad+\frac{1}{2}(k^c-k^c_\infty)\,(\tilde\varepsilon^c-\tilde\varphi^0)^2\Big)\,(x,t)\,dx+(\tilde y\,g(\tilde y_t))(1,t)\\
&=\int_0^1\Big(\varrho\,\tilde y_t^2+\frac{1}{2}(k^c-k^c_\infty)\,(\tilde\varphi^0)^2\Big)(x,t)\,dx\,.
\end{aligned}$$

We have for (almost) all $t>0$ that

$$\begin{aligned}
\int_0^1\tilde y_t^2(x,t)\,dx &\le \int_0^1\tilde\varepsilon_t^2(x,t)\,dx\,, &(4.22)\\
|\tilde y(1,t)|^2 &\le \int_0^1\tilde\varepsilon^2(x,t)\,dx\,, &(4.23)\\
g^2(\tilde y_t)(1,t) &\le C\,(\tilde y_t\,g(\tilde y_t))(1,t)\,. &(4.24)
\end{aligned}$$

For some small $c>0$ we now put

$$\begin{aligned}
\mathcal{E}(t)\ =\ &\int_0^1\Big(\frac{\varrho}{2}\tilde y_t^2+c\varrho\tilde y_t\,\tilde y+\frac{E_p+c\mu}{2}\tilde\varepsilon^2+F(\varepsilon)-F(\varepsilon_\infty)-\tilde\varepsilon\,f(\varepsilon_\infty)+\frac{E_s}{2}(\tilde\varepsilon-\tilde\varepsilon^c)^2 \quad (4.25)\\
&+\frac{k^c_\infty+c\mu_c}{2}\,(\tilde\varepsilon^c)^2+\frac{1}{2}(k^c-k^c_\infty)\,(\tilde\varepsilon^c-\tilde\varphi^0)^2\Big)(x,t)\,dx\,,
\end{aligned}$$

and combining (4.20) with (4.21) we obtain (for a possibly smaller $c > 0$)

$$\dot{\mathcal{E}}(t) + c\,\mathcal{E}(t) \ \le\ C \int_0^1 (k^c - k^c_\infty)(x,t)\,dx\,. \tag{4.26}$$

The right-hand side of (4.26) tends to 0 as $t \to \infty$, hence the convergences (1.12), (1.13) are verified, while (1.14) follows from (4.17) – (4.18).

The remaining part of Theorem 1.3 is obtained by differentiating formally Eq. (4.2) as mentioned at the beginning of this section.

Estimate 7. Differentiate (4.2) by t and test by y_{tt}. This yields

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \Big(\frac{\varrho}{2}\,y_{tt}^2 + \frac{E_p}{2}\,\varepsilon_t^2 + \frac{E_s}{2}(\varepsilon_t - \varepsilon_t^c)^2 + \frac{k^c}{2}(\varepsilon_t^c)^2\Big)(x,t)\,dx \\ &\qquad + \int_0^1 \Big(\mu\varepsilon_{tt}^2 + \mu_c(\varepsilon_{tt}^c)^2 + f'(\varepsilon)\,\varepsilon_t\,\varepsilon_{tt} + \frac{\alpha}{2}k^c|\varepsilon_t^c|^3\Big)(x,t)\,dx + (y_{tt}^2\,g'(y_t))(1,t) \\ &\quad = \int_0^1 \big(\alpha\,k^c\,(\varepsilon^c - \varphi^0)\,|\varepsilon_t^c|\,\varepsilon_{tt}^c\big)(x,t)\,dx\,. \end{aligned} \tag{4.27}$$

By virtue of (3.46), it follows from (4.27) that

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \Big(\frac{\varrho}{2}\,y_{tt}^2 + \frac{E_p}{2}\,\varepsilon_t^2 + \frac{E_s}{2}(\varepsilon_t - \varepsilon_t^c)^2 + \frac{k^c}{2}(\varepsilon_t^c)^2\Big)(x,t)\,dx \\ &\qquad + \int_0^1 \Big(\frac{\mu}{2}\varepsilon_{tt}^2 + \frac{\mu_c}{2}(\varepsilon_{tt}^c)^2 + \frac{\alpha}{2}k^c|\varepsilon_t^c|^3\Big)(x,t)\,dx + (y_{tt}^2\,g'(y_t))(1,t) \\ &\quad \le\ C\int_0^1 \big((\varepsilon_t^c)^2 + \varepsilon_t^2\big)(x,t)\,dx\,. \end{aligned} \tag{4.28}$$

This is some sort of “higher order energy balance”. The cubic dissipation term $|\varepsilon_t^c|^3$ is typical for equations with convex hysteretic constitutive laws, cf. [16].

Estimate 8. Differentiate (4.2) by t and test by y_t. We obtain

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \Big(\varrho\,y_{tt}\,y_t + \frac{\mu}{2}\,\varepsilon_t^2 + \frac{\mu_c}{2}\,(\varepsilon_t^c)^2\Big)(x,t)\,dx \\ &\qquad + \int_0^1 \big(E_p\,\varepsilon_t^2 + E_s(\varepsilon_t - \varepsilon_t^c)^2 + f'(\varepsilon)\,\varepsilon_t^2 + k^c(\varepsilon_t^c)^2\big)(x,t)\,dx + (y_{tt}\,y_t\,g'(y_t))(1,t) \\ &\quad = \int_0^1 \big(\varrho\,y_{tt}^2 + \alpha\,k^c\,(\varepsilon^c - \varphi^0)\,|\varepsilon_t^c|\,\varepsilon_t^c\big)(x,t)\,dx\,. \end{aligned} \tag{4.29}$$

For some small $c > 0$ we now put

$$\mathcal{F}(t) \ =\ \int_0^1 \Big(\frac{\varrho}{2}\,y_{tt}^2 + c\varrho\,y_{tt}\,y_t + \frac{E_p + c\mu}{2}\,\varepsilon_t^2 + \frac{E_s}{2}(\varepsilon_t - \varepsilon_t^c)^2 + \frac{k^c + c\mu_c}{2}(\varepsilon_t^c)^2\Big)(x,t)\,dx\,. \tag{4.30}$$

Using the inequalities

$$\int_0^1 y_{tt}^2(x,t)\,dx \ \le\ \int_0^1 \varepsilon_{tt}^2(x,t)\,dx\,, \tag{4.31}$$

$$(g'(y_t)\,y_t\,y_{tt})(1,t) \ \le\ C\,\Big(\int_0^1 \varepsilon_t^2(x,t)\,dx\Big)^{1/2}\,\Big(\int_0^1 \varepsilon_{tt}^2(x,t)\,dx\Big)^{1/2}\,. \tag{4.32}$$

we proceed as above to obtain that

$$\dot{\mathcal{F}}(t) + c\,\mathcal{F}(t) \;\leq\; C \int_0^1 \left((\varepsilon_t^c)^2 + \varepsilon_t^2\right)(x,t)\,dx\,. \tag{4.33}$$

Invoking (4.20), we see that the function $\Psi(t) = C\,\int_0^1 \left((\varepsilon_t^c)^2 + \varepsilon_t^2\right)(x,t)\,dx$ on the right-hand side of (4.33) belongs to $L^1(0,\infty)$, that is, $\int_0^\infty \Psi(t)\,dt < \infty$. This fact and (4.33) imply that $\lim_{t\to\infty} \mathcal{F}(t) = 0$, which completes the proof of Theorem 1.3. ■

5 Example: A mass-spring system

It is natural to ask the question if the limit stiffness k_∞^c in Theorem 1.3 may vanish or remains positive. We have no answer in the general case. In order to have an idea about what can be expected, we consider in this section a mass-spring system associated with our constitutive law as a simplification of (4.2) – (4.6). In other words, we solve the ODEs

$$\begin{aligned}
\varrho\ddot{\varepsilon} &= -(\mu\dot{\varepsilon} + \sigma)\,, && (5.1)\\
\sigma &= E_p\varepsilon + f(\varepsilon) + E_s(\varepsilon - \varepsilon^c)\,, && (5.2)\\
\mu_c\dot{\varepsilon}^c &= E_s(\varepsilon - \varepsilon^c) - k^c(\varepsilon^c - \varphi^0)\,, && (5.3)\\
\dot{k}^c &= -\alpha|\dot{\varepsilon}^c|k^c && (5.4)
\end{aligned}$$

with given constants $\varrho, \mu, \mu_c, E_p, E_s, \alpha$ which are all positive and $\varphi^0 \in \mathbb{R}$. We prescribe the initial conditions

$$\varepsilon(0) = \varepsilon^0,\ \dot{\varepsilon}(0) = \varepsilon^1,\ \varepsilon^c(0) = \varepsilon^{oc},\ k^c(0) = k^{oc} > 0\,. \tag{5.5}$$

In addition to Hypothesis 1.1 (i), we assume that the derivative f' of f is locally Lipschitz continuous. The above system actually corresponds to (3.1) – (3.8) for $n = 2$, $u \equiv 0$ and $g \equiv 0$, where $\varepsilon = \varepsilon_1 = 2y_1$ and ϱ stands for $\varrho/4$, while τ^c is eliminated as in (4.1) – (4.6). Let

$$k_\infty^c \;=\; \lim_{t\to\infty} k^c(t) \;\geq\; 0 \tag{5.6}$$

and let ε_∞ and ε_∞^c be solutions to Eqs. (4.8) – (4.9). Repeating the argument of Estimates 5 and 6 in Section 4, we obtain that

$$\lim_{t\to\infty} \varepsilon(t) = \varepsilon_\infty\,,\ \lim_{t\to\infty} \varepsilon^c(t) = \varepsilon_\infty^c\,,\ \lim_{t\to\infty} \Big(|\sigma(t)| + |\dot{\varepsilon}(t)| + |\dot{\varepsilon}^c(t)| + |\dot{k}^c(t)|\Big) = 0\,. \tag{5.7}$$

The counterparts of identities (4.27) and (4.29) read

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{\varrho}{2}\ddot{\varepsilon}^2 + \frac{E_p}{2}\dot{\varepsilon}^2 + \frac{E_s}{2}(\dot{\varepsilon} - \dot{\varepsilon}^c)^2 + \frac{k^c}{2}(\dot{\varepsilon}^c)^2\Big)(t) + \Big(\mu\ddot{\varepsilon}^2 + \mu_c(\ddot{\varepsilon}^c)^2 + f'(\varepsilon)\,\dot{\varepsilon}\,\ddot{\varepsilon} + \frac{\alpha}{2}\,k^c|\dot{\varepsilon}^c|^3\Big)(t) \qquad (5.8)\\
&\quad= \;\left(\alpha\,k^c(\varepsilon^c - \varphi^0)|\dot{\varepsilon}^c|\,\ddot{\varepsilon}^c\right)(t)\,,
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dt}\Big(\varrho\,\dot{\varepsilon}\,\ddot{\varepsilon} + \frac{\mu}{2}\dot{\varepsilon}^2 + \frac{\mu_c}{2}(\dot{\varepsilon}^c)^2\Big)(t) + \Big(E_p\dot{\varepsilon}^2 + E_s(\dot{\varepsilon} - \dot{\varepsilon}^c)^2 + f'(\varepsilon)\,\dot{\varepsilon}^2 + k^c(\dot{\varepsilon}^c)^2\Big)(t) \qquad (5.9)\\
&\quad= \;\left(\varrho\,\ddot{\varepsilon}^2 + \alpha\,k^c(\varepsilon^c - \varphi^0)|\dot{\varepsilon}^c|\,\dot{\varepsilon}^c\right)(t)\,.
\end{aligned}$$

We now use the integration-by-parts formulæ

$$f'(\varepsilon)\,\dot\varepsilon\,\ddot\varepsilon \;=\; \frac{d}{dt}\left(\frac{1}{2}f'(\varepsilon)\,\dot\varepsilon^2\right) - \frac{1}{2}f''(\varepsilon)\,\dot\varepsilon^3\,, \tag{5.10}$$

$$\alpha\, k^c(\varepsilon^c-\varphi^0)|\dot\varepsilon^c|\,\ddot\varepsilon^c \;=\; \frac{d}{dt}\left(\frac{\alpha}{2}\,k^c(\varepsilon^c-\varphi^0)|\dot\varepsilon^c|\,\dot\varepsilon^c\right) - \frac{\alpha}{2}\,k^c|\dot\varepsilon^c|^3 + \frac{\alpha^2}{2}\,k^c(\varepsilon^c-\varphi^0)\,(\dot\varepsilon^c)^3\,, \tag{5.11}$$

and and choose an appropriate (small) constant $c>0$. Putting

$$\begin{aligned}\mathcal{E}_0(t) \;=\; &\Big(\frac{\varrho}{2}\,\ddot\varepsilon^2 + c\varrho\,\dot\varepsilon\,\ddot\varepsilon + \frac{E_p+f'(\varepsilon)+c\mu}{2}\,\dot\varepsilon^2 + \frac{E_s}{2}\,(\dot\varepsilon-\dot\varepsilon^c)^2 + \frac{k^c+c\mu_c}{2}\,(\dot\varepsilon^c)^2 \\ &-\frac{\alpha}{2}\,k^c(\varepsilon^c-\varphi^0)|\dot\varepsilon^c|\,\dot\varepsilon^c\Big)(t)\end{aligned} \tag{5.12}$$

we take possibly a smaller $c>0$ and obtain

$$\dot{\mathcal{E}}_0(t) + c\,\mathcal{E}_0(t) \;\le\; C\,\left(|\dot\varepsilon(t)|^3 + |\dot\varepsilon^c(t)|^3\right) \tag{5.13}$$

for some $C>0$. We now keep the constants c and C fixed and prove the following result.

Proposition 5.1 *The limit value k^c_∞ in (5.6) is positive.*

Proof. We find $\varrho^*>0$ and $E^*>0$ such that for all $p,q,r\in\mathbb{R}$ we have

$$\varrho p^2 + 2c\varrho pq + E_p q^2 + E_s(q-r)^2 \;\ge\; \varrho^* p^2 + E^*(q^2+r^2)\,. \tag{5.14}$$

If $E^*\le \alpha k^c_\infty|\varphi^0|$, then $k^c_\infty>0$ and we are done. Assume now that $E^*>\alpha k^c_\infty|\varphi^0|$. We find $t_0>0$ sufficiently large and $\delta>0$ such that for $t\ge t_0$ we have

$$E^* - \alpha k^c(t)|\varphi^0| \;\ge\; \delta\,, \tag{5.15}$$

$$|\varepsilon^c(t)-\varphi^0| \;\le\; |\varphi^0| + \frac{1}{\alpha}\,, \tag{5.16}$$

$$|\dot\varepsilon(t)| + |\dot\varepsilon^c(t)| \;\le\; \frac{c}{4C}\delta\,, \tag{5.17}$$

where we used (5.7) and (4.13). For $t\ge t_0$ we then have

$$\begin{aligned}\mathcal{E}_0(t) \;&\ge\; \frac{\varrho^*}{2}\ddot\varepsilon^2(t) + \frac{E^*}{2}\left(\dot\varepsilon^2(t) + (\dot\varepsilon^c(t))^2\right) + \frac{k^c(t)}{2}(\dot\varepsilon^c(t))^2 - \frac{\alpha}{2}k^c(t)\left(|\varphi^0| + \frac{1}{\alpha}\right)(\dot\varepsilon^c(t))^2 \\ &\ge\; \frac{\delta}{2}\left(\dot\varepsilon^2(t) + (\dot\varepsilon^c(t))^2\right)\,,\end{aligned} \tag{5.18}$$

$$C\,\left(|\dot\varepsilon(t)|^3 + |\dot\varepsilon^c(t)|^3\right) \;\le\; \frac{c\delta}{4}\left(\dot\varepsilon^2(t) + (\dot\varepsilon^c(t))^2\right)\,. \tag{5.19}$$

From (5.13) we then obtain

$$\dot{\mathcal{E}}_0(t) + \frac{c}{2}\,\mathcal{E}_0(t) \;\le\; 0 \qquad \text{for } \; t\ge t_0\,. \tag{5.20}$$

We thus have $\mathcal{E}_0(t)\le \mathcal{E}_0(t_0)\,\mathrm{e}^{-(c/2)(t-t_0)}$, hence

$$|\dot\varepsilon^c(t)| \;\le\; \sqrt{\frac{2}{\delta}\mathcal{E}_0(t)} \;\le\; C_1\,\mathrm{e}^{-(c/4)(t-t_0)} \tag{5.21}$$

for some $C_1>0$. Integrating Eq. (5.4) yields

$$k^c_\infty \;\ge\; k^c(t_0)\,\mathrm{e}^{-4\alpha C_1/c} \;>\; 0 \tag{5.22}$$

and the proof of Proposition 5.1 is complete. ∎

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