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Asymptotic stability of continual sets of periodic solutions to systems with hysteresis

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Abstract

We consider hysteresis perturbations of the system of ODEs which has an asymptotically stable periodic solution z_*. It is proved that if the oscillation of the appropriate projection of z_* is smaller than some threshold number defined by the hysteresis nonlinearity, then the perturbed system has a continuum of periodic solutions with a rather simple structure in a vicinity of z_*. The main result is the theorem on stability of this continuum.

1 Introduction

The analysis of the long-term behavior of systems with hysteresis nonlinearities was essentially developed in the last decade on the basis of the mathematical formalism of the hysteresis operator approach [8]. This formalism allows to use classical tools of the nonlinear functional analysis and methods of the theory of differential equations for the study of models of systems with hysteresis, in particular in problems on periodic and almost periodic oscillations, cycles, stability, bifurcations, and others (for some results, open problems, and bibliography see, e.g. [2, 10, 13]).

In this paper, we consider small hysteresis perturbations of nonautonomous systems of ordinary differential equations supposing that the unperturbed system has an exponentially stable periodic solution $z_* = z_*(t)$ of the period T and restricting our analysis mainly to some neighborhood of this solution. Here one can distinguish between the two situations. The first of them, studied in [3, 4], is determined by the condition $\operatorname{osc} x_* := \max_{t',t''\in\mathbb{R}} |x_*(t') - x_*(t'')| > r_*$ or its analogs, where the function x_* is an appropriate projection of z_* and the number r_* is defined by the hysteresis nonlinearity. This condition implies that the flow in the state space of the hysteresis nonlinearity with the input x_* is a contraction for $t \geq T$. As it is proved in [3], a consequence of such a contracting property is that for any sufficiently small value of the perturbation parameter ε the perturbed system has an exponentially stable periodic solution and this solution approaches the solution z_* of the unperturbed system as $\varepsilon \to 0$. The contracting property and the threshold number r_* are well-known for various particular classes of hysteresis models (typically, r_* has a simple interpretation of some characteristic size and its definition is straightforward). Main results of [3] are formulated for systems with vector stop and play hysteresis operators. Another example of hysteresis operators, for which these results are valid, are the scalar Preisach nonlinearities and similar classes of hysteresis models. Here the contracting property becomes trivial, since the estimate $\operatorname{osc} x_* > r_*$ implies that all the outputs corresponding to the input x_* and arbitrary admissible initial states of the Preisach nonlinearity are equal on the semiaxis $t \geq T$.

In this paper, we consider the situation determined by the relation $\operatorname{osc} x_* < r_*$ (i.e., complementary to the case above). The results are formulated for systems with the Preisach hysteresis nonlinearities. It is proved that if $\operatorname{osc} x_* < r_*$, then for each sufficiently small ε the perturbed system has a continuum of periodic solutions in a small neighborhood of z_*. The main result is the theorem on the stability of this continuum.

In the proof of the existence of the continual set of periodic solutions we follow the method of [12] based on the reduction to operator equations of the periodic problem in functional spaces. These equations include some special operators acting in the space of periodic functions and describing periodic responses of the Preisach nonlinearity to periodic inputs. The method allows to observe the simple structure of the continual set of periodic solutions. It is proved that in the main situation this set consists of the disjoint subsets that depend continuously on a scalar parameter, each subset being the join of periodic solutions with the same component $z = z(t)$ in the phase variable space $\mathbb{R}^d$. Actually, continua of periodic solutions and cycles with the similar structure can be observed in various problems on systems with the Preisach and some other hysteresis terms (cf. [6, 7]); in this sense, it is hysteresis that accounts for the existence of such continua, rather than a specific type of the periodic problem considered. Particularly, here the continual set of periodic solutions arises from the isolated periodic solution z_* of the system of ODE (which has a rather general form) due to the hysteresis nature of the perturbation.

To analyze stability of the set of periodic solutions, we use constructions in the product of the phase variable space $\mathbb{R}^d$ and the state space of the hysteresis nonlinearity. The main result is that the set of initial values of the z-components of periodic solutions in $\mathbb{R}^d$ has some basin of attraction such that every periodic solution with the initial value of its z-component from this basin and with an arbitrary initial value of the component in the state space of the hysteresis nonlinearity converges to some periodic solution as t increases. The results of this type are known from [14] for an elastoplastic oscillator (the methods presented here are different from that of [14]).

The paper is organized as follows. In the next section we define the system with a hysteresis term and discuss the problem on periodic solutions. In subsection 2.1, some description and common properties of hysteresis nonlinearities are considered. Subsection 2.2 contains a more detailed discussion of the Preisach models; an interested reader can find numerous references to the literature concerning the foundations, properties, and applications of such hysteresis models in magnetism, plasticity, economics, agriculture, etc., in [2, 10, 13]. In subsection 2.3, we introduce the operators of the periodic problem which play the main role in the theorems on the existence of periodic solutions. The main results are formulated and proved in Section 3. In subsection 3.1, we recall simple facts about systems with general continuous perturbations (some proves are given in the Appendix for convenience of the reader). Theorems 1 and 2 of Subsection 3.2 present conditions for the existence of a continuum of periodic solutions for the system with the hysteresis perturbation and describe the structure of this continuum. The main result on stability of the continuum of periodic solutions is Theorem 3 of Subsection 3.3. This theorem is proved in Subsection 3.4. Subsection 3.5 contains some remarks on the results and their possible extensions.

2 Problem statement

2.1 Systems with hysteresis perturbations

Consider the system

$$\dot{z} = f(t, z, 0), \qquad z \in \mathbb{R}^d, \tag{1}$$

where dot denotes differentiation with respect to the time t, the function $f(\cdot,\cdot,\cdot)$ is continuous and $f(t,z,\xi) = f(t+T,z,\xi)$ for all $t \in \mathbb{R}_+$, $z \in \mathbb{R}^d$, $\xi \in \mathbb{R}^\ell$ with $\mathbb{R}_+ = [0,\infty)$. Everywhere below it is assumed that system (1) has a T-periodic solution z_* which is locally exponentially stable.

Let $g(\cdot) : \mathbb{R}^d \to \mathbb{R}^\ell$ be a smooth function and let $f(\cdot,\cdot,\cdot)$ be continuously differentiable in z, ξ. Consider the perturbation $\dot{z} = f(t, z, \varepsilon g(z))$ of system (1), where $0 < \varepsilon \le 1$. It is well-known that for each sufficiently small ε the system $\dot{z} = f(t, z, \varepsilon g(z))$ has an exponentially stable T-periodic solution z_ε such that $\|z_\varepsilon - z_*\|_C \to 0$ as $\varepsilon \to 0$. In this paper, we consider another class of perturbations of system (1), namely perturbations by hysteresis nonlinearities. Generically, a nonlinearity of this type can be described as the composition of the input-state and state-output relations, which possess some typical properties. To define these relations, one introduces the functional space $\mathbb{E}_1$ of inputs $x : \mathbb{R}_+ \to \mathbb{R}^k$, the functional space $\mathbb{E}_2$ of outputs $u : \mathbb{R}_+ \to \mathbb{R}^\ell$, and some metric space Ω of states of the hysteresis nonlinearity; elements of Ω represent the memory. The input-state relation is determined by the operator Γ that sends pairs (w_0, x) from some set $\mathcal{D} \subseteq \Omega \times \mathbb{E}_1$ to the functions $w : \mathbb{R}_+ \to \Omega$ satisfying $w(0) = w_0$. Here w_0 is called the initial state and the value $w(\cdot) = (\Gamma[w_0]x)(\cdot)$ of the operator Γ is called the variable state of the hysteresis nonlinearity. The definition of the output $u \in \mathbb{E}_2$ for each variable state $w = w(t)$ completes the construction. Usually, u and w are related by the simple formula $u(t) = \Phi(w(t))$, $t \in \mathbb{R}_+$, where $\Phi : \Omega \to \mathbb{R}^\ell$ is a given functional (we shall write $u(\cdot) = (\Phi(w))(\cdot)$ as well).

The input-state operators Γ of various phenomenological hysteresis models arising from physics, mechanics, engineering, etc., have important common basic properties. From them, we recall the *Volterra* property

$$x(t) = y(t) \quad \text{for all} \quad 0 \le t \le t_1 \quad \Rightarrow \quad (\Gamma[w_0]x)(t) = (\Gamma[w_0]y)(t) \quad \text{for all} \quad 0 \le t \le t_1,$$

the *Semi-group* property

$$x(t) = y(t - t_1) \quad \text{for all} \quad t \geq t_1 \quad \Rightarrow \quad (\Gamma[(\Gamma[w_0]x)(t_1)]y)(t - t_1) = (\Gamma[w_0]x)(t) \quad \text{for all} \quad t \geq t_1,$$

systematically used below (here $t_1 \geq 0$), and the *Rate-independence* property

$$x(t) = y(\tau(t)) \quad \text{for all} \quad t \geq 0 \quad \Rightarrow \quad (\Gamma[w_0]x)(t) = (\Gamma[w_0]y)(\tau(t)) \quad \text{for all} \quad t \geq 0,$$

where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is any increasing function of the appropriate regularity with $\tau(0) = 0$. The continuity, compactness, and other more specific properties useful in different problems may vary from one class of hysteresis nonlinearities to another. Here we do not pursue the goal to formulate theorems on hysteresis perturbations of system (1) for a possibly larger class of hysteresis nonlinearities described by a set of abstract requirements to the operator Γ and the functional Φ. Instead, we consider the particular type of hysteresis nonlinearities called the Preisach models; they are defined in the next subsection. For these nonlinearities $k = \ell = 1$, their inputs and outputs are scalar continuous functions. Formulated originally as models of magnetic hysteresis, the Preisach models are now used in many fields; the reason is that they are identified by the two rather natural properties of the memory and the input-output loops (the so-called wiping-out and congruency properties, cf. [11]).

The main system considered below is

$$\begin{gathered} \dot{z} = f(t, z, \varepsilon u(t)), \\ u(t) = (\Phi(w))(t), \quad w(t) = (\Gamma[w_0]Dz)(t) \end{gathered} \tag{2}$$

with $0 < \varepsilon \leq 1$, $t \geq 0$. Here the $d \times 1$ row-matrix D sends the function z to the scalar continuous input $x(\cdot) = (Dz)(\cdot)$ of the Preisach hysteresis nonlinearity; u and w are the scalar continuous output and the changing state of this nonlinearity; the continuous T-periodic in t function $f(\cdot, \cdot, \cdot) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is supposed to be continuously differentiable with respect to the variable z. By definition, solutions of (2) are pairs (z, w) with the continuously differentiable first component. Remark that the Vollterra property allows to consider inputs, variable states and outputs of the hysteresis nonlinearity on finite intervals; therefore solutions of system (2) may be defined on the intervals $[0, \tau)$, $[0, \tau]$ and $[0, \infty)$.

We consider system (2) as a perturbation of system (1). It will be proved that if the parameter $\varepsilon > 0$ is sufficiently small, then under appropriate assumptions system (2) has a continual connected set of periodic solutions which has a simple structure and attracts all the solutions starting in its neighborhood. This continual set arises from the isolated exponentially stable periodic solution z_* of system (1) as a result of the Preisach hysteresis perturbation.

2.2 Definition of the Preisach nonlinearity

We shall use the space $C([0, T]; \mathbb{R})$ of continuous functions $x : [0, T] \to \mathbb{R}$ with the uniform norm $\|\cdot\|_{C[0,T]}$ and the space $C(\mathbb{R}_+; \mathbb{R})$ of continuous functions $x : \mathbb{R}_+ \to \mathbb{R}$

with the seminorms $\|x\|_{C[t_1,t_2]} = \max_{t\in[t_1,t_2]} |x(t)|$. For bounded $x \in C(\mathbb{R}_+;\mathbb{R})$, we denote $\|x\|_{C[t_1,\infty)} = \sup_{t\geq t_1} |x(t)|$. The same notations $\|\cdot\|_{C[t_1,t_2]}$, $\|\cdot\|_{C[t_1,\infty)}$ will be used for the vector-valued continuous functions $z = z(t)$ from the spaces $C([0,T];\mathbb{R}^d)$ and $C(\mathbb{R}_+;\mathbb{R}^d)$.

Consider the elementary hysteresis nonlinearity called the nonideal relay with the two states ± 1 and with the thresholds α, β, $\alpha > \beta$. We denote its variable state by $\eta(t) = (\mathcal{R}_{\alpha\beta}[\eta_0]x)(t)$. Here the input is an arbitrary continuous function $x : \mathbb{R}_+ \to \mathbb{R}$ and the initial state $\eta_0 = \eta(0)$ should be *admissible* for the input, which means that $\eta_0 \in \{-1, 1\}$ if $\beta < x(0) < \alpha$; $\eta_0 = -1$ if $x(0) \leq \beta$; $\eta_0 = 1$ if $x(0) \geq \alpha$. For such initial states the values $\eta : \mathbb{R}_+ \to \{-1, 1\}$ of the operator $\mathcal{R}_{\alpha\beta}$ are defined by

$$(\mathcal{R}_{\alpha\beta}[\eta_0]x)(\tau) = \begin{cases} \eta_0, & \text{if } \beta < x(t) < \alpha \text{ for all } t \in [t_0,\tau]; \\ 1, & \text{if there is a } t_1 \in [t_0,\tau] \text{ such that} \\ & x(t_1) \geq \alpha \text{ and } x(t) > \beta \text{ for all } t \in [t_1,\tau]; \\ -1, & \text{if there is a } t_1 \in [t_0,\tau] \text{ such that} \\ & x(t_1) \leq \beta \text{ and } x(t) < \alpha \text{ for all } t \in [t_1,\tau]. \end{cases} \tag{3}$$

This formula implies that the function η has at most finite number of jumps (relay switches) on every segment $[0,t]$ and that $\eta(\tau) = 1$ whenever $x(\tau) \geq \alpha$ as well as $\eta(\tau) = -1$ whenever $x(\tau) \leq \beta$. By definition, the output of the relay equals its variable state, i.e., Φ is the identity in terms of the previous subsection.

The Preisach hysteresis nonlinearity may be described as a collection of relays (with all possible thresholds) that have a common input and function independently. For the strict definition, consider the set W of nonideal relays parameterized by the pairs (α, β), each relay is represented by the point of the half-plane $\Pi = \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > \beta\}$. Denote by $\Omega(x_0)$ the set of all measurable functions $w_0 : \Pi \to \{-1, 1\}$ satisfying

$$\begin{aligned} & w_0(\alpha,\beta) = -1 \quad \text{for} \quad \beta \geq x_0; \qquad w_0(\alpha,\beta) = 1 \quad \text{for} \quad \alpha \leq x_0; \\ & |w_0(\alpha,\beta)| = 1 \quad \text{for} \quad \beta < x_0 < \alpha \end{aligned} \tag{4}$$

and define $\Omega = \cup_{x_0\in\mathbb{R}}\Omega(x_0)$. The states of the Preisach nonlinearity are all functions $w_0 \in \Omega$. The further construction is based on the fact that for every continuous input $x : \mathbb{R}_+ \to \mathbb{R}$ and every initial state $w_0 \in \Omega(x(0))$ (such initial states are called *admissible* for the input $x = x(t)$) the function

$$w(t,\alpha,\beta) = (\mathcal{R}_{\alpha\beta}[w_0(\alpha,\beta)]x)(t) \tag{5}$$

satisfies $w(t,\cdot,\cdot) \in \Omega(x(t)) \subset \Omega$ for each $t \geq 0$. This allows to use the formula (5) as the definition of the input-state operator $w(\cdot) = (\Gamma[w_0]x)(\cdot)$ of the Preisach nonlinearity that assigns variable states $w : \mathbb{R}_+ \to \Omega$ to continuous inputs x and admissible initial states w_0. The outputs $u(\cdot) = (\Phi(\Gamma[w_0]x))(\cdot)$ of the Preisach nonlinearity are defined with the help of the functional $\Phi : \Omega \to \mathbb{R}$ given by

$$\Phi(w_0(\cdot,\cdot)) = \iint\limits_{\alpha>\beta} \mu(\alpha,\beta) w_0(\alpha,\beta)\, d\alpha d\beta, \tag{6}$$

where $\mu : \Pi \to \mathbb{R}$ is a fixed integrable function. It means that

$$u(t) = \iint\limits_{\alpha>\beta} \mu(\alpha,\beta)(\mathcal{R}_{\alpha\beta}[w_0(\alpha,\beta)]x)(t)\, d\alpha\, d\beta, \qquad t \geq 0;$$

this is usually interpreted in the sense that the outputs of the individual relays of the set W are averaged over the domain $\Pi \ni (\alpha,\beta)$. All the outputs $u : \mathbb{R}_+ \to \mathbb{R}$ of the Preisach nonlinearity are continuous functions (although the outputs of individual relays have jumps) and satisfy the uniform estimate $\sup_{t\in\mathbb{R}_+} |u(t)| \leq M$ with

$$M = \iint\limits_{\alpha>\beta} |\mu(\alpha,\beta)|\, d\alpha\, d\beta < \infty. \tag{7}$$

A usual way to metrize the state space Ω of the Preisach nonlinearity is to use the L_1-type metric $\chi(w_1^0, w_2^0) = \mathrm{mes}_\mu \{(\alpha,\beta) :\ w_1^0(\alpha,\beta) \neq w_2^0(\alpha,\beta)\}$, where the measure is defined by

$$\mathrm{mes}_\mu\, E = \iint\limits_{E} |\mu(\alpha,\beta)|\, d\alpha\, d\beta \tag{8}$$

for all Lesbegue measurable sets $E \subseteq \Pi$. We follow this standard scheme, which suggests that we use the identification procedure and start to consider as states w_0 the classes of a.e. equal (with respect to the measure (8)) functions, rather than individual measurable functions. The correctness of the identification is justified in [8], where it is shown that if the functions $w_1^0, w_2^0 \in \Omega(x(0))$ satisfy $\chi(w_1^0, w_2^0) = 0$ then $\chi(w_1(t), w_2(t)) = 0$ at every moment $t \geq 0$ for the variable states $w_j(\cdot) = (\Gamma[w_j^0]x)(\cdot)$, which also implies trivially the equality $u_1 = u_2$ of the outputs $u_j(\cdot) = (\Phi(w_j))(\cdot)$. The state space Ω with the metric $\chi(\cdot,\cdot)$ obtained as a result of this construction is a complete metric space; moreover, every variable state $w : \mathbb{R}_+ \to \Omega$ is a continuous function.

For simplicity, we suppose throughout the paper that the function μ is bounded and

$$|\mu(\alpha,\beta)| \leq \mu_* \quad \text{for all} \quad (\alpha,\beta) \in \Pi; \qquad \mu(\alpha,\beta) = 0 \quad \text{whenever} \quad \alpha > \beta + h \tag{9}$$

for some positive μ_*, h. As it is proved in [8], these relations imply the global Lipschitz continuity of the Preisach nonlinearity. More precisely, for every pair of continuous inputs $x_j : \mathbb{R}_+ \to \mathbb{R}$ and admissible initial states w_j^0 the variable states $w_j(\cdot) = (\Gamma[w_j^0]x_j)(\cdot)$ satisfy for each $t \geq 0$

$$\chi(w_1(t), w_2(t)) \leq \chi(w_1^0, w_2^0) + L_\Gamma \|x_1 - x_2\|_{C[0,t]} \tag{10}$$

with $L_\Gamma = 2\mu_* h$. The proof of this estimate is based on a simple explicit algorithm to construct variable states $w_j(\cdot)$; we do not consider it here. From the definition (6) of the functional Φ it follows that

$$|\Phi(w_1^0) - \Phi(w_2^0)| \leq L_\Phi\, \chi(w_1^0, w_2^0) \quad \text{for all} \quad w_1^0, w_2^0 \in \Omega$$

with $L_\Phi = 2$, which together with (10) implies the estimate $\|u_1 - u_2\|_{C[0,t]} \leq 2\chi(w_1^0, w_2^0) + 2L_\Gamma \|x_1 - x_2\|_{C[0,t]}$ for the outputs $u_j(\cdot) = (\Phi(w_j))(\cdot)$ of the Preisach nonlinearity at each moment $t \geq 0$.

2.3 Periodic solutions

A solution (z, w) of system (2) is called periodic if it is defined on the whole semiaxis $t \in \mathbb{R}_+$ and both its components are periodic with the period T (other periods are not considered), i.e., $z(t) = z(t+T)$, $w(t) = w(t+T)$ for all $t \in \mathbb{R}_+$. These relations imply the periodicity $x(t) = x(t+T)$, $u(t) = u(t+T)$ of the input $x = Dz$ and the output $u = \Phi(w)$ of the Preisach hysteresis nonlinearity for all $t \in \mathbb{R}_+$ as well. Remark that due to the semi-group property of the operator Γ, for each periodic input x the variable state $w(\cdot) = (\Gamma[w_0]x)(\cdot)$ and the output $u(\cdot) = (\Phi(w))(\cdot)$ are periodic iff the initial state satisfies $w_0 = (\Gamma[w_0]x)(T)$.

Everywhere below the periodic functions defined on the semiaxis $\mathbb{R}_+$ are identified with their restrictions to the segment $[0, T]$. By C_{per} we denote the subspace $C_{per} = \{x \in C : x(0) = x(T)\}$ of the space $C = C([0,T]; \mathbb{R})$. Our analysis of the structure of the set of periodic solutions will be based on the simple results proved in [12], where the class of continuous operators $\mathcal{J}_\lambda : C_{per} \to C_{per}$, $\lambda \in [-1, 1]$, was introduced, which assign periodic outputs of the Preisach nonlinearity to its periodic inputs. We formulate these results (adapting the formulation to our purposes) in the form of the following lemma.

Lemma 1 *There is an operator $Q : C_{per} \to \Omega$ which assigns to each continuous periodic input x of the Preisach nonlinearity an admissible initial state $w_*^0 = Q(x) \in \Omega(x(0))$ such that the following statements are valid.*

(i) *The function u is a periodic output of the Preisach nonlinearity for the periodic input x and some initial state w_0 iff $u = \mathcal{J}_\lambda x$ for some $\lambda \in [-1, 1]$, where the operator $\mathcal{J}_\lambda$ is defined by*

$$(\mathcal{J}_\lambda x)(t) = \Phi(w_*(t)) + \lambda\, \mathrm{mes}_\mu\, E(x_m, x_M), \qquad w_*(t) = (\Gamma[Q(x)]x)(t), \qquad t \in [0, T],$$

with

$$x_m = \min_{t \in [0,T]} x(t), \quad x_M = \max_{t \in [0,T]} x(t), \quad E(x_m, x_M) = \{(\alpha, \beta) \in \Pi : \beta < x_m \leq x_M < \alpha\}.$$

(ii) *The operators $\mathcal{J}_\lambda : C_{per} \to C_{per}$ are globally Lipschitz continuous and satisfy the estimate*

$$\|\mathcal{J}_{\lambda_1} x_1 - \mathcal{J}_{\lambda_2} x_2\|_{C[0,T]} \leq L_J \|x_1 - x_2\|_{C[0,T]} + M|\lambda_1 - \lambda_2| \tag{11}$$

with $L_J = 2L_\Gamma = 4\mu_* h$ and M defined by (7) for every $x_j \in C_{per}$ and $\lambda_j \in [-1,1]$, $j = 1,2$.

(iii) *Let $w_*^0 = Q(x)$ for a periodic input x; then the variable state $(\Gamma[w_0]x)(\cdot)$ is periodic iff $w_0(\alpha,\beta) = w_*^0(\alpha,\beta)$ for all $(\alpha,\beta) \notin E(x_m,x_M)$. Moreover, the function $w : \mathbb{R}_+ \to \Omega$ is a periodic variable state of the Preisach nonlinearity for the periodic input x and some initial state w_0 iff for each $t \geq 0$*

$$\begin{aligned} &w(t,\alpha,\beta) = w_*(t,\alpha,\beta) \quad \textit{for} \quad (\alpha,\beta) \notin E(x_m,x_M); \\ &w(t,\alpha,\beta) = \eta(\alpha,\beta) \quad \textit{for} \quad (\alpha,\beta) \in E(x_m,x_M), \end{aligned} \tag{12}$$

where $w_* = \Gamma[Q(x)]x$ and $\eta : E(x_m,x_M) \to \mathbb{R}$ is an arbitrary measurable function with the absolute value $|\eta| \equiv 1$. If relations (12) are valid, then the output $u(\cdot) = (\Phi(w))(\cdot)$ of the Preisach nonlinearity is periodic and equals $u = \mathcal{J}_\lambda x$ with λ defined by

$$\lambda \operatorname{mes}_\mu E(x_m,x_M) = \iint\limits_{E(x_m,x_M)} \mu(\alpha,\beta)\eta(\alpha,\beta)\, d\alpha\, d\beta \tag{13}$$

(if $\mu = 0$ a.e. in $E(x_m,x_M)$, then $\lambda \in [-1,1]$ is arbitrary).

For example, Q may be defined as follows. For a given input $x \in C_{per}$, set $w_T^0 = (\Gamma[w_0]x)(T)$ with any $w_0 \in \Omega(x(0))$. Set

$$\begin{aligned} &\eta(\alpha,\beta) = 1 \ \text{ for } \ s(\alpha - x_M) + (1-s)(\beta - x_m) < 0, \\ &\eta(\alpha,\beta) = -1 \ \text{ for } \ s(\alpha - x_M) + (1-s)(\beta - x_m) \geq 0 \end{aligned}$$

with the smallest $s \in [0,1]$ such that

$$\iint\limits_{E(x_m,x_M)} \mu(\alpha,\beta)\eta(\alpha,\beta)\, d\alpha\, d\beta = 0.$$

Then the value $w_*^0 = Q(x)$ of the operator Q is given by

$$\begin{aligned} &w_*^0(\alpha,\beta) = w_T^0(\alpha,\beta) \quad \text{for} \quad (\alpha,\beta) \in \Pi \setminus E(x_m,x_M), \\ &w_*^0(\alpha,\beta) = \eta(\alpha,\beta) \quad \text{for} \quad (\alpha,\beta) \in E(x_m,x_M). \end{aligned}$$

This definition does not depend on the choice of the initial state $w_0 \in \Omega(x(0))$ and is based on the so-called monocyclicity property of the Preisach nonlinearity (inherited from the analogous property of nonideal relays), which can be expressed as the identity

$$(\Gamma[w_0]x)(t+T) = (\Gamma[w_0]x)(t) \qquad \text{for all} \qquad t \geq T \tag{14}$$

valid for each T-periodic continuous input $x : \mathbb{R}_+ \to \mathbb{R}$ and each admissible initial state w_0.

In what follows, we shall use the operators whose fixed points define periodic solutions of system (2). We first introduce the linear operator

$$(Pz)(t) = Dz(t) + tT^{-1}D(z(0) - z(T)),$$

which maps the space $C([0,T];\mathbb{R}^d)$ onto $C_{per} \subset C([0,T];\mathbb{R})$ and satisfies $(Pz)(\cdot) = Dz(\cdot)$ whenever $z(0) = z(T)$, and then define the operator

$$(H_{\lambda,\varepsilon}z)(t) = z(T) + \int_0^t f(s, z(s), \varepsilon\,(\mathcal{J}_\lambda Pz)(s))\,ds \tag{15}$$

in the space $C([0,T];\mathbb{R}^d)$. From $z = H_{\lambda,\varepsilon}z$ it follows $z(0) = z(T)$ and $\dot{z}(\cdot) = f(\cdot, z(\cdot), \varepsilon\,u(\cdot))$ with $u(\cdot) = (\mathcal{J}_\lambda Pz)(\cdot) = (\mathcal{J}_\lambda Dz)(\cdot)$. By Lemma 1, it means that $z = H_{\lambda,\varepsilon}z$ iff z is the first component of a periodic solution of system (2); the second component is defined by relations (12) with any $\eta = \eta(\alpha,\beta)$ satisfying (13) (this is discussed in more detail in Subsection 3.2 below). Remark that for $\varepsilon = 0$ and any $\lambda \in [-1,1]$ we have $H_{\lambda,0} = H_0$, where

$$(H_0 z)(t) = z(T) + \int_0^t f(s, z(s), 0)\,ds \tag{16}$$

is the so-called operator of the periodic problem for the unperturbed equation (1); the fixed points $z \in C([0,T];\mathbb{R}^d)$ of this operator are periodic solutions of (1).

3 Main results

3.1 Lemma on perturbed systems

Let us linearize the unperturbed system (1) along its periodic solution z_*. Consider the Jacobi matrix $A(t,z,\xi) = \partial f(t,z,\xi)/\partial z$ and denote by $Y = Y(t)$ the fundamental solution matrix of the linearization $\dot{z} = A(t, z_*(t), 0)z$ of system (1). Our assumption about the exponential stability of the solution z_* means that the spectrum of the matrix $Y(T)$ belongs to the open unit disc of the complex plain (in other notation, the eigenvalues of $Y(T)$ are called characteristic multipliers of the periodic linear system $\dot{z} = A(t, z_*(t), 0)z$). Therefore there is a number $q_0 < 1$ and a norm $|\cdot|$ in $\mathbb{R}^d$ such that

$$|Y(T)z| \le q_0|z|, \qquad z \in \mathbb{R}^d. \tag{17}$$

Everywhere below we use this norm in $\mathbb{R}^d$. We assume that the nonlinearity satisfies the local Lipschitz condition

$$\begin{array}{rcl} |f(t,z_1,\xi_1) - f(t,z_2,\xi_2)| & \le & L_z(r)|z_1 - z_2| + L_\xi(r)|\xi_1 - \xi_2| \\ & & \text{if} \quad \min\limits_{t\in[0,T]} |z_j - z_*(t)| \le r,\ |\xi_j| \le M \end{array} \tag{18}$$

with $j=1,2$ and M defined by (7).

Consider the system

$$\dot{z}=f(t,z,\varepsilon\xi(t)),\qquad z(0)=z_0 \tag{19}$$

where $\xi:\mathbb{R}_+\to\mathbb{R}^d$ represents a general continuous perturbation. A unique solution of (19) we denote by $z^\varepsilon(\cdot;z_0,\xi)$. We consider the functions ξ from the ball $\|\xi\|_{C[0,\infty)}\le M$ only.

Lemma 2 *Let $q_0<q_*<1$. Then there are positive numbers ε_0, r_0, ρ_0 with $r_0>\rho_0$ such that for every $\varepsilon<\varepsilon_0$, every ξ from the ball $\|\xi\|_{C[0,\infty)}\le M$ and every initial value z_0 from the ball $|z_0-z_*(0)|\le\rho_0$ the solution $z^\varepsilon(\cdot;z_0,\xi)$ of equation (19) satisfies*

$$|z^\varepsilon(T;z_0,\xi)-z_*(0)|<\rho_0,\qquad \|z^\varepsilon(\cdot;z_0,\xi)-z_*(\cdot)\|_{C[0,\infty)}<r_0. \tag{20}$$

Moreover, there is a number $c_1>0$ such that for every pair of solutions $z^\varepsilon(\cdot;z_j^0,\xi_j)$ with $\varepsilon<\varepsilon_0$, $\|\xi_j\|_{C[0,\infty)}\le M$, $|z_j^0-z_*(0)|\le\rho_0$ the estimates

$$\|z^\varepsilon(\cdot;z_1^0,\xi_1)-z^\varepsilon(\cdot;z_2^0,\xi_2)\|_{C[0,T]}\le a|z_1^0-z_2^0|+b\varepsilon\|\xi_1-\xi_2\|_{C[0,T]}, \tag{21}$$

$$|z^\varepsilon(T;z_1^0,\xi_1)-z^\varepsilon(T;z_2^0,\xi_2)|\le q_*|z_1^0-z_2^0|+(1-q_*)c_1\,\varepsilon\|\xi_1-\xi_2\|_{C[0,T]} \tag{22}$$

are valid with $a=e^{L_z(r_0)T}$, $b=(e^{L_z(r_0)T}-1)L_\xi(r_0)/L_z(r_0)$.

This lemma is well-known. Nevertheless, we give its proof in the Appendix to show, how to estimate the constants ε_0, r_0, ρ_0, etc. introduced in the lemma. These constants will be used below systematically.

Remark that iterating (22) we obtain

$$|z^\varepsilon(kT;z_1^0,\xi_1)-z^\varepsilon(kT;z_2^0,\xi_2)|\le q_*^k|z_1^0-z_2^0|+c_1\,\varepsilon\|\xi_1-\xi_2\|_{C[0,kT]} \tag{23}$$

and this together with (21) implies

$$\|z^\varepsilon(\cdot;z_1^0,\xi_1)-z^\varepsilon(\cdot;z_2^0,\xi_2)\|_{C[kT,\infty)}\le aq_*^k|z_1^0-z_2^0|+(ac_1+b)\,\varepsilon\|\xi_1-\xi_2\|_{C[0,\infty)}. \tag{24}$$

3.2 Structure of the set of periodic solutions

Denote by $[z,u]$ the class of periodic solutions (z,w) of system (2) such that $u(\cdot)=(\Phi(w))(\cdot)$. By Lemma 1, for every such class the function z and the output u of the Preisach nonlinearity are related by the equalities $u=\mathcal{J}_\lambda x$, $x=Dz$ for some $\lambda\in[-1,1]$ (due to the condition (25) below, this λ is determined uniquely). The first of relations (12) implies that the second components $w=w(t,\alpha,\beta)$ of all the solutions $(z,w)\in[z,u]$ equal each other in the domain $(\alpha,\beta)\in\Pi\setminus E(x_m,x_M)$ at every moment $t\ge 0$. These components are different only in the domain $(\alpha,\beta)\in E(x_m,x_M)$, where they are constant in t, since by the second of relations (12) the inclusion $(z,w)\in[z,u]$ implies $w(t,\alpha,\beta)\equiv\eta(\alpha,\beta)$ for all $t\ge 0$. Here the only requirement to the measurable function $\eta : E(x_m, x_M) \to \{-1, 1\}$ is that it should satisfy the equality (13).

Let all the conclusions of Lemma 2 hold for some ε_0, r_0, ρ_0. Let $\varepsilon < \varepsilon_0$. Set

$$G = \{z \in C([0,T];\mathbb{R}^d) : \ |z(0) - z_*(0)| \le \rho_0, \ \|z - z_*\|_{C[0,T]} \le r_0\}.$$

We are interested in the situation when the set of periodic solutions of system (2) with $z \in G$ is larger than a unique class $[z, u]$. The main condition for this is

$$\operatorname{mes}_\mu\{(\alpha,\beta) : \beta < r_m \le r_M < \alpha\} > 0 \quad \text{for} \quad \begin{aligned} r_m &= \min_{t\in[0,T]} Dz_*(t) - r_0|D|, \\ r_M &= \max_{t\in[0,T]} Dz_*(t) + r_0|D|; \end{aligned} \tag{25}$$

we assume it to hold. By statement (i) of Lemma 1, the Preisach nonlinearity has a unique periodic output for a periodic input x iff $\operatorname{mes}_\mu E(x_m, x_M) = 0$ (i.e., $\mu(\alpha,\beta) = 0$ a.e. in $E(x_m, x_M)$), otherwise the periodic outputs $u_\lambda = \mathcal{J}_\lambda x$ are different for different λ. Therefore estimate (25) implies that the Preisach nonlinearity has a one-parametric set of different periodic outputs $u_\lambda = \mathcal{J}_\lambda x$ for every periodic input $x = Dz$ with $z \in G$.

Theorem 1 *Let $\varepsilon < \varepsilon_0$. Then for each $\lambda \in [-1,1]$ system (2) has a class $[z_\lambda, u_\lambda]$ of T-periodic solutions such that $z_\lambda \in G$, $u_\lambda = \mathcal{J}_\lambda(Dz_\lambda)$. Estimate (25) implies that these classes do not intersect for different λ.*

Proof. We use the rotation of vector fields and the Affinity Theorem [1, 9] that relates the rotations of different vector fields associated with the periodic problem for the unperturbed system (1).

First, consider the vector fields $I - H_{\lambda,\varepsilon}$ with $|\lambda| \le 1$, $0 \le \varepsilon < \varepsilon_0$ in the space $C([0,T];\mathbb{R}^d)$ (I denotes the identity in all spaces). Let $z \in G$ be the first component of a periodic solution of system (2). Then z is a solution of (19) with ξ satisfying $\|\xi\|_{C[0,T]} \le M$ and therefore estimates (20) imply $|z(T) - z_*(0)| < \rho_0$, $\|z - z_*\|_{C[0,T]} < r_0$. Due to $z(0) = z(T)$, it means that z belongs to the interior of G and we conclude that there are no fixed points of the operators $H_{\lambda,\varepsilon}$ on the boundary of G. Consequently, the rotation $\gamma(I - H_{\lambda,\varepsilon}, G)$ of the vector field $I - H_{\lambda,\varepsilon}$ on the boundary of G is defined for every λ and ε considered. Moreover, since for each fixed λ the operators $H_{\lambda,\varepsilon}$ are continuous in ε uniformly with respect to $z \in G$, it follows that the rotation $\gamma(I - H_{\lambda,\varepsilon}, G)$ does not depend on ε. Finally, the equality $H_{\lambda,0} = H_0$ implies that the rotation $\gamma(I - H_{\lambda,\varepsilon}, G)$ is the same for all $|\lambda| \le 1$, $\varepsilon < \varepsilon_0$ and $\gamma(I - H_{\lambda,\varepsilon}, G) = \gamma(I - H_0, G)$, where H_0 is the operator (16) of the periodic problem for the unperturbed system (1).

Now consider the translation operator S of system (1) defined by $S(z_0) = z(T; z_0)$, where $z(\cdot; z_0)$ denotes a unique solution of the initial value problem for the unperturbed system (1) with the initial condition $z(0) = z_0 \in \mathbb{R}^d$. Consider the ball

$B = \{z_0 \in \mathbb{R}^d : |z_0 - z_*(0)| \le \rho_0\}$ with the center at the initial value of the periodic solution z_*. Estimate (22) with $\xi_1 \equiv \xi_2 \equiv 0$ implies that S is a contraction on B. Therefore $z_*(0)$ is a unique fixed point of S in the ball B and the rotation of the vector field $I - S$ on the boundary of B equals 1. According to the Affinity Theorem, this rotation is equal to the rotation of the completely continuous vector field $I - H_0$ on the boundary of any open bounded domain $V \subset C([0,T];\mathbb{R}^d)$ such that $z_* \in V$ and z_* is a unique periodic solution of system (1) in the closure $\overline{V}$ of V. In particular, this is true for $\overline{V} = G$, since $z(0) \in B$ for every $z \in G$. Thus, $\gamma(I - H_0, G) = 1$, therefore $\gamma(I - H_{\lambda,\varepsilon}, G) = 1$ and the principle of the non-zero rotation implies that $H_{\lambda,\varepsilon}$ has at least one fixed point $z \in G$. As we know, this fixed point defines the class $[z,u]$ of periodic solutions of system (2) with $u = \mathcal{J}_\lambda(Dz)$. □

Let us show that for appropriate range of ε every periodic solution with the first component in G belongs to one of the classes $[z_\lambda, u_\lambda]$ considered in Theorem 1 and moreover, $\lambda \to [z_\lambda, u_\lambda]$ is a one-to-one Lipschitz continuous map of the segment $[-1,1]$ to the product $C([0,T];\mathbb{R}^d) \times C([0,T];\mathbb{R})$. The corresponding estimate of ε follows from (11), (21) and (22).

Theorem 2 *Let $\varepsilon < \varepsilon_0$ and*

$$L_J L_D (ac_1 + b)\, \varepsilon < 1 \tag{26}$$

with $L_D = |D|$. Then for each $\lambda \in [-1,1]$ there is a unique class $[z,u] = [z_\lambda, u_\lambda]$ of T-periodic solutions of system (2) such that $z_\lambda \in G$, $u = \mathcal{J}_\lambda(Dz)$ and the estimates

$$\|z_{\lambda_1} - z_{\lambda_2}\|_{C[0,T]} \le L_1|\lambda_1 - \lambda_2|, \qquad \|u_{\lambda_1} - u_{\lambda_2}\|_{C[0,T]} \le L_2|\lambda_1 - \lambda_2| \tag{27}$$

hold for all $\lambda_1, \lambda_2 \in [-1,1]$.

Proof. Let $[z_j, u_j] = [z_j, \mathcal{J}_{\lambda_j}(Dz_j)]$ with $z_j \in G$, $j = 1,2$ be two classes of periodic solutions of system (2). Estimate (22) implies

$$|z_1(T) - z_2(T)| \le q_*|z_1(0) - z_2(0)| + (1 - q_*)c_1\,\varepsilon\|u_1 - u_2\|_{C[0,T]},$$

hence $|z_1(0) - z_2(0)| \le c_1\,\varepsilon\|u_1 - u_2\|_{C[0,T]}$. Substituting this estimate in the relation $\|z_1 - z_2\|_{C[0,T]} \le a|z_1(0) - z_2(0)| + b\varepsilon\|u_1 - u_2\|_{C[0,T]}$, which follows from (21), we obtain

$$\|z_1 - z_2\|_{C[0,T]} \le (ac_1 + b)\varepsilon\|u_1 - u_2\|_{C[0,T]}.$$

On the other hand, (11) implies $\|u_1 - u_2\|_{C[0,T]} \le L_J L_D\|z_1 - z_2\|_{C[0,T]} + M|\lambda_1 - \lambda_2|$. Therefore

$$\|z_1 - z_2\|_{C[0,T]} \le L_J L_D(ac_1 + b)\varepsilon\|z_1 - z_2\|_{C[0,T]} + M(ac_1 + b)\varepsilon|\lambda_1 - \lambda_2|,$$

$$\|u_1 - u_2\|_{C[0,T]} \le L_J L_D(ac_1 + b)\varepsilon\|u_1 - u_2\|_{C[0,T]} + M|\lambda_1 - \lambda_2|$$

and the conclusion of the theorem follows from the estimate (26). In particular, estimates (27) are valid with $L_2 = M(1 - L_J L_D(ac_1 + b)\varepsilon)^{-1}$, $L_1 = (ac_1 + b)\varepsilon L_2$. □

3.3 Stability of the set of periodic solutions

Let $\tilde{z} : \mathbb{R}_+ \to \mathbb{R}^d$ be a solution of system (1) and U be some neighborhood of the point $\tilde{z}(0)$. Following [5], we say that the solution $\tilde{z}$ is U-uniformly stable if every solution $z = z(\cdot; z_0)$ of (1) with the initial value $z_0 = z(0) \in U$ is defined on the semiaxis $\mathbb{R}_+$ and

$$\lim_{\tau\to\infty} \sup_{z_0\in U,\ t\geq\tau} |z(t; z_0) - \tilde{z}(t)| = 0.$$

Evidently, this implies the asymptotic stability of the solution $\tilde{z}$.

We formulate the similar concept for system (2), taking into account that its periodic solutions form a continual set.

The Lipschitz condition (18) implies the *wellposedness* of the initial value problem for system (2). It means that the initial value problem for (2) with any initial conditions $z(0) = z_0$, $w(0) = w_0$ such that $w_0 \in \Omega(Dz_0)$ (this inclusion ensures that the initial state w_0 is admissible for the input $x = Dz$ of the hysteresis nonlinearity, i.e., the pair (w_0, x) belongs to the domain of the input-state operator Γ) has a unique solution

$$(z(t), w(t)) = (z^\varepsilon(t; z_0, w_0), w^\varepsilon(t; z_0, w_0)), \quad 0 \leq t < \delta,$$

which moreover depends continuously on (z_0, w_0). Consider a set Z of solutions (z, w) defined for all $t \geq 0$. Let some set $U \subset \mathbb{R}^d$ contain the initial values $z(0)$ as its internal points for all solutions $(z, w) \in Z$. We say that the set Z of solutions of system (2) is *U-uniformly stable* if for every $z_0 \in U$ and every initial state $w_0 \in \Omega(Dz_0)$ there is a solution $(z, w) \in Z$ such that

$$\lim_{\tau\to\infty} \|z^\varepsilon(\cdot; z_0, w_0) - z(\cdot)\|_{C[\tau,\infty)} = \lim_{\tau\to\infty} \sup_{t\geq\tau} \chi(w^\varepsilon(t; z_0, w_0), w(t)) = 0 \tag{28}$$

and this convergence in τ is uniform with respect to all the initial values $z_0 \in U$, $w_0 \in \Omega(Dz_0)$. Particularly, if Z consists of the periodic solutions, then the solution $(z, w) \in Z$ satisfying (28) is uniquely determined for each z_0, w_0. The requirement that the convergence is uniform with respect to arbitrary admissible initial states w_0 is natural from the point of view that the initial state of the hysteresis nonlinearity can usually be neither controlled nor observed.

Theorem 3 *Let $\varepsilon < \varepsilon_0$ and*

$$2L_\Gamma L_D L_\Phi(ac_1 + b)\,\varepsilon < 1 \tag{29}$$

(recall that $L_\Gamma = 2\mu_* h$, $L_D = |D|$, $L_\Phi = 2$ with μ_*, h defined by (9)). Then the continual set of all periodic solutions (z, w) of system (2) such that $z \in G$ is U-uniformly stable for $U = B = \{z_0 \in \mathbb{R}^d : |z_0 - z_*(0)| \leq \rho_0\}$. Moreover, let a real number q and a positive integer $k = k(q)$ satisfy

$$2L_\Gamma L_D L_\Phi(ac_1 + b)\,\varepsilon < q < 1, \tag{30}$$

$$aq_*^k\big(1+2L_\Gamma L_D L_\Phi c_1\,\varepsilon(1+1/q)\big) \le q - 2L_\Gamma L_D L_\Phi(ac_1+b)\,\varepsilon. \tag{31}$$

Then for every solution $(\tilde z(\cdot),\tilde w(\cdot)) = (z^\varepsilon(\cdot;z_0,w_0), w^\varepsilon(\cdot;z_0,w_0))$ *with the initial value* $z_0 \in B$ *there is a unique periodic solution* $(\tilde z_*,\tilde w_*)$ *with* $\tilde z_* \in G$ *such that*

$$|\tilde z(t)-\tilde z_*(t)| \le 2R\,q^{-1}\exp\{-\kappa t\} \qquad \textit{for all} \qquad t\ge 0, \tag{32}$$

$$\chi(\tilde w(t),\tilde w_*(t)) \le 4L_\Gamma L_D R\,q^{-1}\exp\{-\kappa(t-T)\} \qquad \textit{for all} \qquad t\ge T, \tag{33}$$

$$|\tilde u(t)-\tilde u_*(t)| \le 4L_\Gamma L_D L_\Phi R\,q^{-1}\exp\{-\kappa(t-T)\} \qquad \textit{for all} \qquad t\ge T \tag{34}$$

with $\kappa = -\ln q/((k+1)T) > 0$ *and* $R = \max\{r_0, 2\rho_0\}$, *where* $\tilde u = \Phi(\tilde w)$, $\tilde u_* = \Phi(\tilde w_*)$.

Due to $L_J = 2L_\Gamma$, estimate (29) implies (26) and therefore the conditions of Theorem 2 are satisfied. Remark that Theorem 2 guarantees that the set of all periodic solutions is arcwise connected and consequently any its proper subset is not U-uniformly stable for any closed $U \subseteq B$.

3.4 Proof of Theorem 3

The proof is in three steps. In the first step, we define by induction sequences of periodic functions $z_n : \mathbb{R}_+ \to \mathbb{R}^d$, $u_n : \mathbb{R}_+ \to \mathbb{R}$ for $n \ge 0$ and periodic variable states $w_n : \mathbb{R}_+ \to \Omega$ for $n \ge 1$ for a given solution $(\tilde z,\tilde w)$ of system (2) with $\tilde z(0) \in B$ and prove some estimates for these sequences. At the same time, we construct auxiliary sequences of functions x_n, y_n, initial states $\tilde w_n^0$ and variable states $\tilde w_n$. Note that by Lemma 2 the inclusion $\tilde z(0) \in B$ implies that the solution $(\tilde z,\tilde w)$ is defined on the whole semiaxis $t \in \mathbb{R}_+$ and $\|\tilde z\|_{C[0,\infty)} < r_0$.

Let us fix a real q and a positive integer $k = k(q)$ satisfying (30) and (31) and denote $t_n = n(k+1)T$. For $n = 0$ set $z_0 = z_*$, $u_0 = 0$.

Suppose that the functions z_n, u_n are already defined for some integer $n \ge 0$. Then set $y_n(t) = (1-t)D\tilde z(t_n) + tDz_n(t_n)$ for $t \in [0,1]$ and define $\tilde w_n^0 = (\Gamma[\tilde w(t_n)]y_n)(1)$,

$$x_n(t) = Dz_n(t+t_n), \quad \tilde w_n(t) = (\Gamma[\tilde w_n^0]x_n)(t), \quad t \ge 0.$$

From the semi-group property of Γ it follows $\tilde w(t+t_n) = (\Gamma[\tilde w(t_n)]v)(t)$ with $v(t) = (D\tilde z)(t+t_n)$ for $t \ge 0$, hence the Lipschitz condition (10) implies

$$\begin{aligned}\chi(\tilde w(t+t_n),\tilde w_n(t)) &\le \chi(\tilde w(t_n),\tilde w_n^0) + L_\Gamma\|v-x_n\|_{C[0,t]}\\ &\le \chi(\tilde w(t_n),\tilde w_n^0) + L_\Gamma L_D\|\tilde z - z_n\|_{C[t_n,t_n+t]}.\end{aligned}$$

Since $(\Gamma[w^0]x)(\tau) = w^0$ whenever $x(t) = x(0)$ for all $t \in [0,\tau]$, estimate (10) also implies

$$\chi(\tilde w(t_n),\tilde w_n^0) \le L_\Gamma|y_n(0)-y_n(1)| \le L_\Gamma L_D|\tilde z(t_n) - z_n(t_n)|$$

and therefore

$$\chi(\tilde w(t+t_n),\tilde w_n(t)) \le 2L_\Gamma L_D\|\tilde z - z_n\|_{C[t_n,t_n+t]}, \quad t \ge 0. \tag{35}$$

Set

$$w_{n+1}(t) = \tilde{w}_n(t+T), \quad u_{n+1}(t) = (\Phi(w_{n+1}))(t), \quad t \geq 0.$$

From the semi-group property of the operator Γ and the periodicity of z_n, it follows

$$w_{n+1}(t) = (\Gamma[\tilde{w}_n(T)]v)(t) \quad \text{with } v(t) = x_n(t+T) = Dz_n(t+t_n+T) = Dz_n(t), \ t \geq 0,$$

i.e. $w_{n+1}(t) = (\Gamma[\tilde{w}_n(T)]Dz_n)(t)$. Consequently w_{n+1} and u_{n+1} are some variable state and output of the hysteresis nonlinearity for the input Dz_n. By the monocyclicity property (14), $\tilde{w}_n(t) = \tilde{w}_n(t+T)$ for all $t \geq T$. This implies the periodicity

$$w_{n+1}(t) = w_{n+1}(t+T), \qquad u_{n+1}(t) = u_{n+1}(t+T), \qquad t \geq 0$$

of w_{n+1} and u_{n+1}; the estimate (35) implies

$$\begin{aligned} &\chi(\tilde{w}(t_n+T+t), w_{n+1}(t)) \leq 2L_\Gamma L_D \|\tilde{z} - z_n\|_{C[t_n, t_n+T+t]}, \qquad t \geq 0, \\ &|\tilde{u}(t_n+T+t) - u_{n+1}(t)| \leq 2L_\Gamma L_D L_\Phi \|\tilde{z} - z_n\|_{C[t_n, t_n+T+t]}, \quad t \geq 0. \end{aligned} \tag{36}$$

Now note that from the first of estimates (20) and from the estimate (22) it follows that the translation operator S_ξ of equation (19) defined by $S_\xi(z_0) = z(T; z_0, \xi)$ maps the ball $B \subset \mathbb{R}^d$ into itself and is a contraction on this ball. Therefore equation (19) has a unique periodic solution $z \in G$ for each periodic ξ satisfying $\|\xi\|_{C[0,T]} \leq M$. Denote by $z_{n+1} \in G$ a unique periodic solution of (19) with $\xi = u_{n+1}$. By this, the definition of the sequences $\{z_n\}$, $\{u_n\}$ and $\{w_n\}$ is complete.

In the next step, we prove the estimate

$$\|\tilde{z} - z_n\|_{C[t_n,\infty)} \leq Rq^n \qquad \text{for all} \qquad n = 0, 1, 2, \ldots \tag{37}$$

From (24), we see that

$$\begin{aligned} &\|\tilde{z}(t_n+T+\cdot) - z_{n+1}(\cdot)\|_{C[kT,\infty)} \leq \\ &\leq aq_*^k|\tilde{z}(t_n+T) - z_{n+1}(0)| + (ac_1+b)\,\varepsilon\|\tilde{u}(t_n+T+\cdot) - u_{n+1}(\cdot)\|_{C[0,\infty)} \end{aligned}$$

and due to $z_{n+1}(\cdot) = z_{n+1}(t_n+T+\cdot)$ and (36)

$$\begin{aligned} &\|\tilde{z}(t_n+T+\cdot) - z_{n+1}(t_n+T+\cdot)\|_{C[kT,\infty)} \leq \\ &\leq aq_*^k|\tilde{z}(t_n+T) - z_{n+1}(0)| + 2L_\Gamma L_D L_\Phi(ac_1+b)\,\varepsilon\|\tilde{z} - z_n\|_{C[t_n,\infty)}, \end{aligned}$$

hence

$$\|\tilde{z}-z_{n+1}\|_{C[t_n+T+kT,\infty)} \leq aq_*^k|\tilde{z}(t_n+T)-z_{n+1}(0)|+2L_\Gamma L_D L_\Phi(ac_1+b)\,\varepsilon\|\tilde{z}-z_n\|_{C[t_n,\infty)}. \tag{38}$$

Equation (22) implies

$$|z_{n+1}(T) - z_n(T)| \leq q_*|z_{n+1}(0) - z_n(0)| + (1-q_*)c_1\,\varepsilon\|u_{n+1} - u_n\|_{C[0,T]}$$

and due to the periodicity of z_n and z_{n+1}

$$|z_{n+1}(0) - z_n(0)| \le c_1\,\varepsilon\|u_{n+1} - u_n\|_{C[0,T]}.$$

¿From (36) it follows

$$\|u_{n+1} - u_n\|_{C[0,T]} \le 2L_\Gamma L_D L_\Phi\big(\|\tilde z - z_n\|_{C[t_n,\infty)} + \|\tilde z - z_{n-1}\|_{C[t_{n-1},\infty)}\big),$$

consequently

$$|z_{n+1}(0) - z_n(0)| \le 2L_\Gamma L_D L_\Phi c_1\,\varepsilon\big(\|\tilde z - z_n\|_{C[t_n,\infty)} + \|\tilde z - z_{n-1}\|_{C[t_{n-1},\infty)}\big)$$

and therefore

$$\begin{aligned}&|\tilde z(t_n+T) - z_{n+1}(0)| \le |\tilde z(t_n+T) - z_n(0)| + \\ &+2L_\Gamma L_D L_\Phi c_1\,\varepsilon\big(\|\tilde z - z_n\|_{C[t_n,\infty)} + \|\tilde z - z_{n-1}\|_{C[t_{n-1},\infty)}\big).\end{aligned}$$

Here $|\tilde z(t_n+T) - z_n(0)| = |\tilde z(t_n+T) - z_n(t_n+T)| \le \|\tilde z - z_n\|_{C[t_n,\infty)}$, hence

$$\begin{aligned}&|\tilde z(t_n+T) - z_{n+1}(0)| \le (1 + 2L_\Gamma L_D L_\Phi c_1\,\varepsilon)\|\tilde z - z_n\|_{C[t_n,\infty)} + \\ &+2L_\Gamma L_D L_\Phi c_1\,\varepsilon\|\tilde z - z_{n-1}\|_{C[t_{n-1},\infty)}.\end{aligned}$$

Substituting this estimate in (38), we obtain

$$\begin{aligned}\|\tilde z - z_{n+1}\|_{C[t_{n+1},\infty)} \;\le\;& \big(aq_*^k(1 + 2L_\Gamma L_D L_\Phi c_1\,\varepsilon) + 2L_\Gamma L_D L_\Phi(ac_1 + b)\,\varepsilon\big) \\ & \|\tilde z - z_n\|_{C[t_n,\infty)} \\ +\;& 2aq_*^k L_\Gamma L_D L_\Phi c_1\,\varepsilon\|\tilde z - z_{n-1}\|_{C[t_{n-1},\infty)}.\end{aligned} \tag{39}$$

It follows from (39) and (31) that the estimates $\|\tilde z - z_{n-1}\|_{C[t_{n-1},\infty)} \le Rq^{n-1}$ and $\|\tilde z - z_n\|_{C[t_n,\infty)} \le Rq^n$ imply $\|\tilde z - z_{n+1}\|_{C[t_{n+1},\infty)} \le Rq^{n+1}$. By induction we conclude that if

$$\|\tilde z - z_0\|_{C[t_0,\infty)} \le R, \qquad \|\tilde z - z_1\|_{C[t_1,\infty)} \le Rq, \tag{40}$$

then (37) holds. Since $\|\tilde z - z_0\|_{C[t_0,\infty)} = \|\tilde z - z_*\|_{C[0,\infty)} \le r_0$, the first of the estimates (40) is valid for $R \ge r_0$. Furthermore, (38) implies

$$\|\tilde z - z_1\|_{C[t_1,\infty)} \le aq_*^k|\tilde z(T) - z_1(0)| + 2L_\Gamma L_D L_\Phi(ac_1 + b)\,\varepsilon\|\tilde z - z_0\|_{C[t_0,\infty)},$$

therefore if $\|\tilde z - z_0\|_{C[t_0,\infty)} \le R$, then $\|\tilde z - z_1\|_{C[t_1,\infty)} \le 2aq_*^k\rho_0 + 2L_\Gamma L_D L_\Phi(ac_1 + b)\,\varepsilon R$ and due to (31),

$$\|\tilde z - z_1\|_{C[t_1,\infty)} \le \frac{2\rho_0(q - 2L_\Gamma L_D L_\Phi(ac_1 + b)\varepsilon)}{1 + 2L_\Gamma L_D L_\Phi c_1\,\varepsilon(1+q)/q} + 2L_\Gamma L_D L_\Phi(ac_1 + b)\,\varepsilon R.$$

For $R \ge 2\rho_0$, the right-hand side here is not greater than Rq and hence the second of the estimates (40) is valid. Thus, both the estimates (40) are valid for $R = \max\{r_0, 2\rho_0\}$ and consequently (37) holds with this R.

Now we are ready for the last step of the proof. Since the functions z_n are T-periodic, the estimate (37) implies $\|z_n - z_\ell\|_{C[0,T]} \le R(q^n + q^\ell)$ for any nonnegative integers

n, ℓ. Therefore the sequence $\{z_n\}$ is fundamental in the space of all continuous T-periodic functions with the norm $\|\cdot\|_{C[0,T]}$ and converges to a T-periodic continuous function $\tilde{z}_* : \mathbb{R}_+ \to \mathbb{R}^d$ such that $\|z_n - \tilde{z}_*\|_{C[0,T]} \le Rq^n$ for all $n = 0, 1, 2, \dots$ Similarly, the estimates

$$\chi(\tilde{w}(t), w_{n+1}(t)) \le 2L_\Gamma L_D R q^n, \quad |\tilde{u}(t) - u_{n+1}(t)| \le 2L_\Gamma L_D L_\Phi R q^n \text{ for all } t \ge t_n + T \tag{41}$$

with every $n = 0, 1, 2, \dots$ (which follow from (36)) imply that the sequences $\{w_n\}$ and $\{u_n\}$ converge uniformly to some T-periodic continuous functions $\tilde{w}_* : \mathbb{R}_+ \to \Omega$ and $\tilde{u}_* : \mathbb{R}_+ \to \mathbb{R}$ and

$$\chi(\tilde{w}_*(t), w_{n+1}(t)) \le 2L_\Gamma L_D R q^n, \qquad |\tilde{u}_*(t) - u_{n+1}(t)| \le 2L_\Gamma L_D L_\Phi R q^n \quad \text{for all} \quad t \in [0, T] \tag{42}$$

for any $n = 0, 1, 2, \dots$ Recall that by construction w_{n+1} and $u_{n+1} = \Phi(w_{n+1})$ are some periodic variable state and periodic output of the hysteresis nonlinearity for the periodic input Dz_n and that z_{n+1} is the periodic solution of equation (19) for $\xi = u_{n+1}$. Therefore the relations

$$\max_{t \in [0,T]} \chi(w_n(t), \tilde{w}_*(t)) \to 0, \quad \|u_n - \tilde{u}_*\|_{C[0,T]} \to 0, \quad \|z_n - \tilde{z}_*\|_{C[0,T]} \to 0 \quad \text{as} \quad n \to \infty$$

imply that $\tilde{w}_*$ and $\tilde{u}_* = \Phi(\tilde{w}_*)$ are a periodic variable state and periodic output of the hysteresis nonlinearity for the periodic input $D\tilde{z}_*$ and $\tilde{z}_*$ is the periodic solution of equation (19) for $\xi = \tilde{u}_*$. Equivalently, it means that $(\tilde{z}_*, \tilde{w}_*)$ is a T-periodic solution of system (2) which belongs to the class $[\tilde{z}_*, \tilde{u}_*]$.

Relations (37), (41), (42) and $\|z_n - \tilde{z}_*\|_{C[0,T]} \le Rq^n$ imply for all $n = 0, 1, 2, \dots$

$$\|\tilde{z} - \tilde{z}_*\|_{C[t_n, \infty)} \le 2Rq^n, \qquad \sup_{t \ge t_n + T} \chi(\tilde{w}(t), \tilde{w}_*(t)) \le 4L_\Gamma L_D R q^n,$$
$$\|\tilde{u} - \tilde{u}_*\|_{C[t_n + T, \infty)} \le 4L_\Gamma L_D L_\Phi R q^n$$

and due to $t_n = n(k+1)T$, the estimates (32), (33) and (34) are valid with $\kappa = -\ln q/((k+1)T)$. This completes the proof of the theorem.

3.5 Remarks

a. Let $z : \mathbb{R}_+ \to \mathbb{R}^d$ be a U-uniformly stable solution of the unperturbed system (1) for some neighborhood U of the initial value $z(0)$. The union of all bounded open sets U such that z is U-uniformly stable is called the basin of attraction of the solution z. It is easy to see by the continuity argument that if a bounded closed set $U_0 \supseteq B$ belongs to the basin of attraction of the periodic solution z_* of (1), then under the conditions of Theorem 3 the continual set of all periodic solutions (z, w) of system (2) with $z \in G$ is U_0-uniformly stable whenever $\varepsilon > 0$ is sufficiently small (the interval of ε depends on U_0).

b. Denote by $Z(\varepsilon, G)$ the set of all T-periodic solutions (z, w) of system (2) with $z \in G$. From (22) it follows that $|z(T) - z_*(T)| \le q_*|z(0) - z_*(0)| + (1 - q_*)c_1 M\varepsilon$ for each solution $(z, w) \in Z(\varepsilon, G)$. Equivalently, $|z(0) - z_*(0)| \le c_1 M\varepsilon$, which together with (21) implies $\|z - z_*\|_{C[0,T]} \le (ac_1 + b)M\varepsilon$. Thus, the continual B-uniformly stable set $Z(\varepsilon, G)$ of periodic solutions of (2) enjoys the property

$$\sup\{\|z - z_*\|_{C[0,T]} : \ (z, w) \in Z(\varepsilon, G)\} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

c. Let us say that some set Z of solutions (z, w) of system (2) is *globally asymptotically stable* if there is a sequence of open bounded sets $U_1 \subset U_2 \subset \cdots$ such that the union of U_n is $\mathbb{R}^d$ and Z is U_n-uniformly stable for each n. Local Theorems 1 – 3 can be sometimes extended to statements on the global asymptotic stability of the set of periodic solutions. As a simple example, consider system (2) with the first equation of the form $\dot z = Az + g(t, z, \varepsilon u(t))$, where A is a stable matrix (the spectrum of A belongs to the open left half-plane of the complex plane) and the function $g(\cdot, \cdot, \cdot)$ satisfies the global Lipschitz condition

$$|g(t, z_1, \xi_1) - g(t, z_2, \xi_2)| \le L_z|z_1 - z_2| + L_\xi|\xi_1 - \xi_2|, \qquad z_j \in \mathbb{R}^d, \ |\xi_j| \le M.$$

This estimate and the relation $\dot z - \dot z_* = A(z - z_*) + (g(t, z, \varepsilon u) - g(t, z_*, 0))$ imply that there is a number $L_0 = L_0(A)$ determined by the matrix A such that if $L_z < L_0$ then the sequence $z(0)$, $z(T)$, $z(2T), \dots$ converges exponentially to the ball $\{z_0 \in \mathbb{R}^d : \ |z_0 - z_*(0)| \le \delta_\varepsilon\}$ with $\delta_\varepsilon = O(\varepsilon)$ for any solution (z, w) of (2). Since this ball is contained in the ball B for small ε and (due to Theorem 3) the continual set $Z(\varepsilon, G)$ of periodic solutions satisfying $z \in G$ is B-uniformly stable, it follows that the estimate $L_z < L_0$ implies the global asymptotic stability of the set $Z(\varepsilon, G)$ for any sufficiently small ε.

d. Our main assumption (25) means that $\text{mes}_\mu E(x_m, x_M) > 0$ for all periodic continuous functions $x = Dz$ with z from the ball $\|z - z_*\|_{C[0,T]} \le r_0$. Suppose that $\text{mes}_\mu E(x_m, x_M) = 0$ for all such x, i.e., instead of (25) the relation

$$\begin{aligned} &\text{mes}_\mu\{(\alpha, \beta) \in \Pi : \beta < r'_m, \alpha > r'_M\} = 0; \\ &r'_m = \min_{t \in [0,T]} Dz_*(t) + r_0|D|, \quad r'_M = \max_{t \in [0,T]} Dz_*(t) - r_0|D| \end{aligned}$$

holds. The equality $\text{mes}_\mu E(x_m, x_M) = 0$ implies that formulas (12) of Lemma 2 define the same periodic variable state $w_* = \Gamma[Qx]x$ of the Preisach nonlinearity for all η (recall that the functions that coincide almost everywhere with respect to the measure mes_μ represent the same state) and the formula $u = \mathcal{J}_\lambda x$ defines the same periodic output $u_* = \Phi(w_*)$ for all λ. In other words, there is a unique periodic variable state and periodic output for each periodic input $x = Dz$ such that $\|z - z_*\|_{C[0,T]} \le r_0$. In this situation, the conclusions of Theorems 1 – 3 should be modified as follows. Estimate $\varepsilon < \varepsilon_0$ ensures the existence of at least one periodic solution $(\tilde z_*, \tilde w_*)$ of system (2) with $\tilde z_* \in G$. If in addition condition (26) of Theorem 2 is valid, then the periodic solution $(\tilde z_*, \tilde w_*)$ with $\tilde z_* \in G$ is unique. Estimate (29) of Theorem 3 implies that this solution is B-uniformly stable and for all solutions

$(\tilde z,\tilde w) = (z^\varepsilon(\cdot;z_0,w_0), w^\varepsilon(\cdot;z_0,w_0))$ with the initial values $z_0\in B$ relations (32) – (34) hold. These assertions continue the results of [3].

e. Analogs of the results presented in the paper are valid for systems with hysteresis nonlinearities of some other types, in particular the Prandtl – Ishlinskii models, vector generalizations of the Preisach and Prandtl – Ishlinskii nonlinearities, scalar and vector stops and plays, final sets of such nonlinearities, etc. The results can also be developed and modified in other directions. For example, natural modifications of Theorems 1 – 3 are valid for the autonomous system $\dot z = f(z,\varepsilon u)$, $u(t) = (\Phi(w))(t)$, $w(t) = (\Gamma[w_0]Dz)(t)$. Assuming that the unperturbed equation $\dot z = f(z,0)$ has an asymptotically orbitally stable cycle and all $d-1$ characteristic multipliers of this cycle belong to the open unit disc of the complex plane, one can formulate statements on the existence and U-uniform orbital stability of a continual set of cycles for the system with hysteresis perturbation. The proofs for the autonomous problem are more complicated.

4 Appendix: proof of Lemma 2

Fix a $c_0\ge 1$ such that $c_0\ge T|Y(T)|\max_{t\in[0,T]}|Y^{-1}(t)|$, where $|\cdot|$ denotes the matrix norm generated by the norm in $\mathbb{R}^d$. Define

$$\varepsilon(r) = \frac{(1-q_*)(q_*-q_0)r}{(1-q_0)ML_\xi(r)(c_0e^{L_z(r)T} + (q_*-q_0)(e^{L_z(r)T}-1)/L_z(r))},$$

$$\rho(r) = \frac{(q_*-q_0)r}{(1-q_0)e^{L_z(r)T}},\qquad c(r) = \frac{L_\xi(r)(c_0L_z(r) + (q_*-q_0)(1-e^{-L_z(r)T}))}{L_z(r)(1-q_*)}.$$

Consider the Jacobi matrix $A(t,z,\xi) = \partial f(t,z,\xi)/\partial z$. Since $A(\cdot,\cdot,\cdot)$ depends continuously on its arguments, we can choose a $r_0>0$ such that

$$\begin{aligned} &c_0e^{L_z(r_0)T}\max_{t\in[0,T]}|A(t,z_*(t)+y,\xi) - A(t,z_*(t),0)| \le q_*-q_0 \\ &\text{for all}\quad |y|\le r_0,\ |\xi|\le M\varepsilon(r_0) \end{aligned}\tag{43}$$

and in addition $\varepsilon(r_0)\le 1$. Let us show that the conclusions of the lemma are valid for such a r_0 and for $\varepsilon_0=\varepsilon(r_0)$, $\rho_0=\rho(r_0)$ and $c_1=c(r_0)$.

Consider a pair of solutions $z_j(\cdot) = z^\varepsilon(\cdot;z_j^0,\xi_j)$ of system (19) with $\varepsilon<\varepsilon_0$, $\|\xi_j\|_{C[0,\infty)}\le M$, $|z_j^0 - z_*(0)|\le\rho_0$. By the Lipschitz condition (18), on any segment $[0,t]$ such that $\|z_j-z_*\|_{C[0,t]}\le r_0$ for both $j=1,2$, the difference z_1-z_2 satisfies $\dot z_1-\dot z_2\le L_z(r_0)|z_1-z_2| + \varepsilon L_\xi(r_0)\|\xi_1-\xi_2\|_{C[0,t]}$ and from Gronwall's inequality it follows that

$$\|z_1-z_2\|_{C[0,t]}\le e^{L_z(r_0)t}|z_1^0-z_2^0| + \varepsilon\|\xi_1-\xi_2\|_{C[0,t]}(e^{L_z(r_0)t}-1)L_\xi(r_0)/L_z(r_0).\tag{44}$$

Particularly, for $z_1 = z(\cdot) = z^{\varepsilon}(\cdot; z_0, \xi)$ and $z_2 = z_*(\cdot)$ we have

$$\begin{array}{rcl} \|z - z_*\|_{C[0,t]} & \leq & e^{L_z(r_0)t}|z_0 - z_*(0)| + \varepsilon\|\xi\|_{C[0,t]}L_\xi(r_0)(e^{L_z(r_0)t} - 1)/L_z(r_0) \\ & < & e^{L_z(r_0)t}\rho_0 + \varepsilon_0 M L_\xi(r_0)(e^{L_z(r_0)t} - 1)/L_z(r_0) \end{array}$$

as long as $\|z - z_*\|_{C[0,t]} \leq r_0$, therefore the estimate

$$e^{L_z(r_0)T}\rho_0 + \varepsilon_0 M L_\xi(r_0)(e^{L_z(r_0)T} - 1)/L_z(r_0) \leq r_0$$

(which follows from the definition of $\varepsilon_0 = \varepsilon(r_0)$ and $\rho_0 = \rho(r_0)$) implies

$$\|z - z_*\|_{C[0,T]} < r_0 \quad \text{whenever} \quad |z_0 - z_*(0)| \leq \rho_0 \tag{45}$$

for every solution z of (19) with $\varepsilon < \varepsilon_0$, $\|\xi\|_{C[0,\infty)} \leq M$. This allows to conclude that (44) holds for $t = T$, i.e., the estimate (21) is valid with $a = e^{L_z(r_0)T}$, $b = (e^{L_z(r_0)T} - 1)L_\xi(r_0)/L_z(r_0)$ whenever $|z_j^0 - z_*(0)| \leq \rho_0$, $j = 1, 2$.

Set

$$\Psi(t) = f(t, z_1(t), \varepsilon\xi_1(t)) - f(t, z_2(t), \varepsilon\xi_1(t)) - A(t, z_*(t), 0)(z_1(t) - z_2(t)),$$

$$\Xi(t) = f(t, z_2(t), \varepsilon\xi_1(t)) - f(t, z_2(t), \varepsilon\xi_2(t)).$$

Using this notation, we can rewrite the equation $\dot{z}_1 - \dot{z}_2 = f(t, z_1, \varepsilon\xi_1) - f(t, z_2, \varepsilon\xi_2)$ as $\dot{z}_1 - \dot{z}_2 = A(t, z_*(t), 0)(z_1 - z_2) + (\Psi(t) + \Xi(t))$. This implies

$$z_1(t) - z_2(t) = Y(t)(z_1^0 - z_2^0) + \int_0^t Y(t)Y^{-1}(s)(\Psi(s) + \Xi(s))\, ds, \qquad 0 \leq t \leq T.$$

By definition of c_0 and due to the estimate (17), it follows

$$|z_1(T) - z_2(T)| \leq q_0|z_1^0 - z_2^0| + c_0(\|\Psi + \Xi\|_{C[0,T]}). \tag{46}$$

Consider the equality

$$\Psi(t) = \int_0^1 (A(t, z_2(t) + s(z_1(t) - z_2(t)), \varepsilon\xi_1(t)) - A(t, z_*(t), 0))(z_1(t) - z_2(t))\, ds.$$

Here $\|z_j - z_*\|_{C[0,T]} < r_0$, $\varepsilon\|\xi_1\|_{C[0,T]} \leq M\varepsilon(r_0)$ and therefore (43) implies $\|\Psi\|_{C[0,T]} \leq \|z_1 - z_2\|_{C[0,T]}(q_* - q_0)e^{-L_z(r_0)T}/c_0$. Combining this estimate with (44) for $t = T$, we obtain

$$c_0\|\Psi\|_{C[0,T]} \leq (q_* - q_0)(|z_1^0 - z_2^0| + \varepsilon\|\xi_1 - \xi_2\|_{C[0,T]}(1 - e^{-L_z(r_0)T})L_\xi(r_0)/L_z(r_0)).$$

From the Lipschitz estimate (18), it follows that $\|\Xi\|_{C[0,T]} \leq \varepsilon L_\xi(r_0)\|\xi_1 - \xi_2\|_{C[0,T]}$, therefore

$$\begin{array}{rcl} c_0\|\Psi + \Xi\|_{C[0,T]} & \leq & (q_* - q_0)|z_1^0 - z_2^0| + \varepsilon\|\xi_1 - \xi_2\|_{C[0,T]}L_\xi(r_0) \\ & & (c_0 + (q_* - q_0)(1 - e^{-L_z(r_0)T})/L_z(r_0)) \\ & = & (q_* - q_0)|z_1^0 - z_2^0| + (1 - q_*)\varepsilon c_1\|\xi_1 - \xi_2\|_{C[0,T]} \end{array}$$

and (46) implies the estimate (22). Furthermore, substituting $z_1 = z(\cdot) = z^\varepsilon(\cdot; z_0, \xi)$ and $z_2 = z_*(\cdot)$ in (22), we obtain

$$|z(T) - z_*(T)| \leq q_*|z_0 - z_*(0)| + (1 - q_*)c_1\varepsilon\|\xi\|_{C[0,T]} < q_*\rho_0 + (1 - q_*)c_1\varepsilon_0 M$$

and this together with the estimate $c_1\varepsilon_0 M \leq \rho_0$, which follows from the definitions of the constants c_1, ε_0 and ρ_0, implies the first of the relations (20). Finally, the second of these relations follows from the first one and from the estimate (45). This completes the proof.

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