Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Homoclinic tangencies of arbitrarily high orders in conservative and dissipative two-dimensional maps

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submitted: September 28th 2004

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No. 968
Berlin 2004

WIAS

Key words and phrases. non-hyperbolic dynamics, global bifurcations, Hamiltonian chaos, elliptic islands.

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Abstract

We show that maps with infinitely many homoclinic tangencies of arbitrarily high orders are dense among real-analytic area-preserving diffeomorphisms in the Newhouse regions.

1 Introduction

In [1, 2, 3, 5] we have established that an arbitrarily small smooth perturbation of a two-dimensional map with a quadratic homoclinic tangency can produce homoclinic tangencies of arbitrarily high orders and, as a consequence, arbitrarily degenerate periodic orbits (see also [4]). These results have shown that global bifurcations of codimension 1 can be accompanied by bifurcations of arbitrarily high codimension, i.e. the unfolding of global bifurcations can lead to the increase of the level of degeneracy, contrary to the usual logic coming from the singularity theory.

Based on this we made a conclusion that a complete description of dynamics and bifurcations of systems with homoclinic tangencies is impossible in principle (see more discussion in [3, 4, 5]). We recall that systems with homoclinic tangencies are dense in open regions in the space of smooth dynamical systems [6, 7, 9]. Moreover, these regions (Newhouse regions) exist near any system with a homoclinic tangency [9, 10, 11, 13, 14]. In fact, homoclinic tangencies and, hence, Newhouse regions in the parameter space have been found in a huge variety of different models with chaotic dynamics. Thus, they exist in the Hénon map (see discussion in [15]), in the standard map [12] and in "soft billiards" [16], they obviously appear in the process of the development of a Smale horseshoe (after period-doubling), they play a central role in in the transition from quasiperiodicity to chaos (the destruction of invariant tori) [17, 45, 18, 19, 22], they are present in Lorenz-like models beyond the boundary of the region of existence of Lorenz attractor [20, 21], in systems with "spiral chaos", like Chua circuit of Rössler model (see [23, 24]), and with wild spiral attractor [25]. According to our results in [1]-[5], in all these models one should expect an incomprehensibly complex behavior.

Recently, it has been realized that the density of systems with homoclinic tangencies of arbitrarily high orders in the Newhouse regions is a useful working tool for proving that many seemingly exotic dynamical phenomena are, in fact, generic. Thus, it was shown in [26] that the results of [1]-[5] disprove a Smale's conjecture on the genericity of the exponential growth of the number of periodic orbits with period. In [27], our results were used to show that generic two-dimensional C^r-diffeomorphisms from the Newhouse regions, with r finite, cannot be topologically conjugate to any C^∞-diffeomorphism, and that they have transitive sets of full Hausdorff dimension. In the same manner, in [28] the ultimate topological complexity of such sets was established, and it was shown in [29] that the probability for a system to have infinitely many coexisting stable periodic orbits is positive for a dense set of finite-parameter families.

The fact that systems with homoclinic tangencies of arbitrarily high orders are dense in the Newhouse regions was proven in [3, 5] for the space of general smooth maps, and one of our genericity conditions excluded area-preserving maps. Therefore, the validity of the result (and the above cited results based on it) in the area-preserving case can be questioned. In the present paper we close the problem and provide a unified proof which works in the area-preserving case as well. Moreover, we enhance our perturbation technique so that the new proof covers the real analytic case too.

Let f be a diffeomorphism of a two-dimensional manifold. We assume f to be C^r ($r = 2, \ldots, \infty$) or C^ω (i.e. real-analytic). Let f have a saddle periodic orbit L whose stable and unstable manifolds have a quadratic tangency at some point M. This is a tangency of invariant manifolds, therefore they are tangent at each point of the orbit of M. By construction, this orbit Γ is homoclinic to L, i.e. it closes on L both at forward and backward iterations of f.

Fix any, sufficiently large compact subset K of the phase space (in the theorem below we assume that K contains a neighborhood of $\Gamma \cup L$). Fix also some small complex neighborhood Q of K. For small δ we will say that two real analytic diffeomorphisms are δ-close if they are δ-close at every point of Q. For finite r we will say that two C^r-smooth diffeomorphisms are δ-close if they are δ-close on K in a C^r-metric. Two C^∞-diffeomorphisms will be called δ-close if they are $(r \cdot \delta)$-close on K in a C^r-metric for every $r < \frac{1}{\delta}$. We can now formulate our main theorem.

Theorem 1. *Arbitrarily close to f there exists a diffeomorphism f^* (an area-preserving one if f itself is area-preserving) for which there exist infinitely many orbits of homoclinic tangency of every order between the stable and unstable manifolds of L.*

The proof occupies Sections 2-4. In Section 2 we give necessary formulas for the Poincaré return maps near the periodic and homoclinic orbits and describe the form of the perturbations which we use (Lemma 1). In Section 3 we prove certain key lemmas and in Section 4 we construct the sequence of perturbations which leads from f to f^*.

The main reason why the homoclinic tangency can be perturbed in such a way that a tangency of a higher order is created is the presence of a "hidden degeneracy" in the system. Thus, it was established in [34, 35, 36] that non-conservative systems with a homoclinic tangency of the "third class" in the terminology of [31] have a modulus (i.e. a continuous invariant) of local Ω-conjugacy (i.e. the topological conjugacy on the set of nonwandering orbits which lie entirely in a small neighborhood of the orbit of homoclinic tangency). Such a modulus is, for example, the ratio θ of the logarithms of the multipliers of the saddle periodic orbit to which the given homoclinic orbit converges. This means that two such systems cannot be locally Ω-conjugate if the corresponding values of θ are different. As a result, θ can be taken as an additional bifurcation parameter, by changing which more degenerate homoclinic tangencies can be obtained (see more discussion in [37, 5]).

Two-dimensional area-preserving maps with homoclinic tangencies have no moduli [33] (thus $\theta \equiv 1$ for such systems). Therefore, in order to prove the main theorem in the area-preserving case, we first prove (Lemmas 2,3) that a small perturbation of a map with a homoclinic tangency can produce a heteroclinic cycle with two different saddle periodic orbits, one transverse heteroclinic orbit, and one orbit of heteroclinic tangency; moreover, such heteroclinic cycle belongs to the third class of [38]. Since the heteroclinic cycles of the third class have local Ω-moduli both in dissipative and conservative case [38], we can prove that systems with homoclinic tangencies of arbitrarily high orders are dense among the systems with such heteroclinic cycles, by applying a refined version of the machinery (here Lemmas 4,5) developed in [3, 5].

In fact, a stronger statement is proved in Section 4. Namely, we show that the diffeomorphism f^* constructed in Theorem 1 has a non-trivial hyperbolic set (a horseshoe) which includes the original saddle periodic orbit L, and there exists infinitely many orbits of tangency of every order between stable and unstable manifolds of every periodic orbit in this hyperbolic set.

We might as well assume from the very beginning that the original diffeomorphism f has a zero-dimensional transitive hyperbolic set Λ whose stable and unstable manifolds have a tangency. Then, our main theorem is reformulated as follows (the proof is also given in Section 4):

Theorem 2. *Arbitrarily close to f there exists a diffeomorphism f^* (an area-preserving one if f itself is area-preserving) for which there exist infinitely many orbits of tangency of every order between the stable and unstable manifolds of every pair of periodic orbits of Λ.*

As we mentioned, the C^r-closure ($r = 2, ..., \infty, \omega$) of the set of C^r-maps with homoclinic tangencies contains open (Newhouse) regions. For the space of all two-dimensional C^r-maps this statement was proved in [9], while extending this result onto the space of two-dimensional area-preserving C^r-maps was a long-standing open problem, until the proof was obtained in [13, 14]. It also follows from [8, 13, 14] that the C^1-closure of the Newhouse regions in the space of two-dimensional area-preserving diffeomorphisms coincides with the set of all non-Anosov area-preserving diffeomorphisms (whether the same remains true in the C^r-topology with $r \geq 2$ occurs to be a so far intractable question). Our main theorem immediately implies

Theorem 3. *Maps with infinitely many homoclinic tangencies of all orders are dense in the Newhouse regions.*

2 Preliminary constructions.

2.1 Local map.

Consider a C^r-diffeomorphism f of a two-dimensional manifold, $r = 2, ..., \infty, \omega$, where $r = \omega$ stands for real-analytic diffeomorphisms. Let f have a saddle periodic orbit L. This means that there is a point O such that $f^m O = O$ (the positive integer m is the period of L, and $L = \{O, fO, ..., f^{m-1}O\}$), and that one can introduce coordinates (x, y) with the origin at O such that the map $f^m : (x, y) \mapsto (\bar{x}, \bar{y})$ will have the following form near O:

$$\bar{x} = \lambda x + o(x, y), \qquad \bar{y} = \gamma y + o(x, y), \tag{1}$$

where $|\lambda| < 1$ and $|\gamma| > 1$. In the case of area-preserving map f, we have additionally

$$|\lambda\gamma| = 1. \tag{2}$$

We will denote the map f^m near the point O as T_0 and will call it the local map. The saddle fixed point $O(0, 0)$ of T_0 has the stable and unstable invariant C^r-manifolds which have, locally, the form $y = \varphi(x)$ and $x = \varphi(y)$, respectively, with $\varphi(0) = 0$, $\varphi'(0) = 0$, $\psi(0) = 0$, $\psi'(0) = 0$.

Let $\phi(x)$ be the inverse function to $x \mapsto x - \varphi(\psi(x))$, i.e. $\phi(x - \varphi(\psi(x))) = x$ at small x. The area-preserving coordinate transformation

$$(x, y)^{new} = (x - \varphi(y), y - \psi(\phi(x^{new}))) \tag{3}$$

straightens the local invariant manifolds, i.e. they take the form $x^{new} = 0$ and $y^{new} = 0$. Hence, the local map (1) takes the following form in the new coordinates

$$\bar{x} = \lambda x + p(x, y)x, \qquad \bar{y} = \gamma y + q(x, y)y, \tag{4}$$

where the functions p and q vanish at the origin. Note that if we consider a family f_ε of maps, C^r with respect to both (x, y) and the parameters ε, then the invariant manifolds of saddle periodic orbits are C^r with respect to the parameters as well, so the local map near a saddle periodic point can be brought to the form (4) by a C^r transformation for all ε.

When proving the results of Sections 3 and 5 we will use the fact (see [32, 33, 36]) that by an additional, close to identity coordinate transformation, one may achieve that the functions p and q will vanish identically both at $x = 0$ and $y = 0$:

$$\begin{aligned} p(x, 0) &\equiv 0, \qquad p(0, y) \equiv 0, \\ q(x, 0) &\equiv 0, \qquad q(0, y) \equiv 0. \end{aligned} \tag{5}$$

If the map f is area-preserving, then this coordinate transformation can be chosen area-preserving too [40]. Indeed, according to [36], there exists a function $\zeta(x,y)$, vanishing identically at $x=0$, such that the first line in (5) is satisfied after the implicitly defined change $x = x^{new} + \zeta(x^{new}, y)$ of the coordinate x. Let us make the area-preserving coordinate transformation defined by the formulas

$$x = x^{new} + \zeta(x^{new}, y), \quad y^{new} = y + \int_0^y \zeta'_x(x^{new}, s)ds.$$

By construction, after this change of coordinates, the map (4) will satisfy the first line of identities (5), and it will remain area-preserving. The latter means that

$$\det\begin{pmatrix} \lambda + p(x,y) + p'_x(x,y)x & p'_y(x,y)x \\ q'_x(x,y)y & \gamma + q(x,y) + q'_y(x,y)y \end{pmatrix} = \lambda\gamma$$

for all small (x,y). At $x=0$ or $y=0$ this identity reduces, respectively, to $q(0,y) + q'_y(0,y)y = 0$ and $q(x,0) = 0$, which gives us the second line of identities (5) indeed.

In the case where the smoothness r of f is finite, the given coordinate transformation is C^{r-1} and it is, in general, only C^{r-2} with respect to ε if f depends on parameters ε (see [40, 41]).In the case $r=\infty$ or $r=\omega$, the coordinate transformation is also C^∞ or, respectively, C^ω with respect to (x,y), and we may ensure any finite smoothness with respect to the parameters.

According to [42, 43], for any small $x^{(0)}$ and $y^{(k)}$ and for any $k \geq 0$ there exist uniquely defined, small $x^{(k)}$ and $y^{(0)}$ such that $(x^{(k)}, y^{(k)}) = T_0^k(x^{(0)}, y^{(0)})$ and all the points in the orbit $\{(x^{(0)}, y^{(0)}), T_0(x^{(0)}, y^{(0)}), ..., T_0^k(x^{(0)}, y^{(0)})\}$ lie in a small neighborhood of zero. We denote

$$x^{(k)} = \lambda^k x^{(0)} + \lambda^k \xi_k(x^{(0)}, y^{(k)}), \qquad y^{(0)} = \gamma^{-k} y^{(k)} + \gamma^{-k}\eta_k(x^{(0)}, y^{(k)}) \tag{6}$$

(in the case where the map depends on parameters, ξ_k and η_k are functions of ε as well). By [32, 33, 36, 41], when the identities (5) are satisfied, the functions ξ_k and η_k are uniformly small along with all the derivatives up to the order $(r-1)$ with respect to (x_0, y_k) and up to the order $(r-2)$ with respect to parameters:

$$||\xi_k, \eta_k|| = o(1)_{k\to+\infty} \tag{7}$$

(in the case of infinite r we have the uniform smallness for all the derivatives up to any given finite order).

2.2 Global map.

Let us now assume that the map f has an orbit of homoclinic tangency. It means that in the local unstable manifold W^u_{loc} of the point O there is a point $M^-(0, y^-)$ such that its image $M^+ = f^l M^-$ for some positive integer l lies in the local stable manifold W^s_{loc} of O, and the curve $f^l W^u_{loc}$ is tangent to W^s_{loc} at the point M^+. The orbit of the point M^- is homoclinic, because all its iterations tend to O both at forward and backward iterations of f.

We call the map f^l in a small neighborhood of M^- the global map and denote it by T_1. It can be written as

$$\begin{aligned} \bar{x} - x^+ &= ax + b(y - y^-) + g_1(x, y), \\ \bar{y} &= cx + \Phi(y) + g_2(x, y), \end{aligned} \tag{8}$$

where the functions g_1 and g_2 do not contain linear terms, and g_2 vanish identically at $x = 0$. By (8), the equation of the curve $T_1 W^u_{loc}$ is

$$\bar{x} = x^+ + b(y - y^-) + g_1(0, y), \qquad \bar{y} = \Phi(y). \tag{9}$$

The condition of the tangency of $T_1 W^u_{loc}$ and W^s_{loc} at $y = y^-$ reads as

$$\Phi(y^-) = 0, \qquad \Phi'(y^-) = 0. \tag{10}$$

Note that f is a diffeomorphism, hence det $T_1'(M^-) \neq 0$; i.e.

$$bc \neq 0. \tag{11}$$

Note that the same holds true when we consider a heteroclinic tangency, i.e. when we have two saddle periodic points, O_1 and O_2, and the image of some piece of $W^u_{loc}(O_1)$ by some iteration of the map f is tangent to $W^s_{loc}(O_2)$ at some point: if the local invariant manifolds are straightened, i.e. the local maps are brought to the form (4) near O_1 and O_2, then the global map T_1 acting from a small neighborhood of some point $M^- \in W^u_{loc}(O_1)$ into a small neighborhood of some point $M^+ \in W^s_{loc}(O_2)$ is defined, and formulas (8)–(11) hold.

The homoclinic or heteroclinic tangency has the order n if $\Phi^{(n+1)}(y^-) \neq 0$ while $\Phi^{(j)}(y^-) = 0$ for all $j \leq n$ (so the quadratic tangency is the tangency of order 1). Of course, to define the tangency of order n, we should require from our map at least the smoothness $r \geq n + 1$. When the map f depends on parameters ε, the global map T_1 can still be written in the form (8), but the functions g_1, g_2, Φ and the coefficients a, b, c, x^+ and y^- may now depend on ε. If we have a homoclinic (or heteroclinic) tangency of order n at $\varepsilon = 0$, we may choose $y^-(\varepsilon)$ in such a way, that $\Phi^{(n)}(y^-) = 0$ for all small ε. We will always fix this choice of $y^-(\varepsilon)$, and we denote, under this assumption,

$$\mu_j(\varepsilon) = \Phi^{(j)}(y^-)/j! \quad (j = 0, \ldots, n-1), \tag{12}$$

so that

$$\Phi(y, \varepsilon) = \mu_0 + \ldots + \mu_{n-1}(y - y^-)^{n-1} + d(y - y^-)^{n+1} + o((y - y^-)^{n+1}), \tag{13}$$

with $d \neq 0$. The tangency is said to be split generically if

$$\text{rank } \partial(\mu_0, \ldots, \mu_{n-1})/\partial\varepsilon = n$$

at $\varepsilon = 0$.

Let us rewrite the parametric equation (9) for the curve $T_1 W^u_{loc}$ in the explicit form: $\bar{y} = \Psi_\varepsilon(\bar{x})$. When there is a tangency of order n, we have $\Psi^{(n+1)}(x^+) \neq 0$, while $\Psi^{(j)}(x^+) = 0$ for all $j \leq n$ at $\varepsilon = 0$. It follows that we may choose $x^+(\varepsilon)$ in such a way, that $\Psi^{(n)}(x^+) = 0$ for all small ε. If we denote, under this assumption,

$$\tilde{\mu}_j(\varepsilon) = \Psi^{(j)}(x^+)/j! \quad (j = 0, \dots, n-1), \tag{14}$$

then, obviously, the vector $(\tilde{\mu}_0, \dots, \tilde{\mu}_{n-1})$ and the vector of the functionals μ_j defined by (12) are related by a diffeomorphism. Hence, the equivalent condition for the tangency to be split generically is

$$\operatorname{rank} \partial(\tilde{\mu}_0, ..., \tilde{\mu}_{n-1})/\partial\varepsilon = n.$$

Analogously, one can easily see from (8) that the curve $T_1^{-1} W^s_{loc}$ near the point M^- is given by the equation

$$x = \hat{\mu}_0 + ... + \hat{\mu}_{n-1}(y - y^-)^{n-1} + \hat{d}(y - y^-)^{n+1} + o((y - y^-)^{n+1}), \tag{15}$$

with $\hat{d} \neq 0$, and with $(\hat{\mu}_0, \dots, \hat{\mu}_{n-1})$ related to the vector of μ_j by a diffeomorphism. This gives us one more equivalent condition for the tangency to be split generically:

$$\operatorname{rank} \partial(\hat{\mu}_0, ..., \hat{\mu}_{n-1})/\partial\varepsilon = n.$$

2.3 Splitting of homoclinic and heteroclinic tangencies.

Below we will frequently use the existence of an n-parameter family of maps (area-preserving maps, if the original map f is area-preserving) in which a given homoclinic or heteroclinic tangency of order n is split generically. Let us first recall the construction for the case of finite smoothness. Let $\bar{y} = \Psi(\bar{x})$ be the equation of the curve $T_1 W^u_{loc}$. Include the function Ψ into any C^r-smooth n-parameter family of functions Ψ_ε, such that $\Psi_0 = \Psi$. Fix a small $\delta > 0$ and denote

$$H_\varepsilon(x, y) = -\chi_\delta(x - x^+, y) \int_{x^+}^{x} (\Psi_\varepsilon(s) - \Psi_0(s)) ds \tag{16}$$

where $\chi_\delta(u, v)$ is a C^{r+1}-smooth cut-off function which vanish identically at $\|u, v\| \geq 2\delta$ and equals to 1 at $\|u, v\| \leq \delta$. Let F_ε be the time-1 map by the orbits of the Hamiltonian system

$$\dot{x} = \frac{\partial H_\varepsilon}{\partial y}, \qquad \dot{y} = -\frac{\partial H_\varepsilon}{\partial x}.$$

By construction, F_ε is a C^r-smooth area-preserving map, which equals to identity outside a small neighborhood of the point M^+ for all ε; at $\varepsilon = 0$ it is equal to identity everywhere. Near the point M^+ the map F_ε acts as

$$(x, y) \mapsto (x, y + \Psi_\varepsilon(x) - \Psi_0(x)). \tag{17}$$

Consider the family $F_\varepsilon \circ f$, which includes our original map f at $\varepsilon = 0$. Since every map of the family coincides with f outside a small neighborhood of $f^{-1}M^+$, it follows that the global map $(F_\varepsilon \circ f)^l$ from a small neighborhood of the point M^- to a neighborhood of M^+ equals to $F_\varepsilon \circ T_1$ (where T_1 is the global map for the map f). By (17), the equation of the curve $F_\varepsilon \circ T_1 W^u_{loc}$ near this point is

$$\bar{y} = \Psi_\varepsilon(\bar{x}).$$

Now, take any Ψ_ε such that

$$\det \frac{\partial(\tilde{\mu}_0, \dots, \tilde{\mu}_{n-1})}{\partial \varepsilon} \neq 0 \tag{18}$$

(where the functionals $\tilde{\mu}_j$ are given by (14)). In particular, we may take

$$\Psi_\varepsilon(\bar{x}) = \sum_{j=0}^{n-1} \varepsilon_j (\bar{x} - x^+)^j,$$

which would correspond to

$$H_\varepsilon(x, y) = -\chi_\delta(x - x^+, y) \sum_{j=0}^{n-1} \varepsilon_j \frac{(\bar{x} - x^+)^{j+1}}{j+1}, \tag{19}$$

and $\tilde{\mu}_j = \varepsilon_j$ in this case. Now recall, that inequality (18) means exactly that the tangency between $T_1 W^u_{loc}$ and W^s_{loc} is split generically in the family $F_\varepsilon \circ f$, as required.

Note that this construction allows us to transform locally the piece of the unstable manifold $T_1 W^u_{loc}$ near the point M^+ into any sufficiently close curve by a small perturbation which does not destroy the area-preservation property of the map f. Note also that our perturbation is localized in a small neighborhood of one homoclinic (heteroclinic) point, so it does not affect any other homoclinic or heteroclinic tangencies which are bounded away from this point.

In the same way one can show that the multiplier λ of the saddle periodic orbit O can be changed by a small smooth localized perturbation of the map f (the perturbation is area-preserving, if f is area-preserving), without destroying any finite number of given homoclinic or heteroclinic tangencies. Indeed, consider a one-parameter family $f_\varepsilon = F_\varepsilon \circ f$, where the area-preserving diffeomorphism F_ε is the time-1 shift by the flow defined by the Hamiltonian

$$H_\varepsilon(x, y) = -\varepsilon \chi_\delta(x, y) x y, \tag{20}$$

where (x, y) are the coordinates near O for which the local invariant manifolds are straightened, and χ_δ is the cut-off function (like in (16)) with some sufficiently small and fixed $\delta > 0$. By construction, $F_0 = id$, hence $f_0 = f$. At non-zero ε, the map F_ε can differ from identity only in the δ-neighborhood of O, so if δ is small enough, then the new local map is $F_\varepsilon \circ T_0$. Direct computation of the multiplier gives then $\lambda_\varepsilon = e^{-\varepsilon}\lambda$, so

$$\frac{\partial \lambda_\varepsilon}{\partial \varepsilon} = -\lambda_\varepsilon \neq 0.$$

The lines $x = 0$ and $y = 0$ are invariant with respect to the map F_ε, hence they remain local unstable and, respectively, stable invariant manifolds of the point O for all small ε. Since the position of the local invariant manifolds is not changed and since the perturbation is localized in a sufficiently small neighborhood of the point O, any given number of homoclinic or heteroclinic tangencies is not split by such perturbation, as required.

These results can be generalized as follows. Let a two-dimensional C^r-diffeomorphism f have a number of saddle periodic orbits $L_1, \dots, L_s$ and a number of homoclinic or heteroclinic to them orbits $\Gamma_1, \dots, \Gamma_m$, corresponding to the tangency of the stable and unstable manifolds of orders $n_1, \dots, n_m$ (we assume $r \geq r_0 \equiv 1 + \max(n_1, \dots, n_m)$). Thus, for all diffeomorphisms C^{r_0}-close to f we can define $s + n_1 + \dots n_m$ smooth functionals: the multipliers λ_i ($i = 1, \dots, s$) of the periodic orbits L_i, and the functionals μ_j^i ($j = 0, \dots, n_i - 1; i = 1, \dots, m$) which determine the splitting of the homoclinic and heteroclinic tangencies (see (13)). Note that the functionals μ_j^i depend on the choice of the coordinate transformation which straightens the local invariant manifolds, so we assume that this transformation is canonically given by formula (3).

Consider the Hamiltonian

$$H_\varepsilon(x, y) = \varepsilon_1 \zeta_1(x, y) + \dots + \varepsilon_{\bar{n}} \zeta_{\bar{n}}(x, y) \tag{21}$$

where ζ_l, ($l = 1, \dots, \bar{n}$) are the functions given by the right-hand sides of (20) and (19), localized in sufficiently small neighborhoods of the appropriately chosen periodic and homo/heteroclinic points, respectively. As it follows from our considerations above, the family $f_\varepsilon = F_\varepsilon \circ f$ where F_ε is the time 1 map by the flow defined by the Hamiltonian (21) satisfies

$$\det \frac{\partial(\lambda_1(f_\varepsilon), \dots, \lambda_s(f_\varepsilon), \mu_0^1(f_\varepsilon), \dots, \mu_{n_m-1}^m(f_\varepsilon))}{\partial(\varepsilon_1, \dots, \varepsilon_{\bar{n}})} \neq 0. \tag{22}$$

Now note that inequality (22) preserves under any, sufficiently small in the C^{r_0}-topology, perturbations of the family f_ε. In particular, the inequality (22) will still be valid if we replace the functions ζ_l in (21) by their sufficiently close (in the C^{r_0+1}-topology) polynomial approximations $\bar{\zeta}_l$. Thus, we have now established that there exists a polynomial Hamiltonian

$$\bar{H}_\varepsilon(x, y) = \varepsilon_1 \bar{\zeta}_1(x, y) + \dots + \varepsilon_{\bar{n}} \bar{\zeta}_{\bar{n}}(x, y) \tag{23}$$

such that the family $F_\varepsilon \circ f$ satisfies the inequality (22), with F_ε being the time 1 map corresponding to $\bar{H}_\varepsilon$.

Thus, we have proved the following lemma.

Lemma 1. *There exists an analytic $\bar{n}$-parameter family of real-analytic area-preserving diffeomorphisms F_ε such that in the family $F_\varepsilon \circ f$ the tangencies $\Gamma_1, \dots, \Gamma_m$*

split generically and independently, and the multipliers of the periodic orbits $L_1, \dots, L_s$ change independently as well.

This lemma is proven for all diffeomorphisms f whose smoothness r exceeds the maximal order of the homo/heteroclinic tangencies under consideration ($r \geq r_0$). In particular, it holds true for real-analytic f. Thus, we have now the existence of finite-parameter families in which any given finite number of homoclinic or heteroclinic tangencies is split generically and independently in the real-analytic case as well.

The main facts which we will use below in the proof of the main theorem and which follow from this lemma are that given any sufficiently smooth, or real analytic, diffeomorphism with any given finite number of homoclinic or heteroclinic tangencies, we can split any of these tangencies generically by a small (in a smooth or, respectively, real analytic topology) perturbation which does not split the other tangencies, neither it changes the values of the multipliers of the saddle periodic orbits involved. As well, we can change a multiplier of either of the saddles, without changing the multipliers of the other ones, nor destroying the tangencies. By construction, if the original map f is area-preserving, the perturbed maps will be area-preserving too.

3 Secondary homoclinic and heteroclinic tangencies

Here we show how perturbations of homoclinic and heteroclinic cycles can produce secondary homoclinic and heteroclinic tangencies of various types. Statements analogous to the following lemma were given in [9, 10, 46] for different situations. Here we give a unified proof which includes the area-preserving case.

Lemma 2. *Let f_μ be a one-parameter family of two-dimensional C^r-diffeomorphisms ($r \geq 3$). Let f_0 have a saddle periodic point O with an orbit of quadratic homoclinic tangency which splits generically as μ varies. Then, arbitrarily close to $\mu = 0$ there exists a value of μ for which the map f_μ has an orbit of quadratic homoclinic tangency (which splits generically as μ varies) and a homoclinic orbit corresponding to a transverse intersection of the invariant manifolds of O.*

Proof. Note that orbits of transverse homoclinic intersection may exist already at the moment of the original homoclinic tangency; in this case the sought value of μ is 0. Otherwise, we have to find a converging to zero sequence of non-zero values of μ for which the map has new, secondary homoclinic tangencies accompanied by transverse homoclinic orbits. We will explore both possibilities in the proof.

By (9), (13), the equation of the curve $T_1(W^u_{loc})$ is

$$y = \mu + \frac{d}{b^2}(x - x^+)^2 + o((x - x^+)^2). \tag{24}$$

At $\mu d < 0$ this curve intersects $W^s_{loc} \cap \Pi^+ : \{y = 0\}$ transversely at two points.

Assume that the coordinates near O are chosen such that (4), (5) hold. Then, by (6),(24), the image $(\hat{x}, \hat{y})$ of a point $(x, y) \in T_1(W^u_{loc})$ by the map T_0^k gets in a small neighborhood of the homoclinic point M^- (the domain of the global map T_1) if and only if

$$\hat{x} = \lambda^k x + \lambda^k \xi_k(x, \hat{y}), \qquad y = \gamma^{-k} \hat{y} + \gamma^{-k} \eta_k(x, \hat{y}). \tag{25}$$

The point (x, y) will be homoclinic if $T_1(\hat{x}, \hat{y}) \in W^s_{loc}$, which means (see (8))

$$0 = c\hat{x} + \mu + d(\hat{y} - y^-)^2 + o(|\hat{x}| + (\hat{y} - y^-)^2).$$

Thus, a homoclinic point (x, y): $T_1 T_0^k(x, y) \in W^s_{loc}$, $T_1^{-1}(x, y) \in W^u_{loc}$ corresponds to a solution of the system

$$\begin{aligned} 0 &= \mu + c\lambda^k x^+ + c\lambda^k X + dY^2 + \phi_1(X, Y, \mu) + \phi_2(Y, \mu), \\ 0 &= \mu - \gamma^{-k} y^- - \gamma^{-k} Y + \tfrac{d}{b^2} X^2 + \phi_3(X, Y, \mu) + \phi_4(X, \mu), \end{aligned} \tag{26}$$

where we denote $X = x - x^+, Y = \hat{y} - y^-$. The functions ϕ satisfy

$$\begin{aligned} \phi_1 &= o(\lambda^k), \quad &\phi_2 = o(Y^2), \\ \phi_3 &= o(\gamma^{-k}), \quad &\phi_4 = o(X^2). \end{aligned} \tag{27}$$

Note that the right-hand sides of system (26) are at least C^2 with respect to X and Y: the map f is at least C^3 by assumption, but we lose one smoothness when introduce the coordinates bringing the local map to the form (4),(5). Accordingly, the first derivative of the right-hand side with respect to (X, Y) is C^1 with respect to X, Y and μ. In particular, the coefficients λ, γ, x^+, y^-, b and c are C^1-functions of μ (while d is a constant).

Note that $d \neq 0$ implies that we may shift the origin of coordinates to a small constant: $(X, Y) \mapsto (X + o(\gamma^{-k}), Y + o(\lambda^k))$, so that the first derivative of the right-hand side of the first equation in (26) with respect to Y and of the right-hand side of the second equation with respect to X will vanish at $(X, Y) = 0$. After that, the system will take the form

$$\begin{aligned} 0 &= \mu + \nu_k^1 + c\lambda^k X + dY^2 + \tilde{\phi}_1(X, Y, \mu) + \tilde{\phi}_2(Y, \mu), \\ 0 &= \mu + \nu_k^2 - \gamma^{-k} Y + \tfrac{d}{b^2} X^2 + \tilde{\phi}_3(X, Y, \mu) + \tilde{\phi}_4(X, \mu), \end{aligned} \tag{28}$$

where we denote as $\nu_k^{1,2}$ the independent of X and Y terms:

$$\nu_k^1 = c\lambda^k x^+ + o(\lambda^k), \qquad \nu_k^2 = -\gamma^{-k} y^- + o(\gamma^{-k}); \tag{29}$$

and the functions $\tilde{\phi}$ satisfy

$$\begin{aligned} \tilde{\phi}_1 &= o(\lambda^k X), \quad &\tilde{\phi}_2 = o(Y^2), \\ \tilde{\phi}_3 &= o(\gamma^{-k} Y), \quad &\tilde{\phi}_4 = o(X^2). \end{aligned} \tag{30}$$

The nondegenerate solutions of (28) correspond to transverse homoclinics, and the degenerate ones correspond to homoclinic tangencies. We will consider system (28) for even k only, so $\lambda^k > 0$ and $\gamma^{-k} > 0$, no matter what are the signs of λ and γ. It is easy to see then, that when $cx^+d < 0$ and $y^-d > 0$ the system has non-degenerate solutions

$$Y = \pm\sqrt{\frac{cx^+}{d} + o(1)}\ |\lambda|^{k/2}, \quad X = \pm b\sqrt{\frac{y^-}{d} + o(1)}\ |\gamma|^{-k/2}$$

at $\mu = 0$ for all sufficiently large k. This means that at $\mu = 0$, in addition to the original orbit of homoclinic tangency, we have also transverse homoclinic orbits, which gives us the lemma in this case.

If, on the contrary, $cx^+d > 0$ or $y^-d < 0$, we will search for secondary homoclinic tangencies at small $\mu \neq 0$. They correspond to solutions of (26) for which the Jacobian of the right-hand side vanishes. Thus, they solve the system

$$\begin{aligned} 0 &= c\lambda^k\gamma^{-k} + 4\frac{d^2}{b^2}(X + \varphi_1(X,Y))(Y + \varphi_2(X,Y)) + o(\lambda^k\gamma^{-k}), \\ 0 &= \nu_k + c\lambda^k X + \gamma^{-k}Y + dY^2 - \frac{d}{b^2}X^2 + o(\lambda^k|X| + Y^2 + X^2 + \gamma^{-k}|Y|), \end{aligned} \tag{31}$$

where the constant term ν_k is given by

$$\nu_k = c\lambda^k x^+\gamma^{-k}y^- + o(\lambda^k + \gamma^{-k}), \tag{32}$$

and

$$\varphi_1 = o(|X| + \gamma^{-k}|Y|), \qquad \varphi_2 = o(|Y| + \lambda^k|X|). \tag{33}$$

We obtained the second equation in (31) as follows: from the first equation of (28) we expressed μ as a function of (X,Y), and plugged the result into the second equation. Thus, in (31) there is no dependence on μ, so by λ, γ, c, x^+, y^- in (31), (32) we mean the values of these coefficients at $\mu = 0$.

The non-degenerate solutions of (31) correspond to quadratic homoclinic tangencies for μ found from either one of the two equations in (28). Note that the value of parameter μ is found uniquely for any given small X and Y, and this remains true for an arbitrary small perturbation of the map under consideration, which means that the corresponding tangency splits indeed generically as μ varies.

Consider, first, the case $|\lambda\gamma| < 1$ (hence $\lambda^k\gamma^k \to 0$ as $k \to +\infty$). If $y^-d > 0$ and $cx^+d > 0$ (the case $y^-d > 0$ and $cx^+d < 0$ has already been considered), we scale the variables as follows:

$$(X,Y) \mapsto b|\gamma|^{-k/2}\sqrt{\frac{y^-}{d}}\ (X^{new}, -\frac{c}{4y^-d}\lambda^k\ Y^{new}).$$

In the new variables, system (31) recasts as

$$1 = XY + o(1)_{k\to+\infty}, \qquad 1 = X^2 + o(1)_{k\to+\infty}. \tag{34}$$

For all k large enough, this system has nondegenerate solutions $X = Y = \pm 1 + o(1)$. In the non-rescaled variables it corresponds to $X = O(|\gamma|^{-k/2}), Y = O(\lambda^k|\gamma|^{-k/2})$.

The first equation of (28) gives us then, that the corresponding homoclinic tangencies happen at $\mu = \mu_k = -cx^+\lambda^k(1 + o(1))$. By our current assumptions $cx^+d > 0$, hence $\mu_k d < 0$, i.e. the found secondary homoclinic tangencies coexist with primary transverse homoclinic orbits, as required.

In the case $y^-d < 0$, we use the following scaling:

$$(X, Y) \mapsto \sqrt{|y^-/d|}|\gamma|^{-k/2}\ (-\frac{cb^2}{4|y^-d|}\lambda^k(X^{new} - \varphi_1(0, Y)), Y^{new}).$$

In the new variables, system (31) recasts as

$$1 = XY + o(1)_{k\to+\infty}, \qquad 1 = Y^2 + o(1)_{k\to+\infty}. \tag{35}$$

For all k large enough, this system has nondegenerate solutions $X = Y = \pm 1 + o(1)$. In the non-rescaled variables it corresponds to $X = o(|\gamma|^{-3k/2}), Y = O(|\gamma|^{-k/2})$. The second equation of (28) gives us then, that the corresponding homoclinic tangencies happen at $\mu = \mu_k = y^-\gamma^{-k}(1 + o(1))$. Since we assume here $y^-d < 0$, it follows that $\mu_k d < 0$, i.e. the found secondary homoclinic tangencies coexist with primary transverse homoclinic orbits in this case too. This finishes the proof of the lemma in the case $|\lambda\gamma| < 1$.

The case $|\lambda\gamma| > 1$ is reduced to the previous one if we consider the map f^{-1} instead of f. Thus, it remains to prove the lemma in the case $|\lambda\gamma| = 1$, which includes the area-preserving maps. Let us choose the sequence of (even) values of k such that there exists the limit (finite or infinite)

$$\lim_{k\to+\infty} \nu_k/\lambda^{2k} = M. \tag{36}$$

If $M = +\infty$, we scale

$$(X, Y) \mapsto b\sqrt{\frac{\nu_k}{d}}\ (X^{new}, -\frac{c}{4d}\frac{\lambda^{2k}}{\nu_k}Y^{new})$$

if $\nu_k d > 0$, and

$$(X, Y) \mapsto \sqrt{|\nu_k/d|}\ (-\frac{cb^2}{4|d|}\frac{\lambda^{2k}}{|\nu_k|}X^{new}, Y^{new})$$

if $\nu_k d < 0$. In the first case the system (31) takes the form (34). Its non-degenerate solutions correspond to $(X = O(|\lambda|^{k/2}), Y = O(|\lambda|^{3k/2}))$, and the first equation of (28) gives $\mu = \mu_k = -cx^+\lambda^k(1 + o(1))$ for the moments of homoclinic tangencies. The assumption $\nu_k d > 0$ implies (see (32)) that either $cx^+d > 0$ or $y^-d > 0$. The latter implies $cx^+d > 0$ as well (because the case $cx^+d < 0$, $y^-d > 0$ has already been considered). Thus $\mu_k d < 0$, i.e. the found secondary homoclinic tangencies coexist with primary transverse homoclinic orbits.

In the second case the system (31) reduces to the form (35). Its non-degenerate solutions correspond to $(X = O(|\lambda|^{3k/2}), Y = O(|\lambda|^{k/2}))$, and the second equation of (28) gives $\mu = \mu_k = y^-\lambda^k(1 + o(1))$ for the moments of homoclinic tangencies.

The assumption $\nu_k d < 0$ implies that either $cx^+d < 0$ or $y^-d < 0$, but the former inequality implies $y^-d < 0$ anyway. Thus $\mu_k d < 0$ in this case too.

It remains to consider the case where M is finite in (36). It follows, in particular, that $cx^+ + y^- = 0$, which implies $cx^+d > 0$. We make the following rescaling:

$$(X,Y) \mapsto \frac{1}{d}\lambda^k(b^2cX^{new}, -Y^{new}).$$

The system (31) takes the form

$$\begin{aligned} 0 &= 1 - 4XY + o(1)_{k\to+\infty}, \\ 0 &= Md - (bc)^2(X^2 - X) + Y^2 - Y + o(1)_{k\to+\infty}. \end{aligned} \tag{37}$$

It is easy to see that this system has a nondegenerate solution in the region $\{X < 0, Y < 0\}$ for any Md and $bc \neq 0$. In the non-rescaled variables this solution gives $(X,Y) = O(\lambda^k)$, and from (28) we have $\mu = \mu_k = -cx^+\lambda^k(1+o(1))$ for the moments of homoclinic tangencies. Again we have $\mu_k d < 0$, so the coexistence of the secondary homoclinic tangencies with primary transverse homoclinics is established in this last remaining case as well. The lemma is proven.

The existence of a transverse homoclinic to O implies [42] the existence of a nontrivial zero-dimensional transitive hyperbolic set Λ (a Smale horseshoe) which includes O. Thus, Lemma 2 can be reformulated as the existence of a nontrivial hyperbolic set whose stable and unstable sets have a quadratic homoclinic tangency at the values of μ arbitrarily close to zero. Since the stable and unstable manifolds of any periodic orbit in Λ approximate, in the C^r topology, any leaf in, respectively, $W^s(\Lambda)$ and $W^u(\Lambda)$, and since the found homoclinic tangency splits generically as μ changes, it follows that for any two periodic points in Λ a quadratic heteroclinic tangency of their invariant manifolds can be obtained by an arbitrarily small variation of μ, and this tangency also splits generically.

Let O_1, O_2 be two different saddle periodic points in Λ, different from O and with all multipliers positive (such periodic points always exist in any horseshoe). Let us fix some small μ for which $W^u(O_2)$ and $W^s(O_1)$ have a quadratic heteroclinic tangency. A heteroclinic orbit corresponding to a transverse intersection of $W^u(O_1)$ and $W^s(O_2)$ also exists at the same μ, because O_1 and O_2 belong to the same transitive hyperbolic set Λ.

We introduce the coordinates (x_i, y_i) near the points O_i $(i = 1, 2)$ such that the local invariant manifolds are straightened, i.e. the local maps T_{0i} take the form (see (4)):

$$\bar{x}_i = \lambda_i x_i + p_i(x_i, y_i)x_i, \qquad \bar{y}_i = \gamma_i y_i + q_i(x_i, y_i)y_i,$$

where $0 < \lambda_i < 1 < \gamma_i$. Note that the point O_i divides its stable and unstable manifolds into two invariant components each, we will denote these components as $W^{u+}(O_i)$, $W^{u-}(O_i)$ and $W^{s+}(O_i)$, $W^{s-}(O_i)$.

Choose a pair of heteroclinic points: $M_1^+(x_1^+, 0)$ in a small neighborhood of O_1 and $M_2^+(0, y_2^-)$ in a small neighborhood of O_2, which belong to the same orbit of heteroclinic tangency. The global map T_{21} from a small neighborhood of M_2^- into a small neighborhood of M_1^+ is written as follows (see (8),(13)):

$$x_1 - x_1^+ = ax_2 + b(y_2 - y_2^-) + \dots, \qquad y_1 = cx_2 + d(y_2 - y_2^-)^2 + \dots. \tag{38}$$

Choose also a pair of heteroclinic points $M_1^-(0, y_1^-)$ in a small neighborhood of O_1 and $M_2^+(x_2^+, 0)$ in a small neighborhood of O_2, which belong to a transverse heteroclinic orbit. In the terminology of [38, 39], the heteroclinic cycle belongs to the "third class", when

$$cy_1^- x_2^+ > 0. \tag{39}$$

Note that we can always choose the points O_1 and O_2 among the "inner" points of the horseshoe Λ, i.e. the leaves of $W^s(\Lambda)$ and $W^u(\Lambda)$ converge, respectively, to $W^s(O_1)$ and $W^u(O_1)$ from both sides, and the same holds true for O_2. In other words, we can always choose the points O_1 and O_2 such that each of the manifolds $W^{u+}(O_1)$, $W^{u-}(O_1)$, has, in Λ, an orbit of transverse intersection with each of the manifolds $W^{s+}(O_2)$, $W^{s-}(O_2)$. These four orbits correspond to different combinations of the signs of y_1^- and x_2^+. Thus, no matter what is the sign of c, we can always choose a transverse heteroclinic in such a way that (39) holds.

Lemma 3. *Under the conditions of Lemma 2, arbitrarily close to $\mu = 0$ there exist values of μ for which the map f_μ has a non-trivial transitive hyperbolic set Λ which includes the point O and two saddle periodic points O_1 and O_2 such that $W^u(O_2)$ and $W^s(O_1)$ have a quadratic heteroclinic tangency which splits generically as μ varies. In Λ there exists also an orbit of transverse intersection of $W^u(O_1)$ and $W^s(O_2)$ such that the corresponding heteroclinic cycle belongs to the third class.*

The peculiarity of the diffeomorphisms with the heteroclinic cycles of the third class is that they have moduli of local Ω-conjugacy [38]. In particular, the value

$$\alpha = -\frac{\ln \gamma_1}{\ln \lambda_2} \tag{40}$$

is such a modulus: if two diffeomorphisms with a heteroclinic cycle of the third class have different values of α, they are not locally Ω-conjugate (it is well known that α is an invariant of topological conjugacy for maps with a quadratic heteroclinic tangency [44, 45, 46], however the fact that it is also an invariant of local Ω-conjugacy holds true only for maps with heteroclinic cycles of the third class [38]). It follows that any change in α must lead to bifurcations in the set of orbits lying in a small neighborhood of the heteroclinic cycle. Thus, we have the following result [39]:

Lemma 4. *Let f_ε be any smooth family of C^r-diffeomorphisms ($r \geq 3$) such that all diffeomorphisms in the family have a heteroclinic cycle of the third class, i.e. for all ε there are two periodic points O_1 and O_2, an orbit of transverse intersection of $W^u(O_1)$ and $W^s(O_2)$, and an orbit of quadratic heteroclinic tangency between*

$W^u(O_2)$ and $W^s(O_1)$ (it does not split as ε varies), plus sign condition (39) holds. If the value of α changes monotonically with ε, i.e.

$$\frac{\partial}{\partial\varepsilon}\,\alpha(f_\varepsilon) \neq 0,$$

then there is a dense set of values of ε for which the map f_ε has a quadratic heteroclinic tangency (which splits generically as ε varies) between $W^u(O_1)$ and $W^s(O_2)$.

Let us give an idea of the proof. Note that by lambda-lemma, since $W^u(O_1)$ and $W^s(O_2)$ intersect transversely, there are pieces of $W^s(O_2)$ which converge in the C^r-topology to $W^s_{loc}(O_1)$ and pieces of $W^u(O_1)$ which converge to $W^u_{loc}(O_2)$. By (6), these pieces of $W^s(O_2)$ near the point M_1^+ form an infinite sequence of curves

$$W_i^s : y_1 \sim \gamma_1^{-i} y_1^-, \tag{41}$$

and the pieces of $W^u(O_1)$ near the point M_2^- form an infinite sequence of curves

$$W_j^u : x_2 \sim \lambda_2^j x_2^+, \tag{42}$$

where y_1^- and x_2^+ are the coordinates of the points, respectively, M_1^- and M_2^+ on the transverse heteroclinic orbit. By (38), the curves $T_1 W_j^u$ form a sequence of parabola-like curves, extended towards positive y_1 if $d > 0$ and towards negative y_1 if $d < 0$, with the tops at $y_1 \sim c x_2^+ \lambda_2^j$. Thus, W_i^s and $T_1 W_j^u$ intersect transversely when

$$d\left(\frac{\gamma_1^{-i} y_1^-}{c x_2^+ \lambda_2^j} - 1\right) \gg 0, \tag{43}$$

and have no intersection when

$$d\left(\frac{\gamma_1^{-i} y_1^-}{c x_2^+ \lambda_2^j} - 1\right) \ll 0. \tag{44}$$

Now take any two arbitrarily close values $\varepsilon_1 \neq \varepsilon_2$. By assumption, $\alpha(\varepsilon_1) \neq \alpha(\varepsilon_2)$, and we may assume $\alpha(\varepsilon_1) > \alpha(\varepsilon_2)$. Hence, we can find two sufficiently large integers i and j such that

$$i \ln\gamma_1 - j|\ln\lambda_2| \gg 0$$

at $\varepsilon = \varepsilon_1$ and

$$i \ln\gamma_1 - j|\ln\lambda_2| \ll 0$$

at $\varepsilon = \varepsilon_2$. Taking into account (39), it follows that one of inequalities (43), (44) is fulfilled at $\varepsilon = \varepsilon_1$, and another one at $\varepsilon = \varepsilon_2$. We see that for such chosen i and j the intersection of the curves $T_1 W_i^u$ and W_j^s disappears when ε runs the interval between ε_1 and ε_2. Hence, the two curves must have a tangency at some ε from this interval, which is the required heteroclinic tangency between $W^u(O_1)$ and $W^s(O_2)$. A formal proof of the lemma can be found in [39].

The following lemma deals with secondary heteroclinic or homoclinic tangencies of high orders. It is a version of Lemma 2 from [5]. Since our formulation here is slightly different, we give a complete proof.

Lemma 5. *Let f_ε, $\varepsilon = (\mu_0, \dots, \mu_{n-1}, \nu)$, be a smooth $(n+1)$-parameter family of two-dimensional C^r-diffeomorphisms ($r > n+2$) which have saddle periodic points O_1, O_2, O_3 (not necessarily different) such that at $\mu = 0$ the manifolds $W^u(O_1)$ and $W^s(O_2)$ have a tangency of order n, and at $\nu = 0$ the manifolds $W^u(O_2)$ and $W^s(O_3)$ have a quadratic tangency. Suppose that the tangency between $W^u(O_1)$ and $W^s(O_2)$ splits generically as μ varies, and the tangency between $W^u(O_2)$ and $W^s(O_3)$ splits generically as ν varies. Then there exists a sequence $\varepsilon_k \to 0$ such that the map f_ε has an orbit of tangency of order $(n+1)$ between $W^u(O_1)$ and $W^s(O_3)$ at $\varepsilon = \varepsilon_k$.*

Proof. Let (x, y) be the C^{r-1}-coordinates near O_2 for which formulas (4),(5) hold for the local map T_0, hence formulas (6) hold for its iterations T_0^k. Let $M^+(x^+, 0)$ be a point at which $W^u(O_1)$ has the tangency of order n with $W^s_{loc}(O_2)$ at $\mu = 0$, and $M^-(0, y^-)$ be a point at which $W^s(O_3)$ has the quadratic tangency with $W^u_{loc}(O_2)$ at $\nu = 0$. As it was explained in Section 2, we may choose parameters μ in such a way (see (14)) that the equation of the piece of $W^u(O_1)$ near M^+ will be

$$y = \mu_0 + \dots + \mu_{n-1}(x - x^+)^{n-1} + d(x - x^+)^{n+1} + o((x - x^+)^{n+1}). \tag{45}$$

The parameter ν can be chosen in such a way that the equation of the piece of $W^s(O_3)$ near M^- will be (see (15)):

$$x = \nu + \hat{d}(y - y^-)^2 + o((y - y^-)^2). \tag{46}$$

By (6), the image of the curve (45) by the map T_0^k has the sought tangency of order $(n+1)$ with the curve (46) if and only if the following curves have a tangency of order $(n+1)$:

$$\gamma^{-k}(Y + y^-) + \gamma^{-k}\eta_k(X + x^+, Y + y^-) = \mu_0 + \dots + \mu_{n-1}X^{n-1} + dX^{n+1} + o(X^{n+1}), \tag{47}$$

and

$$\lambda^k(X + x^+) + \lambda^k\xi_k(X + x^+, Y + y^-) = \nu + \hat{d}Y^2 + o(Y^2) \tag{48}$$

(here $(x^+ + X)$ is the x-coordinate of the point of tangency and $(y^- + Y)$ is the y-coordinate of the image of the same point by the map T_0^k).

One can rewrite equations (47),(48) in the explicit form:

$$\gamma^{-k}Y = \bar{\mu}_0 + \dots + \bar{\mu}_{n-1}X^{n-1} + \bar{\mu}_n X^n + dX^{n+1} + o(X^{n+1}), \tag{49}$$

and

$$\lambda^k X = \bar{\nu} + \bar{\nu}_1 Y + \hat{d}Y^2 + o(Y^2), \tag{50}$$

where

$$\begin{aligned} &\bar{\mu}_0 = \mu_0 - \gamma^{-k}y^- + o(\gamma^{-k}), \\ &\bar{\mu}_j = \mu_j + o(\gamma^{-k}) \qquad (j = 1, \dots, n-1), \\ &\bar{\mu}_n = o(\gamma^{-k}) \end{aligned}$$

and

$$\bar{\nu} = \nu - \lambda^k x^+ + o(\lambda^k), \qquad \bar{\nu}_1 = o(\lambda^k).$$

After the change of variables

$$X^{new} = X - \frac{\bar{\nu}_1}{2\hat{d}}, \qquad Y^{new} = Y - \frac{\bar{\mu}_n}{(n+1)d}$$

equations (49), (50) recast as

$$\gamma^{-k} Y = \mu'_0 + ... + \mu'_{n-1} X^{n-1} + d X^{n+1} + o(X^{n+1}), \tag{51}$$

and

$$\lambda^k X = \nu' + \hat{d} Y^2 + o(Y^2), \tag{52}$$

where

$$\begin{array}{l} \mu'_0 = \mu_0 - \gamma^{-k} y^- + o(\gamma^{-k}), \\ \mu'_j = \mu_j + o(\gamma^{-k}) \qquad (j = 1, \ldots, n-1), \\ \nu' = \nu - \lambda^k x^+ + o(\lambda^k). \end{array} \tag{53}$$

Let us rescale the variables:

$$X = -\frac{1}{(\hat{d} d^2)^{\frac{1}{2n+1}}} \lambda^{\frac{k}{2n+1}} \gamma^{-\frac{2k}{2n+1}} X^{new}, \quad Y = (-1)^{n+1} \frac{1}{(\hat{d}^{n+1} d)^{\frac{1}{2n+1}}} \lambda^{k\frac{n+1}{2n+1}} \gamma^{-\frac{k}{2n+1}} Y^{new}.$$

Equations (51),(52) change to

$$Y = M_0 + ... + M_{n-1} X^{n-1} + X^{n+1} + o(1)_{k \to +\infty}, \tag{54}$$

and

$$X = N - Y^2 + o(1)_{k \to +\infty}, \tag{55}$$

where

$$\begin{array}{ll} M_j = \mu'_j \dfrac{(-1)^{n+1-j}}{d} (\hat{d} d^2)^{\frac{n+1-j}{2n+1}} \lambda^{-\frac{k(n+1-j)}{2n+1}} \gamma^{\frac{2k(n+1-j)}{2n+1}} & (j = 0, \ldots, n-1), \\ N = \nu' (\hat{d} d^2)^{\frac{1}{2n+1}} \lambda^{-k\frac{2n+2}{2n+1}} \gamma^{\frac{2k}{2n+1}}. & \end{array} \tag{56}$$

As it is proven in [5], for all sufficiently large k there are uniquely defined, uniformly (with respect to k) bounded values of the rescaled parameters N and $M_0, \ldots, M_{n-1}$ for which the curves (54) and (55) have a tangency of order $(n+1)$. The proof is quite straightforward. The equation of the curve (55) can be rewritten as

$$Y = \pm\sqrt{N - X + o(1)}. \tag{57}$$

We obtain the sought tangency of order $(n+1)$ if and only if the first $(n+2)$ terms of the Taylor expansion of (54) coincide at some X^* with the corresponding terms of the Taylor expansion of one of the two branches of (57). Given any N and X^*, we can always achieve that initial segments of length n of the two expansions coincide by an appropriate choice of $M_0, \dots, M_{n-1}$. The coincidence conditions for the two remaining terms read as

$$
\begin{aligned}
(n+1)X^* &= \pm\frac{\sigma_n}{n!}(N - X^*)^{\frac{1}{2}-n} + o(1)_{k\to+\infty}, \\
1 &= \pm\frac{\sigma_{n+1}}{(n+1)!}(N - X^*)^{-\frac{1}{2}-n} + o(1)_{k\to+\infty},
\end{aligned}
\tag{58}
$$

where

$$
\sigma_n = \frac{-1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-3}{2} < 0.
$$

Equalities (58) show that there can be no tangency of order $(n+1)$ with the "plus" branch of (57), while for the "minus" branch the uniquely defined values of N and X^* corresponding to the sought tangency exist indeed:

$$
N = X^*(n+\frac{1}{2}) + o(1)_{k\to+\infty}, \quad X^* = \frac{2}{2n-1}\left(\frac{|\sigma_{n+1}|}{(n+1)!}\right)^{2/(2n+1)} + o(1)_{k\to+\infty}.
$$

The found tangency of order $(n+1)$ between the curves (54) and (55) corresponds to the tangency of order $(n+1)$ between the curves (47) and (48). Since the corresponding values of $N, M_0, \dots, M_{n-1}$ remain uniformly bounded for all large k, the respective values of $\varepsilon = (\nu, \mu_0, \dots, \mu_{n-1})$ tend to zero as $k \to +\infty$ (see (53), (56)). This completes the proof of the lemma.

4 Homoclinic tangencies of arbitrarily high orders

In this section we prove our main theorem. Let f be a two-dimensional C^r-diffeomorphism ($r = 2, \dots, \infty, \omega$) having a saddle periodic orbit O and an orbit of a quadratic homoclinic tangency of $W^u(O)$ and $W^s(O)$. By Lemma 1 we may include f into a one-parameter family f_μ of C^r-diffeomorphisms (all area-preserving if f itself is area-preserving) for which the homoclinic tangency splits generically. Then, by Lemma 2, arbitrarily close to f in this family we find a diffeomorphism $\tilde{f}$ which has a non-trivial transitive hyperbolic set Λ which includes the point O, and some leaf of the unstable set of Λ has a quadratic tangency (which splits generically as μ varies) with some leaf of the stable set of Λ.

In another setting we may assume the existence of such set Λ from the very beginning. In any case, as Lemma 3 gives it to us, we may achieve by an additional arbitrarily small perturbation that the diffeomorphism $\tilde{f}$ will have a quadratic heteroclinic tangency between stable and unstable manifolds of two periodic points O_1 and O_2 in Λ, and the orbit of this tangency will be a part of the heteroclinic cycle of third class.

This tangency is split generically as the parameter μ varies. It follows that for any family of maps which approximates f_μ sufficiently closely, at least in the C^2-topology, there will exist a value of the parameter corresponding to a quadratic heteroclinic tangency between $W^s(O_1)$ and $W^u(O_2)$. Hence, even if our original map f were not analytic, by choosing a sufficiently close real analytic approximation to f_μ (note that if f_μ is a family of area-preserving maps, it can always be approximated by a family of real analytic, and even polynomial [30] area-preserving maps) we find near f a real analytic (and area-preserving if f is area-preserving) map $\tilde{f}$ with a quadratic tangency between $W^s(O_1)$ and $W^u(O_2)$. Thus, from now on, all the diffeomorphisms we obtain as small perturbations of f will be real-analytic. Moreover, all of them will be area-preserving if f is. Thus, the term "perturbation" means now a small real analytic perturbation which does not destroy the area-preservation property (all the perturbations below will be of the type given by Lemma 1).

Let $n_1, n_2, \ldots$ be an arbitrary infinite sequence of positive integers, and $((L_{11}, L_{12}), (L_{21}, L_{22}), (L_{31}, L_{32}), \ldots)$ be an arbitrary sequence of pairs of periodic points from the hyperbolic set Λ. We will prove that $\tilde{f}$ can be perturbed in such a way that the perturbed map will have an infinite sequence of homoclinic/heteroclinic tangencies, and these will be exactly the tangencies of orders n_k between $W^u(L_{k1})$ and $W^s(L_{k2})$, $k = 1, 2, \ldots$. The perturbation which we construct can be as small as we need, and it does not lead out of the class of area-preserving maps if the original map f is area-preserving. This will, obviously, give us a proof of the main theorem (by letting the numbers n_k take all natural values infinitely many times, and by choosing all L_{kj} equal to the same periodic point O we will have infinitely many homoclinic tangencies of every order).

Take an arbitrarily small $\delta > 0$ and let $\delta_k > 0$ be such that

$$\delta_1 + \delta_2 + \ldots = \delta.$$

We will construct a sequence of real analytic maps f_k, $f_0 \equiv \tilde{f}$, such that each of them retains the heteroclinic tangency between $W^s(O_1)$ and $W^u(O_2)$, and f_k has k additional orbits of tangency: between $W^u(L_{11})$ and $W^s(L_{12})$ of order n_1, between $W^u(L_{21})$ and $W^s(L_{22})$ of order n_2, etc.. We will construct maps f_k in such a way that the distance between f_{k+1} and f_k will be less than δ_k. Hence, the sequence f_k will have a limit f^* which lies on the distance less than δ from $\tilde{f}$ and has the required infinite sequence of tangencies.

Thus, in order to prove the main theorem, we need to prove that given a diffeomorphism f_k which has the heteroclinic cycle of the third class and $k \geq 0$ additional orbits of homoclinic/heteroclinic tangencies, one can perturb it, destroying neither the heteroclinic tangency between $W^u(O_2)$ and $W^s(O_1)$ nor the k additional tangencies, nor changing the order of these tangencies, such that the new diffeomorphism f_{k+1} will have one more orbit of tangency, between $W^u(L_{k+1,1})$ and $W^s(L_{k+1,2})$, of the given order n_{k+1}. We will construct such perturbation from f_k to f_{k+1} as a finite sequence of perturbations each of which can be made arbitrarily small, so the total size of the resulting perturbation will be less than δ_{k+1}, as required.

By Lemma 1, we can include f_k into a one-parameter family $f_{k\varepsilon}$ such that neither of the given heteroclinic and homoclinic tangencies splits as ε varies, nor the multipliers of the point O_1 change, while the multiplier λ_2 of the point O_2 changes with non-zero velocity. It follows that the Ω-modulus α is changing with non-zero velocity as well. Hence, by Lemma 4, arbitrarily close to f_k we find in this family a diffeomorphism which has an additional quadratic heteroclinic tangency between $W^u(O_1)$ and $W^s(O_2)$. When ε changes, this tangency splits generically. Recall that the points O_1, O_2, $L_{k+1,1}$ and $L_{k+1,2}$ belong to the same transitive hyperbolic set Λ, therefore $W^u(L_{k+1,1})$ accumulates onto $W^u(O_1)$ and $W^s(L_{k+1,1})$ accumulates onto $W^s(O_2)$. Thus, by an additional arbitrarily small change in ε, when splitting the tangency between $W^u(O_1)$ and $W^s(O_2)$ we can obtain a new quadratic homoclinic tangency between $W^u(L_{k+1,1})$ and $W^s(L_{k+1,1})$.

If $n_{k+1} > 1$, we repeat the procedure n_{k+1} times, obtaining each time a new quadratic homoclinic tangency between $W^u(L_{k+1,1})$ and $W^s(L_{k+1,1})$, without perturbing the other tangencies. Next, we include the map into a two-parameter family of maps for which two of the newly obtained quadratic homoclinic tangencies split generically and independently, while all the other tangencies are kept in place. By Lemma 5, by an arbitrarily small variation of parameters within any such family we can obtain a cubic homoclinic tangency. If $n_{k+1} > 2$, we include then the map into a three-parameter family of maps for which the cubic tangency and one of the remaining quadratic homoclinic tangencies split generically and independently, while all the other tangencies remain unperturbed. According to Lemma 5 again, by an arbitrarily small variation of the parameters we obtain now a quartic tangency, etc.. Thus, repeating the procedure, after a finite number of arbitrarily small consecutive perturbations we obtain a new homoclinic tangency of order n_{k+1} between $W^u(L_{k+1,1})$ and $W^s(L_{k+1,1})$, in addition to the heteroclinic tangency between $W^u(O_2)$ and $W^s(O_1)$ and the k tangencies between $W^u(L_{11})$ and $W^s(L_{12})$, $W^u(L_{21})$ and $W^s(L_{22})$, ..., $W^u(L_{k1})$ and $W^s(L_{k2})$ which the map f_k already had. If $L_{k+1,2} = L_{k+1,1}$, it means that we have found the tangency we sought, and the map f_{k+1} is constructed. If $L_{k+1,2} \neq L_{k+1,1}$, we note that $L_{k+1,1}$ and $L_{k+1,2}$ belong to the same transitive hyperbolic set Λ, therefore $W^s(L_{k+1,2})$ accumulates onto $W^s(L_{k+1,1})$. Thus, by an arbitrarily small perturbation which splits the homoclinic tangency $W^u(L_{k+1,1})$ and $W^s(L_{k+1,1})$ generically (and does not split the other tangencies) we obtain the sought tangency between $W^u(L_{k+1,1})$ and $W^s(L_{k+1,2})$.

End of the proof.

Acknowledgments

This work was partially supported by the grants CRDF No.Ru-M1-2583-MO-04, RFFI No. 02-01-00273, "Universitety Rossii" No. 1905, RFFI No. 04-01-00487. Leonid Shilnikov acknowledges support of the Alexander von Humboldt foundation. The authors acknowledge the hospitality of Klaus Schneider and WIAS.

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