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On existence of a bounded solution in a problem with a control parameter

E. Shchetinina

Weierstrass Institute for Applied

Analysis and Stochastics

Mohrenstraße 39

10117 Berlin, Germany

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

This paper is devoted to the problem of existence of bounded solutions for non-autonomous differential equations in the case when the linear part has a pair of simple complex conjugate eigenvalues crossing the imaginary axis for increasing t. By introducing a control parameter into the system we derive conditions for the existence of a global uniformly bounded solution.

1 Introduction

We consider in $\mathbb{R}^2$ the system of non-autonomous ordinary differential equations

$$\frac{dz}{dt} = B(t)z + f(t, x, a), \tag{1}$$

where $a \in \mathbb{R}^2$ is a parameter vector. Our goal is to establish the existence of a nontrivial solution of (1) which exists for all $t \in \mathbb{R}$ and is uniformly bounded. Usually, the reason to look for bounded solutions is to study the longtime behavior of system (1). In particular, stationary and periodic solutions as well as homoclinic and heteroclinic solutions are of special interest.

In the case $B(t)$ has an exponential dichotomy the existence of uniformly bounded solutions was intensively studied (see e.g. [2]). Different from this situation, in what follows we suppose that the matrix $B(t)$ has a pair of complex conjugate eigenvalues which crosses the imaginary axes from left to right at some point $t = t_0$. This assumption can be interpreted as some variant of the phenomenon of delayed loss of stability which is intimately related to the existence of canard solutions in singularly perturbed systems. Therefore, the goal of this paper is to find canard like solutions for system (1).

The paper is organized as follows. Next section contains the hypotheses on the r.h.s. of (1). Also, we shall show that like in the situation of canard in singularly perturbed systems we need a parameter to guarantee the existence of bounded solutions. In Sections 3, we shall state and proof the existence of this type solutions.

2 Problem statement

In this section we study the problem of existence of bounded solutions for systems of the form

$$\frac{dz}{dt} = B(t)z + Z(t, z), \tag{2}$$

where $z \in \Omega_z$, $\Omega_z := \{z \in \mathbb{R}^2 : \|z\| \leq \Delta\}$, $B(t)$ is the matrix

$$B(t) = \begin{pmatrix} \alpha t & \beta \\ -\beta & \alpha t \end{pmatrix}. \tag{3}$$

We note that the eigenvalues of B have negative real parts for $t < 0$ and positive ones for $t > 0$. Concerning the function $Z(t, z)$ we suppose

(A$_1$). $Z(t, y)$ is continuous on $\mathbb{R} \times \Omega_z$ and satisfies the following conditions

$$\begin{aligned} \|Z(t, z)\| &\leq M, \\ \|Z(t, z) - Z(t, \bar{z})\| &\leq \mu\|z - \bar{z}\|. \end{aligned} \tag{4}$$

Here and elsewhere, $\| \cdot \|$ denotes the Euclidean norm and the corresponding norm of matrices. We shall assume that

$$\frac{\sqrt{2\pi}}{\sqrt{\alpha}} M(1 + e^{\beta^2/2\alpha}) < \Delta. \tag{5}$$

Let $W(t)$ be the matrix

$$W(t) = \begin{pmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{pmatrix}. \tag{6}$$

Then

$$V(t, t_0) := e^{\frac{\alpha(t^2 - t_0^2)}{2}} W(t - t_0) \tag{7}$$

is a fundamental matrix of the linear system

$$\frac{dz}{dt} = B(t)z. \tag{8}$$

Here, $z \equiv 0$ is the only bounded solution of (8). Other solutions satisfy

$$\|z(t)\| = \|z(t_0)\| e^{\frac{\alpha(t^2 - t_0^2)}{2}}.$$

Since the matrix $B(t)$ is stable for $t < 0$ and unstable for $t > 0$, this relation shows that the behaviour of the trajectories of the system is similar to that, typical for problems on delayed loss of stability.

From (A$_1$) it follows that for any pair (t_0, z_0) the Cauchy problem for equation (2) with the initial condition $z(t_0) = z_0$ has a unique solution. This problem is equivalent to the integral equation

$$z(t) = V(t, t_0) \left(z_0 + \int_{t_0}^{t} V^{-1}(s, t_0) Z(s, z(s)) ds \right) \tag{9}$$

that can be rewritten as

$$V^{-1}(t,t_0)z(t) = z_0 + \int_{t_0}^{t} V^{-1}(s,t_0)Z(s,z(s))ds. \tag{10}$$

If there exists a bounded solution $z(t)$ of (2), then from (10) it follows that

$$\|V^{-1}(t,t_0)z(t)\| \leq ce^{\frac{\alpha(t_0^2-t^2)}{2}} \tag{11}$$

and we get

$$\lim_{t\to\pm\infty} \|V^{-1}(t,t_0)z(t)\| = 0. \tag{12}$$

Since $W(t-s) = W(t)W^{-1}(s)$, from (10) and (12) it follows that the initial value z_0 has to fulfil the conditions

$$\begin{aligned} z_0 &= \int_{-\infty}^{t_0} e^{\frac{\alpha(t_0^2-s^2)}{2}} W(t_0-s)Z(s,z(s))ds, \\ z_0 &= -\int_{t_0}^{+\infty} e^{\frac{\alpha(t_0^2-s^2)}{2}} W(t_0-s)Z(s,z(s))ds, \end{aligned} \tag{13}$$

Substituting these formulas into (9) we get for a bounded solution of (2)

$$z(t) = \begin{cases} \displaystyle\int_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)Z(s,z(s))ds, & t<0 \\ \displaystyle -\int_{t}^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)Z(s,z(s))ds, & t \geq 0. \end{cases} \tag{14}$$

From the condition of continuity of the bounded solution we get the condition

$$\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)Z(s,z(s))ds = 0 \tag{15}$$

on the function Z. It is clear that (15) is not fulfilled for arbitrary function $Z(t,z)$.

Let us consider some examples.

Example 2.1 Consider the system

$$\frac{dz}{dt} = B(t)z + a, \tag{16}$$

where $a = (a_1, a_2)^T$ is a parameter vector.

For system (16)

$$z^+(t) = -\int_t^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)a\,ds,$$

is the solution bounded for $t > 0$ and

$$z^-(t) = \int_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)a\,ds,,$$

is the solution bounded for $t < 0$.

Between these solutions there is a "step"

$$z^-(0) - z^+(0) = \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W(t-s)a\,ds = \frac{\sqrt{2\pi}}{\sqrt{\alpha}}\, e^{\frac{-\beta^2}{2\alpha}}\, a.$$

Taking the vector $a = 0$ we can remove this step and "glue" these solutions. Then under the condition that $a = 0$ system (16) has the solution $z \equiv 0$ bounded for all t.

In this example the vector a plays a role of a control or "gluing" parameter: by changing the value of a we are able to "glue" together solutions bounded on negative and positive semi-axes.

Example 2.2 Consider the system

$$\frac{dz}{dt} = B(t)z + f(t) + a, \tag{17}$$

where $f(t)$ is continuous and bounded for all $t \in \mathbb{R}$.

In order to have the uniformly bounded solution, we use (15) to get the equation for determining the vector a and arrive at

$$\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)\,(f(s) + a)\;ds = 0. \tag{18}$$

Let us introduce the following notation

$$J := \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)\,ds = \frac{\sqrt{2\pi}}{\sqrt{\alpha}}\, e^{\frac{-\beta^2}{2\alpha}}\, I, \tag{19}$$

where I is the identity matrix. From (18), we get

$$a_0 := -J^{-1} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) f(s)\,ds.$$

Therefore system (17) with $a = a_0$ has a unique solution bounded for all t. This solution is defined by

$$z(t) = \begin{cases} \int\limits_{-\infty}^{t} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\left(f(s)+a_0\right) ds, & t<0, \\ -\int\limits_{t}^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} W(t-s)\left(f(s)+a_0\right) ds, & t \geq 0. \end{cases}$$

For example, let us take in (18)

$$f(t) = (\cos t,\ 0)^T. \tag{20}$$

Then

$$a_0 = -\frac{e^{1/2}}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{\frac{-s^2}{2}} W^{-1}(s) Z(s,y) ds = -\left(\frac{e^{1/2}}{2}(1+e^{-2}), 0\right)^T,$$

and the bounded solution is defined by

$$z(t) = \begin{cases} \int\limits_{-\infty}^{t} e^{\frac{t^2-s^2}{2}} (Z(s)+a_0) ds & t<0, \\ -\int\limits_{t}^{+\infty} e^{\frac{t^2-s^2}{2}} (Z(s)+a_0) ds & t \geq 0. \end{cases}$$

Its graph is shown on Figure 1.

The idea of gluing attracting and repelling parts is applied in [1, 4] for obtaining integral manifolds with variable attractivity and canard solutions.

Let us apply this approach to system (2). For this purpose we introduce a gluing parameter into the system. Thus, we consider a system of the form

$$\frac{dz}{dt} = B(t)z + Z(t,z) + a. \tag{21}$$

In the next section we establish conditions under which (21) has a global uniformly bounded solution.

3 Main result

We consider a system of the type

$$\frac{dz}{dt} = B(t)z + Z(t,z) + a, \tag{22}$$

where $B(t)$ is defined in (3) and a is a vector of parameters.

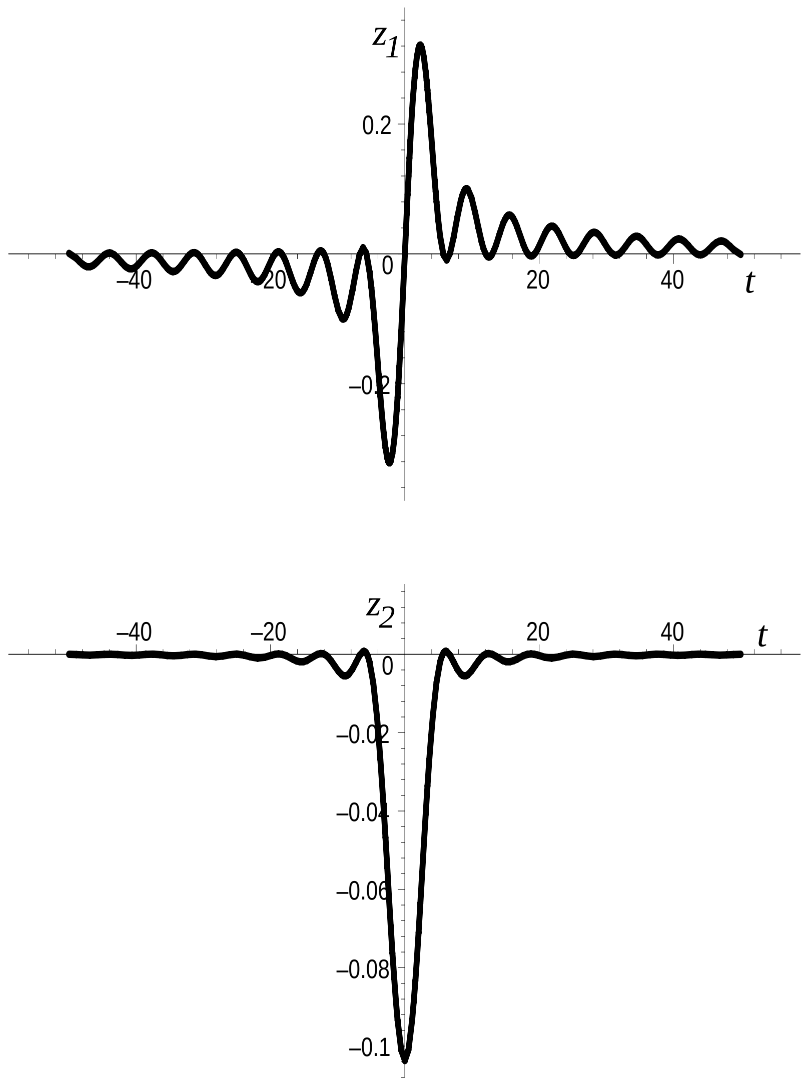

Figure 1: The two components of the bounded solution.

Theorem 3.1 *Let the function $Z(t,z)$ in the r.h.s. of (22) satisfy the assumption (A_1). Let*

$$\frac{\sqrt{2\pi}}{\sqrt{\alpha}}\mu(1+e^{\beta^2/2\alpha})<1. \tag{23}$$

Then there exists a unique vector a such that (22) has a global uniformly bounded solution.

Generally, solutions of (22) exhibit the same type of behaviour as that of (8). More precisely, the trajectory of system (22) starting for $t=t_0<0$ at any initial point z_0 enters after a short time interval a small neighbourhood of the uniformly bounded solution and stays in it until some time $t=t^*(t_0,z_0)>0$, where t^* increases with respect to $|t_0|$. For $t>|t_0|$ the trajectory jumps away. This phenomenon is similar to the effect of delayed loss of stability for singularly perturbed systems [3, 5].

Theorem 3.1 can be generalized to systems of the form (22) with the crucial feature that the matrix $B(t)$ has a pair of simple complex conjugate eigenvalues crossing the imaginary axis for some value $t=t^*$ at some points $\pm i\,\omega$, $\omega\neq 0$.

4 Proof of the main result

Let H be the complete metric space of functions $h(t)$ mapping continuously $\mathbb{R}$ into Ω_z and satisfying the inequality

$$\|h(t)\|\leq N, \tag{24}$$

with $N\leq\Delta$, equipped with the uniform metric

$$\rho(h,\bar{h})=\sup_{t\in\mathbb{R}}\|h(t)-\bar{h}(t)\|.$$

On the space H we define the operator T_a

$$T_ah(t)=\begin{cases}-\int\limits_t^{+\infty}e^{\frac{\alpha(t^2-s^2)}{2}}W(t-s)\left[Z\left(s,h(s)\right)+a\right]\,ds, & t\geq 0,\\ \int\limits_{-\infty}^{t}e^{\frac{\alpha(t^2-s^2)}{2}}W(t-s)\left[Z\left(s,h(s)\right)+a\right]\,ds, & t<0,\end{cases}$$

depending on a. By definition, if $h=T_ah$ for $t>0$ then $z=h(t)$ is a bounded solution of (22) on the positive semi-axis. Similarly, $h=T_ah$ for $t<0$ implies that $z=h(t)$ is a bounded solution of (22) on the negative semi-axis.

We shall prove that for every $h\in H$ there exists a unique $a=P(h)$ such that the function $T_{P(h)}h$ is continuous. We define the operator T by

$$Th(t)=T_{P(h)}h(t)$$

and show that T maps H into itself and is a contraction. Therefore, there exists a unique fixed point h^* of T in H. By construction, the fixed point is a the solution of (22) bounded for all $t\in\mathbb{R}$, which implies the conclusion of the theorem.

4.1 Continuity of Th and estimates of P

It is easy to check that $T_a h$ is continuous for $t < 0$ and $t > 0$ for each $h \in H$. The continuity of $T_a h$ at $t = 0$ is considered in the following lemma.

Lemma 4.1 *For any function $h \in H$ there exist a unique vector a such that the function $T_a h$ is continuous.*

Proof.

The condition of continuity of the function $T_a h$ at the point $t = 0$ is equivalent to the following condition

$$\int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s)[Z(s, h(s)) + a]\, ds \;=\; 0. \tag{25}$$

Let us rewrite (25) in the form

$$J_1 \;+\; J\, a \;=\; 0,$$

where J is defined by (19) and

$$J_1 := \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) Z(s, h(s))\; ds.$$

The integral J_1 converges due to the assumption (A_1) on the function Z. Therefore, $a := -J^{-1} J_1$, that is

$$a = -\frac{\sqrt{\alpha} e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} W^{-1}(s) Z(s, h(s))\; ds. \tag{26}$$

It completes the proof.

We use (26) as the definition of the functional $a = P(h)$ acting from H to $\mathbb{R}^2$. To complete the proof we need the next lemma.

Lemma 4.2 *The following estimates are valid*

$$\|a\| \leq e^{\beta^2/2\alpha} M,$$

$$\|a - \bar{a}\| \leq e^{\beta^2/2\alpha} \mu \rho(h, \bar{h}),$$

where $a = P(h)$ and $\bar{a} = P(\bar{h})$ for any $h, \bar{h} \in H$.

Proof. From (26) and the assumption (A_1) it follows that

$$\|a\| \leq \|J^{-1}\| \int_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}}\, \|Z(s, h(s))\|\; ds \leq e^{\beta^2/2\alpha} M.$$

For the difference between a and $\bar{a}$ we have

$$\|a-\bar{a}\| \leq \|J^{-1}\| \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \|Z\left(s,h(s)\right) - Z\left(s,\bar{h}(s)\right)\| \, ds \leq$$

$$\leq \frac{\sqrt{\alpha} e^{\beta^2/2\alpha}}{\sqrt{2\pi}} \mu \int\limits_{-\infty}^{+\infty} e^{\frac{-\alpha s^2}{2}} \|h(s)-\bar{h}(s)\| \, ds \leq e^{\beta^2/2\alpha} \mu \rho(h,\bar{h}).$$

Thus,

$$\|a-\bar{a}\| \leq e^{\beta^2/2\alpha} \mu \rho(h,\bar{h}). \tag{27}$$

This completes the proof of Lemma 4.2.

4.2 Existence of the bounded solution

Let $t \geq 0$. By the assumption (A_1) and Lemma 4.2 we have

$$\|Th(t)\| \leq \int\limits_{t}^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \left[\|Z\left(s,h(s)\right)\| + \|a\|\right] \, ds \leq \frac{\sqrt{2\pi}}{\sqrt{\alpha}} M(1+e^{\beta^2/2\alpha}).$$

Analogously, one sees that the same estimate holds for $t \leq 0$. It means that Th is uniformly bounded. From (5) it follows that the function Th belongs to the space H, that is T maps H into itself.

From the assumption (A_1) and Lemma 4.2 it follows for $t \geq 0$

$$\|Th(t) - T\bar{h}(t)\| \leq \int\limits_{t}^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} \|Z\left(s,h(s)\right) - Z\left(s,\bar{h}(s)\right)\| + \|a-\bar{a}\| \leq$$

$$\leq \int\limits_{t}^{+\infty} e^{\frac{\alpha(t^2-s^2)}{2}} [\mu\rho(h,\bar{h}) + e^{\beta^2/2\alpha}\mu\rho(h,\bar{h})] \, ds \;=\; \frac{\sqrt{2\pi}}{\sqrt{\alpha}} \mu(1+e^{\beta^2/2\alpha})\rho(h,\bar{h}).$$

The same estimate is valid for $t \leq 0$, therefore

$$\rho(Th, T\bar{h}) \leq \frac{\sqrt{2\pi}}{\sqrt{\alpha}} \mu(1+e^{\beta^2/2\alpha})\rho(h,\bar{h}),$$

and the condition (23) implies that T is a contraction in H. This completes the proof of Theorem 3.1.

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