für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Existence and uniquenes results for reaction–diffusion processes of electrically charged species

Herbert Gajewski[1], Igor V. Skrypnik[2]

submitted: 25th May 2004

[1] Weierstrass Institute for Applied
Analysis and Stochastics
Mohrenstr. 39
D-10117 Berlin
Germany
E-Mail: gajewski@wias-berlin.de

[2] Institute for Applied Mathematics
and Mechanics
Rosa Luxemburg Str. 74
83114 Donetsk
Ukraine
E-Mail: skrypnik@iamm.ac.donetsk.ua

No. 938
Berlin 2004

WIAS

2000 *Mathematics Subject Classification.* 35B45, 35K15, 35K20, 35K65.

Key words and phrases. Nonlinear elliptic-parabolic systems, nonlocal drift, global bounded solutions, uniqueness, nonstandard assumptions, degenerate type.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We study initial– boundary value problems for elliptic–parabolic systems of nonlinear partial differential equations describing drift–diffusion processes of electrically charged species in N–dimensional bounded Lipschitzian domains. We include Fermi–Dirac statistics and admit nonsmooth material coefficients. We prove existence and uniqueness of bounded global solutions.

1 Introduction

We study a mathematical model describing drift–diffusion processes of electrically charged species. Such processes play an important role in many branches of modern technology (see [4], [13], [14], [17]). The classical drift–diffusion model of charged carrier transport in semiconductors was established by van Roosbroeck [16]. It consists of a Poisson equation for the electrostatic potential v_0 and continuity equations for the densities u_1, u_2 of electrons and holes. The clasical van Roosbroeck model describes processes in homogeneous semiconductor materials (e.g. silicon). Modern devices are often heterostructures where complex reactions take place. By this reason we admit nonsmooth data and $n\,(\geq 2)$ species with densities u_i and specific charges q_i.

The mathematical model is formulated below in Section 2. It consists of a Poisson equation (2.1) and n continuity equations (2.2). The equation (2.1) expresses the Gauss law, the system (2.2) means local carrier conservation. The system (2.1), (2.2) is completed by current relations (2.3), which suppose the antigradients of the electrochemical potentials ζ_i from (2.5) to be driving forces for carrier transport. We consider the system (2.1), (2.2) in a bounded Lipschitzian domain $\Omega \subset R^N$, $N \geq 3$, completed by boundary and initial conditions (2.7) – (2.9).

The initial-boundary value problem (2.1), (2.2), (2.7) – (2.9) was formulated and studied in [8]. In that paper the free energy was recovered to be Lyapunov functional of the system and was used for proving a priori estimates, existence and uniqueness results. However, for space dimension $N \geq 3$ a gap remained between existence and uniqueness results in particular for the physically most relevant case that the state relation (2.4) is according to Fermi–Dirac statistics. Actually for $N \geq 3$ the paper [8] rests on following restricting hypotheses:
The existence result holds for dielectric pemittivity ϵ from (2.1) and conductivity functions d_i from (2.3) such that

$$\epsilon = constant \ , \quad d_i(t,x,z,\xi) = \sigma_i(z)\ \xi \ , \tag{1.1}$$

that means, $J_i = -\sigma_i(v_i)\,\nabla\zeta_i$. Moreover, except for the special case of Boltzmann statististics (i.e., e_i in (2.4) is specified as exponential function), the uniqueness result in [8] supposes the electrostatic potential v_0 to satisfy the regularity condition

$$\nabla v_0 \in L^\infty\big((0,T);L^p(\Omega)\big) \quad \text{for some} \quad p > N\ . \tag{1.2}$$

(As to the validity of (1.2) in some nonsmooth situations comp. [5].)

The present paper mainly aims to fill that gap by proving global existence and uniqueness results without the restricting hypotheses (1.1), (1.2). To this end we apply to problem (2.1), (2.2), (2.7) – (2.9) an approach developed for model situations in our papers [9], [10], [11], [12]. The key role play sophisticated test functions in integral identities for proving a priori estimates and the uniqueness result.

The paper is organized as follows. Formulations of all hypotheses and main results are contained in Section 2. Integral estimates for the chemical potentials v_i and the electrostatical potential v_0 are proved in Section 3. In Section 4 we study the boundedness of the potentials v_i and v_0. A proof of the existence result is sketched in Section 5. The detailed proof of our main result, the uniqueness theorem, is given in Section 5. Finally, Section 6 is devoted to the spcecial case of functions d_i being linear with respect to ζ. This case is studied without growth conditions for the reaction terms.

2 Mathematical model and formulation of main results

The drift–diffusion model describing n species with densities u_i and specific charges q_i was formulated in [6], [7], [8] and reads as follows

$$-\nabla\cdot(\epsilon\nabla v_0) = f + \sum_{i=1}^{n} q_i\, u_i \quad on \quad Q_T = \Omega\times(0,T)\ , \tag{2.1}$$

$$\frac{\partial u_i}{\partial t} + \nabla\cdot J_i + R_i = 0 \quad \text{on} \quad Q_T\ , \quad i = 1,\dots,n\ , \tag{2.2}$$

where T is a finite time and Ω is a bounded Lipschitzian domain in R^N. We suppose later on that $N \geq 3$. In (2.1) v_0 is the electrostatical potential, ϵ is the dielectric permittivity, f describes external sources (impurities). The currents J_i are given in the form

$$J_i = -d_i(\cdot, v_i, \nabla\zeta_i), \quad i = 1,\dots,n\ , \tag{2.3}$$

where v_i are chemical potentials related to the densities u_i by the state equations

$$u_i = u_i^*\, e_i(v_i), \quad i = 1,\dots,n\ , \tag{2.4}$$

with given, strictly positive state densities $u_i^* \in L^\infty(\Omega)$. The electrochemical potentials ζ_i from (2.3) are defined by

$$\zeta_i = q_i\, v_0 + v_i, \quad i = 1,\dots,n. \tag{2.5}$$

Remark 2.1 *The state equations 2.4 are choosen for simplicity. The results of the paper remain true for state equations like*

$$u_i = u_i^* \, e_i(v_i + q_i g_i), \quad i = 1, \ldots, n \,,$$

with given band edges $g_i \in H^{1,\infty}(\Omega)$. *This can be seen by replacing the argument* $v_i = \zeta_i - q_i \, v_0$ *of* e_i *and its derivatives by* $v_i = \zeta_i - q_i \, \bar{v}_0$, $\bar{v}_0 = v_0 - g_i$. *An extension of our results to piece-wise constant* g_i*'s, desirable in view of heterostructures, is not trivial. However, note that for Boltzmann statistics, i.e.,* $e_i = \exp$, *even the case* $g_i \in L^\infty(\Omega)$ *can be included by setting* $u_i^* := u_i^* \exp(q_i g_i) \in L^\infty(\Omega)$.

The reaction terms R_i in (2.2) have the form:

$$R_i(\cdot, v, \zeta) = \sum_{(\alpha,\beta)\in\mathcal{R}} \big[r_{\alpha\beta}(\cdot, v, \alpha\cdot\zeta) - r_{\alpha\beta}(\cdot, v, \beta\cdot\zeta) \big] (\alpha_i - \beta_i) \,, \tag{2.6}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathcal{R} \subset \mathbb{R}^n$ are vectors of stoichiometric coefficients and the finite set $\mathcal{R}$ denotes the reactions actually taking place in the volume Ω occupied by the species.

Remark 2.2 *There are modified drift–diffusion models of charged species. So in the papers [3] and [4] Poisson's equation (2.1) is replaced by the neutrality condition*

$$f + \sum_{i=1}^{n} q_i \, u_i \, = 0 \quad \textit{on} \quad Q_T = \Omega \times (0,T) \,.$$

We complete system (2.1), (2.2) by boundary and initial conditions :

$$\nu \cdot J_i + R_i^\Gamma = 0 \quad \text{on} \quad \Gamma_T = (0,T) \times \partial\Omega, \tag{2.7}$$

$$\nu \cdot (\epsilon \nabla v_0) + \kappa v_0 = f^\Gamma \quad \text{on} \quad \Gamma_T, \tag{2.8}$$

$$u_i(0,\cdot) = h_i \quad \text{on} \quad \Omega \,, \quad i = 1, \ldots, n, \tag{2.9}$$

where $\nu(x')$ is the outer unit normal at $x' \in \partial\Omega$, R_i^Γ represents reactions taking place on the boundary $\partial\Omega$ of Ω. We assume that

$$R_i^\Gamma = \sum_{(\alpha,\beta)\in\mathcal{R}^\Gamma} \big(r_{\alpha\beta}^\Gamma(\cdot, v, \alpha\cdot\zeta) - r_{\alpha\beta}^\Gamma(\cdot, v, \beta\cdot\zeta) \big) (\alpha_i - \beta_i) \,, \tag{2.10}$$

where $\mathcal{R}^\Gamma$ is a finite set of vector pairs of stoichiometric coefficients and the functions $r_{\alpha\beta}^\Gamma$ model surface reaction rates.

Remark 2.3 *As a special feature the boundary condition (2.8) with (2.10) allows thermal equilibria, i. e. steady states with vanishing driving forces* $\nabla\zeta_i$. *However, the results of the paper remain true for other kinds of boundary conditions, for example*

$$\nu \cdot J_i + \kappa_i \, (\zeta_i - f_i^\Gamma) = 0 \quad \textit{on} \quad \Gamma_T = (0,T) \times \partial\Omega,$$

with κ_i, $f_i^\Gamma \in L^\infty(\partial\Omega)$, $\kappa_i \geq 0$.

The system (2.1), (2.2), (2.7) – (2.9) will be solved for the unknown vector $v = (v_0, v_1, \dots, v_n)$ taking into account the relations (2.3) – (2.5) between v and $J = (J_1, \dots, J_n)$, $u = (u_1, \dots, u_n)$, $\zeta = (\zeta_1, \dots, \zeta_n)$, respectively.

We assume the data of problem (2.1), (2.2), (2.7) – (2.9) to satisfy following hypotheses:

i) $d_{ij}(t,x,z,\xi)$, $i = 1, \dots, n$, $j = 1, \dots, N$, $r_{\alpha\beta}(t,x,v,y)$, $r^{\Gamma}_{\gamma\delta}(t,x',v,y)$, $(\alpha,\beta) \in \mathcal{R}$, $(\gamma,\delta) \in \mathcal{R}^{\Gamma}$, are measurable functions of $(t,x) \in Q_T$, $(t,x') \in \Gamma_T$ with respect to Lebesgue and surface measures respectively for every $z, y \in \mathbb{R}^1$, $\xi \in R^N$, $v \in \mathbb{R}^{n+1}$ and continuous functions with respect to y, z, ξ, v for almost every $(t,x) \in Q_T$, $(t,x') \in \Gamma_T$, $d_i(t,x,z,0) = 0$ for $i = 1, \dots, n$;
ϵ and u_i^* are measurable functions on Ω; $\kappa \in L^\infty(\partial\Omega)$; $\kappa \geq 0$, $\kappa \neq 0$;
q_i is equal to 1 or to -1 ;

ii) $e_i \in (\mathbb{R}^1 \to \mathbb{R}^1)$ is continuously differentiable such that $e_i'(z) > 0$, $z \in \mathbb{R}^1$; $\lim\limits_{z\to-\infty} e_i(z) = 0$, $\lim\limits_{z\to+\infty} e_i(z) = +\infty$, $\int_{-\infty}^{0} e_i(z)\, dz < \infty$, $i = 1, \dots, n$;

iii) there exist positive constants ν_1, ν_2 such that for arbitrary $\xi, \xi', \xi'' \in R^N$, $(t,x) \in Q_T, z \in \mathbb{R}^1$

$$\sum_{j=1}^{N} \big[d_{ij}(t,x,z,\xi') - d_{ij}(t,x,z,\xi'')\big](\xi_j' - \xi_j'') \geq \nu_1\, e_i'(z)|\xi' - \xi''|^2,$$
$$\big|d_{ij}(t,x,z,\xi)\big| \leq \nu_2(1 + |\xi|)e_i'(z);\; e_i'(z) \leq \nu_2\, e_i(z) \quad \text{for} \quad z < 0,$$
$$\nu_1 \leq \epsilon(x) \leq \nu_2,\; \nu_1 \leq u_i^*(x) \leq \nu_2, \quad i = 1, \dots, n; \quad j = 1, \dots, N;$$

iv) the functions $r_{\alpha\beta}(t,x,v,y)$, $r^{\Gamma}_{\gamma\delta}(t,x',v,y)$, $(\alpha,\beta) \in \mathcal{R}$, $(\gamma,\delta) \in \mathcal{R}^{\Gamma}$, are increasing in $y \in \mathbb{R}^1$ for $(t,x) \in Q_T$, $(t,x') \in \Gamma_T$, $v \in \mathbb{R}^{n+1}$ and there exist convex functions $M : \mathbb{R}^1 \to \mathbb{R}^1_+ = \{z \in \mathbb{R}^1 : z > 0\}$, $M^{\Gamma} : \mathbb{R}^1 \to \mathbb{R}^1_+$ such that

$$[r_{\alpha\beta}(t,x,v,\alpha\cdot\zeta) - r_{\alpha\beta}(t,x,v,\beta\cdot\zeta)](\alpha - \beta)\cdot\zeta \leq M(|v|),$$
$$[r^{\Gamma}_{\gamma\delta}(t,x',v,\gamma\cdot\zeta) - r^{\Gamma}_{\gamma\delta}(t,x',v,\delta\cdot\zeta)](\gamma - \delta)\cdot\zeta \leq M^{\Gamma}(|v|), \quad \zeta_i = v_i + q_i v_0.$$

Finally, we assume the data f, f^{Γ}, h_i to satisfy:

$$\begin{gathered} f \in C\big([0,T]; L^{p_1}(\Omega)\big), \quad \frac{\partial f}{\partial t} \in L^2\big(0,T; [W^{1,2}(\Omega)]^*\big), \qquad p_1 > \frac{N}{2}, \\ f^{\Gamma} \in C\big([0,T]; L^{p_2}(\partial\Omega)\big), \quad \frac{\partial f^{\Gamma}}{\partial t} \in L^2\big(0,T; [W^{\frac{1}{2},2}(\partial\Omega)]^*\big), \quad p_2 > N-1\,, \\ \log(h_i) \in L^\infty(\Omega), \quad i = 1, \dots, n. \end{gathered} \tag{2.11}$$

Definition 2.1 *A vector $v = (v_0, \dots, v_n)$ is called solution of problem (2.1), (2.2), (2.7) – (2.9), if for $i = 1, \dots, n$:*

i) $v_0 \in C\big([0,T];W^{1,2}(\Omega)\big), \quad v_i \in L^2\big(0,T;W^{1,2}(\Omega)\big)$,
$u_i = u^* e_i(v_i) \in C\left([0,T];L^2(\Omega)\right)$, $\frac{\partial}{\partial t} u_i \in L^2\left(0,T;[W^{1,2}(\Omega)]^*\right)$,
where the time derivative is to be understood in the sense of distributions,

$$\begin{gathered}\iint\limits_{Q_T} \Big\{ e_i'(v_i)\Big[\left|\frac{\partial v_i}{\partial x}\right|^2 + \left|\frac{\partial v_0}{\partial x}\right|^2 \Big] + M(|v|) \Big\}\, dx\, dt < \infty \ , \\ \iint\limits_{\Gamma_T} M^\Gamma(|v|)\, dx\, dt < \infty \ ; \end{gathered} \tag{2.12}$$

ii) *for arbitrary test functions $\varphi \in C^\infty(\overline{Q}_T)$, $\psi \in C^\infty(\overline{\Omega})$, almost every $\tau \in (0,T)$ and $i = 1,\dots,n$ the following integral identities hold:*

$$\begin{aligned} \int_0^\tau \Big\{ < \frac{\partial u_i}{\partial t}, \varphi > + \int_\Omega \Big[\sum_{j=1}^N d_{ij}\Big(t,x,v_i,\frac{\partial(v_i + q_i v_0)}{\partial x}\Big) \frac{\partial \varphi}{\partial x_j} + \\ + R_i(t,x,v,\zeta)\,\varphi \Big]\, dx + \int_{\partial\Omega} R_i^\Gamma(t,x,v,\zeta)\,\varphi\, ds \Big\}\, dt = 0 \ , \end{aligned} \tag{2.13}$$

$$\begin{aligned} \int_\Omega \Big\{ \epsilon(x) \sum_{j=1}^N \frac{\partial v_0}{\partial x_j}\frac{\partial \psi}{\partial x_j} - \Big[\sum_{i=1}^n q_i\ u_i + f(t,x) \Big] \psi \Big\}\, dx + \\ + \int_{\partial\Omega} \big(\kappa(x) v_0 - f^\Gamma\big)\psi\, ds = 0 \ , \end{aligned} \tag{2.14}$$

where $\zeta(t,x) = \big(\zeta_1(t,x),\dots,\zeta_n(t,x)\big)$, $\zeta_i(t,x) = v_i(t,x) + q_i\, v_0(t,x)$;

iii) *for test functions $\varphi \in C^\infty\left(\overline{Q}_T\right)$ with $\varphi(\tau,x) = 0$, $x \in \Omega$, the integral identity*

$$\int_0^\tau < \frac{\partial u_i}{\partial t}, \varphi > dt \ + \int_0^\tau \int_\Omega [u_i - h_i] \frac{\partial \varphi}{\partial t}\, dx\, dt = 0 \tag{2.15}$$

holds for $\tau \in (0,T)$, $\quad i = 1,\dots,n$.

Besides of (2.2), (2.7) we shall consider for $\delta \in [0,1]$ the regularized equations

$$\frac{\partial u_i}{\partial t} + \nabla \cdot J_i^{(\delta)} + R_i = 0 \quad \text{on} \quad Q_T \ , \tag{2.16}$$

$$\nu \cdot J_i^{(\delta)} + R_i^\Gamma = 0 \ , \quad \text{on} \quad \Gamma_T \ , \tag{2.17}$$

$$J_i^{(\delta)} = -d_i\Big(\cdot, v_i^{(\delta)}, \nabla\zeta_i\Big) \ , \quad v_i^{(\delta)} = \max\Big\{v_i, -\frac{1}{\delta}\Big\}, \ \ J_i^{(0)} = J_i \ , \quad i = 1,\dots,n. \tag{2.18}$$

Solutions of problem (2.1), (2.16), (2.17), (2.8), (2.9) are defined as in Definition 2.1.

In what follows we understand as known parameters the numbers ν_1, ν_2, n, N, T, vectors in $\mathcal{R}, \mathcal{R}^\Gamma$, norms of the data f, f^Γ, h_i in respective spaces and numbers that depend only on Ω, M, M^Γ and κ. Moreover, we denote by c_k, $k = 1,\dots,$ constants depending only on known parameters.

Theorem 2.1 *Let the conditions i) – iv), (2.11) be satisfied. Then there exists a constant* K_1 *depending only on known parameters and independent of* $\delta \in [0,1]$ *such that each solution* v *of problem (2.1), (2.16), (2.17), (2.8), (2.9) satisfies*

$$\begin{aligned} &ess \sup_{t\in(0,T)} \left\{ \int_\Omega \left[\Lambda_i\big(v_i(t,x)\big) + \left|\frac{\partial v_0(t,x)}{\partial x}\right|^2 \right] dx \ + \int_{\partial\Omega} \kappa(x) v_0^2(t,x)\, ds \right\} + \\ &+ \iint\limits_{Q_T} e_i'(v_i) \left| \frac{\partial(v_i + q_i v_0)}{\partial x} \right|^2 dx\, dt \le K_1 \ , \end{aligned} \tag{2.19}$$

where

$$\Lambda_i(v) = \int_0^v s\, e_i'(s)\, ds, \quad i = 1, \ldots, n. \tag{2.20}$$

For establishing further integral estimates we need growth conditions for the functions e_i', R_i, R_i^T, $i = 1, \ldots, n$:

$$\nu_3(v^{\gamma_i} + 1) \le e_i'(v) \le \nu_4(v^{\gamma_i} + 1), \quad v > 0, \quad 0 \le \gamma_i < \frac{4}{N-2}\ , \tag{2.21}$$

$$R_i(t,x,v,\zeta) \ge -\nu_4 \left(\sum_{j=1}^n [v_j]_+^{p_3} + |v_0|^{p_3} \right) - \alpha_1(t,x), \quad \text{for} \quad v_i > 0\ , \tag{2.22}$$

$$R_i^\Gamma(t,x',v,\zeta) \ge -\nu_4 \left(\sum_{j=1}^n [v_j]_+^{p_4} + |v_0|^{p_4} \right) - \alpha_2(t,x'), \quad \text{for} \quad v_i > 0\ , \tag{2.23}$$

where ν_3, ν_4 are positive constants and

$$\begin{aligned} &p_3 < \gamma_* + 1 + \frac{2}{N}(\gamma_* + 2), \quad p_4 < \gamma_* + 1 + \frac{1}{N}(\gamma_* + 2), \quad \gamma_* = \min(\gamma_1, \ldots, \gamma_n), \\ &\alpha_1 \in L^{r_1}(Q_T), \quad r_1 > \frac{N+2}{2}, \quad \alpha_2 \in L^{r_2}(\Gamma_T), \quad r_2 > N+1. \end{aligned}$$

Remark 2.4 *The growth condition (2.21) is satisfied by functions* e_i *according to Fermi-Dirac statistics, i.e. by Fermi Integrals:*

$$e_i(v) = \mathcal{F}_{\gamma_i}(v) = \frac{1}{\Gamma(\gamma_i + 1)} \int_0^\infty \frac{s^{\gamma_i}\, ds}{1 + \exp(s - v)}\ .$$

Note that the exponential function (Boltzmann statistics) violates(2.21).
Standard reaction terms like Shockley-Read and Auger recombination/generation [8] satisfy (2.22)- (2.23).

We understand numbers from conditions (2.21) – (2.23) and norms of the functions α_1, α_2 as known parameters too.

Theorem 2.2 *Let the assumptions of Theorem 2.1 and the conditions (2.21) – (2.23) be satisfied. Then there exists a constant* K_2 *depending only on known parameters and independent of* $\delta \in [0,1]$ *such that each solution* v *of problem (2.1), (2.16), (2.17), (2.8), (2.9) satisfies*

$$\iint\limits_{Q_T} e_i'(v_i)\left\{\left|\frac{\partial v_i}{\partial x}\right|^2 + \left|\frac{\partial v_0}{\partial x}\right|^2\right\} dx\,dt \le K_2, \quad i = 1,\dots,n. \tag{2.24}$$

Theorem 2.3 *Let the assumptions of Theorem 2.2 be satisfied. Then there exist constants* K_3 *and* $\eta \in (0,1)$ *depending only on known parameters and independent of* δ *such that for arbitrary* $t \in [0,T]$, $x, y \in \Omega$

$$\|v_0\|_{L^\infty(Q_T)} \le K_3, \quad |v_0(t,x) - v_0(t,y)| \le K_3\,|x-y|^\eta\,. \tag{2.25}$$

In view of controlling $v_i(t,x)$ from below we suppose additionally to (2.22) and (2.22) that for $v_i < 0,\ i = 1,\dots,n$:

$$R_i(t,x,v,\zeta) \le \nu_4\, e_i(v_i)\big[F\big(v_0, e(v)\big) + \alpha_1(t,x)\big]\,, \tag{2.26}$$

$$R_i^\Gamma(t,x',v,\zeta) \le \nu_4\, e_i(v_i)\big[F\big(v_0, e(v)\big) + \alpha_2(t,x')\big] \tag{2.27}$$

with $\nu_4,\ \alpha_1(t,x),\ \alpha_2(t,x')$ as in (2.21), (2.22), $e(v) = (e_1(v_1),\dots,e_n(v_n))$ and some continuous function $F : \mathbb{R}^{n+1} \to \mathbb{R}^1$.

Theorem 2.4 *Let the assumptions of Theorem 2.2 and the conditions (2.26), (2.27) be satisfied. Then there exists a constant* K_4 *depending only on known parameters and independent of* $\delta \in [0,1]$ *such that for each solution* $v = (v_0, v_1, \dots, v_n)$ *of problem (2.1), (2.16), (2.17), (2.8), (2.9)*

$$\text{ess sup}\,\big\{|v_i(t,x)| : (t,x) \in Q_T\big\} \le K_4\,, \quad i = 0,\dots,n\,. \tag{2.28}$$

Theorem 2.5 *Let the conditions i) – iv), (2.11), (2.21) – (2.23), (2.26), (2.27) be satisfied. Then the initial–boundary value problem (2.1), (2.2), (2.7) – (2.9) has at least one solution in the sense of the Definition 2.1.*

Theorem 2.6 *Let the conditions of Theorem 2.5 be satisfied. Assume additionally that for* $i = 1,\dots,n,\ j = 1,\dots,N$*:*

(i) the functions $d_{ij}(t,x,z,\xi)$ *have the special structure*

$$d_{ij}(t,x,z,\xi) = e_i'(z)\gamma_{ij}(t,x,\xi) \tag{2.29}$$

where $e_i' \circ e_i^{-1} : (0,\infty) \to (0,\infty)$ *is piece–wise differentiable and concave ;*

(ii) the functions e_i'', $\gamma_{ij}(t,x,\xi)$, $r_{\alpha\beta}(t,x,v,y)$, $r^\Gamma_{\gamma\delta}(t,x',v,y)$ are locally Lipschitzian with respect to ξ, v, y.

Then the initial–boundary value problem (2.1), (2.2), (2.7) – (2.9) has a unique solution in the sense of the Definition 2.1.

Remark 2.5 *The Fermi integrals from Remark 2.4 satisfy the respective assumptions of Theorem 2.6. In particular the concavity property follows easily from Jensens's inequality [1].*

Corollary 2.1 *Let the conditions of Theorem 2.6 be satisfied and assume additionally that the functions f_i, F^Γ, d_{ij}, $r_{\alpha\beta}$, $r^\Gamma_{\alpha\beta}$ are Lipschitzian with respect to t. Then the solution v of problem (2.1), (2.2),(2.7) – (2.9) is regular in the sense that*

$$t \to t\,\frac{\partial v_i}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)),\ i = 1,\dots,n.$$

Remark 2.6 *Corollary 2.1 and Theorem 2.4 imply that $t \to t\,\frac{\partial u_i}{\partial t} \in L^\infty(0,T;L^2(\Omega))$. Consequently, (2.2) can be understood not only in the sense of distributions, but even as an equation in $L^2(0,T;L^2(\Omega))$.*

We conclude this Section considering the special case that the currents J_i are linear with respect to the gradients of the electrochemical potentials ζ_i. This case is interesting in so far as we don't need the growth restrictions (2.22), (2.23) for the reaction terms.

Theorem 2.7 *Let the conditions i) – iv), (2.11), (2.21), (2.26), (2.27) be satisfied. Suppose that the reference densities from (2.4) and the exponents γ_i from (2.21) satisfy*

$$u_i^* = u^*\ ,\quad \gamma_i = \gamma\ ,\quad i = 1,\dots,n \tag{2.30}$$

and that for $(\alpha,\beta) \in \mathcal{R}$, $(\gamma,\delta) \in \mathcal{R}^\Gamma$

$$\alpha\cdot q = \beta\cdot q,\quad \gamma\cdot q = \delta\cdot q\ . \tag{2.31}$$

Moreover, assume the functions d_{ij} to have the structure

$$d_{ij}(t,x,z,\xi) = \sum_{k=1}^{N} e_i'(z)a_{kj}(t,x)\xi_k\ ,\quad i = 1,\dots,n,\quad j = 1,\dots,N. \tag{2.32}$$

Then all assertions of the Theorems 2.2–2.6 are valid.

Remark 2.7 *We assumed the conincidence of the γ_i's for simplicity. It is possible to replace it by some restriction on $\max\{|\gamma_i - \gamma_j|,\ 1 \le i,\ j \le n\}$. Analogously to [11] it is possible to prove Theorem 2.7 for γ_i satisfying only $0 \le \gamma_i < \frac{2}{N-2}$, $i = 1,\dots,n$.*

We shall prove the Theorems 2.1, 2.2 in Section 3, the Theorems 2.3, 2.4 in Section 4 and the Theorems 2.5, 2.6 in Section 5. Finally we shall make some comments with respect to the proof of Theorem 2.7 in Section 6.

3 Proof of integral estimates

The proof of a priori estimates in this section rests on testing the integral identities (2.13), (2.14) by suitable functions. For that purpose the following remark is useful:

Remark 3.1 *Let $F : \mathbb{R}^{n+1} \to \mathbb{R}^1$ be an arbitrary piece–wise differentiable function with bounded gradient and let $v(t,x)$ be a solution of problem (2.1), (2.16), (2.17), (2.8), (2.9). Then the equality (2.13) holds for $\varphi(t,x) = F(v(t,x))$. Moreover, (2.14) holds for arbitrary functions $\psi \in W^{1,2}(\Omega)$. That follows from (2.12) after approximating $v(t,x)$ by smooth functions.*

Proof of Theorem 2.1. Let v be a solution of problem (2.1), (2.16), (2.17), (2.8), (2.9). Denote by $g_0(x)$ a solution of the problem

$$-\nabla \cdot (\epsilon \nabla g_0) = f(0,x) + \sum_{i=1}^{n} q_i h_i(x) \quad \text{on} \quad \Omega \, , \tag{3.1}$$

$$\nu \cdot (\epsilon \nabla g_0) + \kappa g_0 = f^{\Gamma}(0,x') \quad \text{on} \quad \partial\Omega \, . \tag{3.2}$$

We extend $v_i(t,x)$ for $t < 0, \ x \in \Omega$ by setting $v_i(t,x) = g_i(x)$, where $g_i(x) = e_i^{-1}\left(\frac{h_i(x)}{u_i^*(x)}\right)$, $i = 1, \dots, n$. In analogous way we extend $f(t,x)$ and $f^{\Gamma}(t,x')$. Testing the integral identity (2.14) with $\psi(x) = v_0(t+s,x) - v_0(t,x)$ and integrating on t, we obtain for $\tau \in (0,T), \ s \in (0, T-\tau)$,

$$\begin{aligned}
&\int_{-s}^{\tau}\int_{\Omega}\Big\{\epsilon(x)\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\Big[v_0(t+s,x)+v_0(t,x)\Big]\frac{\partial}{\partial x_j}\big[v_0(t+s,x)-v_0(t,x)\big]-\\
&\Big[\sum_{i=1}^{n} q_i\big(u_i(t+s,x)+u_i(t,x)\big)+f(t+s,x)+f(t,x)\Big]\times\\
&\times\big[v_0(t+s,x)-v_0(t,x)\big]\Big\}\,dx\,dt+\int_{-s}^{\tau}\int_{\partial\Omega}\Big\{\kappa(x)\big[v_0(t+s,x)+v_0(t,x)\big]-\\
&-f^{\Gamma}(t+s,x)-f^{\Gamma}(t,x)\Big\}\big[v_0(t+s,x)-v_0(t,x)\big]\,ds\,dt=0\,.
\end{aligned}\tag{3.3}$$

Arguing as in the proof of Theorem 2.3 in [11], we infer from (3.3)

$$\begin{aligned}
&\int_{\Omega}\epsilon(x)\left|\frac{\partial v_0(\tau,x)}{\partial x}\right|^2 dx+\int_{\partial\Omega}\kappa(x)v_0^2(\tau,x)\,ds-\sum_{i=1}^{n}q_i\int_0^{\tau}<\frac{\partial u_i}{\partial t},v_0>\,dt\le\\
&\le c_1\Big\{1+\int_0^{\tau}\int_{\Omega}\epsilon(x)\left|\frac{\partial v_0(\tau,x)}{\partial x}\right|^2 dx\,dt+\int_0^{\tau}\int_{\partial\Omega}\kappa(x)v_0^2(t,x)\,ds\,dt\Big\}.
\end{aligned}\tag{3.4}$$

Note that $W^{1,2}(\Omega)$ can be normed equivalently by

$$\left\{\int_{\Omega}\left|\frac{\partial u(x)}{\partial x}\right|^2 dx+\int_{\partial\Omega}\kappa(x)u^2(x)\,ds\right\}^{\frac{1}{2}}.$$

Remark 3.1 allows us to test the regularized version of 2.13 with $\varphi = v_i + q_i v_0$:

$$\int_0^\tau \Big\{ < \frac{\partial u_i}{\partial t}, \varphi > + \int_\Omega \Big[\sum_{j=1}^N d_{ij}\Big(t, x, v_i^\delta, \frac{\partial (v_i + q_i v_0)}{\partial x}\Big) \frac{\partial \varphi}{\partial x_j} + \\ + R_i(t,x,v,\zeta)\varphi \Big] \, dx \Big\} \, dt + \int_0^\tau \int_{\partial\Omega} R_i^\Gamma(t,x,v,\zeta)\varphi \, ds \, dt = 0. \tag{3.5}$$

So, using (3.4), we get

$$\int_\Omega \epsilon(x) \left| \frac{\partial v_0(\tau,x)}{\partial x} \right|^2 \, dx + \int_{\partial\Omega} \kappa(x) v_0^2(\tau,x) \, ds \; + \sum_{i=1}^n \int_0^\tau \Big\{ < \frac{\partial u_i}{\partial t}, v_i > + \\ + \int_\Omega \Big[\sum_{j=1}^N d_{ij}\Big(t, x, v_i^{(\delta)}, \frac{\partial (v_i + q_i v_0)}{\partial x}\Big) \frac{\partial}{\partial x_j}(v_i + q_i v_0) + \\ + R_i(t,x,v,\zeta)(v_i + q_i v_0) \Big] \, dx \Big\} \, dt \; + \int_0^\tau \int_{\partial\Omega} R_i^\Gamma(t,x,v,\zeta)(v_i + q_i v_0) \, ds \, dt \; \le \\ \le c_2 \Big\{ 1 + \int_0^\tau \int_\Omega \epsilon(x) \left| \frac{\partial v_0(t,x)}{\partial x} \right|^2 \, dx \, dt \; + \int_0^\tau \int_{\partial\Omega} \kappa(x) v_0^2(t,x) \, ds \, dt \Big\}. \tag{3.6}$$

We transform the integral with $\frac{\partial u_i}{\partial t}$ by means of Lemma 1 and Lemma 3 from [10] and obtain

$$\int_0^\tau < \frac{\partial u_i}{\partial t}, v_i > \; dt \; = \int_\Omega u_i^*(x) \big[\Lambda_i\big(v_i(\tau,x)\big) - \Lambda_i\big(g_i(x)\big) \big] \, dx. \tag{3.7}$$

Estimating terms with R_i, R_i^Γ by means of condition iv), we get

$$\sum_{i=1}^n R_i(t,x,v,\zeta)(v_i + q_i v_0) = \\ = \sum_{(\alpha,\beta)\in R} \big[r_{\alpha,\beta}(t,x,v,\alpha\cdot\zeta) - r_{\alpha\beta}(t,x,v,\beta\cdot\zeta) \big] \cdot \cdot (\alpha - \beta)\cdot\zeta \ge 0, \\ \sum_{i=1}^n R_i^\Gamma(t,x',v,\zeta)(v_i + q_i v_0) = \\ = \sum_{(\alpha,\beta)\in R^\Gamma} \big[r_{\alpha\beta}^\Gamma(t,x',v,\alpha\cdot\zeta) - r_{\alpha\beta}^\Gamma(t,x',v,\beta\cdot\zeta) \big] (\alpha - \beta)\cdot\zeta \ge 0. \tag{3.8}$$

By condition iii) we obtain from (3.6) – (3.8)

$$\int_\Omega \epsilon(x) \left| \frac{\partial v_0(\tau,x)}{\partial x} \right|^2 \, dx + \int_{\partial\Omega} \kappa(x) v_0^2(\tau,x) \, dx \; + \\ + \sum_{i=1}^n \int_0^\tau \int_\Omega e_i'(v_i) \left| \frac{\partial (v_i + q_i v_0)}{\partial x} \right|^2 \, dx \, dt \le \\ \le c_2 \Big\{ 1 + \int_0^\tau \int_\Omega \epsilon(x) \left| \frac{\partial v_0(t,x)}{\partial x} \right|^2 \, dx \, dt \; + \int_0^\tau \int_{\partial\Omega} \kappa(x) v_0^2(t,x) \, ds \, dt \Big\}. \tag{3.9}$$

The last inequality and Gronwall's lemma imply (2.19) and the proof of Theorem 2.1 is complete. □

Lemma 3.1 *Let the conditions of Theorem 2.1 be satisfied. Suppose that*

$$\text{ess} \sup_{t\in(0,T)} \int_\Omega u_i^r(t,x)\, dx \le L_1 \quad \text{for} \quad i = 1,\dots,n\ , \tag{3.10}$$

with numbers $r \in \left(\frac{2N}{N+2}, \frac{N}{2}\right)$ *and* L_1 *depending only on known parameters. Then*

$$\begin{aligned}\text{ess} \sup_{t\in(0,T)} \Bigg\{ \int_\Omega \left(\left|v_0(t,x)\right|^{\frac{pN}{N-2}} + \left|v_0(t,x)\right|^{p-2} \left|\frac{\partial v_0(t,x)}{\partial x}\right|^2 \right) dx + \\ + \int_{\partial\Omega} \left|v_0(t,x)\right|^{\frac{p(N-1)}{N-2}} ds \Bigg\} \le L_2\ ,\end{aligned} \tag{3.11}$$

where the constant L_2 *depends only on known parameters and* p *is defined by*

$$p \cdot \frac{N}{N-2} = (p-1)\frac{r}{r-1}\ . \tag{3.12}$$

Proof. For arbitrary functions w we define

$$w_k(t,x) = \min\big\{w(t,x), k\big\}, \quad k \in \mathbb{R}^1,\ (t,x) \in Q_T\ . \tag{3.13}$$

Testing the integral identity (2.14) with $\psi(t,x) = |v_0(t,x)|_k^{p-1}$ sign $v_0(t,x),\ k > 0$, using the conditions iii), (2.11), (3.10) and Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega |v_0|_k^{p-2} \left|\frac{\partial |v_0|_k}{\partial x}\right|^2 dx + \int_{\partial\Omega} \kappa(x)|v_0|_k^p\, ds \le \\ \le c_4\bigg(\int_\Omega |v_0|_k^{(p-1)\frac{r}{r-1}} dx\bigg)^{\frac{r-1}{r}} + c_4\bigg(\int_{\partial\Omega} |v_0|_k^{(p-1)\frac{r(N-1)}{N(r-1)}} ds\bigg)^{\frac{N(r-1)}{r(N-1)}}.\end{aligned} \tag{3.14}$$

Hence Sobolev's embedding theorem yields

$$\begin{aligned}\bigg(\int_\Omega |v_0|_k^{p\frac{N}{N-2}} dx\bigg)^{\frac{N-2}{N}} + \bigg(\int_{\partial\Omega} |v_0|_k^{p\frac{N-1}{N-2}} ds\bigg)^{\frac{N-2}{N-1}} \le \\ \le c_5\bigg(\int_\Omega |v_0|_k^{(p-1)\frac{r}{r-1}} dx\bigg)^{\frac{r-1}{r}} + c_5\bigg(\int_{\partial\Omega} |v_0|_k^{(p-1)\frac{r(N-1)}{N(r-1)}} ds\bigg)^{\frac{N(r-1)}{r(N-1)}}.\end{aligned} \tag{3.15}$$

In view of the restriction on r and (3.12) we infer (3.11) from (3.14) and (3.15) letting $k \to \infty$. The proof of Lemma 3.1 is completed. □

In what follows we suppose the conditions (2.21) – (2.23) to be satisfied. We fix a $\Delta \in (0,1)$ such that

$$\begin{aligned}&\Delta \le 1 + \gamma_* + \frac{2}{N}(\gamma_* + 2) - p_3, \quad \Delta \le 1 + \gamma_* + \frac{1}{N}(\gamma_* + 2) - p_4, \\ &\gamma^* = \max\{\gamma_1,\dots,\gamma_n\} \le \frac{4}{N-2} - \frac{\Delta N}{N-2}\end{aligned} \tag{3.16}$$

and define

$$r(m) = \Delta m\ , \quad m = 0, 1, 2 \dots . \tag{3.17}$$

Lemma 3.2 *Let the conditions of Theorem 2.2 be satisfied. Suppose that for some nonnegative integer* m

$$\begin{gathered}\iint\limits_{Q_T} \bigl|v_0(t,x)\bigr|^{r(m)} \left|\frac{\partial v_0}{\partial x}\right|^2 \,dx\,dt\ + \iint\limits_{\Gamma_T} \kappa(x)\bigl|v_0(t,x)\bigr|^{2+r(m)}\,ds\,dt \le L_3\ , \\ \iint\limits_{Q_T} \bigl[v_i(t,x)\bigr]_+^{r(m)} \left|\frac{\partial v_i}{\partial x}\right|^2 \,dx\,dt \le L_3, \quad i = 1, \dots, n\ ,\end{gathered} \tag{3.18}$$

with $[v_i(t,x)]_+ = \max\{v_i(t,x), 0\}$ *and a constant* L_3 *depending only on known parameters and* m. *Then there exists a constant* L_4 *depending only on known parameters and* m *such that*

$$\iint\limits_{Q_T} \bigl|v_0(t,x)\bigr|^{r(m+1)} \left|\frac{\partial v_0}{\partial x}\right|^2 \,dx\,dt\ + \iint\limits_{\Gamma_T} \kappa(x)\bigl|v_0(t,x)\bigr|^{r(m+1)+2}\,ds\,dt \le L_4\ . \tag{3.19}$$

Proof. Remark that by condition (2.20)

$$e_i(v) \le c_6\, v^{\gamma_i+1}, \quad \Lambda_i(v) \ge c_6\, v^{\gamma_i+2}, \quad v \ge 1, \quad i = 1, \dots, n, \tag{3.20}$$

where the function Λ_i is defined by (2.20). From (2.19), (3.20) we have

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \bigl[v_i(t,x)\bigr]_+^{\gamma_i+2}\,dx \le c_7, \quad i = 1, \dots, n. \tag{3.21}$$

Testing the integral identity (2.14) with $\psi(t,x) = \bigl|v_0(t,x)\bigr|_k^{r(m+1)+1} \operatorname{sign} v_0(t,x)$ and using condition iii) and (2.11) we have

$$\begin{aligned}&\iint\limits_{Q_T} |v_0|_k^{r(m+1)} \left|\frac{\partial |v_0|_k}{\partial x}\right|^2 \,dx\,dt\ + \iint\limits_{\Gamma_T} \kappa(x)|v_0|_k^{r(m+1)+2}\,ds\,dt \le \\ &\le c_8\biggl\{ \sum_{i=1}^{n} \iint\limits_{Q_T} u_i |v_0|_k^{r(m+1)+1}\,dx\,dt\ + \int_0^T \Bigl\{\int_\Omega \bigl|v_0(t,x)\bigr|_k^{[r(m+1)+1]p_1'}\,dx\Bigr\}^{\frac{1}{p_1'}}\,dt + \\ &+ \int_0^T \Bigl\{\int_{\partial\Omega} \bigl|v_0(t,x)\bigr|_k^{[r(m+1)+1]p_2'}\,ds\Bigr\}^{\frac{1}{p_2'}}\,dt\biggr\}\end{aligned} \tag{3.22}$$

with $p_i' = \frac{p_i}{p_i-1}$, $i = 1, 2$. The embedding theorem and (3.18) imply

$$\int_0^T \Bigl\{\int_\Omega |v_0|^{[r(m)+2]\frac{N}{N-2}}\,dx\Bigr\}^{\frac{N-2}{N}}\,dt + \int_0^T \Bigl\{\int_{\partial\Omega} |v_0|^{[r(m)+2]\frac{N-1}{N-2}}\,ds\Bigr\}^{\frac{N-2}{N-1}}\,dt \le c_9. \tag{3.23}$$

Hence we can estimate the second and the third integral on the right hand side of (3.22) by a constant depending only on known parameters.
In order to estimate the first integral on the right hand side of (3.22) we derive firstly an auxiliary estimate for $v_i(t,x)$. By Hölder's inequality, the embedding theorem, (3.18) and (3.21) we obtain with an arbitrary number $q \in \big(0, \frac{N}{N-2}\big)$:

$$\begin{aligned} &\int_0^T \left\{ \int_\Omega \big[v_i(t,x)\big]_+^{(\gamma_i+2)[1-q\frac{N-2}{N}]+[r(m)+2]q}\, dx \right\}^{\frac{1}{q}} dt \;\le \\ &\le \int_0^T \left\{ \int_\Omega \big[v_i(t,x)\big]_+^{\gamma_i+2}\, dx \right\}^{\frac{1}{q}-\frac{N-2}{N}} \cdot \left\{ \int_\Omega \big[v_i(t,x)\big]^{\frac{[r(m)+2]N}{N-2}}\, dx \right\}^{\frac{N-2}{N}} dt \le \\ &\int_0^T \left\{ \int_\Omega \big[v_i(t,x)\big]_+^{\gamma_i+2}\, dx \right\}^{\frac{1}{q}-\frac{N-2}{N}} \int_\Omega \big[v_i(t,x)\big]_+^{r(m)} \left| \frac{\partial v_i}{\partial x} \right|^2 dx\;\, dt \;\le c_{10}\;. \end{aligned} \tag{3.24}$$

Let us choose the number q_* such that

$$\big[r(m+1)+1\big] q'_* = \big[r(m)+2\big]\frac{N}{N-2}, \quad q'_* = \frac{q_*}{q_*-1}. \tag{3.25}$$

Since $\Delta \in (0,1)$ and $r(m) = m\Delta$, we have $q'_* > \frac{N}{N-2}$. Using Hölder's inequality, (2.4) and (2.21), we get

$$\begin{aligned} &\iint\limits_{Q_T} u_i |v_0|_k^{r(m+1)+1}\, dx\; dt \le \\ &\qquad \le c_{11} \int_0^T \left\{ \int_\Omega |v_0|_k^{[r(m)+2]\frac{N}{N-2}}\, dx \right\}^{\frac{1}{q'_*}} \left\{ 1 + \int_\Omega [v_i]_+^{(\gamma_i+1)q_*}\, dx \right\}^{\frac{1}{q_*}} dt \le \\ &\qquad \le c_{11} \left\{ \int_0^T \left\{ \int_\Omega |v_0|_k^{[r(m)+2]\frac{N}{N-2}}\, dx \right\}^{\frac{N-2}{N}} dt \right\}^{\frac{N}{q'_*(N-2)}} \times \\ &\qquad\qquad \times \left\{ \int_0^T \left\{ 1 + \int_\Omega [v_i]_+^{(\gamma_i+1)q_*}\, dx \right\}^{\frac{1}{q_* - \frac{N(q_*-1)}{N-2}}} dt \right\}^{1-\frac{N}{q'_*(N-2)}} . \end{aligned} \tag{3.26}$$

Let $q = q_* - \frac{N}{N-2}(q_*-1) = \frac{N}{N-2} - \frac{2}{N-2}\, q_* \in \big(0, \frac{N}{N-2}\big)$ with q_* defined by (3.25). Since

$$\big[r(m)+2\big] q = q_* \big[r(m)+2\big] \Big(1 - \frac{N}{N-2} \cdot \frac{1}{q'_*} \Big) = q_* \big(r(m) - r(m+1) + 1 \big) = q_*(1-\Delta)\;,$$

we have by (3.16)

$$\begin{aligned} &(\gamma_i+2)\Big[1 - q\frac{N-2}{N}\Big] + \big[r(m)+2\big] q - (\gamma_i+1) q_* = \\ &= \Big[\frac{2}{N}(\gamma_i+2) - \gamma_i - \Delta\Big] q_* = \frac{q_*}{N}\big[4 - N\Delta - \gamma_i(N-2)\big] \ge 0\;. \end{aligned} \tag{3.27}$$

The inequalities (3.18), (3.21), (3.24), (3.26), (3.27) imply

$$\iint\limits_{Q_T} u_i |v_0|_k^{r(m+1)+1}\, dx\; dt \le c_{12}\;. \tag{3.28}$$

So we obtain the desired estimate (3.19) from (3.22), (3.23) and (3.28). This ends the proof of Lemma 3.2. □

Lemma 3.3 *Suppose that the assumptions of Theorem 2.2 and the inequalities (3.18) are satisfied for a nonnegative integer m such that*

$$\gamma_{i_0} \ge r(m+1)\ , \quad i_0 \in \{1,\dots,n\}\ . \tag{3.29}$$

Then there exists a constant L_5 depending only on known parameters such that

$$\iint\limits_{Q_T} \big[v_{i_0}(t,x)\big]_+^{r(m+1)} \left|\frac{\partial v_{i_0}(t,x)}{\partial x}\right|^2 \ dx\ dt \le L_5\ . \tag{3.30}$$

Proof. For arbitrary functions $w_1(t,x)$, $w_2(t,x)$ defined on Q_T we define the set

$$\big\{w_1 \le w_2\big\} = \big\{(t,x) \in Q_T : w_1(t,x) \le w_2(t,x)\big\}\ .$$

By (2.19) and (3.19) we have

$$\iint\limits_{\{[v_{i_0}]_+ \le 2|v_0|\}} \big[v_{i_0}(t,x)\big]_+^{r(m+1)} \left|\frac{\partial v_{i_0}(t,x)}{\partial x}\right|^2 \ dx\ dt \le c_{13}. \tag{3.31}$$

To complete the proof we need an analogous estimate with respect to $\{[v_{i_0}]_+ > 2|v_0|\}$. Testing the identity (2.14) with

$$\psi = |v_0|_k \Big\{ \big[[v_{i_0} - |v_0|_k]_+\big]_k + |v_0|_k \Big\}^{r(m)+\varepsilon} \ \operatorname{sign} v_0\ , \quad \varepsilon \in (0,\Delta],\ k > 1,$$

and using condition iii) and (2.11), we obtain

$$I_1 + I_2 \le c_{14}\big[I_3(1) + I_4 + I_5\big] \tag{3.32}$$

where

$$I_1 = \iint\limits_{\{|v_0|<k\}} \Big\{ \big[[v_{i_0} - |v_0|_k]_+\big]_k + |v_0|_k \Big\}^{r(m)+\varepsilon} \left|\frac{\partial v_0}{\partial x}\right|^2 \ dx\ dt,$$

$$I_2 = \iint\limits_{\Gamma_T} \kappa(x)|v_0|_k^2 \Big\{ \big[[v_{i_0} - |v_0|_k]_+\big]_k + |v_0|_k \Big\}^{r(m)+\varepsilon} \ ds\ dt,$$

$$I_3(l) = \iint\limits_{\{|v_0|_k < v_{i_0}\}} |v_0|_k^l \Big\{ \big[[v_{i_0} - |v_0|_k]_+\big]_k + |v_0|_k \Big\}^{r(m)+\varepsilon-l} \left|\frac{\partial v_{i_0}}{\partial x}\right| \left|\frac{\partial v_0}{\partial x}\right| \ dx\ dt,$$

$$I_4 = \iint\limits_{Q_T} \Big[\sum_{i=1}^n u_i + \big|f(t,x)\big|\Big] |v_0|_k \Big\{ \big[[v_{i_0} - |v_0|_k]_+\big]_k + |v_0|_k \Big\}^{r(m)+\varepsilon} \ dx\ dt,$$

$$I_5 = \iint\limits_{\Gamma_T} \big|f^\Gamma(t,x)\big| |v_0|_k \Big\{ \big[[v_{i_0} - |v_0|_k]_+\big]_k + |v_0|_k \Big\}^{r(m)+\varepsilon} \ ds\ dt.$$

Up to the end of Lemma 3 we choose $\varepsilon = \Delta$.
We estimate $I_3(l)$ for natural numbers $l < r(m+1)$ by Young's inequality

$$I_3(l) \le \varepsilon_1 I_1 + \frac{c_{15}}{\varepsilon_1} \iint\limits_{\{|v_0|_k < v_{i_0}\}} |v_0|_k^{2l} \Big\{ \big[[v_{i_0} - |v_0|_k]_+\big]_k + |v_0|_k \Big\}^{r(m+1)-2l} \left| \frac{\partial v_{i_0}}{\partial x} \right|^2 dx\, dt$$
$$+ c_{15} \iint\limits_{\{k \le |v_0| < v_{i_0}\}} \left\{ |v_{i_0}|^{r(m+1)} \left| \frac{\partial (v_{i_0} + q_{i_0} v_0)}{\partial x} \right|^2 + |v_0|^{r(m+1)} \left| \frac{\partial v_0}{\partial x} \right|^2 \right\} dx\, dt, \tag{3.33}$$

where ε_1 is an arbitrary positive number. Using the simple inequality

$$\left| \frac{\partial v_{i_0}}{\partial x} \right|^2 \le c_{16} \left(\left| \frac{\partial (v_{i_0} + q_{i_0} v_0)}{\partial x} \right|^2 + \left| \frac{\partial v_{i_0}}{\partial x} \right| \left| \frac{\partial v_0}{\partial x} \right| \right),$$

we have from (3.33), (2.19), (3.19) and (3.29)

$$I_3(l) \le \varepsilon_1 I_1 + \frac{c_{17}}{\varepsilon_1} \Big\{ 1 + I_3(2l) + \iint\limits_{Q_T} |v_0|^{r(m+1)} \left| \frac{\partial v_0}{\partial x} \right|^2 dx\, dt +$$
$$+ \iint\limits_{Q_T} [v_{i_0}]_+^{\gamma_i} \left| \frac{\partial (v_{i_0} + q_{i_0} v_0)}{\partial x} \right|^2 dx\, dt \Big\} \le \varepsilon_1 I_1 + \frac{c_{18}}{\varepsilon_1} \big(1 + I_3(2l)\big). \tag{3.34}$$

The inequalities (2.19), (3.19) imply also $I_3(l) \le c_{19}$ for $l > r(m+1)$. Therefore, iterating (3.34), we get

$$I_3(1) < \frac{1}{2c_{14}} I_1 + c_{19}. \tag{3.35}$$

Next we estimate the term I_4 by Hölder's inequality and condition (2.11):

$$I_4 \le c_{20} \Big\{ \sum_{i=1}^{n} \iint\limits_{Q_T} |v_i|^{\gamma_i + 2 + r(m+1)} dx\, dt + \int_0^T \Big\{ \int_\Omega |v_i|^{[r(m+1)+1]p_1'} dx \Big\}^{\frac{1}{p_1'}} dt +$$
$$+ \sum_{i=1}^{n} \iint\limits_{Q_T} |v_0|_k^{r(m+1)+1} u_i\, dx\, dt + \int_0^T \Big\{ \int_\Omega |v_0|_k^{[r(m+1)+1]p_1'} dx \Big\}^{\frac{1}{p_1'}} dt \Big\}. \tag{3.36}$$

Now all integrals in (3.36) can be estimated from above by a constant depending only on known parameters. Indeed, since by (3.16)

$$\frac{2}{N}(\gamma_i + 2) + r(m) + 2 - \big[\gamma_i + 2 + r(m+1)\big] =$$
$$= \frac{2}{N}(\gamma_i + 2) - \gamma_i - \Delta = \frac{1}{N}\big[4 - (N-2)\gamma_i - N\Delta\big] \ge 0,$$

an estimate of the first integral in (3.36) follows from (3.24) with $q = 1$. In analogous way the second integral in (3.36) can be estimated by means of (3.24) with $q = p_2'$.

Estimates for the third and the fourth integral in (3.36) follow from (3.28) and (3.23), respectively. So we have shown that

$$I_4 \leq c_{21}. \tag{3.37}$$

Further, condition (2.11), (3.18), (3.23), (3.24) and the embedding theorem yield

$$\begin{aligned} I_5 &\leq c_{22}\Big\{ \int_0^T \Big[\int_{\partial\Omega} \big[|v_{i_0}| + |v_0|_k\big]^{[r(m)+2]\frac{N-1}{N-2}} \, ds \Big]^{\frac{N-2}{N-1}} dt + 1 \Big\} \leq \\ &\leq c_{23}\Big\{ \iint\limits_{Q_T} \Big[|v_{i_0}|^{r(m)} \Big(\Big|\frac{\partial v_{i_0}}{\partial x}\Big|^2 + |v_{i_0}|^2 \Big) + \\ &+ |v_0|^{r(m)} \Big(\Big|\frac{\partial v_0}{\partial x}\Big|^2 + |v_0|^2 \Big) \Big] \, dx\, dt + 1 \Big\} \leq c_{24}. \end{aligned} \tag{3.38}$$

Now (3.32), (3.35), (3.37), (3.38) and (3.19) imply

$$\iint\limits_{\{v_{i_0} > |v_0|\}} [v_{i_0}]^{r(m+1)} \left|\frac{\partial v_0}{\partial x}\right|^2 dx\, dt \leq c_{25}. \tag{3.39}$$

Finally, the desired inequality (3.30) follows from (2.19), (3.31), (3.39) and the proof of Lemma 3.3 is completed. □

Lemma 3.4 *Let the assumptions of Theorem 2.2 be satisfied and suppose that the inequalities (3.18) hold for some nonnegative integer m. Moreover, let m and i_0 be such that*

$$\gamma_{i_0} < r(m+1) \tag{3.40}$$

and suppose that

$$ess \sup_{t \in (0,T)} \int_\Omega \big[v_i(t,x)\big]_+^{r(m)+2} dx + \iint\limits_{Q_T} \big[v_i(t,x)\big]_+^{r(m)} \left|\frac{\partial v_0}{\partial x}\right|^2 dx\, dt \leq L_6 \,, \tag{3.41}$$

with a constant L_6 depending only on m and known parameters. Then there exists a constant L_7 depending only on the same parameters such that

$$\begin{aligned} &ess \sup_{t \in (0,T)} \int_\Omega \big[v_{i_0}(t,x)\big]_+^{r(m+1)+2} dx + \\ &+ \iint\limits_{Q_T} \big[v_{i_0}(t,x)\big]_+^{r(m+1)} \left\{ \left|\frac{\partial v_0}{\partial x}\right|^2 + \left|\frac{\partial v_{i_0}}{\partial x}\right|^2 \right\} dx\, dt \leq L_7 \,. \end{aligned} \tag{3.42}$$

Proof. We start by proving that

$$\iint\limits_{Q_T} \big[v_i(t,x)\big]_+^{r(m)+\varepsilon} \left|\frac{\partial v_0}{\partial x}\right|^2 dx\, dt \leq c_{26} \tag{3.43}$$

for $i = i_0$ and $\varepsilon = \varepsilon_1 = \frac{\Delta}{r(m)+2}$. We want to apply (3.32) with this ε. Since in the proof of Lemma 3.3 I_4, I_5 have been estimated without using assumption (3.29), we can suppose (3.38) to be hold. Further, (3.19) holds also true. To estimate $I_3(l)$ with the chosen ε we apply Young's inequality, (3.18), (3.41) and (3.19):

$$\begin{aligned} I_3(1) \le c_{27} \iint\limits_{Q_T} \Big\{ [v_{i_0}]_+^{r(m)} \left(\left| \frac{\partial v_{i_0}}{\partial x} \right|^2 + \left| \frac{\partial v_0}{\partial x} \right|^2 \right) + \\ + |v_0|_k^{r(m)+1} \Big\{ \big[[v_{i_0} - |v_0|_k]_+ \big]_k + |v_0|_k \Big\}^{[r(m)+1]\varepsilon - 1} \left| \frac{\partial v_0}{\partial x} \right|^2 \Big\} \, dx \, dt \le c_{28}. \end{aligned} \tag{3.44}$$

Now the inequalities (3.31), (3.32), (3.37), (3.38) and (3.44) imply (3.43) for $i = i_0$. Note that this estimate follows in the same way for $i = 1, \dots, n$.

The key for continuing our previous discussions is following estimate

$$\operatorname{ess} \sup_{t \in (0,T)} \int_\Omega \big[v_i(t,x) \big]_+^{r(m)+\varepsilon+2} dx + \iint\limits_{Q_T} \big[v_i(t,x) \big]^{r(m)+\varepsilon} \left| \frac{\partial v_i}{\partial x} \right|^2 dx \, dt \le c_{29}. \tag{3.45}$$

Indeed, it can be seen from the proof of the Lemma 3.2, that (3.19) and (3.45) imply (3.19) with $r(m+1)+\varepsilon$ instead of $r(m+1)$. This ensures that (3.43) remains true for $\varepsilon = \varepsilon_2 = \frac{2\Delta}{r(m)+2}$ and even for further steps.

The estimate (3.45) follows immediately from (3.43) and (2.19) provided $r(m) + \varepsilon \le \gamma_i$. So it remains to prove (3.45) for the case that $r(m) + \varepsilon > \gamma_i$. Tho this end we test integral identity (3.5) with

$$\varphi = \Big[\big[e_i(v_i) - e_i(m_0) \big]_+ \Big]_{k(i)} \Big\{ a(\rho) + \Big[\big[e_i(v_i) - e_i(m_0) \big]_+ \Big]_{k(i)}^2 \Big\}^\rho , \quad \rho > -\frac{1}{2} ,$$

where

$$\begin{aligned} m_0 &= \operatorname{ess} \sup \left\{ \left| e_i^{-1} \left(\frac{h_i(x)}{u_i^*(x)} \right) \right| ; \quad x \in \Omega, \quad i = 1, \dots, n \right\}, \\ z_+ &= \max(z, 0), \ [s]_{k(i)} = \min(s, k(i)), \\ k(i) &= e_i(k) - e_i(m_0) \text{ for } k > m_0; \quad a(\rho) = 1 \text{ for } \rho \le 1, \ a(\rho) = 0 \text{ for } \rho > 1 \, . \end{aligned}$$

Then, using Lemma 2 from [10], we can evaluate the first term:

$$\int_0^\tau < \frac{\partial u_i}{\partial t}, \varphi > dt = \int_\Omega u^*(x) \Lambda_{k,i}^{(\rho)} \Big(e_i\big(v_i(\tau, x)\big) - e_i(m_0) \Big) \, dx \, , \tag{3.46}$$

where

$$\Lambda_{k,i}^{(\rho)}(z) = \left[\int_0^z [s]_{k(i)} \Big\{ a(\rho) + [s]_{k(i)}^2 \Big\}^\rho ds \right]_+ \ge \frac{1}{2(\rho+1)} [z_+]_{k(i)}^{2\rho+2}. \tag{3.47}$$

We write the space derivative of φ in the form

$$\frac{\partial \varphi}{\partial x_j} = \Phi_{k,i}^{(\rho)}(v_i) \frac{\partial v_i}{\partial x_j} \chi(m_0 < v_i < k) \, , \tag{3.48}$$

where $\chi(m_0 < v_i < k)$ is the characteristic function of the set $\{m_0 < v_i < k\}$ and the function $\Phi_{k,i}^{(\rho)}(v_i)$ satisfies for $\rho > -\frac{1}{2}$ the estimate

$$\begin{aligned}\rho_* e_i'(v_i)\Big\{a(\rho) + \big[e_i(v_i) - e_i(m_0)\big]^2\Big\}^\rho \le \Phi_{k,i}^{(\rho)}(v_i) \le \\ \le c_{31} e_i'(v_i)\Big\{a(\rho) + \big[e_i(v_i) - e_i(m_0)\big]^2\Big\}^\rho\end{aligned} \tag{3.49}$$

for $m_0 < v_i < k$ with $\rho_* = \min(1; 1 + 2\rho)$.
Using (3.46) – (3.49) and the conditions iii), (2.20) – (2.22), we obtain from (3.5) with the chosen test function φ

$$\begin{aligned}&\int_\Omega \Big[\big[v_i(\tau,x)\big]_+\Big]_k^{2(\gamma_i+1)(\rho+1)} dx + \\ &\quad \int_0^\tau \int_\Omega \Big[\big[v_i(t,x)\big]_+\Big]_k^{2\gamma_i+2(\gamma_i+1)\rho} \left|\frac{\partial v_i}{\partial x}\right|^2 \chi(m_0 < v_i < k)\, dx\, dt \le \\ &\quad \le c_{32}\left(\frac{\rho+1}{\rho_*}\right)^2 \Bigg\{ \int_0^\tau \int_\Omega \Big[\big[v_i(t,x)\big]_+\Big]_k^{2\gamma_i+2(\gamma_i+1)\rho} \left|\frac{\partial v_0}{\partial x}\right|^2 \times \\ &\quad \times \chi(m_0 < v_i < k)\, dx\, dt + \\ &\quad + \sum_{j=1}^{n} \int_0^\tau \int_\Omega \Big(\big[v_j(t,x)\big]_+^{(\gamma_i+1)(1+2\rho)+p_3} + \big|v_0(t,x)\big|^{(\gamma_i+1)(1+2\rho)+p_3}\Big)\, dx\, dt + \\ &\quad + \Big\{\int_0^\tau \int_\Omega \Big[\big[v_i(t,x)\big]_+\Big]_k^{(\gamma_i+1)(1+2\rho)r_1'} dx\, dt\Big\}^{\frac{1}{r_1'}} + \\ &\quad + \int_0^\tau \int_{\partial\Omega} \sum_{j=0}^{N} \big[v_j(t,x)\big]_+^{(\gamma_i+1)(1+2\rho)+p_4} ds\, dt + \\ &\quad + \Big\{\int_0^\tau \int_{\partial\Omega} \Big[\big[v_i(t,x)\big]_+\Big]_k^{(\gamma_i+1)(1+2\rho)r_2'} ds\, dt\Big\}^{\frac{1}{r_2'}} + 1\Bigg\}.\end{aligned} \tag{3.50}$$

To continue the proof of the inequality (3.45) we choose ρ such that

$$2\gamma_i + 2(\gamma_i+1)\rho = r(m) + \varepsilon \tag{3.51}$$

and estimate the right hand side of (3.50) integral by integral. An estimation of the first one follows from (3.43). Note that by (3.51) and (3.16)

$$\begin{aligned}(\gamma_i+1)(1+2\rho) + p_3 = \Big[r(m) + 2 + \frac{2}{N}(\gamma_*+2)\Big] + \\ + \Big[p_3 - 1 - \gamma_i - \frac{2}{N}(\gamma_*+2)\Big] + \varepsilon < r(m) + 2 + \frac{2}{N}(\gamma_*+2).\end{aligned}$$

Hence estimates for the v_j terms, $j = 1, \ldots, n$, of the second integral on the right hand side of (3.50) follow from (3.24) with $q = 1$. Taking into account (3.11), the v_0 term can be estimated by the same arguments.

In order to estimate the third integral we use the next inequality that follows analogously to the inequality (3.24):

$$\begin{aligned} &\iint\limits_{Q_T} \big[v_i(t,x)\big]^{(r(m)+2)(1+\frac{2}{N})}\, dx\; dt \le \\ &\le \int_0^T \left\{ \int_\Omega \big[v_i(t,x)\big]_+^{r(m)+2}\, dx \right\}^{\frac{2}{N}} \left\{ \int_\Omega \big[v_i(t,x)\big]_+^{(r(m)+2)\frac{N}{N-2}} dx \right\}^{\frac{N-2}{N}} dt \le \\ &\le c_{33}\big[r(m)+2\big]^2 \operatorname*{ess\,sup}_{t\in(0,T)} \left\{ \int_\Omega \big[v_i(t,x)\big]_+^{r(m)+2}\, dx \right\}^{\frac{2}{N}} \times \\ &\times \iint\limits_{Q_T} \left\{ \big[v_i(t,x)\big]_+^{r(m)} \left|\frac{\partial v_i}{\partial x}\right|^2 + \big[v_i(t,x)\big]_+^{r(m)+2} \right\} dx\; dt. \end{aligned} \tag{3.52}$$

It is simple to check that $(\gamma_i+1)(1+2\rho)r_1' < \big(r(m)+2\big)(1+\frac{2}{N})$, such that the third integral can be estimated by means of (3.52), (3.41), (3.18).
To estimate the last integrals in (3.50) we note firstly following auxiliary inequality that follows analogously to the inequality (3.52):

$$\begin{aligned} &\iint\limits_{\Gamma_T} \big[v_i(t,x)\big]_+^{(r(m)+2)(1+\frac{1}{N})}\, ds\; dt \le c_{34}\big[r(m)+2\big]^2 \times \\ &\qquad \times \int_0^T \left\{ \int_\Omega \big[v_i(t,x)\big]_+^{r(m)+\frac{2}{N+1}} \left(\left|\frac{\partial v_i}{\partial x}\right| + [v_i]_+ \right)^{\frac{2N}{N+1}} dx \right\}^{\frac{N+1}{N}} dt \le \\ &\qquad \le c_{35}\big[r(m)+2\big]^2 \operatorname*{ess\,sup}_{t\in(0,T)} \left\{ \int_\Omega \big[v_i(t,x)\big]_+^{r(m)+2} dx \right\}^{\frac{1}{N}} \times \\ &\qquad \times \iint\limits_{Q_T} \big[v_i(t,x)\big]_+^{r(m)} \left(\left|\frac{\partial v_i}{\partial x}\right|^2 + \big[v_i(t,x)\big]_+^2 \right) dx\; dt. \end{aligned} \tag{3.53}$$

Then analogous arguments as used for proving (3.24) and (3.53) lead to

$$\iint\limits_{\Gamma_T} \big[v_i(t,x)\big]_+^{r(m)+2+\frac{1}{N}(\gamma_i+2)}\, ds\; dt \le c_{36}. \tag{3.54}$$

Since by (3.51) and (3.16)

$$(\gamma_i+1)(1+2\rho) + p_4 < r(m) + 2 + \frac{1}{N}(\gamma_* + 2)\ ,$$

(3.54) implies an estimation for the fourth integral on the right hand side of (3.50). Finally, (3.53) implies an estimate for the last integral in (3.50). With (3.50) the key estimate (3.45) is fully proved. This ends the proof of Lemma 3.4. □

Proof of Theorem 2.2. Remark that for $m = 0$ the conditions (3.18), (3.41) follow from Lemma 3.1 and Theorem 2.1. Starting from $m = 0$, we can iterate the application of the Lemmas 3.2 – 3.4. After $M+1$ steps we arrive at the inequalities (3.19) and (3.42) with $m = M$. Taking M so large that $\Delta \cdot (M+1) \ge \gamma^*$, we get Theorem 2.2. □

4 L^∞–estimate of solution

Proof of Theorem 2.3. We apply Lemma 3.4 with $m = M$ and M such that $r(M+1)+2 > \frac{N}{2}\gamma^*$, $\gamma^* = \max(\gamma_1, \ldots, \gamma_n)$. Then Theorem 2.3 follows immediately from (3.42), conditions i), iii), (2.11) and well known results on the regularity of solutions of linear elliptic equations (see, for example [15]) to Poisson's equation (2.1). □

In what follows we assume the conditions of Theorem 2.4 to be satisfied. We shall estimate for v_i, $i = 1, \ldots, n$, separately on the sets $\{v_i > 0\}$ (Lemma 4.1) and $\{v_i < 0\}$ (Lemma 4.2).

Lemma 4.1 *Let the condition of Theorem 2.4 be satisfied. Then there exists a constant L_8 depending only on known parameters such that for $i = 1, \ldots, n$,*

$$ess\ \sup\big\{v_i(t,x) : (t,x) \in Q_T\big\} \le L_8\ . \tag{4.1}$$

Proof. Using Lemma 3.4 and (3.50) we get for $r \ge r_* = 2 + 4\max(\gamma_1, \ldots, \gamma_n)$

$$\begin{aligned}
&\int_\Omega \big[v_i(\tau,x)\big]_+^{r+2}\,dx\ + \int_0^\tau\int_\Omega \big[v_i(t,x)\big]_+^{r}\left|\frac{\partial v_i}{\partial x}\right|^2 dx\ dt \le \\
&\le c_{37}\, r^2\Bigg\{1 + \int_0^\tau\int_\Omega \big[v_i(t,x)\big]_+^{r}\left|\frac{\partial v_0}{\partial x}\right|^2 dx\ dt\ + \\
&+\ \bigg[\int_0^\tau\int_\Omega \big[v_i(t,x)\big]_+^{(r+1)r_1'}\,dx\ dt\bigg]^{\frac{1}{r_1'}} + \bigg[\int_0^\tau\int_{\partial\Omega} \big[v_i(t,x)\big]_+^{(r+1)r_2'}\,ds\ dt\bigg]^{\frac{1}{r_2'}}\Bigg\}\ .
\end{aligned} \tag{4.2}$$

Remark only that Lemma 3.4 gives us the estimate of $[v_j]_+^{p_3}$ in $L^{r_1}(Q_T)$, $j = 1, \ldots, n$. We start estimating the first integral on the right hand side of (4.2). Let

$$\{\varphi_j \in C^\infty(R^N),\ j = 1, \ldots, J,\}$$

be a partition of unity such that

$$\begin{gathered}
\sum_{j=1}^{J}\varphi_j^2(x) = 1, \quad \left|\frac{\partial\varphi_j}{\partial x}\right| \le \frac{c_0}{R} \quad \text{for} \quad x \in \Omega, \quad \operatorname{supp}\varphi_j \subset B(x_j, R), \\
JR^N \le c_0\big[d(\Omega)\big]^N, \quad R < 1, \quad \sum_{j=1}^{J}\chi\big(B(x_j,R)\big) \le c_0\ ,
\end{gathered} \tag{4.3}$$

where $B(x_j, R)$ is the ball of radius R with centre $x_j \in \Omega$, c_0 is a constant depending only on N, $d(\Omega)$ is the diameter of Ω, $\chi(B(x_j, R))$ is the characteristic function of $B(x_j, R)$. The radius R will be fixed later.
We test the integral identity (2.14) with

$$\Psi(t,x) = \big[v_i(t,x)\big]_+^{r}\cdot\big[v_0(t,x) - v_{0,l}(t)\big]\varphi_l^2(x),\ v_{0,l}(t) = v(t,x_l). \tag{4.4}$$

Integration with respect to t and summing up on l yield

$$\begin{aligned} \int_0^\tau \int_\Omega \big[v_i(t,x)\big]_+^r \left|\frac{\partial v_0}{\partial x}\right|^2 dx\,dt \le c_{38}\, r \Big\{ I_1(r) + I_2(r) + \\ + \int_0^\tau \Big\{ \int_\Omega \big[v_i(t,x)\big]_+^{rp_1'}\, dx \Big\}^{\frac{1}{p_1'}} dt \; + \int_0^\tau \Big\{ \int_{\partial\Omega} \big[v_i(t,x)\big]_+^{rp_2'}\, ds \Big\}^{\frac{1}{p_2'}} dt \Big\}\,, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} I_1(r) &= \sum_{l=1}^{J} \int_0^\tau \int_\Omega \big[v_i(t,x)\big]_+^{r-1} \big|v_0(t,x) - v_{0,l}(t)\big| \varphi_l^2(x) \left|\frac{\partial v_i}{\partial x}\right| \left|\frac{\partial v_0}{\partial x}\right| dx\,dt\;, \\ I_2(r) &= \frac{1}{R} \int_0^\tau \int_\Omega \big[v_i(t,x)\big]_+^r \left|\frac{\partial v_0}{\partial x}\right| dx\,dt. \end{aligned}$$

Since by (2.25)

$$\big|v_0(t,x) - v_{0,l}(t)\big| \le K_3\, R^\eta \quad \text{for} \quad x \in B(x_l, R)\;, \tag{4.6}$$

we obtain

$$\begin{aligned} c_{38}\, r I_1(r) \le \frac{1}{2} \int_0^\tau \int_\Omega \big[v_i(t,x)\big]_+^r \left|\frac{\partial v_0}{\partial x}\right|^2 dx\,dt\, + \\ + c_{39} \Big[r^2 R^{2\eta} \int_0^\tau \int_\Omega \big[v_i(t,x)\big]_+^r \left|\frac{\partial v_i}{\partial x}\right|^2 dx\,dt\; + 1 \Big]. \end{aligned} \tag{4.7}$$

We fix R such that $4\; c_{39}\; c_{37}\; r^4\; R^{2\eta} = 1$. Estimating $I_2(r)$ by Cauchy's inequality and using (4.5), (4.7), we deduce from (4.2)

$$\begin{aligned} &\int_\Omega \big[v_i(\tau,x)\big]_+^{r+2} dx \; + \int_0^\tau \int_\Omega \big[v_i(t,x)\big]_+^r \left|\frac{\partial v_i}{\partial x}\right|^2 dx\,dt \le \\ &\quad \le c_{40}\; r^{2+\frac{4}{\eta}} \Big\{ 1 + \Big[\int_0^\tau \int_\Omega \big[v_i(t,x)\big]_+^{(r+1)r_1'} dx\,dt \Big]^{\frac{1}{r_1'}} + \\ &\quad + \int_0^\tau \Big\{ \int_\Omega \big[v_i(t,x)\big]_+^{rp_1'}\, dx \Big\}^{\frac{1}{p_1'}} dt \; + \Big[\int_0^\tau \int_{\partial\Omega} \big[v_i(t,x)\big]^{(r+1)r_2'} ds\,dt \Big]^{\frac{1}{r_2'}} + \\ &\quad + \int_0^\tau \Big\{ \int_{\partial\Omega} \big[v_i(t,x)\big]_+^{rp_2'}\, ds \Big\}^{\frac{1}{p_2'}} dt \Big\}. \end{aligned}$$

Hence Sobolev's embedding theorem and standard Moser iteration lead to (4.1) and the proof of Lemma 4.1 is completed. □

For $\varepsilon > 0$ and arbitrary functions g defined on Q_T we use the notations

$$g^{(\varepsilon)}(t,x) = \max\big\{g(t,x), \varepsilon\big\},\;\; g_-(t,x) = \min\big\{g(t,x), 0\big\}\;. \tag{4.8}$$

Lemma 4.2 *Let the conditions of the Theorem 2.4 be satisfied. Then there exists a constant L_9 depending only on known parameters such that*

$$\textit{ess}\ \inf \big\{ v_i(t,x) : (t,x) \in Q_T \big\} \geq -L_9, \quad i = 1,\dots,n. \tag{4.9}$$

Proof. Denote

$$\begin{aligned} m_0 &= \text{ess sup} \left\{ \left| e_i^{-1}\left(\frac{h_i(x)}{u_i^*(x)} \right) \right| : x \in \Omega, \quad i = 1,\dots,n \right\}, \quad \widetilde{e}_i(v) = \frac{e_i(v)}{e_i(-m_0)}\ , \\ \Psi^{(r)}(z) &= -\big(\frac{1}{z^2}\,|ln\,z|^r + \frac{r}{z^2}\,|ln\,z|^{r-1} \big) e_i^2(-m_0)\ , z > 0\ . \end{aligned}$$

We test the integral identity (3.5) with

$$\varphi = \frac{1}{e_i^{(\varepsilon)}(v_i)} \big| ln_-\ \widetilde{e}_i^{(\varepsilon)}(v_i) \big|^r, \quad 0 < \varepsilon < 1, \quad r \geq 1\ ,$$

to get

$$\begin{aligned} &\int_0^\tau < \frac{\partial u_i}{\partial t}, \varphi > dt + \sum_{j=1}^n \iint\limits_{Q_\tau} d_{ij}\Big(t, x, v_i^{(\delta)}, \frac{\partial (v_i + q_i v_0)}{\partial x} \Big) \Psi^{(r)}\big(\widetilde{e}_i(v_i) \big) \times \\ &\times e_i'(v_i) \frac{\partial v_i}{\partial x_j} \chi\big(\widetilde{e}_i^{-1}(\varepsilon) < v_i < -m_0 \big)\ dx\ dt\ + \iint\limits_{Q_\tau} R_i(t,x,v,\zeta) \frac{1}{e_i^{(\varepsilon)}(v_i)} \times \\ &\times \big| ln_-\ \widetilde{e}_i^{(\varepsilon)}(v_i) \big|^r\ dx\ dt\ + \iint\limits_{\Gamma_\tau} R_i^\Gamma(t,x,v,\zeta) \frac{1}{e_i^{(\varepsilon)}(v_i)} \big| ln_-\ \widetilde{e}_i^{(\varepsilon)}(v_i) \big|^r\ ds\ dt = 0\ . \end{aligned} \tag{4.10}$$

Evaluating the first integral in (4.10) analogously to equality (40) in [10] yields

$$\begin{aligned} \int_0^\tau < \frac{\partial u_i}{\partial t}, \varphi >\ dt = \frac{1}{r+1} &\int\limits_{\{\widetilde{e}_i(v_i) < \varepsilon\}} \big| ln_-\ \widetilde{e}_i^{(\varepsilon)}\big(v_i(\tau,x) \big) \big|^{r+1} u_i^*(x)\ dx\ - \\ - &\int\limits_{\{\widetilde{e}_i(v_i) \geq \varepsilon\}} |ln\,\varepsilon|^r \big| ln_-\ \widetilde{e}_i^{(\varepsilon)}\big(v_i(\tau,x) \big) \big| u_i^*(x)\ dx\ \leq \\ \leq -\frac{1}{r+1} &\int_\Omega \big| ln_-\ \widetilde{e}_i^{(\varepsilon)}\big(v_i(\tau,x) \big) \big|^{r+1} u_i^*(x)\ dx\ . \end{aligned} \tag{4.11}$$

We estimate the second integral in (4.10) by using the condition iii) to obtain

$$\begin{aligned} \sum_{j=1}^n \iint\limits_{Q_\tau} d_{ij}\Big(t{,}x, v_i^{(\delta)}, \frac{\partial (v_i + q_i v_0)}{\partial x} \Big) & e_i'(v_i) \psi^{(r)}\big(\widetilde{e}_i(v_i) \big) \frac{\partial v_i}{\partial x_j} \times \\ & \times \chi\big(\widetilde{e}_i^{-1}(\varepsilon) < v_i < -m_0 \big)\ dx\ dt \leq \\ & \leq -c_{41}\, r \iint\limits_{Q_\tau} \big| ln_-\ \widetilde{e}_i^{(\varepsilon)}(v_i) \big|^{r-1} \Big| \frac{\partial}{\partial x} ln_-\ \widetilde{e}_i^{(\varepsilon)}(v_i) \Big|^2\ dx\ dt\ + \\ & + c_{42}\, r \iint\limits_{Q_\tau} \Big(1 + \big| ln_-\ \widetilde{e}_i^{(\varepsilon)}(v_i) \big|^r \Big) \Big| \frac{\partial v_0}{\partial x} \Big|^2\ dx\ dt\ . \end{aligned} \tag{4.12}$$

Estimating the two last integrals in (4.10) by using (2.26), (2.27) and Lemma 4.1, we get

$$\begin{aligned}
&\iint\limits_{Q_\tau} R_i(t,x,v,\zeta)\frac{1}{e_i^{(\varepsilon)}(v_i)}\left|ln_-\,\widetilde{e}_i^{(\varepsilon)}(v_i)\right|^r\,dx\,dt\,+\\
&\qquad\qquad + \iint\limits_{\Gamma_\tau} R_i^{\Gamma}(t,x,v,\zeta)\frac{1}{e_i^{(\varepsilon)}(v_i)}\left|ln_-\,\widetilde{e}_i^{(\varepsilon)}(v_i)\right|^r\,ds\,dt \le\\
&\qquad\qquad \le c_{43}\Bigg\{\iint\limits_{Q_\tau}\big[1+\alpha_1(t,x)\big]\left|ln_-\,\widetilde{e}_i^{(\varepsilon)}(v_i)\right|^r\,dx\,dt\,+\\
&\qquad\qquad + \iint\limits_{\Gamma_\tau}\big[1+\alpha_2(t,x)\big]\left|ln_-\,\widetilde{e}_i^{(\varepsilon)}(v_i)\right|^r\,ds\,dt\Bigg\}.
\end{aligned} \tag{4.13}$$

By (4.10), (4.11) and (4.13) we find for $w^{(\varepsilon)}(t,x)=\left|ln_-\,\widetilde{e}_i^{(\varepsilon)}\big((v_i)(t,x)\big)\right|$

$$\begin{aligned}
&\int_\Omega\big[w^{(\varepsilon)}(\tau,x)\big]^{r+1}\,dx+\iint\limits_{Q_\tau}\big[w^{(\varepsilon)}(t,x)\big]^{r-1}\left|\frac{\partial w^{\varepsilon}}{\partial x}\right|^2\,dx\,dt\le\\
&\qquad \le c_{44}\,r^2\Bigg\{\iint\limits_{Q_\tau}\Big(1+\big[w^{(\varepsilon)}(t,x)\big]^r\Big)\Big(\left|\frac{\partial v_0}{\partial x}\right|^2+\alpha_1(t,x)+1\Big)\,dx\,dt\,+\\
&\qquad + \iint\limits_{\Gamma_\tau}\big[w^{(\varepsilon)}(t,x)\big]^r\big(\alpha_2(t,x)+1\big)\,ds\,dt\Bigg\}\;.
\end{aligned} \tag{4.14}$$

To estimate that term in (4.14) with the derivative of v_0, we test the integral identity (2.14) with

$$\Psi(t,x)=\big[w^{(\varepsilon)}(t,x)\big]^2\big[v_0(t,x)-v_{0,l}(t)\big]\varphi_l^2(x)\;, \tag{4.15}$$

where $v_{0,l}(t),\varphi_l(x)$ are the functions from (4.4). By integration on t and taking the sum on l we get

$$\begin{aligned}
&\iint\limits_{Q_\tau}\big[w^{(\varepsilon)}(t,x)\big]^r\left|\frac{\partial v_0}{\partial x}\right|^2\,dx\,dt\le c_{45}\,r\Bigg\{I(r)+\frac{1}{R^2}\iint\limits_{Q_\tau}\big[w^{\varepsilon}(t,x)\big]^r\,dx\,dt\,+\\
&+\int_0^\tau\Big\{\int_\Omega\big[w^{(\varepsilon)}(t,x)\big]^{rp_1'}\,dx\Big\}^{\frac{1}{p_1'}}\,dt\;+\int_0^\tau\Big\{\int_{\partial\Omega}\big[w^{(\varepsilon)}(t,x)\big]^{rp_2'}\,ds\Big\}^{\frac{1}{p_2'}}\,dt\Bigg\}\;,
\end{aligned} \tag{4.16}$$

where

$$I(r)=\sum_{l=1}^{J}\iint\limits_{Q_\tau}\big[w^{(\varepsilon)}(t,x)\big]^{r-1}\big|v_0(t,x)-v_{0,l}(t)\big|\varphi_l^2(x)\left|\frac{\partial w^{(\varepsilon)}}{\partial x}\right|\left|\frac{\partial v_0}{\partial x}\right|\,dx\;dt.$$

Using (4.7), we can estimate the last integral

$$\begin{aligned} c_{45}\, r\, I(r) \le & \frac{1}{2} \iint\limits_{Q_\tau} \left[w^{(\varepsilon)}(t,x)\right]^r \left|\frac{\partial v_0}{\partial x}\right|^2 dx\, dt + \\ & + c_{46}\left\{1 + r^2 R^{2\eta} \iint\limits_{Q_\tau} \left|w^{(\varepsilon)}(t,x)\right|^r \left|\frac{\partial w^{(\varepsilon)}}{\partial x}\right|^2 dx\, dt\right\}. \end{aligned} \tag{4.17}$$

Fixing the number R such that $4\, c_{44}\, c_{46}\, r^4\, R^{2\eta} = 1$, we obtain from (4.14), (4.16) and (4.17)

$$\begin{aligned} & \int_\Omega \left[w^{(\varepsilon)}(\tau,x)\right]^{r+1} dx \; + \iint\limits_{Q_\tau} \left[w^{(\varepsilon)}(t,x)\right]^{r-1} \left|\frac{\partial w^{(\varepsilon)}}{\partial x}\right|^2 dx\, dt \le \\ & \le c_{47}\, r^{2+\frac{4}{\eta}} \left\{1 + \left[\iint\limits_{Q_\tau} \left[w^{(\varepsilon)}(t,x)\right]^{r\cdot r_1'} dx\, dt\right]^{\frac{1}{r_1'}} + \right. \\ & + \left[\iint\limits_{\Gamma_\tau} \left[w^{(\varepsilon)}(t,x)\right]^{r\, r_2'} ds\, dt\right]^{\frac{1}{r_2'}} + \int_0^\tau \left[\int_\Omega \left[w^{(\varepsilon)}(t,x)\right]^{r\, p_1'} dx\right]^{\frac{1}{p_1'}} dt + \\ & \left. + \int_0^\tau \left[\int_{\partial\Omega} \left[w^{(\varepsilon)}(t,x)\right]^{r\, p_2'} ds\right]^{\frac{1}{p_2'}} dt\right\} . \end{aligned} \tag{4.18}$$

Remark also that (4.18) implies

$$\int_\Omega \left[w^{(\varepsilon)}(\tau,x)\right]^2 dx \; + \iint\limits_{Q_\tau} \left|\frac{\partial w^{(\varepsilon)}(t,x)}{\partial x}\right|^2 dx\, dt \; \le c_{48} \tag{4.19}$$

with a constant c_{48} depending only on known parameters. To verify (4.19) we have to estimate the integrals on the right hand side of (4.18) with $r = 1$ and then to apply Gronwall's Lemma. As an example we consider the third integral. Define $\overline{p}$ by $p_1' = \frac{N}{N-\overline{p}}$. Then $\overline{p} < 2$ and we can assume $\overline{p} > 1$. Using Sobolev's embedding theorem we have with $\overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$

$$\begin{aligned} & c_{47} \int_0^\tau \left[\int_\Omega \left[w^{(\varepsilon)}(t,x)\right]^{p_1'} dx\right]^{\frac{1}{p_1'}} dt \\ & \qquad \le c_{49}\left\{1 + \int_0^\tau \left[\int_\Omega \left[w^{(\varepsilon)}(t,x)\right]^{\overline{p}^*} dx\right]^{\frac{\overline{p}}{\overline{p}^*}} dt\right\} \\ & \qquad \le c_{50}\left\{1 + \iint\limits_{Q_\tau} \left(\left|\frac{\partial w^{(\varepsilon)}}{\partial x}\right|^{\overline{p}} \left|w^{(\varepsilon)}\right|^{\overline{p}}\right) dx\, dt\right\} \\ & \qquad \le \frac{1}{8} \iint\limits_{Q_\tau} \left|\frac{\partial w^{(\varepsilon)}}{\partial x}\right|^2 dx\, dt \; + c_{51}\left\{1 + \iint\limits_{Q_\tau} \left|w^{(\varepsilon)}(t,x)\right|^2 dx\, dt\right\}. \end{aligned} \tag{4.20}$$

Now (4.18), (4.19) and standard Moser iterations give

$$w^{(\varepsilon)}(t,x) \leq c_{52} \ , \tag{4.21}$$

with a constant c_{52} depending only on known parameters and independent of ε. The estimate (4.21) implies that the measure of the set $\{\widetilde{e}_i(v_i(t,x)) < \varepsilon\}$ is equal to zero if $|ln\, \varepsilon| > c_{52}$, i.e.,

$$v_i(t,x) > \widetilde{e}_i^{-1}\big(e^{-c_{52}-1}\big)$$

and the proof of Lemma 4.2 is complete. $\Box$

Proof of Theorem 2.4. Theorem 2.4 follows immediately from the inequalities (4.1) and (4.9). $\Box$

5 Proof of existence and uniqueness

Proof of Theorem 2.5. We modify the functions $e_i(z)$, $d_{ij}(t,x,z,\xi)$, $r(t,x,v,y)$, $r^{\Gamma}_{\gamma\delta}(t,x',v,y)$ in following way:

$$\begin{aligned}
\widetilde{e}_i(z) &= \int_{-\infty}^{z} e_i'(\min[s,K_4])\, ds \ , \\
\widetilde{d}_{ij}(t,x,z,\xi) &= d_{ij}\big(t,x,\min[z,K_4],\xi\big), \\
\widetilde{r}_{\alpha\beta}(t,x,v,y) &= r_{\alpha\beta}\big(t,x,\min[v,K_3+K_4],\min[y,K_{\alpha\beta}]\big), \\
\widetilde{r}^{\Gamma}_{\gamma\delta}(t,x',v,y) &= r^{\Gamma}_{\gamma\delta}\big(t,x',\min[v,K_3+K_4],\min[y,K_{\gamma\delta}]\big),
\end{aligned} \tag{5.1}$$

where K_3, K_4 are the constants from Theorem 2.3, 2.4 and
$\min[v,K_3+K_4] = \min_{j=0,\ldots,n}[v_j, K_3+K_4]$, $\quad K_{\alpha\beta} = \sum_{i=1}^{n}\big(|\alpha_i| + |\beta_i|\big)(K_3+K_4)$.
Now we consider the system

$$-\nabla\cdot(\epsilon\nabla v_0) + f + \sum_{i=1} q_i\widetilde{u}_i \quad \text{on} \quad Q_T, \tag{5.2}$$

$$\frac{\partial\widetilde{u}_i}{\partial t} + \nabla\cdot\widetilde{J}_i^{(\delta)} + \widetilde{R}_i = 0, \quad i=1,\ldots,n \quad \text{on} \quad Q_T \ , \tag{5.3}$$

where $\widetilde{u}_i$, $\widetilde{J}_i^{\delta}$, $\widetilde{R}_i$ are defined by (2.4), (2.18), (2.6) with $\widetilde{e}_i, \widetilde{d}_i, \widetilde{r}_{\alpha\beta}$ instead of $e_i, d_i, r_{\alpha\beta}$.
We assume further that $\delta = \frac{1}{K_4}$.
In analogous way we modify the boundary condition (2.17):

$$\nu\cdot\widetilde{J}_i^{(\delta)} + \widetilde{R}_i^{\Gamma} = 0, \quad i=1,\ldots,n \quad \text{on} \quad \Gamma_T. \tag{5.4}$$

The solvability of the nondegenerate problem (5.2) – (5.4), (2.8), (2.9) can be simply shown by using backward time discretization (see, for example [2]). By Theorems 2.3, 2.4 each solution $v = (v_0, v_1, \ldots, v_n)$ of that nondegenerate problem, satisfies the a priori estimates (2.25), (2.28). But, because of (5.1), v is automatically solution of the original problem (2.1), (2.2), (2.7) – (2.9). So theorem 2.5 is proved. $\Box$

We want now to prepare the proof of the uniqueness result. Let us to this aim suppose contradictionarily the existence of two solutions $v^{(1)} = \big(v_0^{(1)}, v_1^{(1)}, \dots, v_n^{(1)}\big)$, $v^{(2)} = \big(v_0^{(2)}, v_1^{(2)}, \dots, v_n^{(2)}\big)$ of problem (2.1), (2.2), (2.7) – (2.9). Remark that both solutions necessarily fulfill the a priori estimates (2.25), (2.28). We shall show that $v^{(1)} = v^{(2)}$.
We start by proving auxiliary Lemmas.

Lemma 5.1 *Let the assumptions of Theorem 2.6 be satisfied. Then there exists a constant L_{10} depending only on known parameters such that for arbitrary $\tau \in (0,T]$*

$$\sum_{i=1}^{n} \Big\{ \int_\Omega \big|v_i^{(1)}(\tau,x) - v_i^{(2)}(\tau,x)\big|^2\, dx \;+ \iint\limits_{Q_\tau} \Big|\frac{\partial(v_i^{(1)} - v_i^{(2)})}{\partial x}\Big|^2\, dx\, dt \Big\} \leq$$
$$\leq L_{10} \iint\limits_{Q_\tau} \Big[\Big|\frac{\partial(v_0^{(1)} - v_0^{(2)})}{\partial x}\Big|^2 + \sum_{l=0}^{n} \big|v_l^{(1)} - v_l^{(2)}\big|^2 + \tag{5.5}$$
$$+ \sum_{i=1}^{n} \big|v_i^{(1)} - v_i^{(2)}\big|^2 \Big(1 + \Big|\frac{\partial(v_i^{(1)} + q_i v_0^{(1)})}{\partial x}\Big|\Big) \Big|\frac{\partial v_0^{(1)}}{\partial x}\Big| \Big]\, dx\, dt\ .$$

Proof. We test the integral identity (2.13) for the solution $v^{(k)},\ k = 1,2$, with $\varphi^{(k)}$, where

$$\varphi^{(1)} = \frac{1}{e_i'(v_i^{(1)})}\big[e_i\big(v_i^{(1)}\big) - e_i\big(v_i^{(2)}\big)\big], \quad \varphi^{(2)} = v_i^{(2)} - v_i^{(1)}.$$

Taking the sum of the obtained equalities, we get

$$\sum_{i=1}^{n}\sum_{k=1}^{2} \Big\{ \int_0^\tau < \frac{\partial u_i^{(k)}}{\partial t}, \varphi^{(k)} > dt \;+ \iint\limits_{Q_\tau} \Big[\sum_{j=1}^{N} e_i'(v_i^{(k)}) \gamma_{ij}\Big(t, x, \frac{\partial \zeta_i^{(k)}}{\partial x}\Big) \frac{\partial \varphi^{(k)}}{\partial x_j} +$$
$$+ R_i\big(t, x, v^{(k)}, \zeta^{(k)}\big)\varphi^{(k)} \Big]\, dx\, dt \;+ \iint\limits_{\Gamma_\tau} R_i^\Gamma\big(t, x, v^{(k)}, \zeta^{(k)}\big)\varphi^{(k)}\, ds\, dt \Big\} = 0. \tag{5.6}$$

We evaluate the first integral in (5.6) analogously to Lemma 2 from [10] and obtain

$$\sum_{k=1}^{2} \int_0^\tau < \frac{\partial u_i^{(k)}}{\partial t}, \varphi^{(k)} > dt = \int_\Omega u_i^*(x) \int_{v_i^{(2)}(\tau,x)}^{v_i^{(1)}(\tau,x)} \big[v_i^{(1)}(\tau,x) - z\big] e_i'(z)\, dz\, dx \geq$$
$$\geq c_{53} \int_\Omega \big|v_i^{(1)}(\tau,x) - v_i^{(2)}(\tau,x)\big|^2\, dx. \tag{5.7}$$

The second one can be estimated by the assumptions (i), (ii) of Theorem 2.6:

$$e_i'(z_1) - \frac{e_i''(z_1)}{e_i'(z_1)}\big[e_i(z_1) - e_i(z_2)\big] \geq e_i'(z_1) - \int_{z_2}^{z_1} \frac{e_i''(s)}{e_i'(s)} e_i'(s)\, ds = e_i'(z_2)\ , \tag{5.8}$$

$$e_i'(z_1) - e_i'(z_2) - \frac{e_i''(z_1)}{e_i'(z_1)}\big[e_i(z_1) - e_i(z_2)\big] \leq c_{54}|z_1 - z_2|^2. \tag{5.9}$$

The last inequalities, conditions (2.29), iii) and the local Lipschitz continuity of γ_{ij} imply

$$\begin{aligned}
&\sum_{k=1}^{2}\sum_{j=1}^{N} e_i'\big(v_i^{(k)}\big)\gamma_{ij}\left(t,x,\frac{\partial\zeta_i^{(k)}}{\partial x}\right)\frac{\partial\varphi^{(k)}}{\partial x_j} \geq \\
&\quad\geq e_i'\big(v_i^{(2)}\big)\sum_{j=1}^{N}\left[\gamma_{ij}\left(t,x,\frac{\partial\zeta_i^{(1)}}{\partial x}\right)-\gamma_{ij}\left(t,x,\frac{\partial\zeta_i^{(2)}}{\partial x}\right)\right]\frac{\partial\big(\zeta_i^{(1)}-\zeta_i^{(2)}\big)}{\partial x_j}- \\
&\quad - c_{55}\big|v_i^{(1)}-v_i^{(2)}\big|^2\left(1+\left|\frac{\partial\zeta_i^{(1)}}{\partial x}\right|\right)\left|\frac{\partial v_0^{(1)}}{\partial x}\right|- \qquad\qquad (5.10)\\
&\quad - c_{55}\left|\frac{\partial\big(\zeta_i^{(1)}-\zeta_i^{(2)}\big)}{\partial x}\right|\left|\frac{\partial\big(v_0^{(1)}-v_0^{(2)}\big)}{\partial x}\right| \geq c_{56}\left|\frac{\partial\big(v_i^{(1)}-v_i^{(2)}\big)}{\partial x}\right|^2- \\
&\quad - c_{57}\left\{\big|v_i^{(1)}-v_i^{(2)}\big|^2\left(1+\left|\frac{\partial\zeta_i^{(1)}}{\partial x}\right|\right)\left|\frac{\partial v_0^{(1)}}{\partial x}\right|+\left|\frac{\partial\big(v_0^{(1)}-v_0^{(2)}\big)}{\partial x}\right|^2\right\}.
\end{aligned}$$

By the local Lipschitz continuity of R_i, R_i^Γ we get

$$\left|\sum_{k=1}^{2} R_i\big(t,x,v^{(k)},\zeta^{(k)}\big)\varphi^{(k)}\right| \leq c_{58}\sum_{l=0}^{n}\big|v_l^{(1)}-v_l^{(2)}\big|^2, \tag{5.11}$$

$$\left|\sum_{k=1}^{2} R_i^\Gamma\big(t,x,v^{(k)},\zeta^{(k)}\big)\varphi^{(k)}\right| \leq c_{58}\sum_{l=0}^{n}\big|v_l^{(1)}-v_l^{(2)}\big|^2. \tag{5.12}$$

Using the interpolation inequality

$$\iint\limits_{\Gamma_\tau} v^2(t,x)\,ds\,dt \leq \iint\limits_{Q_\tau}\left\{\varepsilon\left|\frac{\partial v}{\partial x}\right|^2+c_\varepsilon|v|^2\right\}dx\,dt$$

for functions $v_l^{(1)}-v_l^{(2)}$, $l=0,1,\ldots,n$, and suitable $\varepsilon>0$, we obtain (5.5) from (5.6), (5.7), (5.10), (5.12) and the proof of Lemma 5.1 is completed. □

Lemma 5.2 *Let the conditions of Theorem 2.6 be satisfied. Then a constant* L_{11} *depending only on known parameters exists such that*

$$\int_\Omega\left(\left|\frac{\partial\big(v_0^{(1)}-v_0^{(2)}\big)}{\partial x}\right|^2+\big|v_0^{(1)}-v_0^{(2)}\big|^2\right)dx \leq L_{11}\sum_{i=1}^{n}\int_\Omega\big|v_i^{(1)}-v_i^{(2)}\big|^2\,dx\,. \tag{5.13}$$

Proof. We test the integral identity (2.14) associated with the solutions $v^{(k)}$, $k=1,2$, with $\psi^{(1)}=v_0^{(1)}-v_0^{(2)}$, $\psi^{(2)}=v_0^{(2)}-v_0^{(1)}$. The sum of the obtained equalities reads:

$$\int_\Omega\epsilon(x)\left|\frac{\partial\big(v_0^{(1)}-v_0^{(2)}\big)}{\partial x}\right|^2dx+\int_{\partial\Omega}\kappa(x)\big|v_0^{(1)}-v_0^{(2)}\big|^2\,ds=\sum_{k=1}^{2}\sum_{i=1}^{n}q_i\int_\Omega u_i^{(k)}\psi^{(k)}\,dx. \tag{5.14}$$

Now (5.13) follows from (5.14), Cauchy and embedding inequalities. □

Lemma 5.3 *Let the conditions of Theorem 2.6 be satisfied. Then a constant L_{12} depending only on known parameters exists such that for arbitrary $\tau \in (0,T]$*

$$\begin{aligned}\sum_{i=1}^{n} \Big\{ \int_\Omega \big|v_i^{(1)}(\tau,x) - v_i^{(2)}(\tau,x)\big|^2 dx + \iint\limits_{Q_\tau} \big|v_i^{(1)} - v_i^{(2)}\big|^2 \Big[\Big|\frac{\partial v_i^{(1)}}{\partial x}\Big|^2 + \\ + \Big|\frac{\partial v_i^{(2)}}{\partial x}\Big|^2 \Big] dx\,dt \Big\} \le L_{12} \iint\limits_{Q_\tau} \Big\{ \sum_{i=1}^{n} \Big(\Big| \frac{\partial\big(v_i^{(1)} - v_i^{(2)}\big)}{\partial x}\Big|^2 + \\ + \Big[1 + \Big|\frac{\partial v_0^{(1)}}{\partial x}\Big| + \Big|\frac{\partial v_0^{(2)}}{\partial x}\Big|\Big]^2 \big|v_i^{(1)} - v_i^{(2)}\big|^2 \Big) + \\ + \Big|\frac{\partial}{\partial x}\big(v_0^{(1)} - v_0^{(2)}\big)\Big|^2 + \big|v_0^{(1)} - v_0^{(2)}\big|^2 \Big\}\, dx\, dt\, .\end{aligned} \tag{5.15}$$

Proof. We test the integral identity (2.13) for the solution $v^{(k)}$, $k = 1,2$, with

$$\widetilde{\varphi}^{(1)} = \frac{\exp\big(A\, e_i(v_i^{(1)})\big) - \exp\big(A e_i(v_i^{(2)})\big)}{e_i'(v_i^{(1)})}, \quad \widetilde{\varphi}^{(2)} = A\big[v_i^{(2)} - v_i^{(1)}\big] \exp\big(A\, e_i(v_i^{(2)})\big),$$

where A is a positive number, depending only on known parameters, such that

$$A\big[e_i'(s)\big]^2 + 2\, e_i''(s) \ge 1 \quad \text{for} \quad |s| \le K_4, \quad i = 1, \dots, n\,, \tag{5.16}$$

with K_4 from (2.28). Taking the sum of the obtained equalities, we get

$$\begin{aligned}\sum_{i=1}^{n}\sum_{k=1}^{2} \Big\{ \int_0^\tau < \frac{\partial u_i^{(k)}}{\partial t}, \widetilde{\varphi}^{(k)} > dt + \iint\limits_{Q_\tau} \Big[\sum_{j=1}^{N} e_i'(v_i^{(k)}) \gamma_{ij}\Big(t, x, \frac{\partial \zeta_i^{(k)}}{\partial x}\Big) \frac{\partial \widetilde{\varphi}^{(k)}}{\partial x_j} + \\ + R_i\big(t,x,v^{(k)},\zeta^{(k)}\big)\widetilde{\varphi}^{(k)}\Big]\, dx\, dt + \iint\limits_{\Gamma_\tau} R_i^\Gamma\big(t,x,v^{(k)},\zeta^{(k)}\big)\widetilde{\varphi}^{(k)}\, ds\, dt \Big\} = 0.\end{aligned} \tag{5.17}$$

We transform the first integral in (5.17) analogously to the inequality (5.7) to obtain

$$\begin{aligned}\sum_{i=1}^{2} \int_0^\tau < \frac{\partial u_i^{(k)}}{\partial t}, \widetilde{\varphi}^{(k)} > dt = A \int_\Omega u_i^*(x) \int_{v_i^{(2)}(\tau,x)}^{v_i^{(1)}(\tau,x)} \big[v_i^{(1)}(\tau,x) - z\big] \times \\ \times e_i'(z) \exp A e_i(z)\, dz\, dt \ge c_{59} \int_\Omega \big[v_i^{(1)}(\tau,x) - v_i^{(2)}(\tau,x)\big]^2 dx.\end{aligned} \tag{5.18}$$

To estimate the second term in (5.17) we use the iequality

$$\begin{aligned}- \frac{e_i''(z_1)}{e_i'(z_1)}\big[\exp[Ae_i(z_1)] - \exp[Ae_i(z_2)]\big] \ge -A \int_{z_2}^{z_1} e_i''(z) \exp[Ae_i(z)]\, dz = \\ = A\big[e_i'(z_2)\exp[Ae_i(z_2)] - e_i'(z_1)\exp[Ae_i(z_1)]\big] + A^2 \int_{z_2}^{z_1} \big[e_i'(z)\big]^2 \exp[Ae_i(z)]\, dz\,,\end{aligned} \tag{5.19}$$

that follows for $z_1, z_2 \in \mathbb{R}^1$ from condition 1) of Theorem 2.6. So we obtain

$$\sum_{j=1}^{N} \iint\limits_{Q_\tau} e_i'\big(v_i^{(1)}\big)\gamma_{ij}\left(t, x, \frac{\partial \zeta_i^{(1)}}{\partial x}\right) \frac{\partial \widetilde{\varphi}^{(1)}}{\partial x_j}\, dx\, dt \geq I^{(1)} + I^{(2)} + I^{(3)} \;, \tag{5.20}$$

where

$$\begin{aligned}
I^{(1)} &= A^2 \sum_{j=1}^{N} \iint\limits_{Q_\tau} \gamma_{ij}\left(t, x, \frac{\partial \zeta_i^{(1)}}{\partial x}\right) \frac{\partial \zeta_i^{(1)}}{\partial x_j} \int_{v_i^{(2)}}^{v_i^{(1)}} \big[e_i'(z)\big]^2 \exp[Ae_i(z)]\, dz\, dx\, dt, \\
I^{(2)} &= A \sum_{j=1}^{N} \iint\limits_{Q_\tau} \gamma_{ij}\left(t, x, \frac{\partial \zeta_i^{(1)}}{\partial x}\right) \frac{\partial \big(v_i^{(1)} - v_i^{(2)}\big)}{\partial x_j} e_i'\big(v_i^{(2)}\big) \exp[Ae_i\big(v_i^{(2)}\big]\big)\, dx\, dt, \\
I^{(3)} &= A\, q_i \sum_{j=1}^{N} \iint\limits_{Q_\tau} \gamma_{ij}\left(t, x, \frac{\partial \zeta_i^{(1)}}{\partial x}\right) \frac{\partial v_0^{(1)}}{\partial x_j} \left\{ \frac{e_i''\big(v_i^{(1)}\big)}{e_i'\big(v_i^{(1)}\big)} \int_{v_i^{(2)}}^{v_i^{(1)}} e_i'(z) \exp[A\, e_i(z)]\, dz \; - \right. \\
&\qquad \left. - \, e_i'\big(v_i^{(1)}\big) \exp[Ae_i\big(v_i^{(1)}\big)] + e_i'\big(v_i^{(2)}\big) \exp[Ae_i\big(v_i^{(2)}\big)] \right\} dx\; dt.
\end{aligned}$$

We rewrite the second term in (5.17) with $k = 2$ as follows

$$\sum_{j=1}^{N} \iint\limits_{Q_\tau} e_i'\big(v_i^{(2)}\big)\gamma_{ij}\left(t, x, \frac{\partial \zeta_i^{(2)}}{\partial x}\right) \frac{\partial \widetilde{\varphi}^{(2)}}{\partial x_j}\, dx\, dt = I^{(4)} + I^{(5)} + I^{(6)} + I^{(7)} \;, \tag{5.21}$$

$$I^{(4)} = -A^2 \sum_{j=1}^{N} \iint\limits_{Q_\tau} \gamma_{ij}\left(t, x, \frac{\partial \zeta_i^{(1)}}{\partial x}\right) \frac{\partial \zeta_i^{(1)}}{\partial x_j} \big[e_i'\big(v_i^{(2)}\big)\big]^2 \exp[Ae_i\big(v_i^{(2)}\big)]\big(v_i^{(1)} - v_i^{(2)}\big)\, dx\, dt,$$

$$I^{(5)} = q_i A^2 \sum_{j=1}^{N} \iint\limits_{Q_\tau} \gamma_{ij}\left(t, x, \frac{\partial \zeta_i^{(1)}}{\partial x}\right) \frac{\partial v_0^{(1)}}{\partial x_j} \big[e_i'\big(v_i^{(2)}\big)\big]^2 \exp[Ae_i\big(v_i^{(2)}\big)]\big(v_1^{(1)} - v_i^{(2)}\big)\, dx\, dt,$$

$$\begin{aligned}
I^{(6)} = A^2 \sum_{j=1}^{N} \iint\limits_{Q_\tau} &\left[\gamma_{ij}\left(t, x, \frac{\partial \zeta_i^{(1)}}{\partial x}\right) \frac{\partial v_i^{(1)}}{\partial x_j} - \gamma_{ij}\left(t, x, \frac{\partial \zeta_i^{(2)}}{\partial x}\right) \frac{\partial v_i^{(2)}}{\partial x_j}\right] \times \\
&\times \big[e_i'\big(v_i^{(2)}\big)\big]^2 \exp[Ae_i\big(v_i^{(2)}\big)]\big(v_i^{(1)} - v_i^{(2)}\big)\, dx\, dt,
\end{aligned}$$

$$I^{(7)} = -A \sum_{j=1}^{N} \iint\limits_{Q_\tau} \gamma_{ij}\left(t, x, \frac{\partial \zeta_i^{(2)}}{\partial x}\right) e_i'\big(v_i^{(2)}\big) \exp[Ae_i\big(v_i^{(2)}\big)] \frac{\partial \big(v_i^{(1)} - v_i^{(2)}\big)}{\partial x_j}\, dx\, dt.$$

We want to estimate sums of terms from (5.20) and (5.21). Note that by (5.16)

$$\begin{aligned}
&\int_{z_2}^{z_1} \big[e_i'(z)\big]^2 \exp[Ae_i(z)]\, dz \;-\; \big[e_i'(z_2)\big]^2 \exp[Ae_i(z_2)](z_1 - z_2) = \\
&= \int_{z_2}^{z_1} \int_{z_2}^{z} \Big(2\, e_i''(\theta) + A\big[e_i'(\theta)\big]^2\Big) e_i'(\theta) \exp[Ae_i(\theta)]\, d\theta\, dz \geq \\
&\geq c_{60} |z_1 - z_2|^2 \qquad \text{for} \qquad |z_1|, |z_2| \leq K_4
\end{aligned}$$

and hence

$$I^{(1)} + I^{(4)} \geq c_{61} \iint\limits_{Q_\tau} \left|v_i^{(1)} - v_i^{(2)}\right|^2 \left|\frac{\partial \zeta_i^{(1)}}{\partial x}\right|^2 \, dx\, dt\, . \tag{5.22}$$

The next estimate follows from conditions iii) and (2.29)

$$I^{(2)} + I^{(7)} \geq c_{62} \iint\limits_{Q_\tau} \left|\frac{\partial \zeta_i^{(1)} - \zeta_i^{(2)}}{\partial x}\right|^2 dx\, dt\ - c_{63} \iint\limits_{Q_\tau^{(i)}} \left|\frac{\partial\big(v_0^{(1)} - v_0^{2}\big)}{\partial x}\right|^2 dx\, dt\, . \tag{5.23}$$

The local Lipschitz continuity of the function e_i'' implies

$$\begin{aligned} &\Big|A\,\big[e_i'(z_2)\big]^2 \exp[Ae_i(z_2)](z_1 - z_2) + \frac{e_i''(z_1)}{e_i'(z_1)} \int_{z_2}^{z_1} e_i'(z) \exp[Ae_i(z)]\, dz - \\ &- e_i'(z_1) \exp[Ae_i(z_1)] + e_i'(z_2) \exp[Ae_i(z_2)]\Big| \leq c_{64} |z_1 - z_2|^2 \end{aligned}$$

for arbitrary numbers $z_1, z_2 \in [-K_4, K_4]$ and consequently

$$\left|I^{(3)} + I^{(5)}\right| \leq c_{65} \iint\limits_{Q_\tau} \left(1 + \left|\frac{\partial \zeta_i^{(1)}}{\partial x}\right|\right) \left|\frac{\partial v_0^{(1)}}{\partial x}\right| \cdot \left|v_i^{(1)} - v_i^{(2)}\right|^2 dx\, dt. \tag{5.24}$$

Further, the local Lipschitz condition for γ_{ij} yields:

$$\begin{aligned} \left|I^{(6)}\right| \leq & c_{66} \iint\limits_{Q_\tau} \Bigg[\left|\frac{\partial\big(v_i^{(1)} - v_i^{(2)}\big)}{\partial x}\right| \left(\left|\frac{\partial v_i^{(1)}}{\partial x}\right| + \left|\frac{\partial \zeta_i^{(2)}}{\partial x}\right|\right) + \\ & + \left|\frac{\partial v_i^{(1)}}{\partial x}\right| \cdot \left|\frac{\partial\big(v_0^{(1)} - v_0^{(2)}\big)}{\partial x}\right|\Bigg] \left|v_i^{(1)} - v_i^{(2)}\right| dx\, dt. \end{aligned} \tag{5.25}$$

Finally, we obtain from (5.17), (5.18), (5.22) – (5.25) with view of (5.11), (5.12)

$$\begin{aligned} &\sum_{i=1}^{n} \left\{ \int_\Omega \big[v_i^{(1)}(\tau, x) - v_i^{(2)}(\tau, x)\big]^2 dx + \iint\limits_{Q_\tau} \left|v_i^{(1)} - v_i^{(2)}\right|^2 \left|\frac{\partial v_i^{(1)}}{\partial x}\right|^2 dx\, dt\right\} \leq \\ &\leq c_{67} \iint\limits_{Q_\tau} \left\{ \sum_{i=1}^{n} \left(\left|\frac{\partial\big(v_i^{(1)} - v_i^{(2)}\big)}{\partial x}\right|^2 + \left[1 + \left|\frac{\partial v_0^{(1)}}{\partial x}\right|^2\right] \left|v_i^{(1)} - v_i^{(2)}\right|^2 + \right.\right. \\ &\quad \left. + \left|\frac{\partial\big(v_i^{(1)} - v_i^{(2)}\big)}{\partial x}\right| \left|\frac{\partial \zeta_i^{(2)}}{\partial x}\right| \left|(v_i^{(1)} - v_i^{(2)}\right| \right) + \\ &\quad \left. + \left|\frac{\partial\big(v_0^{(1)} - v_0^{(2)}\big)}{\partial x}\right|^2 + \left|v_0^{(1)} - v_0^{(2)}\right|^2 \right\} dx\, dt. \end{aligned} \tag{5.26}$$

Changing the places of $v_i^{(1)}$ and $v_i^{(2)}$ in (5.26) and applying Cauchy's inequality, we arrive at the desired estimate (5.15) and the proof of Lemma 5.3 is complete. □

Lemma 5.4 *Under the conditions of Theorem 2.6 a constant L_{13} depending only on known parameters exists such that for all $\tau \in (0,T]$, $R \in (0,1]$,*

$$\begin{aligned}\iint\limits_{Q_\tau} |v_i^{(1)} - v_i^{(2)}|^2 \left[\left|\frac{\partial v_0^{(1)}}{\partial x}\right| + \left|\frac{\partial v_0^{(2)}}{\partial x}\right|\right]^2 dx\; dt \le L_{13}\Big\{ R^{2\eta} \iint\limits_{Q_\tau} \left|\frac{\partial\big(v_i^{(1)} - v_i^{(2)}\big)}{\partial x}\right|^2 dx\, dt + \\ + \int_0^\tau \left[\Big\{\int_\Omega \big|v_i^{(1)} - v_i^{(2)}\big|^{2p_1'}\, dx\Big\}^{\frac{1}{p_1'}} + \Big\{\int_{\partial\Omega} \big|v_i^{(1)} - v_i^{(2)}\big|^{2p_2'}\, ds\Big\}^{\frac{1}{p_2'}}\right] dt\Big\}\,,\end{aligned} \tag{5.27}$$

where η is the Hölder exponent from Theorem 2.3.

Proof. Let $\{\varphi_j(x)\}$, $j = 1, \dots, J$, be a partition of unity satisfying (4.3) with a number R to be chosen later on. We test the integral identity (2.14) associateted with the solution $v^{(k)}$, $k = 1, 2$, with

$$\Psi^{(k)}(t,x) = \sum_{l=1}^{J} \big[v_0^{(k)}(t,x) - v_{0,l}^{(k)}(t)\big]\varphi_l^2(x)\big|v_i^{(1)}(t,x) - v_i^{(2)}(t,x)\big|^2, \quad v_{0,l}^{(k)}(t) = v_0^{(k)}(x_l).$$

We obtain after integration with respect to t and using the Hölder inequality

$$\begin{aligned}\iint\limits_{Q_\tau} \epsilon(x)\big|v_i^{(1)} - v_i^{(2)}\big|^2 \left|\frac{\partial v_0^{(k)}}{\partial x}\right|^2 dx\; dt \le I^{(8)}(k) + I^{(9)}(k) + \\ + c_{68}\int_0^\tau \left[\Big\{\int_\Omega \big|v_i^{(1)} - v_i^{(2)}\big|^{2p_1'}\, dx\Big\}^{\frac{1}{p_1'}} + \int_0^\tau \Big\{\int_{\partial\Omega} \big|v_i^{(1)} - v_i^{(2)}\big|^{2p_2'}\, ds\Big\}^{\frac{1}{p_2'}}\right] dt\ ,\end{aligned} \tag{5.28}$$

$$I^{(8)}(k) = -2\sum_{l=1}^{J}\sum_{j=1}^{N} \iint\limits_{Q_\tau} \epsilon(x)\frac{\partial v_0^{(k)}}{\partial x_j}\frac{\partial \varphi_l}{\partial x_j}\varphi_l \cdot \big[v_0^{(k)} - v_{0,l}^{(k)}\big]\big|v_i^{(1)} - v_i^{(2)}\big|^2\, dx\; dt,$$

$$I^{(9)}(k) = -2\sum_{l=1}^{J}\sum_{j=1}^{N} \iint\limits_{Q_\tau} \epsilon(x)\frac{\partial v_0^{(k)}}{\partial x_j} \cdot \frac{\partial\big(v_i^{(1)} - v_i^{(2)}\big)}{\partial x_j}\varphi_l^2\big[v_0^{(k)} - v_{0,l}^{(k)}\big]\big[v_i^{(1)} - v_i^{(2)}\big]dx\, dt.$$

Estimating $I^{(8)}(k), I^{(9)}(k)$ by Cauchy's inequality and using (4.6), we obtain (5.27) immediately from (5.28) and the proof of Lemma 5.4 is complete. ☐

Proof of Theorem 2.6. ¿From (5.5) we get by applying Cauchy's inequality to the second term and using (5.13), (5.15), (5.27) with suitable R

$$\begin{aligned}\sum_{i=1}^{n}\Big\{\int_\Omega \big|v_i^{(1)}(\tau,x) - v_i^{(2)}(\tau,x)\big|^2\, dx\ + \iint\limits_{Q_\tau}\left|\frac{\partial(v_i^{(1)} - v_i^{(2)})}{\partial x}\right|^2 dx\; dt\Big\} \le \\ \le c_{69}\sum_{i=1}^{n}\Big\{\iint\limits_{Q_\tau} \big|v_i^{(1)} - v_i^{(2)}\big|^2\, dx\; dt\ + \int_0^\tau \Big\{\int_\Omega \big|v_i^{(1)} - v_i^{(2)}\big|^{2p_1'}\, dx\Big\}^{\frac{1}{p_1'}} dt + \\ + \int_0^\tau \Big\{\int_{\partial\Omega} \big|v_i^{(1)} - v_i^{(2)}\big|^{2p_2'}\, ds\Big\}^{\frac{1}{p_2'}} dt\Big\}.\end{aligned} \tag{5.29}$$

Here the second integral on the right hand side can by estimated by the interpolation inequality

$$\begin{aligned}&\int_0^\tau \left\{ \int_\Omega \left|v_i^{(1)} - v_i^{(2)}\right|^{2p_1'} dx \right\}^{\frac{1}{p_1'}} dt \le \\ &\le \varepsilon \left\{ \text{ess} \sup_{0<\theta<\tau} \int_\Omega \left|v_i^{(1)}(\theta,x) - v_i^{(2)}(\theta,x)\right|^2 dx + \right. \\ &\left. + \iint\limits_{Q_\tau} \left| \frac{\partial\bigl(v_i^{(1)} - v_i^{(2)}\bigr)}{\partial x} \right|^2 dx\, dt \right\} + c(\varepsilon) \iint\limits_{Q_\tau} \left|v_i^{(1)} - v_i^{(2)}\right|^2 dx\; dt\;,\end{aligned} \tag{5.30}$$

where $p_1' = \frac{p_1}{p_1 - 1} < \frac{N}{N-2}$ and $\varepsilon > 0$. Analogously we get for the last integral in (5.29)

$$\sum_{i=1}^n \int_\Omega \left|v_i^{(1)}(\tau,x) - v_i^{(2)}(\tau,x)\right|^2 dx \le c_{70} \sum_{i=1}^n \iint\limits_{Q_\tau} \left|v_i^{(1)} - v_i^{(2)}\right|^2 dx\, dt. \tag{5.31}$$

Hence Gronwall's lemma yields $v_i^{(1)} = v_i^{(2)}$ for $i = 1, \ldots, n$. Finally, $v_0^{(1)} = v_0^{(2)}$ follows from (5.13) and Theorem 2.6 is proved. □

Proof of Corollary 2.1 Let v be the solution of (2.1), (2.2),(2.7) – (2.9). Set

$$v_i^{(1)} = v_i(t,x), \quad v_i^{(2)}(t,x) = v_i(t+\delta,x), \quad i = 0,\ldots,n\;,\; \delta \in (0, T-t).$$

We test integral identity (3.5) with the functions

$$\varphi^{(1)} = t^2 \frac{1}{e_i'\bigl(v_i^{(1)}\bigr)} \bigl[e_i\bigl(v_i^{(1)}\bigr) - e_i\bigl(v_i^{(2)}\bigr)\bigr], \quad \varphi^{(2)} = t^2\bigl(v_i^{(2)} - v_i^{(1)}\bigr).$$

Then, arguing essentially as in the proof of (5.29) and (5.31), we obtain

$$\begin{aligned}&\tau^2 \sum_{i=1}^n \left\{ \int_\Omega \left|v_i^{(1)}(\tau,x) - v_i^{(2)}(\tau,x)\right|^2 dx \;+ \iint\limits_{Q_\tau} t^2 \left| \frac{\partial\bigl(v_i^{(1)} - v_i^{(2)}\bigr)}{\partial x} \right|^2 dx\; dt \right\} \le \\ &\le c_{71} \left\{ \sum_{i=1}^n \iint\limits_{Q_\tau} t^2 \left|v_i^{(1)} - v_i^{(2)}\right|^2 dx\, dt \;+ \delta^2 \right\}.\end{aligned}$$

Now dividing by δ^2, applying Gronwall's lemma and taking the limit $\delta \to 0$, the corollary follows. □

6 Proof of Theorem 2.7

We start from the proof of (2.24) making some changes in the proof of Theorem 2.2. In the proof of Lemmas 3.1, 3.2, 3.3 we didn't use the conditions (2.22), (2.23).

Consequently, the results of these Lemmas remain valid.
We replace the test function φ in the proof of Lemma 3.4 by

$$\varphi = \Big[a(\rho) + \big[\Lambda(v) - M_0\big]_+\Big]_k^\rho v_i, \quad \Lambda(v) = \sum_{i=1}^n \Lambda_i(v_i), \quad i = 1, \dots, n \ , \tag{6.1}$$

where $\rho > 0$, $k > 1$, the function $\Lambda_i(z)$ is defined by (2.20), $M_0 = \sum_{i=1}^n \Lambda_i(m_0)$, m_0 and $a(\rho)$ are the same numbers as in the proofs of the Lemmas 4.2 and 3.4, respectively.
Using the equalities $\alpha \cdot q = \beta \cdot q$, $\gamma \cdot q = \delta \cdot q$ for $(\alpha, \beta) \in \mathcal{R}$, $(\gamma, \delta) \in \mathcal{R}^\Gamma$ and the monotonicity condition for the functions $r_{\alpha\beta}, r^\Gamma_{\gamma\delta}$, we find

$$\begin{aligned} \sum_{i=1}^n R_i(\cdot, v, \zeta) v_i &= \sum_{i=1}^n \sum_{(\alpha,\beta)\in R} \big[r_{\alpha\beta}(\cdot, v, \alpha\cdot\zeta) - r_{\alpha\beta}(\cdot, v, \beta\cdot\zeta)\big](\alpha_i - \beta_i) v_i = \\ &= \sum_{(\alpha,\beta)\in R} \big[r_{\alpha\beta}(\cdot, v, \alpha\cdot\zeta) - r_{\alpha\beta}(\cdot, v, \beta\cdot\zeta)\big](\alpha - \beta)\cdot\zeta \geq 0 \ , \\ \sum_{i=1}^n R_i^\Gamma(\cdot, v, \zeta) v_i \geq 0 \ , &\quad \zeta = (\zeta_1, \dots, \zeta_n), \quad \zeta_i = q_i v_0 + v_i \ . \end{aligned} \tag{6.2}$$

Using the test function from (6.1) and the inequality (6.2), we obtain from (2.13)

$$\begin{aligned} \sum_{i=1}^n \Bigg\{ \int_0^\tau < \frac{\partial u_i}{\partial t}, \Big[a(\rho) + \big[\Lambda(v) - M_0\big]_+\Big]_k^\rho v_i > \ dt \ + \iint\limits_{Q_\tau} \sum_{j,k=1}^N e_i'(v_i) a_{jk}(t,x) \times \\ \times \frac{\partial}{\partial x_k}(v_i + q_i v_0) \frac{\partial}{\partial x_j}\bigg(\Big[a(\rho) + \big[\Lambda(v) - M_0\big]_+\Big]_k^\rho v_i \bigg) \, dx \, dt \Bigg\} \leq 0. \end{aligned} \tag{6.3}$$

We evaluate the first integral in (6.3) following Lemma 2 in [10] to get

$$\begin{aligned} \sum_{i=1}^n \int_0^\tau < \frac{\partial u_i}{\partial t}, \Big[a(\rho) + \big[\Lambda(v) - M_0\big]_+\Big]_k^\rho v_i > \ dt \ = \\ = \int_\Omega u^* \int_0^{\Lambda(v(\tau,x))} \Big[a(\rho) + \big[z - M_0\big]_+\Big]_k^\rho dz \, dx \geq \\ \geq \frac{1}{\rho+1} \bigg\{ \int_\Omega u^* \Big[a(\rho) + \big[\Lambda(v(\tau,x)) - M_0\big]_+\Big]_k^{\rho+1} dx \ - c_{71} a(\rho) \bigg\}. \end{aligned} \tag{6.4}$$

To estimate the second integral in (6.3) we note that:

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j,k=1}^{N} e_i'(v_i)a_{jk}\frac{\partial v_i}{\partial x_k}\frac{\partial}{\partial x_j}\left(\Big[a(\rho)+\big[\Lambda(v)-M_0\big]_+\Big]_k^{\rho} v_i\right)=\\
&\qquad= \Big[a(\rho)+\big[\Lambda(v)-M_0\big]_+\Big]_k^{\rho}\sum_{i=1}^{n}\sum_{j,k=1}^{N} e_i'(v_i)a_{jk}\frac{\partial v_i}{\partial x_k}\frac{\partial v_i}{\partial x_j}+\\
&\qquad+\rho\Big[a(\rho)+\big[\Lambda(v)-M_0\big]_+\Big]_k^{\rho-1}\sum_{j,k=1}^{N} a_{jk}\left(\sum_{i=1}^{n} e_i'(v_i)v_i\frac{\partial v_i}{\partial x_k}\right)\times\\
&\qquad\times\left(\sum_{l=1}^{n} e_l'(v_l)v_l\frac{\partial v_l}{\partial x_j}\right)\chi\big(0<\Lambda(v)-M_0<k-a(\rho)\big)\geq\\
&\qquad\geq c_{72}\Big[a(\rho)+\big[\Lambda(v)-M_0\big]_+\Big]_k^{\rho}\sum_{i=1}^{n} e_i'(v_i)\left|\frac{\partial v_i}{\partial x}\right|^2 .
\end{aligned}
\tag{6.5}
$$

Now (6.3) – (6.5) and (2.21) imply

$$
\begin{aligned}
&\int_{\Omega} u^*\Big[a(\rho)+\big[\Lambda(v(\tau,x))-M_0\big]_+\Big]_k^{\rho+1}\,dx+\\
&\qquad+\iint\limits_{Q_T}\Big[a(\rho)+\big[\Lambda(v)-M_0\big]_+\Big]_k^{\rho}\sum_{i=1}^{n} e_i'(v_i)\left|\frac{\partial v_i}{\partial x}\right|^2 dx\,dt\ \leq\\
&\qquad\leq c_{73}(\rho+1)^2\Bigg\{\iint\limits_{Q_\tau}\Big\{\Big[a(\rho)+\big[\Lambda(v)-M_0\big]_+\Big]_k^{\rho}+M_0^{\rho}\Big\}\times\\
&\qquad\times\sum_{i=1}^{n} e_i'(v_i)\left|\frac{\partial v_0}{\partial x}\right|^2 dx\,dt\ +a(\rho)\Bigg\}.
\end{aligned}
\tag{6.6}
$$

With view of the proof of Lemma 3.4 we want to reestablish (3.45) as a consequence of (3.43) (and (3.41)) . Note that by assumption $\gamma_i=\gamma$ and let us assume that $r(m)+\varepsilon>\gamma$. Choosing ρ in (6.6) such that $(2+\gamma)\rho+\gamma=r(m)+\varepsilon$ we can estimate the right hand side of (6.6) by (3.43) for $i=1,\ldots,n$. Hence (6.6) implies (3.45). Repeating all another discussions from the proofs of Lemma 3.4 and Theorem 2.2, we obtain the inequality (2.24). The proof of (2.25) in the considered case coincides with that one in the proof of Theorem 2.3.
In order to prove (2.28) we need only to check (4.1). We have from (6.6) with $\rho\geq 1$

$$
\begin{aligned}
&\int_{\Omega} u^*\Big[\Lambda\big(v(\tau,x)\big)-M_0\Big]_+^{\rho+1} dx+\iint\limits_{Q_\tau}\Big[\Lambda(v)-M_0\Big]_+^{\rho}\sum_{i=1}^{n} e_i'(v_i)\left|\frac{\partial v_i}{\partial x}\right|^2 dx\,dt\leq\\
&\qquad\leq c_{73}(\rho+1)^2\Bigg\{\iint\limits_{Q_\tau}\Big[\Lambda(v)-M_0\Big]_+^{\rho}\sum_{i=1}^{n} e_i'(v_i)\left|\frac{\partial v_0}{\partial x}\right|^2 dx\,dt\ +M_0^{\rho}\Bigg\}.
\end{aligned}
\tag{6.7}
$$

To estimate the last integral we test the identity (2.14) with

$$\psi(t,x) = \Big[\Lambda\big(v(t,x)\big) - M_0\Big]_+^{\rho} \sum_{i=1}^{n} \big(1+[v_i]_+\big)^{\gamma} \Big[v_0(t,x) - v_{0,l}(t)\Big] \varphi_l^2(x) \ ,$$

where the notations from (4.4) are used. Integration on t and summing up on l give

$$\begin{aligned}\iint\limits_{Q_\tau} \Big[\Lambda(v) - M_0\Big]_+^{\rho} \sum_{i=1}^{n} e_i'(v_i) \left|\frac{\partial v_0}{\partial x}\right|^2 dx\ dt \le c_{74}\, \rho \sum_{i=1}^{n} \Bigg\{ \sum_{l=1}^{J} \iint\limits_{Q_\tau} e_i'(v_i) \times \\ \times \Big[\Lambda(v) - M_0\Big]_+^{\rho} \Big(|v_0 - v_{0,l}| \varphi_l^2 \left|\frac{\partial v_i}{\partial x}\right| \left|\frac{\partial v_0}{\partial x}\right| + \frac{1}{R^2}\Big)\, dx\ dt + \\ + \int_0^{\tau} \Big\{ \int_{\Omega} \Big(\Big[\Lambda(v) - M_0\Big]_+^{\rho} e_i'(v_i)\Big)^{p_1'} dx \Big\}^{\frac{1}{p_1'}} dt + \\ + \int_0^{\tau} \Big\{ \int_{\partial\Omega} \Big(\Big[\Lambda(v) - M_0\Big]_+^{\rho} e_i'(v_i)\Big)^{p_2'} ds \Big\}^{\frac{1}{p_2'}} dt\ + M_0^{\rho} \Bigg\}.\end{aligned} \tag{6.8}$$

Using (4.6), we obtain from (6.7), (6.8)

$$\begin{aligned}\int_{\Omega} \Big[\Lambda\big(v(\tau,x)\big) - M_0\Big]_+^{\rho+1} dx\ + \iint\limits_{Q_\tau} \Big[\Lambda(v) - M_0\Big]_+^{\rho} \sum_{i=1}^{n} e_i'(v_i) \left|\frac{\partial v_i}{\partial x}\right|^2 dx\ dt \le \\ \le c_{75}\, \rho^{2+\frac{4}{\eta}} \Bigg\{ M_0^{\rho} + \int_0^{\tau} \Big\{ \int_{\Omega} \Big[\Lambda(v) - M_0\Big]_+^{(\rho+1)p_1'} dx \Big\}^{\frac{1}{p_1'}} dt + \\ + \int_0^{\tau} \Big\{ \int_{\partial\Omega} \Big[\Lambda(v) - M_0\Big]_+^{(\rho+1)p_2'} ds \Big\}^{\frac{1}{p_2'}} dt \Bigg\}.\end{aligned} \tag{6.9}$$

The last inequality implies (4.1) by standard Moser iteration.
The proofs of the existence and uniniquenes (Theorems 2.5, 2.6) remain valid under the assumptions of Theorem 2.7. This ends the proof of Theorem 2.7. □

References

[1] G. Albinus, *Convex analysis of the energy model of semiconductor devices*, WIAS Preprint **285**, Berlin (1996).

[2] H.W. Alt, S. Luckhaus, *Quasilinear elliptic-parabolic differential equations*, Math. Z., **183** (1983), 311–341.

[3] H. Amann, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, in: Function spaces, Differential Operators and Nonlinear Analysis (H.J. Scmeisser and H. Tribel, eds), Teubner-Texte zur Mathematik, vol. 133, Teubner, Stuttgart - Leipzig, 1993, pp. 9-116.

[4] H. Amann, M. Renardy, *Rection-diffusion problems in electrolysis*, Nonlinear Differ. Equ. Appl., **1** (1994), 91–117.

[5] C. Ebmeyer, J. Frehse, *Mixed boundary value problems for nonlinear elliptic equations in multidimensional non-smooth domains*, Math. Nachr. **203** (1999) 47–74 .

[6] H. Gajewski, *On the uniqueness of solutions to the drift–diffusion model of semiconductor devices*, Math. Models Methods Appl. Sci., **4** (1994), 121–133.

[7] H. Gajewski, K. Gröger, *Semiconductor equations for variable mobilities based on Boltzmann statistics or Fermi–Dirac statistics*, Math. Nach., **140** (1989), 7–36.

[8] H. Gajewski, K. Gröger, *Reaction–diffusion processes of electrically charged species*, Math. Nachr., **177** (1996), 109–130.

[9] H. Gajewski, I.V. Skrypnik, *To the uniqueness problem for nonlinear elliptic equations*, Nonlinear Anal., Theory Methods Appl., **52** (2003), 291–304.

[10] H. Gajewski, I.V. Skrypnik, *On the uniqueness of solutions for nonlinear parabolic equations*, Discrete Contin. Dyn. Syst., **10** (2003), no. 1 & 2, 315–336.

[11] H. Gajewski, I.V. Skrypnik, *On the uniqueness of solutions for nonlinear elliptic–parabolic equations*, J. Evol. Equ., **3** (2003), 247–281.

[12] H. Gajewski, I.V. Skrypnik, *On the unique solvability of nonlocal drift-diffusion-type problems*, Nonlinear Anal. **56** (2004), 803–830.

[13] A. Glitzky, R. Hünlich, *Electro-reaction-diffusion systems including cluster reactions of higher order*, Math. Nachr. **216** (2000), 95–118.

[14] J.W. Jerome, *Analysis of Charge Transport*, Springer–Verlag, Berlin, (1996) xii+167 pp.

[15] O.A. Ladyzhenskaya, N.N. Uraltseva, *Linear and quasilinear elliptic equations*, Nauka, Moscow (1973) (Russian), 576 pp.

[16] W. van Roosbroeck, *Theory of flow of electrons and holes in germanium and other semiconductors*, Bell System Tech. J., **29** (1950), 560–607.

[17] S. Selberherr, *Analysis and simulation of semiconductor devices*, Springer–Verlag, Wien, New York (1984).