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Analytic-numerical investigation of delayed exchange of stabilities in singularly perturbed parabolic problems

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Abstract

We consider a class of singularly perturbed parabolic problems in case of exchange of stabilities, that is, the corresponding degenerate equation has two intersecting roots. We present an analytic result about the phenomenon of delayed exchange of stabilities and compare it with numerical investigations of some examples.

1 Introduction

Consider an autonomous dynamical system $\mathcal{S}$ depending on some parameter λ. The study of the influence of λ on the long-term behavior of the dynamical system $\mathcal{S}$ represents an essential part of the bifurcation theory. λ^* is called a bifurcation point for $\mathcal{S}$ concerning the region $\mathcal{G}$ in the phase space of $\mathcal{S}$ if in any neighborhood $\mathcal{N}$ of λ^* in the parameter space there exist two points λ_1 and λ_2 such that the phase portrait of $\mathcal{S}$ in $\mathcal{G}$ is not topologically equivalent for λ_1 and λ_2 If we assume that λ is slowly changing in time then we arrive at the so-called dynamic bifurcation theory [1]. As an example we consider the scalar ordinary differential equation

$$\frac{dx}{dt} = f(x, \lambda), \tag{1.1}$$

where we assume $f(0, \lambda) \equiv 0$ for all λ. For definiteness we suppose that $\lambda^* = 0$ is an bifurcation point of (1.1), where $x = 0$ is stable (unstable) for $\lambda < 0$ ($\lambda > 0$). This assumption implies that the bifurcation point $\lambda = 0$ is generically related either to a transcritical bifurcation (see Fig 1.1) or to a pitchfork bifurcation (see Fig. 1.2).

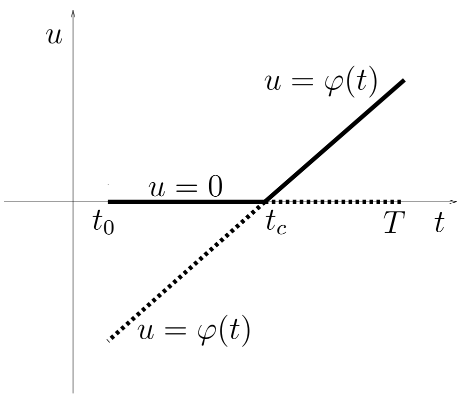

Fig. 1.1. Transcritical bifurcation

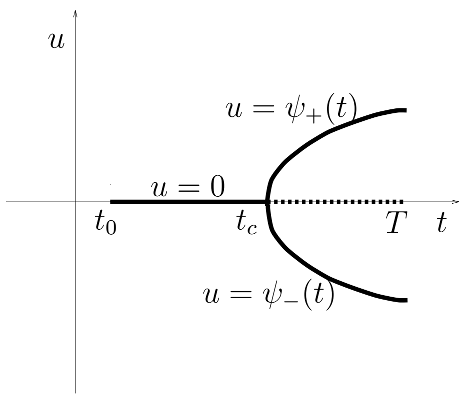

Fig. 1.2. Pitchfork bifurcation

Now we suppose that λ increases slowly with t. For simplicity we set

$$\lambda = \varepsilon t,$$

where ε is a small positive parameter. Introducing the slow time τ by $\tau = \varepsilon t$, the differential equation (1.1) takes the form

$$\varepsilon \frac{dx}{d\tau} = f(x,\tau), \tag{1.2}$$

that is, (1.2) is a singularly perturbed non-autonomous differential equation. Under our assumption, the solution set $f^{-1}(0)$ of the degenerate equation of (1.2)

$$0 = f(x,\tau) \tag{1.3}$$

consists in the $\tau - x$–plane of two curves intersecting for $\tau = 0$, as indicated in Fig. 1.1 and Fig. 1.2. All points of $f^{-1}(0)$ are equilibria of the associated equation to (1.2)

$$\frac{dx}{d\sigma} = f(x,\tau), \tag{1.4}$$

where τ has to be considered as a parameter. The curve $x = 0$ is an invariant manifold of (1.4) which is attracting for $\tau < 0$ and repelling for $\tau > 0$. We call this situation as exchange of stabilities (according to Lebovitz and Schaar [15]).
If we consider for equation (1.2) the initial value problem

$$x(\tau_0) = x_0, \quad \tau_0 < 0, \tag{1.5}$$

and if we assume that x_0 belongs to the region of attraction of the invariant manifold $x = 0$, then it follows from the standard theory of singularly perturbed systems (see, e.g.,[24]) that the solution $x(\tau, \varepsilon)$ of the initial value problem (1.2),(1.5) exists at least for $\tau_0 < \tau < 0$.
For $\tau > 0$ there are the following possibilities for the behavior of the solution $x(\tau, \varepsilon)$:

(i). $x(\tau, \varepsilon)$ follows immediately the new stable branch emerging at $\tau = 0$.

(ii). $x(\tau, \varepsilon)$ follows for some O(1)-time interval (not depending on ε) the repelling part of the invariant manifold $x = 0$ and then jumps to the stable branch.

(iii). $x(\tau, \varepsilon)$ follows for some O(1)-time interval the repelling part of the invariant manifold $x = 0$ and then jumps away from this manifold (possibly blowing up).

The case (ii) is called delayed exchange of stabilities, case (iii) is called delayed loss of stability. The corresponding solutions are said to be canard solutions.
The case of exchange of stabilities for singularly perturbed ordinary differential equations has been treated by several authors using different methods (see, e.g., [11-22, 25-27, 29-31]). In the papers [17, 18], the authors have applied the method of lower and upper solutions to derive conditions for an immediate and for a delayed exchange of stabilities, respectively.
The same technique has been used in the papers [2, 5-10] to derive conditions for an immediate exchange of stabilities for different classes of partial differential equations.

In [16] the authors have shown that the same method can be used to establish the phenomenon of delayed exchange of stabilities for a class of singularly perturbed parabolic problems. In what follows we will compare our analytic estimate of the transition region (interior layer) with numerical investigations of some examples. As a conclusion of this study we get that the space-independent lower and upper solutions should be replaced by space-dependent ones. The paper is organized as follows. Section 1 contains the formulation of the problem, in section 2 we collect our assumptions, section 3 present our analytic result. Section 4 is devoted to the numeriacal study of the phenomenon of delayed exchange of stabilities, the final section 4 contains the conclusions.

2 Formulation of the Problem

We consider the scalar singularly perturbed parabolic differential equation

$$\begin{aligned} &\varepsilon\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\right)=g(u,x,t,\varepsilon),\\ &(x,t)\in Q=\{(x,t):0<x<1,\ 0<t\leq T\}, \end{aligned} \tag{2.1}$$

where $\varepsilon>0$ is a small parameter, and study the initial-boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t,\varepsilon) &= \frac{\partial u}{\partial x}(1,t,\varepsilon)=0 \quad \text{for} \quad t\in(0,T],\\ u(x,0,\varepsilon) &= u^0(x) \quad \text{for} \quad x\in[0,1]. \end{aligned} \tag{2.2}$$

A root $u=\varphi(x,t)$ of the *degenerate equation*

$$g(u,x,t,0)=0 \tag{2.3}$$

represents a family of equilibria of the *associated equation* to (2.1)

$$\frac{du}{d\tau}=g(u,x,t,0), \tag{2.4}$$

where x and t have to be considered as parameters.

We recall that a root $u=\varphi(x,t)$ is referred to as *stable (unstable) in a region* G if $g_u(\varphi(x,t),x,t,0)<0\ (>0)\quad \forall(x,t)\in G$.

As in [9], we consider the case that the degenerate equation (2.3) has exactly two roots $u=\varphi_1(x,t)$ and $u=\varphi_2(x,t)$ intersecting in a curve such that an exchange of stabilities arises. In difference to [9], we treat in this chapter the phenomenon of delayed exchange of stabilities, that is, we derive conditions such that the solution $u(x,t,\varepsilon)$ of (2.1), (2.2) stays in the unstable region of $\varphi_1(x,t)$ arising for $t=t_c(x)$ for some O(1)-time interval near the unstable root $\varphi_1(x,t)$ and then either jumps to the stable root $\varphi_2(x,t)$ (delayed exchange of stability) or escapes from the unstable root (delayed loss of stability).

3 Assumptions

Let I_u be an open bounded interval containing the origin, let $I_{\varepsilon_0} = \{\varepsilon : 0 < \varepsilon < \varepsilon_0 \ll 1\}$, $D = Q \times I_u \times I_{\varepsilon_0}$. Let the functions g and u^0 satisfies the smoothness condition

(A$_0$). $g \in C^2(\overline{D}, R)$, $u^0 \in C^2([0,1], I_u)$.

With respect to the roots of the degenerate equation we suppose

(A$_1$). *The degenerate equation* (2.3) *has in* $\overline{I_u} \times \overline{Q}$ *exactly two roots:* $u \equiv 0$ *and* $u = \varphi(x,t)$, $\varphi(x,t) \in C^2(\overline{Q}, I_u)$. *The roots* $u \equiv 0$ *and* $u = \varphi(x,t)$ *intersect in some smooth curve* $\mathcal{K}$ *with the representation* $t = t_c(x) \in C^1([0,1],(0,T))$. *For definiteness we suppose*

$$\varphi(x,t) < 0 \quad \textit{for} \quad 0 \le t < t_c(x),\ 0 \le x \le 1,$$

$$\varphi(x,t) > 0 \quad \textit{for} \quad t_c(x) < t \le T,\ 0 \le x \le 1$$

(see Fig. 3.1).

From assumption (A$_1$) it follows

$$\varphi(x, t_c(x)) \equiv 0 \quad \text{for} \quad 0 \le x \le 1.$$

Concerning the stability of these roots we assume

(A$_2$).

$$\begin{array}{lll} g_u(0,x,t,0) < 0,\ g_u(\varphi(x,t),x,t,0) > 0 & \textit{for} & 0 \le t < t_c(x),\ 0 \le x \le 1, \\ g_u(0,x,t,0) > 0,\ g_u(\varphi(x,t),x,t,0) < 0 & \textit{for} & t_c(x) < t \le T,\ 0 \le x \le 1. \end{array}$$

Hypothesis (A$_2$) implies that the roots $u \equiv 0$ and $u = \varphi(x,t)$ of the degenerate equation (2.3) considered as families of equilibria of the associated equation (2.4) exchange their stabilities at the curve $\mathcal{K}$.

Furthermore, we suppose

(A$_3$). $g(0,x,t,\varepsilon) \equiv 0$ for $(x,t,\varepsilon) \in \overline{Q} \times \overline{I}_0$.

Assumption (A$_3$) is motivated by applications in reaction kinetics where we are looking for nonnegative solutions.

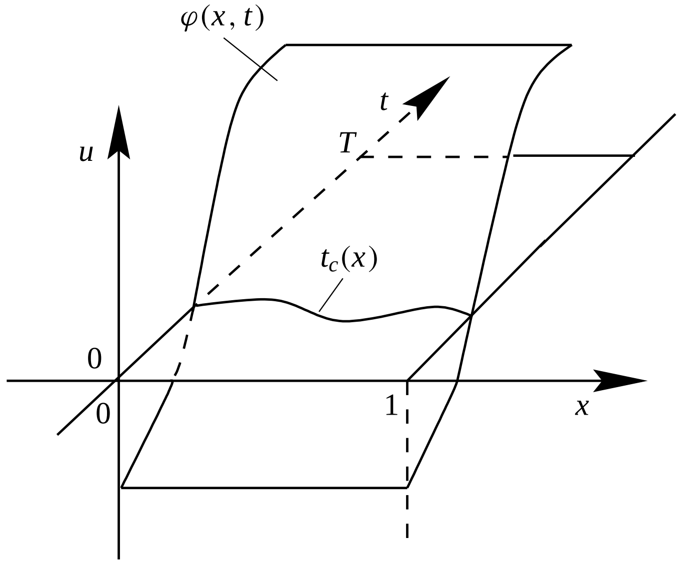

Fig. 3.1. Intersection of $u \equiv 0$ and $u = \varphi(x,t)$ in the curve $t = t_c(x)$.

Now we introduce the functions

$$g_u^{\min}(t) = \min_{x\in[0,1]} g_u(0,x,t,0), \quad g_u^{\max}(t) = \max_{x\in[0,1]} g_u(0,x,t,0) \quad \text{for} \quad 0 \le t \le T.$$

Obviously, we have for $(x,t) \in \overline{Q}$

$$g_u^{\min}(t) \le g_u(0,x,t,0) \le g_u^{\max}(t). \tag{3.5}$$

We need also the primitives of these functions:

$$G^{\min}(t) = \int_0^t g_u^{\min}(s)ds,\ G(x,t) = \int_0^t g_u(0,x,s,0)ds,\ G^{\max}(t) = \int_0^t g_u^{\max}(s)ds.$$

By (3.5) the following inequalities hold for $(x,t) \in \overline{Q}$ (see Fig. 3.2)

$$G^{\min}(t) \ \le\ G(x,t) \ \le\ G^{\max}(t).$$

>From assumption (A_2) we get that the equation $G^{\min}(t) = 0$ has at most one solution in the interval $(0,T)$. We assume that this solution exists.

(A_4). *The equation $G^{\min}(t) = 0$ has a solution $t = t_{max}$ in $(0,T)$.*

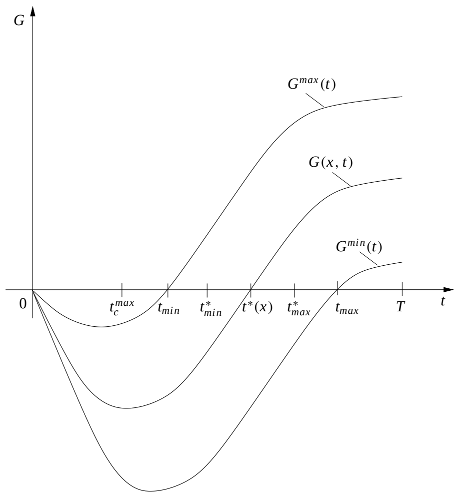

Fig. 3.2. Inclusion of $G(x,t)$ by G^{min} and G^{max} for given x

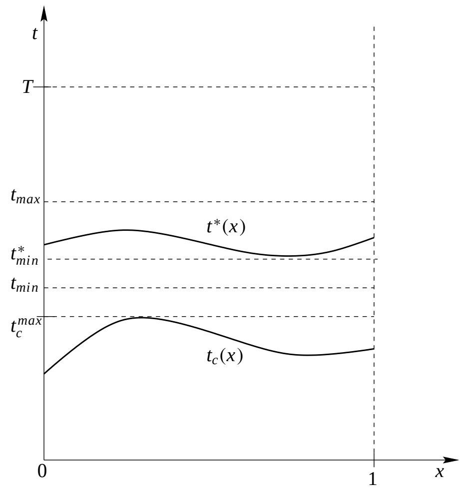

Fig. 3.3. Location of $t_c(x)$ and $t^*(x)$

From hypotheses (A_2) and (A_4) it follows that the equation $G^{\max}(t) = 0$ has a unique solution $t = t_{\min}$ in $(0, T)$, and that for each $x \in [0,1]$ the equation $G(x,t) = 0$ has a unique solution $t = t^*(x)$ in $(0,T)$ (see Fig. 3.2).

Obviously, for $x \in [0,1]$ we have

$$t_{\min} \le t^*(x) \le t_{\max}.$$

Finally we assume that the following conditions hold.

(A_5).

$$t_c^{max} = \max_{x\in[0,1]} t_c(x) < t_{min} \quad \text{(see Fig. 3.3)}.$$

(A_6). *There is a positive number* c_0 *such that* $(-c_0, c_0) \subset I_u$ *where* I_u *is the interval from assumption (A_0), and*

$$g(u,x,t,\varepsilon) \le g_u(0,x,t,\varepsilon)u \quad \textit{for } |u| \le c_0,\ x \in [0,1],\ 0 \le t \le t^*(x),\ \varepsilon \in I_{\varepsilon_0}.$$

We note that assumption (A_6) is satisfied if the second derivative $g_{uu}(0,x,t,\varepsilon)$ is negative for all (x,t,ε) under consideration.

(A_7). $u^0(x)$ *lies in the basin of attraction of the stable root* $u \equiv 0$.

4 Main results

Our main result is concerned with the estimate of the delay time in cases of delayed exchange or delayed loss of stabilities.

Theorem 4.1 *Assume the hypotheses* (A_1)–(A_7) *to be valid and* $u^0(x) > 0$. *Then, for sufficiently small* ε, *there exists a unique solution* $u(x,t,\varepsilon)$ *of* (2.1), (2.2) *which is positive and satisfies*

$$\lim_{\varepsilon\to 0} u(x,t,\varepsilon) = 0 \quad \textit{for} \quad (x,t) \in [0,1]\times(0,t_{min}), \tag{4.6}$$

$$\lim_{\varepsilon\to 0} u(x,t,\varepsilon) = \varphi(x,t) \quad \textit{for} \quad (x,t) \in [0,1]\times(t_{\max},T]. \tag{4.7}$$

In case $u^0(x) < 0$, *the unique solution* $u(x,t,\varepsilon)$ *of* (2.1), (2.2) *is negative and and satisfies*

$$\lim_{\varepsilon\to 0} u(x,t,\varepsilon) = 0 \quad \textit{for} \quad (x,t) \in [0,1]\times(0,t_{\min}),$$

for $t > t_{\min}$ *the solution escapes from* $u \equiv 0$ *at some time* t_{esc} *(escaping time) which can be estimated by* $t_{esc} \leq t_{max}$.

Remark 4.2 *From Theorem* 4.1 *it follows that the solution* $u(x,t,\varepsilon)$ *stays near the unstable root* $u = 0$ *of the degenerate equation at least for the time interval* $(t_c(x), t_{min})$,

Remark 4.3 *In case* $u^0(x) < 0$, *the solution* $u(x,t,\varepsilon)$ *may not exist for all* t *in* $[0,T]$.

The proof is based on the technique of asymptotic lower and upper solutions. Details of the proof can be found in [16].

5 Numerical investigations

The goal of this section is to illustrate Theorem 4.1 by studying an example numerically. For this purpose, we consider the following initial-boundary value problem

$$\begin{aligned} \varepsilon\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial u^2}\right) &= -u\left[u - \left(t - \frac{x}{4} - 1\right)\right], \quad 0 < x < 1, \quad 0 < t \leq 3. \\ \frac{\partial u}{\partial x}(0,t,\varepsilon) &= \frac{\partial u}{\partial x}(1,t,\varepsilon) = 0 \quad \text{for} \quad t \in (0,T]. \\ u(x,0,\varepsilon) &= u^0(x) \quad \text{for} \quad x \in [0,1], \end{aligned} \tag{5.1}$$

The degenerate equation has two roots

$$u = 0 \text{ and } u = \varphi(x,t) := t - \frac{x}{4} - 1$$

which intersect in the curve $t = t_c(x) := \frac{x}{4} + 1$.

The root $u = 0$ ($u = t - \frac{x}{4} - 1$) is stable (unstable) for $0 \leq t < t_c(x)$, $0 \leq x \leq 1$ and unstable (stable) for $t_c(x) < t \leq 3$, $0 \leq x \leq 1$. It can be easily seen that the

assumptions (A_0) - (A_3) of Theorem 4.1 are satisfied.
Furthermore, we have

$$\begin{aligned} g_u^{\min}(t) &:= t - \frac{5}{4}, \quad g_u^{\max}(t) := t - 1, \\ G^{\min}(t) &:= \frac{t}{2}\Big(t - \frac{5}{2}\Big), \quad G^{\max}(t) := t\Big(\frac{t}{2} - 1\Big), \\ G(x,t) &:= t\Big(\frac{t}{2} - 1 - \frac{x}{4}\Big), \end{aligned}$$

and therefore

$$t_{\min} = 2,\ t_{\max} = \frac{5}{2},\ t^*(x) := \frac{x}{2} + 2. \tag{5.2}$$

It follows that the conditions (A_4) and (A_5) of Theorem 4.1 are satisfied. One can also check that assumptions (A_6) and (A_7) are valid. Hence, according to Theorem 4.1, problem (5.1) has a unique solution with a transition layer inside the time interval $[t_{\min}, t_{max}]$.

In order to get more information about the location of the transition layer we investigate the initial-boundary value problem (5.1) numerically. For this purpose, we apply a two layer finite difference scheme of Crank-Nicolson type with a uniform mesh in space (step $h = 1/N$). Taking into account an iterative process to treat the nonlinearity in (5.1), our scheme for the transition from the time layer $j\tau$ to the next time layer $(j+1)\tau$ can be written as follows:

$$\begin{aligned} &\varepsilon\bigg(\frac{\hat{y}_i^{(n)} - y_i}{\tau} - \Delta_{\bar{x}x}\Big(\frac{\hat{y}_i^{(n)} + y_i}{2}\Big)\bigg) = f\bigg(\frac{\hat{y}_i^{(n-1)} + y_i}{2}, ih, (j+0.5)\tau\bigg), \quad i = 0, \ldots, N; \\ &\hat{y}_{-1}^{(n)} = \hat{y}_1^{(n)}, \quad \hat{y}_{N+1}^{(n)} = \hat{y}_{N-1}^{(n)}, \quad n = 1, \ldots; \qquad \hat{y}_i^{(0)} = y_i, \quad \hat{y}_i = \lim_{n\to\infty} \hat{y}_i^{(n)}. \end{aligned} \tag{5.3}$$

Here, we have used the following notation for the grid functions and the operators:

$$y_i \approx u(ih, j\tau), \quad \hat{y}_i \approx u(ih, (j+1)\tau), \qquad \Delta_{\bar{x}x} v_i \stackrel{def}{=} \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}.$$

It is known that for sufficiently smooth initial data and if the solution u and its derivatives remain bounded such a scheme is of a second order precision in time and space, it is unconditionally stable and converges to a solution of problem (5.1) as $\tau, h \to 0$.

5.1 Delayed exchange of stabilities

We consider the initial-boundary value problem (5.1). According to Theorem 4.1 and relation (5.2), problem (5.1) has a unique solution with a transition layer between

$t_{min} = 2.0$ and $t_{max} = 2.5$, where for sufficiently small ε, $u(t, x, \varepsilon)$ stays near $u \equiv 0$ for $0 < t < t_{min}$, and near $\varphi(t, x)$ for $t > t_{max}$. Fig. 5.1 and Fig. 5.2 show the solution of problem (5.1) with $u^0 \equiv 0.5$ for different values of ε.

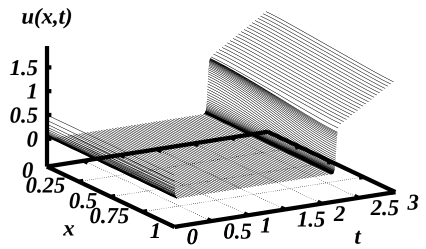

Fig. 5.1. Solution $u(x, t, \varepsilon)$ of (5.1) with $u^0 \equiv 0.5$ for $\varepsilon = 0.01$.

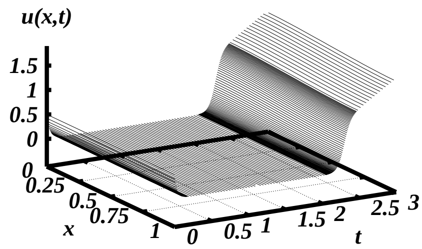

Fig. 5.2. Solution $u(x, t, \varepsilon)$ of (5.1) with $u^0 \equiv 0.5$ for $\varepsilon = 0.05$.

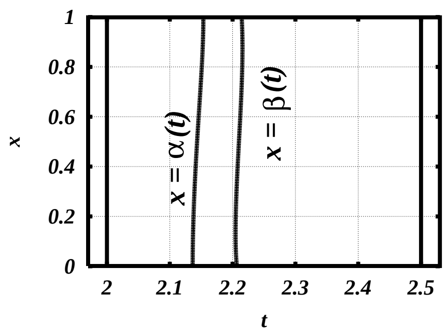

Fig. 5.3. Transition layer for $\varepsilon = 0.01$.

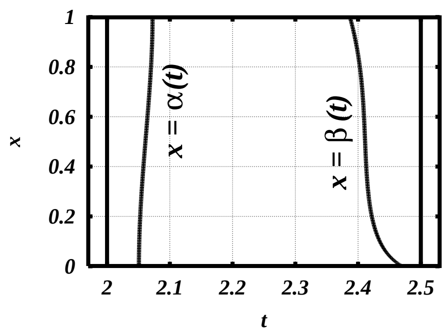

Fig. 5.4. Transition layer for $\varepsilon = 0.05$.

Fig. 5.3 and Fig. 5.4 characterize the time-space region where $u(t, x, \varepsilon)$ exhibits a fast transition from $u \equiv 0$ to some neighborhood of $u = \varphi(t, x)$. The numerical results indicate that the phenomenon of delayed exchange of stabilities arises also for moderate values of ε, where a broadening of the transition layer can be observed for increasing ε. Fig. 5.3 shows that the theoretically derived bounds $t = 2$ and $t = 2.5$ for the transition layer in case of small ε are quite satisfactory. The transition layer is located between the curves $x = \alpha(t)$ and $x = \beta(t)$, where the curve $x = \alpha(t)$ is defined by the condition $u(t, x, \varepsilon) = 0.01$, while $x = \beta(t)$ characterizes the points (t, x) where we have $|u(t, x, \varepsilon) - \varphi(t, x)| = 0.1$.

5.2 Delayed loss of stability

We consider the initial-boundary value problem (5.1) with $u^0(x)$ negative. According to Theorem 4.1, problem (5.1) has a unique solution with a transition layer between $t_{min} = 2.0$ and $t_{max} = 2.5$, where for sufficiently small ε, $u(t, x, \varepsilon)$ stays near $u \equiv 0$

for $0 < t < t_{min}$, for $t_{min} \leq t_{esc} < t < t_{max}$ $u(t,x,\varepsilon)$ exhibits a blowing up. Fig. 5.5 and Fig. 5.6 show the solution of problem (5.1) for $u^0 \equiv -0.5$ and for different values of ε.

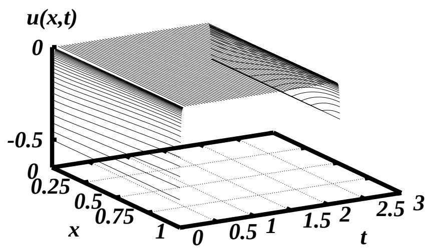

Fig. 5.5. Solution $u(x,t,\varepsilon)$ of (5.1) with $u^0 \equiv -0.5$ for $\varepsilon = 0.01$.

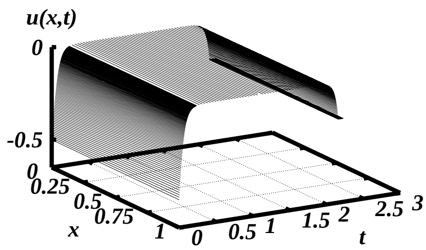

Fig. 5.6. Solution $u(x,t,\varepsilon)$ of (5.1) with $u^0 \equiv -0.5$ for $\varepsilon = 0.05$.

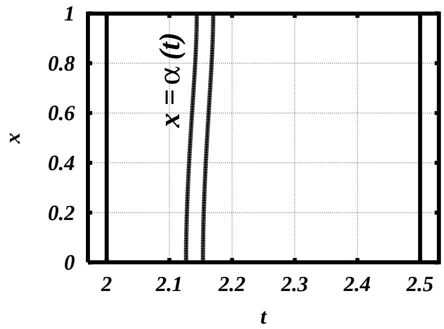

Fig. 5.7. Inclusion of the transition layer of (5.1) with $u^0 \equiv -0.5$ for $\varepsilon = 0.01$.

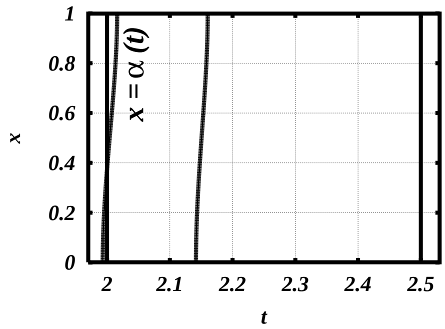

Fig. 5.8. Inclusion of the transition layer of (5.1) with $u^0 \equiv -0.5$ for $\varepsilon = 0.05$.

Fig. 5.7 and Fig. 5.8 characterize the time-space region where $u(t,x,\varepsilon)$ exhibits a fast escaping from $u \equiv 0$. Here again curves $x = \alpha(t)$ show the begining of the transition layer and are defined by the condition $u(t,x,\varepsilon) = -0.01$, while the other curve shows the points (t,x) where blowing up solution satisfy condition $u(t,x,\varepsilon) = -0.2$.

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