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An existence result for the Leray–Lions type operators with discontinuous coefficients

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Abstract

In this paper we prove an existence result for Leray-Lions quasilinear elliptic operator with discontinuous coefficients. The idea of the proof is based on compactness results for sequences of solutions to regularized problems obtained via the Compensated Compactness, Young measures, and Set–Valued Analysis tools.

1 Introduction

Recall the Leray-Lions operator in the form

$$\operatorname{div} A(x,u,\nabla u) + B(x,u,\nabla u)$$

where functions $A\colon \Omega\times\mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m$ and $B\colon \Omega\times\mathbb{R}\times\mathbb{R}^m \to \mathbb{R}$ satisfy the classical assumptions:

- the growth conditions:

$$\begin{aligned} |A(x,s,\xi)| &\le C_1(k(x)+|s|^{p-1}+|\xi|^{p-1}),\\ |B(x,s,\xi)| &\le C_2(k(x)+|s|^{p-1}+|\xi|^{p-1}), \end{aligned}$$

- the monotonicity condition:

$$(A(x,s,\xi_1)-A(x,s,\xi_2))\cdot(\xi_1-\xi_2) > 0$$

  for all $\xi_1 \neq \xi_2$,

- the coercivity conditions:

$$\begin{aligned} A(x,s,\xi)\cdot\xi &\ge c_1|\xi|^p,\\ B(x,s,\xi)\cdot s &\ge c_2|s|^p. \end{aligned}$$

for almost all x in an open subset $\Omega \subset \mathbb{R}^m$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^m$. The constants C_1, C_2, c_1, c_2 are strictly positive, and $k \in L^{p'}(\Omega)$. The other classical assumption is that A and B are Carathéodory functions (i.e. measurable with respect to the first variable and continuous with respect to the second and third). Under these hypotheses the operator is a bounded continuous pseudomonotone operator of Leray-Lions type from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$. (see [3], [5], [12], [10], [13]).

In this paper we consider the equation

$$\mathrm{div}A(x, \nabla u) + B(x, u, \nabla u) = f$$

and we relax the assumption on the continuity of A in the second and B in the second and third variable. Instead, we assume that the functions $A\colon \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ and $B\colon \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}$ are Lebesgue measurable and such that:

1. there exist finitely many open sets U_l such that
   i) $\mathbb{R}^m = \bigcup_{l=1}^k \overline{U_l}$ and $U_i \cap U_l = \emptyset$ for $i \neq l$,
   ii) $A_{|U_l}(x, \cdot)$ is continuous for $l = 1 \dots k$,
   iii) there exists continuous extention $A_l(x, \cdot)$ of $A_{|U_l}(x, \cdot)$ onto $\overline{U_l}$.
2. $\mathrm{div}A(x, \nabla u) = f - B(x, u, \nabla u) \in W^{-1,p'}(\Omega)$,
3. $B(x, \cdot, \cdot)$ is Borel measurable and convolution B with a standard Friedrich's mollifier φ^ε gives a smooth function.
4. $|A(x, \xi)| \leq C_1(k(x) + |\xi|^{p-1})$,
5. $(A(x, \xi_1) - A(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0$ for all $\xi_1 \neq \xi_2$,
6. $A(x, \xi) \cdot \xi \geq c_1 |\xi|^p$.
7. $|B(x, s, \xi)| \leq C_2(k(x) + |s|^{p-1} + |\xi|^{p-1})$,

The assumption $B(x, s, \xi) \cdot s \geq c_2 |s|^p$ is not necessary if we assume that Ω is a bounded set and one can use a Poincaré inequality. The above assumptions are valid throughout the paper and we will not write them in formulations of theorems.

Given functions A and B, we extend their graphs (i.e. we close the graphs and make a set of values in each point convex; this procedure give an unique extension) to obtain multifunctions $\mathcal{A}$ and $\mathcal{B}$. Our main result is the following

Theorem 1.1. *Assume Ω to be an open set with $|\Omega| < +\infty$, and $f \in W^{-1,p'}(\Omega)$. Then there exist a function $u \in W_0^{1,p}(\Omega)$ and measurable selections $\widetilde{A}(x) \in \mathcal{A}(x, \nabla u(x))$, $\widetilde{B}(x) \in \mathcal{B}(x, u, \nabla u(x))$ such that*

$$\int_\Omega \widetilde{A}(x) \nabla \varphi(x) \mathrm{d}x + \int_\Omega \widetilde{B}(x) \varphi(x) \mathrm{d}x = \int_\Omega f(x) \varphi(x) \mathrm{d}x$$

for all $\varphi \in C_0^\infty$.

Malý and Ziemer [7] consider a regularity theory for quasilinear elliptic operators with discontinuous coefficients. They assume that $A(\cdot, s, \xi)$, $B(\cdot, s, \xi)$ are Lebesgue measurable and $A(x, \cdot, \cdot)$, $B(x, \cdot, \cdot)$ are Borel measurable. However, to solve the existence problem, they need to assume that A and B are Carathéodory functions (see p. 162, and p. 253). It is natural to ask if the assumption can be relaxed. It was one of motivations to this work.

The other motivation comes from physical applications: let consider the free energy function of the form

$$E(x, u, \nabla u)$$

with standard polynomial growth and coercivity condition, strictly convex in the last variable but which is only differentiable in the sens of Clarke with respect to the second and third variable. Then Euler-Lagrange equations are of Leray-Lions type but with discontinuous coefficients. Similar motivation one can find in the hemivariational inequalities (see [11], [9]).

The main idea of the proof is based on the application of Young measures in a nonstandard setting, forced by the discontinuity of A, combined with the “biting” div–curl lemma from Zhang [14]. We really need the “biting” version, as the term $A(x, \nabla u) \cdot \nabla u$ is only L^1 under our growth conditions.

Steps of the proof are as follows:

1. We regularize functions A and B by convolution with the function φ^ε and put the term B^ε on the left side of the equation.
2. Using Biting div-curl lemma, and characterization by Young measures we prove a compactness result for a sequence of regularized solutions in $W_0^{1,q}(\Omega)$ for all $q < p$.
3. At the end we use Convegence Theorem from the theory of the Set-Valued Analysis to obtain an existence of weak solutions.

Remark 1. Using similar methods as in [3] and [5] one can show the strong convergence of gradients in the space $L^p(\Omega)$.

Remark 2. In [6] there are considered generalizations of the p–Laplacian in the form

$$\operatorname{div}\{(a|\nabla u|^{p-2} + b|\nabla u|^{q-2})\nabla u\}$$

where $a, b > 0$ and $p, q \in (1, +\infty)$. Our arguments, combined with some more general characterization of Young measures (see [4]) allow us to consider the case $p \in [1, +\infty)$, $q \in (1, +\infty)$.

2 Tools

In the following theorems the notation is slightly different from that of [14], [1] and [8]. We use it to facilitate the reading ot the proof of Theorem 1.1.

Theorem 2.1 (Biting div–curl lemma). *Let Ω be an open set of $\mathbb{R}^m$ and (z^j), (A^j) be sequences in $L^p(\Omega, \mathbb{R}^m)$ and $L^{p'}(\Omega, \mathbb{R}^m)$ respectively, such that*

$$\begin{aligned} z^j &\rightharpoonup z \quad \text{in } L^p(\Omega, \mathbb{R}^m) \\ A^j &\rightharpoonup \widetilde{A} \quad \text{in } L^{p'}(\Omega, \mathbb{R}^m) \\ \text{curl} z^j &\quad \text{is bounded in } L^p(\Omega, \mathbb{R}^m) \\ \text{div} A^j &\quad \text{is compact in } W^{-1,p'}(\Omega)) \end{aligned}$$

Then there exist subsequences, still denoted (z^j), (A^j) and a nonincreasing sequence of measurable sets (E_k), $E_k \subset \Omega$ with $|E_k| \to 0$, such that

$$z^j \cdot A^j \rightharpoonup z \cdot \widetilde{A} \quad \text{in } L^1(\Omega \backslash E_k) \quad \text{for } k = 1, 2, \ldots.$$

The next theorem is a slight modification of the Fundamental Theorem on Young measures (see e.g. [8] Th.3.1).

Theorem 2.2. *Assume $\Omega \subset \mathbb{R}^m$ to be an open set of a finite measure and let $z^j : \Omega \longrightarrow \mathbb{R}^m$ be a sequence of measurable functions. Denote $\nu_x^{j,l} = (\delta_{z^j(x)} * \varphi^j)_{|\overline{U_l}}$. Then there exists a subsequence still denoted by z^j and a family of weak-* measurable maps $\nu^l : \Omega \rightharpoonup \mathcal{M}(\overline{U_l})$ such that*

1. *$\nu_x = \sum_l \nu_x^l \geq 0$, and*

   $$\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = \sum_l \int_{\overline{U_l}} d\nu_x^l \leq 1$$

   for a.e. $x \in \Omega$;

2. *$\nu^{j,l} \overset{*}{\rightharpoonup} \nu^l$ in $L_w^\infty(\Omega, \mathcal{M}(\overline{U_l}))$;*

3. *If for some $p > 0$ there exist constant C such that for all j: $\int_\Omega |z^j|^p \leq C$, then $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} = 1$;*

4. *If 3. holds and a set $E \subset \Omega$ is measurable, $F \in L^1(\Omega, C^0(\overline{U_l}))$, and for every l the family $\{\int_{\overline{U_l}} F(\xi) d\nu_x^{j,l}(\xi)\}$ is relatively weakly compact in $L^1(E)$, then*

   $$\int_{\overline{U_l}} F(\xi) d\nu_x^{j,l}(\xi) \rightharpoonup \int_{\overline{U_l}} F(\xi) d\nu_x^l(\xi) \quad \text{in } L^1(E);$$

5. *If 4. holds and $F \in L^1(\Omega, C^0(\mathbb{R}^m)$ then*

   $$\int_{R^m} F(\xi) d\delta_{z_j(x)}(\xi) \rightharpoonup \int_{\mathbb{R}^m} F(\xi) d\nu_x(\xi) \quad \text{in } L^1(E),$$

   i.e. ν_x is a usual Young measure generated by a sequence $z^j(x)$;

6. If $\nu_x = \delta_{\overline{z}(x)}$ a.e. in Ω then $z^j(x) \to \overline{z}(x)$ a.e. in Ω.

The idea of the proof is based on the fact that $L^\infty_w(\Omega; \mathcal{M}(\overline{U_l}))$ is the dual space to $L^1(\Omega; C_0(\overline{U_l}))$ and the proof is similar to that in [8], pp. 31–34. Passing from the functions z^j which take values in $\mathbb{R}^m$ to maps which take values in the space of measures in $\mathbb{R}^m$, we take instead of $\delta_{z^j(x)}$ measures of the form $\delta_{z^j(x)} * \varphi^j$, and consider them on the sets U_l where the function A is continuous. The assertion 5. comes from the fact: $\{\sum_l \nu^{j,l} - \delta_{z^j}\} \overset{*}{\rightharpoonup} 0$ in the space $L^\infty_w(\Omega; \mathcal{M}(\overline{U_l}))$.

The following theorem is the special case of Convergence Theorem which is well known in Set–Valued Analysis, (see e.g. [1]).

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^m$ be an open set with finite Lebesgue measure, $\mathcal{A} : \mathbb{R}^m \longrightarrow 2^{\mathbb{R}^m}$ be a nontrivial set–valued map, with a closed graph, and such that for any $\xi \in Dom(\mathcal{A}) = \mathbb{R}^m$, a set $\mathcal{A}(\xi)$ is bounded and convex. Let us consider measurable functions $z^j : \Omega \longrightarrow \mathbb{R}^m$ and $A^j : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ single-valued function such that*

i) $\sup_{y\in\mathbb{R}^m} \mathrm{dist}[(y, A^j(y)); \mathrm{Graph}(\mathcal{A})] \longrightarrow 0$, *as $j \to 0$;*

ii) the sequence $\{z^j\}$ converges almost everywhere to a function z;

iii) $A^j(z^j) \in L^1(\Omega)$ and the sequence $\{A^j(z^j)\}$ is convergent weakly in $L^1(\Omega)$ to a function $\widetilde{A} \in L^1(\Omega)$.

Then for almost every $x \in \Omega$ we have $\widetilde{A}(x) \in \mathcal{A}(z(x))$.

The following fact is well known in set–valued analysis (see [1]):

Lemma 2.4. *Let $\mathcal{A}(x, \cdot)$ is a maximal monotone extention of the function $A(x, \cdot)$ satisfying the monotonicity condition 5. If $\widetilde{A}_1 \in \mathcal{A}(x, \xi_1)$ and $\widetilde{A}_2 \in \mathcal{A}(x, \xi_2)$ then*

$$(\widetilde{A}_1 - \widetilde{A}_2) \cdot (\xi_1 - \xi_2) > 0$$

for all $\xi_1 \neq \xi_2$.

Notation 1. Throughout the paper by φ^ε we will denote the standard Friedrich's mollifier with respect to the variable ξ i.e. for a fixed radial (i.e. depending only of $|\xi|$), nonnegative function $\varphi \in C_0^\infty(\mathbb{B}^m)$ with $\int \varphi(y)\mathrm{d}y = 1$, we have $\varphi^\varepsilon(\xi) := \varepsilon^{-m}\varphi(\xi/\varepsilon)$. The symbol $\widetilde{\varphi}^\varepsilon$ will stand for the molifire with respect to the variables s and ξ.

3 Proof of the main result

Proof. Let $A^\varepsilon(x, \xi) = (\varphi^\varepsilon * \mathcal{A})(x, \xi)$ and $B^\varepsilon(x, s, \xi) = (\widetilde{\varphi}^\varepsilon * \mathcal{B})(x, s, \xi)$. These are well defined, continuous regularizations of the functions A and B. The growth, coercivity and monotonicity conditions are preserved under the regularization. Thus, there exists a weak solution u^ε to the problem

$$\operatorname{div} A^\varepsilon(x, \nabla u^\varepsilon) + B^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon) = f,$$

that is a function $u^\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega A^\varepsilon(x, \nabla u^\varepsilon(x))\nabla\varphi(x)\mathrm{d}x + \int_\Omega B^\varepsilon(x, u^\varepsilon(x), \nabla u^\varepsilon(x))\varphi(x)\mathrm{d}x = \\ = \int_\Omega f(x)\varphi(x)\mathrm{d}x$$

for all $\varphi \in C_0^\infty$.

The energy estimates and conditions on A and B yield $\|u^\varepsilon\|_{W_0^{1,p}} \leq C$, where the constant C does not depend on ε. Thus ∇u^ε and $A^\varepsilon(\cdot, \nabla u^\varepsilon)$ are bounded sequences in $L^p(\Omega, \mathbb{R}^m)$ and $L^{p'}(\Omega, \mathbb{R}^m)$ respectively, and up to subsequences we may assume

$$\begin{aligned} u^\varepsilon &\longrightarrow u \quad \text{in } L^p(\Omega, \mathbb{R}^m) \\ u^\varepsilon &\longrightarrow u \quad \text{almost everywhere} \\ \nabla u^\varepsilon &\rightharpoonup \nabla u \quad \text{in } L^p(\Omega, \mathbb{R}^m) \end{aligned}$$

and

$$\begin{aligned} A^\varepsilon(\cdot, \nabla u^\varepsilon) &\rightharpoonup \widetilde{A} \quad \text{in } L^{p'}(\Omega, \mathbb{R}^m) \\ B^\varepsilon(\cdot, u^\varepsilon, \nabla u^\varepsilon) &\rightharpoonup \widetilde{B} \quad \text{in } L^{p'}(\Omega, \mathbb{R}^m). \end{aligned}$$

In order to show that $\widetilde{A}$ is a measurable selection from $\mathcal{A}$ and u is the desired weak solution we use theorem 2.3 and thus we need to show that $\nabla u^\varepsilon \longrightarrow \nabla u$ almost everywhere.

For almost every $x \in \Omega$ set $\mu_x^\varepsilon = \delta_{\{\nabla u^\varepsilon(x)\}}$. Then of course

$$A^\varepsilon(x, \nabla u^\varepsilon(x)) \cdot \nabla u^\varepsilon(x) = \int_{\mathbb{R}^m} A^\varepsilon(x, \xi) \cdot \xi d\mu_x^\varepsilon(\xi).$$

Since $A^\varepsilon(x, \xi) = (\mathcal{A} * \varphi^\varepsilon)(x, \xi)$, one can express the last integral as

$$\begin{aligned} &\int_{\mathbb{R}^m} (\mathcal{A} * \varphi^\varepsilon)(x, \xi) \cdot \xi d\mu_x^\varepsilon(\xi) = \\ &= \int_{\mathbb{R}^m} \Big((\mathcal{A} * \varphi^\varepsilon)(x, \xi) \cdot \xi - ((\mathcal{A} \cdot \mathrm{Id}_{(\xi)}) * \varphi^\varepsilon)(x, \xi)\Big) d\mu_x^\varepsilon(\xi) \\ &+ \int_{\mathbb{R}^m} ((\mathcal{A} \cdot \mathrm{Id}_{(\xi)}) * \varphi^\varepsilon)(x, \xi) d\mu_x^\varepsilon(\xi). \end{aligned} \tag{3.1}$$

The first term of RHS can be written as

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \mathcal{A}(x, \eta)(\xi - \eta)\varphi^\varepsilon(\xi - \eta)\mathrm{d}\eta d\mu_x^\varepsilon(\xi),$$

and then

$$|\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\mathcal{A}(x,\eta)(\xi-\eta)\varphi^\varepsilon(\xi-\eta)\mathrm{d}\eta d\mu_x^\varepsilon(\xi)|\leq$$

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}|\mathcal{A}(x,\eta)||\xi-\eta||\varphi^\varepsilon(\xi-\eta)|\mathrm{d}\eta d\mu_x^\varepsilon(\xi).$$

Using the fact that φ^ε has a support in a ball of a radius ε, and the growth conditions of A we can estimate the last expression by

$$C\varepsilon\int_{\mathbb{R}^m}(|\xi|+\varepsilon)^{p-1}d\mu_x^\varepsilon(\xi)$$

and a convergence to zero follows from the uniformly boudedness of the p–th moment of the measure μ_x^ε.

The second term of RHS of (3.1) can be represent as

$$\int_{\mathbb{R}^m}\mathcal{A}(x,\xi)\cdot\xi d\nu_x^\varepsilon(\xi),$$

where the measure $\nu_x^\varepsilon=\mu_x^\varepsilon*\varphi^\varepsilon$ is absolutely continuous with respect to the Lebesgue measure. Then

$$\int_{\mathbb{R}^m}A^\varepsilon(x,\xi)\cdot\xi d\mu_x^\varepsilon(\xi)=C\varepsilon+\int_{\mathbb{R}^m}\mathcal{A}(x,\xi)\cdot\xi d\nu_x^\varepsilon(\xi).$$

Let $\nu_x^{\varepsilon,l}$ be a restriction of the measure ν_x^ε to the set $\overline{U_l}$ and the function A_l be a continuous extension of $A_{|U_l}$ onto $\overline{U_l}$.

Theorem 2.1 implies

$$A^\varepsilon(x,\nabla u^\varepsilon(x))\cdot\nabla u^\varepsilon(x)\rightharpoonup\widetilde{A}(x)\cdot\nabla u(x)\quad\text{in }L^1(\Omega\backslash E_k)\tag{3.2}$$

and thus the sequence $\int_{\mathbb{R}^m}\mathcal{A}(x,\xi)\cdot\xi d\nu_x^\varepsilon(\xi)$ is relatively weakly compact in $L^1(\Omega\backslash E_k)$. Then Theorem 2.2 yields

$$A^\varepsilon(x,\nabla u^\varepsilon(x))\cdot\nabla u^\varepsilon(x)\rightharpoonup\sum_l\int_{\overline{U_l}}A_l(x,\xi)\cdot\xi d\nu_x^l(\xi)\tag{3.3}$$

in $L^1(\Omega\backslash E_k)$, where the Young measure ν^l is a weak–* limit of $\nu^{\varepsilon,l}$ in $L^\infty_w(\Omega,\mathcal{M}(\overline{U_l}))$ and $\nu_x=\sum_l\nu_x^l$ is a probability measure for almost every $x\in\Omega$.

Therefore combining (3.2) and (3.3) we obtain

$$\int_{\Omega\backslash E_k}\widetilde{A}(x)\cdot\nabla u(x)\,\psi(x)\mathrm{d}x=$$

$$=\sum_l\int_{\Omega\backslash E_k}\int_{\overline{U_l}}A_l(x,\xi)\cdot\xi d\nu_x^l(\xi)\,\psi(x)\mathrm{d}x.\tag{3.4}$$

for every $\psi \in L^\infty(\Omega \backslash E_k)$

Let now ψ be a nonnegative function from $L^\infty(\Omega \backslash E_k)$. It follows from the monotonicity condition on A that the term

$$\sum_l \int_{\Omega \backslash E_k} \Big\{ \int_{\overline{U_l}} [A_l(x,\xi) - A(x, \int_{\mathbb{R}^m} \xi d\nu_x(\xi))] \\ [\xi - \int_{\mathbb{R}^m} \xi d\nu_x(\xi)] d\nu_x^l(\xi) \Big\} \psi(x) \mathrm{d}x \quad (3.5)$$

is nonnegative. We will show that in fact it is equal to zero.

After expanding the expression under the inner integral, using the formula (3.4) and the fact ν_x is a probability measure, we obtain that (3.5) is equal to

$$\int_{\Omega \backslash E_k} \widetilde{A}(x) \cdot \nabla u(x)\, \psi(x) \mathrm{d}x \\ - \int_{\Omega \backslash E_k} \Big(\sum_l \int_{\overline{U_l}} A_l(x,\xi) d\nu_x^l(\xi) \Big) \cdot \int_{\mathbb{R}^m} \xi d\nu_x(\xi)\, \psi(x) \mathrm{d}x \\ - \int_{\Omega \backslash E_k} A(x, \int_{\mathbb{R}^m} \xi d\nu_x(\xi)) \int_{\mathbb{R}^m} [\xi - \int_{\mathbb{R}^m} \xi d\nu_x(\xi)] d\nu_x^l(\xi)\, \psi(x) \mathrm{d}x. \quad (3.6)$$

The last term is equal to zero because

$$\int_{\mathbb{R}^m} [\xi - \int_{\mathbb{R}^m} \xi d\nu_x(\xi)] d\nu_x^l(\xi) = 0.$$

We can represent functions $A^\varepsilon(\cdot, \nabla u^\varepsilon)$ and ∇u^ε also by the Dirac measures μ_x^ε and theorem 2.2 gives

$$\widetilde{A}(x) = \sum_l \int_{\overline{U_l}} A_l(x,\xi) d\nu_x^l(\xi)$$

and

$$\nabla u(x) = \int_{\mathbb{R}^m} \xi d\nu_x(\xi).$$

That however implies (3.6) is equal to zero.

Thus, for every l

$$\int_{\overline{U_l}} \Big[A_l(x,\xi) - A(x, \int_{\mathbb{R}^m} \xi d\nu_x(\xi)) \Big] \cdot \Big[\xi - \int_{\mathbb{R}^m} \xi d\nu_x(\xi) \Big] d\nu_x^l(\xi) = 0$$

and from Lemma 2.4 it follows that

$$\nu_x = \delta_{\{\nabla u(x)\}},$$

and therefore $\nabla u^\varepsilon \longrightarrow \nabla u$ almost everywhere.

Theorem 2.3 now gives $\widetilde{A}$ is a selection from $\mathcal{A}$, $\widetilde{B}$ is a selection from $\mathcal{B}$ and

$$\int_\Omega \widetilde{A}(x)\nabla\varphi(x)\mathrm{d}x + \int_\Omega \widetilde{B}(x)\varphi(x)\mathrm{d}x = \int_\Omega f(x)\varphi(x)\mathrm{d}x$$

for all $\varphi \in C_0^\infty$. Moreover, $\widetilde{A}$ and $\widetilde{B}$ are measurable as weak limits of measurable functions $A^\varepsilon(x, \nabla u^\varepsilon)$ and $B^\varepsilon(x, u^\varepsilon, \nabla u^\varepsilon)$.

□

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