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On immediate-delayed exchange of stabilities and periodic forced canards

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Abstract

We study scalar singularly perturbed non-autonomous ordinary differential equations whose associated equations feature the property of exchange of stabilities, i.e., the set of their equilibria consists of at least two intersecting curves. By means of the method of asymptotic lower and upper solutions we derive conditions guaranteeing that the solution of initial value problems exhibit the phenomenon of immediate exchange of stabilities as well as the phenomenon of delayed exchange of stabilities. We use this result to prove the existence of forced canard solutions.

1 Introduction

Consider the dynamical autonomous system

$$\frac{dx}{d\tau} = f(x, \lambda) \tag{1.1}$$

depending on the parameter λ. The study of the influence of λ on the long-term behavior of system (1.1) represents an essential part of the bifurcation theory. The parameter value λ^* is called a bifurcation point for (1.1) with respect to the region $\mathcal{G}$ in the phase space of (1.1) if in any neighborhood $\mathcal{N}$ of λ^* in the parameter space there exist two points λ_1 and λ_2 such that the phase portrait of (1.1) in $\mathcal{G}$ is not topologically equivalent for λ_1 and λ_2 (see, e.g. [17, 25, 26]).

If we assume that λ changes slowly in time, then we arrive at the so-called dynamic bifurcation theory [1]. In what follows we consider the simplest case that x and λ are scalars and that λ increases slowly with t. For simplicity we set

$$\lambda = \varepsilon\tau,$$

where ε is a small positive parameter. Introducing the slow time t by $t = \varepsilon\tau$, the differential equation (1.1) takes the form ($x(\tau) = x(t/\varepsilon) =: u(t)$)

$$\varepsilon \frac{du}{dt} = f(u, t), \tag{1.2}$$

that is, (1.2) is a singularly perturbed non-autonomous differential equation.

If we suppose $f(0, \lambda) \equiv 0$ for all λ and that $\lambda^* = 0$ is a bifurcation point of (1.1), where $x = 0$ is stable (unstable) for $\lambda < 0$ ($\lambda > 0$), then the solution set $f^{-1}(0)$ of the degenerate equation of (1.2)

$$0 = f(u, t) \tag{1.3}$$

generically consists in the t-u–plane of two curves intersecting for $t = 0$, as depicted in Fig. 1.1 and Fig. 1.2.

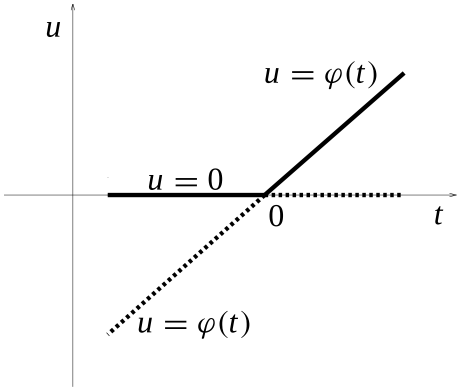

Fig. 1.1. Transcritical bifurcation

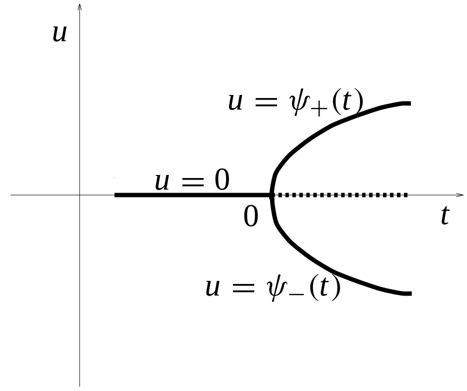

Fig. 1.2. Pitchfork bifurcation

All points of $f^{-1}(0)$ are equilibria of the associated equation to (1.2)

$$\frac{du}{d\sigma} = f(u,t), \tag{1.4}$$

where t has to be considered as a parameter. The curve $u = 0$ is an invariant manifold of (1.4) which is attracting for $t < 0$ and repelling for $t > 0$. We call this situation as exchange of stabilities (according to Lebovitz and Schaar [16]), where Fig. 1.1 represents the case of transcritical bifurcation and Fig. 1.2 the case of pitchfork bifurcation.

If we consider for equation (1.2) the initial value problem

$$u(t_0) = u_0, \quad t_0 < t \leq t_0 + T, \; t_0 < 0, \tag{1.5}$$

and if we assume that u_0 belongs to the region of attraction of the invariant manifold $u = 0$, then it follows from the standard theory of singularly perturbed systems (see, e.g., [30]) that the solution $u(t,\varepsilon)$ of the initial value problem (1.2),(1.5) exists at least for $t_0 < t < 0$. For $t > 0$ there are the following possibilities for the behavior of the solution $u(t,\varepsilon)$:

(i). $u(t,\varepsilon)$ follows immediately the new stable branch emerging at $t = 0$.

(ii). $u(t,\varepsilon)$ follows for some O(1)-time interval (not depending on ε) the repelling part of the invariant manifold $u = 0$ and then jumps to the stable branch.

(iii). $u(t,\varepsilon)$ follows for some O(1)-time interval the repelling part of the invariant manifold $u = 0$ and then jumps away from this manifold (possibly blowing up).

The case (i) is denoted as immediate exchange of stabilities, case (ii) is called delayed exchange of stabilities, and case (iii) is referred to as delayed loss of stability. In the cases (ii) and (iii) the corresponding solutions are said to be canard solutions.
The case of exchange of stabilities for singularly perturbed ordinary differential equations

has been treated by several authors using different methods (see, e.g., [1, 3-10, 12-23 ,25, 26]). The case of immediate exchange of stabilities [4, 21] and the case of delayed exchange of stabilities [4, 20] has been treated by the authors applying the method of lower and upper solutions.

In the following, we derive by the same method conditions on the function f to ensure that the solutions to the initial value problems (1.2),(1.5) exhibit the phenomenon of immediate exchange of stabilities as well as the phenomenon of delayed exchange of stabilities. We emphasize that the results of [20] and [21] concerning the delayed and immediate exchange of stabilities respectively can not be applied to our problem because the conidtions at the points of exchange of stabilities assumed in [20] and [21] are not satisfied. Hence, we have to look for an appropriate modification of the construction of lower and upper solutions.

The paper is organized as follows. Section 2 contains the formulation of the initial value problem. We also collect in this section our assumptions. Section 3 presents our main results for the immediate and delayed exchange of stabilities. In Section 4 we use the results of Section 3 to prove the existence of periodic forced canards and to investigate their asymptotic stability and local uniqueness.

2 Notation. Assumptions

We consider the scalar singularly perturbed non-autonomous differential equation

$$\varepsilon \frac{du}{dt} = g(u, t, \varepsilon), \quad t \in I_T := \{t \in R : t_0 < t \leq t_0 + T\}, \tag{2.1}$$

where ε is a small positive parameter, and study the initial value problem

$$u(t_0, \varepsilon) = u^0. \tag{2.2}$$

Let I_{ε_0} be the interval $I_{\varepsilon_0} := \{\varepsilon \in R : 0 < \varepsilon < \varepsilon_0\}$, where $0 < \varepsilon_0 \ll 1$. Our goal is to derive conditions on g ensuring the existence of a unique solution $u(t, \varepsilon)$ to the initial value problem (2.1), (2.2) for sufficiently small ε and to find an asymptotic representation of $u(t, \varepsilon)$. Our tool for establishing such a result is based on the method of lower and upper solutions. Let us recall the definition of lower and upper solutions for (2.1), (2.2).

Definition 2.1 *The continuous functions $\overline{U}$ and $\underline{U}$ mapping $\overline{I}_T \times \overline{I}_{\varepsilon_0}$ into R and which are piecewise continuously differentiable with respect to t are called ordered lower and upper solutions of the initial value problem* (2.1), (2.2) *for $\varepsilon \in I_{\varepsilon_0}$ provided they satisfy the following conditions for $t \in I_T,\ \varepsilon \in I_{\varepsilon_0}$*

(i) $\quad \underline{U}(t, \varepsilon) \leq \overline{U}(t, \varepsilon)$.

(ii) $\quad L_\varepsilon(\underline{U}) := \varepsilon \frac{d\underline{U}(t,\varepsilon)}{dt} - g(\underline{U}(t, \varepsilon), t, \varepsilon) \leq 0 \leq L_\varepsilon(\overline{U})$.

(iii) $\quad \underline{U}(0, \varepsilon) \leq u^0 \leq \overline{U}(0, \varepsilon)$.

It is known that the existence of ordered lower and upper solutions to the initial value problem (2.1), (2.2) implies the existence of a unique solution located in between the lower and upper solution (see, e.g. [10]).

Let I_u be an open bounded interval containing the origin, let $D := I_u \times I_T \times I_{\varepsilon_0}$. Concerning the smoothness of the function g and the structure of the solution set of the degenerate equation we assume

(A_0). $g \in C^2(\overline{D}, R)$.

(A_1). *The solution set of the degenerate equation*

$$g(u, t, 0) = 0$$

consists in $\overline{I}_u \times \overline{I}_T$ *of the curves* $u \equiv 0$ *and* $u = \varphi(t)$, *where* $\varphi(t)$ *is twice continuously differentiable on* $\overline{I}_T$. *These curves intersect transversally in* $[t_0, T]$ *exactly twice, namely for* $t = t_c^1$ *and for* $t = t_c^2$, *where* $t_0 < t_c^1 < t_c^2 < t_0 + T$. *For definiteness, we suppose* (see Fig. 2.1)

$$\varphi(t) > 0 \quad \text{for} \quad t_0 \le t < t_c^1 \quad \text{and for} \quad t_c^2 < t \le t_0 + T,$$
$$\varphi(t) < 0 \quad \text{for} \quad t_c^1 < t < t_c^2.$$

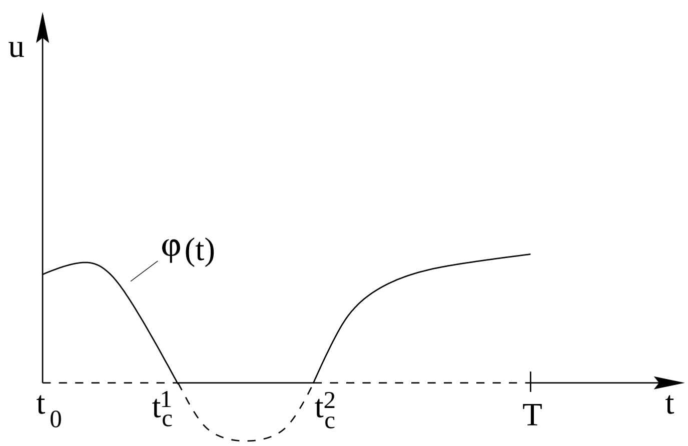

Fig. 2.1. Intersection of the curves $u \equiv 0$ and $u = \varphi(t)$

Assumption (A_1) implies

$$\frac{d\varphi}{dt}(t_c^1) < 0. \tag{2.3}$$

$u = 0$ and $u = \varphi(t)$ are equilibria of the associated equation

$$\frac{du}{d\sigma} = g(u, t, 0), \tag{2.4}$$

where t on the right hand side has to be considered as a parameter. The following assumption determines the stability of these equilibria.

(A_2).
$$g_u(0,t,0) < 0,\ g_u(\varphi(t),t,0) > 0 \quad \text{for} \quad t_c^1 < t < t_c^2,$$
$$g_u(0,t,0) > 0,\ g_u(\varphi(t),t,0) < 0 \quad \text{for} \quad t_0 \le t < t_c^1,\ t_c^2 < t \le t_0 + T.$$

Assumption (A_2) means that the roots $u = 0$ and $u = \varphi(t)$ exchange their stabilities at $t = t_c^1$ and $t = t_c^2$ (see Fig. 2.1).
Now we introduce the function $\hat{u}(t)$, which is called the composed stable solution, by

$$\hat{u}(t) := \begin{cases} \varphi(t) & \text{for} \quad t_0 \le t \le t_c^1, \\ 0 & \text{for} \quad t_c^1 \le t \le t_c^2, \\ \varphi(t) & \text{for} \quad t_c^2 \le t \le t_0 + T. \end{cases} \tag{2.5}$$

From assumption (A_2) and (2.5) we get

$$g_u(\hat{u}(t),t,0) \le 0. \tag{2.6}$$

The following assumption is not generic, but quite natural when we are looking for positive solutions of (2.1), (2.2).

(A_3). $g(0,t,\varepsilon) \equiv 0 \quad$ for $(t,\varepsilon) \in \overline{I}_T \times \overline{I}_{\varepsilon_0}$.

Assumption (A_3) implies that $u \equiv 0$ is a solution of equation (2.1) in $\overline{I}_T$ for all $\varepsilon \in \overline{I}_{\varepsilon_0}$. Consequently, a solution of the initial value problem (2.1), (2.2) with $u_0 > 0$ ($u_0 < 0$) is positive (negative) for all $t \ge t_0$. Another property of the solution $u \equiv 0$ is its attractivity for $t_c^1 < t < t_c^2$, and its repulsivity for $t < t_c^1$ and $t > t_c^2$. In the sequel the function

$$G(t) := \int_{t_c^1}^{t} g_u(0,s,0)\,ds$$

plays a crucial role. From hypothesis (A_2) it follows that $G(t)$ has at most one root in (t_0, t_0+T). We assume

(A_4). *The equation $G(t) = 0$ has a root $t^* \in (t_0, t_0+T)$ (see Fig. 2.2 and Fig. 2.3).*

Concerning the function g we assume

(A_5). *There is a positive number $c_0 \in I_u$ such that*

$$g(u,t,\varepsilon) \le g_u(0,t,\varepsilon)u \quad \text{for} \quad t_c^1 \le t \le t^*,\ \varepsilon \in \overline{I}_{\varepsilon_0},\ 0 \le u \le c_0.$$

Assumption (A_5) is fulfilled if the second derivative of g with respect to u at $u = 0$ is negative for $t \in [t_c^1, t^*)$, $\varepsilon \in I_\varepsilon$.
Finally we suppose

(A_6).
$$\hat{g}_{uu}(t) = g_{uu}(\hat{u}(t),t,0) < 0 \quad \text{for } t = t_c^1.$$

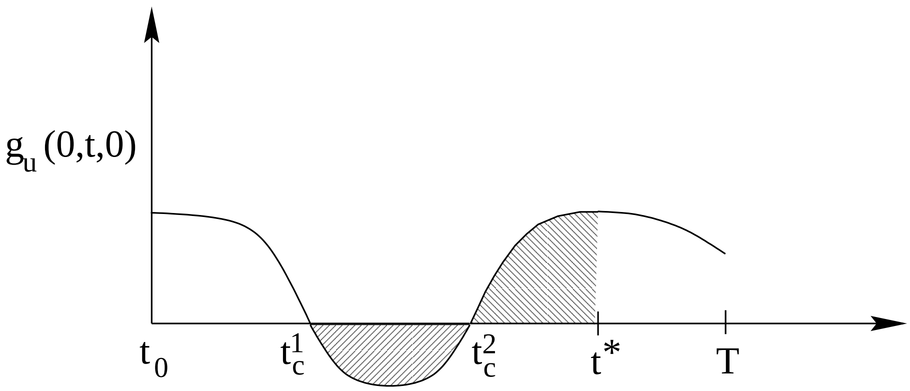

Fig. 2.2. Intersection of the curves $u \equiv 0$ and $u = g_u(0, t, 0)$

3 Immediate and delayed exchange of stabilities

From the assumptions $(A_0) - (A_3)$ it follows that there is an exchange of stabilities for $t = t_c^1$ and $t = t_c^2$. The following theorem states that at the time $t = t_c^1$ there arises an immediate exchange of stabilities, while for $t = t_c^2$ there occurs a delayed exchange of stabilities. Since $u \equiv 0$ is a solution of (2.1) for all ε, it is easy to see that at $t = t_c^1$ there is an immediate exchange of stabilities. Thus, the main result consists in establishing the phenomenon of delayed exchange of stabilities for $t > t_c^2$ and in estimating the delay time by constructing a non-trivial lower solution.

Theorem 3.1 *Assume the hypotheses* $(A_0) - (A_6)$ *to be valid. Then, for sufficiently small* ε*, there exists a unique solution* $u(t, \varepsilon)$ *to the initial value problem* (2.1),(2.2) *with* $u_0 > 0$ *and* $u_0 \in I_u$*, where* $u(t, \varepsilon)$ *is positive and satisfies*

$$\lim_{\varepsilon \to 0} u(t, \varepsilon) = \varphi(t) \quad \textit{for} \quad t \in (t_0, t_c^1) \quad \textit{and} \quad t \in (\, t^*, t_0 + T \,], \tag{3.1}$$

$$\lim_{\varepsilon \to 0} u(t, \varepsilon) = 0 \quad \textit{for} \quad t \in (t_c^1, t^*). \tag{3.2}$$

Proof. The proof of this theorem is based on the technique of lower and upper solutions. As we already mentioned, assumption (A_3) implies that the solution of the initial value problem (2.1), (2.2) is positive provided u_0 is positive. From that assumption it also follows that $u \equiv 0$ is a trivial lower solution for (2.1), (2.2) with $u_0 > 0$.

The proof of Theorem 3.1 proceeds in three steps. In the first step we consider the initial value problem (2.1), (2.2) in the interval $[t_0, t_c^1 - \nu]$, where ν is any sufficiently small positive number independent of ε. Under our hypotheses, to that interval the standard theory of singularly perturbed initial value problems can be applied (see, e.g., [30]). The corresponding asymptotic relation in (3.1) follows immediately from Tikhonov's theorem (see [29]).

For the interval $[t_0 + \delta, t_c^1 - \nu]$, where δ is any sufficiently small number independent of ε, we get from [30] a more precise asymptotic representation of the solution of (2.1), (2.2)

$$u(t, \varepsilon) = \varphi(t) + \varepsilon\, u_1(t) + O(\varepsilon^2), \tag{3.3}$$

where the first order regular term u_1 is defined by

$$u_1(t) := \frac{\frac{d\varphi(t)}{dt} - g_\varepsilon(\varphi(t), t, 0)}{g_u(\varphi(t), t, 0)}. \tag{3.4}$$

The sign of the function u_1 near t_c^1 which we need for the construction of an lower solution can be determined as follows. By hypothesis (A_2) we have $g_u(\varphi(t), t, 0) < 0$ for $t_0 + \delta \le t \le t_c^1 - \nu$. From assumption (A_3) it follows the relation $g_\varepsilon(0, t_c^1, 0) = 0$. Hence, by (2.3) there exist sufficiently small positive numbers ν and κ such that

$$\frac{d\varphi}{dt}(t) - g_\varepsilon(\varphi(t), t, 0) \le -\kappa < 0 \quad \text{for} \quad t \in [t_c^1 - \nu, t_c^1]. \tag{3.5}$$

Therefore, we have

$$u_1(t_c^1 - \nu) > 0. \tag{3.6}$$

Let us introduce the notation

$$u^1 := u(t_c^1 - \nu, \varepsilon). \tag{3.7}$$

Now we construct a nontrivial lower solution $\underline{U}_1(t, \varepsilon)$ to the initial value problem (2.1), (2.2) for $t \in [t_c^1 - \nu, t_c^1]$ in the form

$$\underline{U}_1(t, \varepsilon) = \varphi(t) + \eta\varepsilon^2, \tag{3.8}$$

where η is a positive number independent of ε which will be chosen appropriately later. We have

$$L_\varepsilon(\underline{U}_1) \equiv \varepsilon \frac{d\underline{U}_1}{dt} - g(\underline{U}_1, t, \varepsilon) = \varepsilon \frac{d\varphi}{dt}(t) - g(\varphi(t), t, 0) - \varepsilon g_\varepsilon(\varphi(t), t, 0) + O(\varepsilon^2).$$

Taking into account $g(\varphi(t), t, 0) \equiv 0$ and the relation (3.5) we obtain for sufficiently small ε

$$L_\varepsilon(\underline{U}_1) = \varepsilon \left(\frac{d\varphi}{dt}(t) - g_\varepsilon(\varphi(t), t, 0) \right) + O(\varepsilon^2) \le -\kappa\varepsilon + O(\varepsilon^2) < 0 \quad \text{for } t \in [t_c^1 - \nu, t_c^1].$$

From (3.3), and (3.6)-(3.8) it follows for sufficiently small ε

$$u(t_c^1 - \nu, \varepsilon) - \underline{U}_1(t_c^1 - \nu, \varepsilon) = u_1(t_c^1 - \nu)\varepsilon + O(\varepsilon^2) > 0.$$

Consequently, $\underline{U}_1(t, \varepsilon)$ is a lower solution of (2.1), (2.2) on $[t_c^1 - \nu, t_c^1]$.
Now we construct an upper solution of (2.1), (2.2) for the interval $[t_c^1 - \nu, t_c^1 + \nu]$ in the form

$$\overline{U}_1(t, \varepsilon) = \hat{u}(t) + \gamma\sqrt{\varepsilon}, \tag{3.9}$$

where $\hat{u}(t)$ is the stable composed solution introduced in (2.5), and γ is a positive constant independent of ε which will chosen later. We have

$$L_\varepsilon(\overline{U}_1) \equiv \varepsilon \frac{d\overline{U}_1}{dt} - g(\overline{U}_1, t, \varepsilon) = \varepsilon \frac{d\hat{u}}{dt} - \Big[\hat{g}(t) + \gamma\sqrt{\varepsilon}\, \hat{g}_u(t) + \frac{\varepsilon}{2}\, \hat{g}_{uu}(t)\gamma^2 + \varepsilon\, \hat{g}_\varepsilon(t) + o(\varepsilon) \Big],$$

where $\hat{g}(t) := g(\hat{u}(t), t, 0)$, and analogously $\hat{g}_u(t) := g_u(\hat{u}(t), t, 0)$, ...
By hypothesis (A_1) it holds $\hat{g}(t) \equiv 0$. By (2.6) we have $-\hat{g}_u(t) \geq 0$. According to assumption (A_6) we can choose ν sufficiently small such that there is a small positive number σ satisfying

$$\hat{g}_{uu}(t) \leq -\sigma < 0 \quad \text{for} \quad t \in [t_c^1 - \nu, t_c^1 + \nu].$$

Thus, we have

$$L_\varepsilon(\overline{U}_1) \geq \varepsilon \left(\frac{\sigma \gamma^2}{2} + \frac{d\hat{u}}{dt} - \hat{g}_\varepsilon(t) \right) + o(\varepsilon).$$

Consequently, for sufficiently small ε and sufficiently large γ it holds

$$L_\varepsilon(\overline{U}_1) \geq 0 \quad \text{for} \quad t \in [t_c^1 - \nu, t_c^1 + \nu].$$

If we compare the expressions $\overline{U}_1(t_c^1 - \nu, \varepsilon) = \varphi(t_c^1 - \nu) + \gamma\sqrt{\varepsilon}$ and $u^1 = u(t_c^1 - \nu, \varepsilon) = \varphi(t_c^1 - \nu) + O(\varepsilon)$, then we obtain for sufficiently small ε

$$\overline{U}_1(t_c^1 - \nu, \varepsilon) \geq u^1.$$

Thus, $\overline{U}_1(t, \varepsilon)$ is an upper solution of (2.1), (2.2) for the interval $[t_c^1 - \nu, t_c^1 + \nu]$.
From our investigations above we get that the initial value problem (2.1), (2.2) has a solution $u(t, \varepsilon)$ in the interval $[t_0, t_c^1 + \nu]$ satisfying for $t = t_c^1$

$$\underline{U}_1(t_c^1, \varepsilon) = \varphi(t_c^1) + \eta\varepsilon^2 = \eta\varepsilon^2 \leq u(t_c^1, \varepsilon) \leq \gamma\sqrt{\varepsilon} = \overline{U}_1(t_c^1, \varepsilon). \tag{3.10}$$

To prepare the construction of upper and lower solutions for the next interval we notice that from assumption (A_4) it follows that to any given sufficiently small positive ν there are positive constants $\delta_a(\nu) > 0$ and $\omega(\nu)$ such that the function $a(t, \nu)$ defined by

$$a(t, \nu) := g_u(0, t, 0) + \delta_a(\nu) \tag{3.11}$$

satisfies

$$a(t_c^1 + \nu, \nu) < 0$$

and

$$\int_{t_c^1+\nu}^{t^*-\omega(\nu)} a(t, \nu)dt = 0. \tag{3.12}$$

We note that $\omega(\nu)$ tends to 0 as $\nu \to 0$. For the following we assume that ν is so small that $t^* - 2\omega(\nu) > t_c^1 + \nu$. Thus, we have

$$a(t, \nu) > 0 \quad \text{for} \quad t^* - \omega(\nu) \leq t \leq t_0 + T. \tag{3.13}$$

In order to prove the relation (3.2) we construct an upper solution $\overline{U}_2(t, \varepsilon)$ to (2.1), (2.2) for $t \in [t_c^1 + \nu, t^* - 2\omega(\nu)]$ in the form

$$\overline{U}_2(t, \varepsilon) = \gamma\sqrt{\varepsilon} \exp\Big\{ \frac{1}{\varepsilon} \int_{t_c^1+\nu}^{t} a(s, \nu)ds \Big\}, \tag{3.14}$$

where γ is the same constant as in (3.9). By (3.9) and (2.5) we have

$$u(t_c^1 + \nu, \varepsilon) \le \gamma\sqrt{\varepsilon} = \overline{U}_2(t_c^1 + \nu, \varepsilon).$$

From (3.11), hypothesis (A_2) and (3.12) it follows

$$\int_{t_c^1+\nu}^{t} a(s, \nu)ds < 0 \quad \text{for } t \in \left[t_c^1 + \nu, t^* - 2\omega(\nu)\right].$$

Therefore, we have

$$\lim_{\varepsilon\to 0} \overline{U}_2(t, \varepsilon) = 0 \quad \text{for } t \in \left[t_c^1 + \nu, t^* - 2\omega(\nu)\right].$$

Next we verify that $\overline{U}_2(t, \varepsilon)$ satisfies the differential inequality for an upper solution.
It is easy to check that $\overline{U}_2(t, \varepsilon)$ obeys

$$\varepsilon \frac{d\overline{U}_2}{dt} = a(t, \nu)\,\overline{U}_2. \tag{3.15}$$

From (3.15) we get

$$L_\varepsilon(\overline{U}_2) := \varepsilon \frac{d\overline{U}_2}{dt} - g(\overline{U}_2, t, \varepsilon) = g_u(0, t, \varepsilon)\overline{U}_2 - g(\overline{U}_2, t, \varepsilon) + (a(t, \nu) - g_u(0, t, \varepsilon))\overline{U}_2.$$

By assumption (A_5) and by (3.14) we have for $t \in [t_c^1 + \nu, t^* - 2\omega(\nu)]$ and for sufficiently small ε

$$g_u(0, t, \varepsilon)\overline{U}_2 - g(\overline{U}_2, t, \varepsilon) \ge 0.$$

From (3.11) we obtain for sufficiently small ε

$$a(t, \nu) - g_u(0, t, \varepsilon) = \delta_a(\nu) + g_u(0, t, 0) - g_u(0, t, \varepsilon) \ge 0.$$

Thus,

$$L_\varepsilon(\overline{U}_2) \ge 0,$$

i.e., $\overline{U}_2$ is an upper solution for $t \in [t_c^2 + \nu, t^* - 2\omega(\nu)]$.
It is easy to verify that $\varphi(t) + \beta\varepsilon$, where β does not depend on ε, is an upper solution on $[t^* - \nu, T]$, if we choose β sufficiently large.
By assumption (A_3), $\underline{\tilde{U}} \equiv 0$ is a trivial lower solution for this interval. Hence, the initial value problem (2.1), (2.2) has a solution $u(t, \varepsilon)$ in the interval $[t_0, t_0 + T]$ satisfying

$$\lim_{\varepsilon\to 0} u(t, \varepsilon) = 0 \quad \text{for } t \in [t_c^1 + \nu, t^* - 2\omega(\nu)]. \tag{3.16}$$

Since ν is any small positive number, the relation (3.16) is valid for $t \in (t_c^1, t^*)$. Thus, the validity of relation (3.2) has been proven.
We note that the following relations hold

$$\overline{U}_2(t_c^1 + \nu, \varepsilon) = \overline{U}_2(t^* - \omega(\nu), \varepsilon) = \gamma\sqrt{\varepsilon}.$$

By (3.13) the function $a(t,\nu)$ is positive for $t \geq t^* - \omega(\nu)$. Hence, to given sufficiently small ν and ε, and to given γ and $\gamma_1 01kler$ satisfying $0 < \gamma_1 \leq c_0$, where c_0 is the same constant as in hypothesis (A_5), and where γ_1 is independent of ε, there is a positive constant $\lambda_a(\nu,\varepsilon)$ such that

$$\int_{t_c^1+\nu}^{t^*-\omega(\nu)+\lambda_a(\nu,\varepsilon)} a(s,\nu)ds = \int_0^{\lambda_a(\nu,\varepsilon)} a(t_c^1+\nu+s,\nu)ds = \varepsilon\left(\ln\frac{\gamma_1}{\gamma} - \ln\sqrt{\varepsilon}\right), \tag{3.17}$$

where $\lambda_a(\nu,\varepsilon)$ tends to zero as ε tends to zero. From (3.14) and (3.17) we get

$$\overline{U}_2(t^* - \omega(\nu) + \lambda_a(\nu,\varepsilon), \varepsilon) = \gamma_1. \tag{3.18}$$

We will exploit this relation in the next section.
In order to prove (3.1) we construct a nontrivial lower solution of (2.1), (2.2) on the interval $[t_c^1, t^* + \nu + \lambda_b(\nu,\varepsilon)]$, where the positive number $\lambda_b(\nu,\varepsilon)$ will be defined later.
By hypothesis (A_4) there is to any small $\nu > 0$ a constant $\delta_b(\nu) > 0$ such that the function $b(t,\nu)$ defined by

$$b(t,\nu) := g_u(0,t,0) - \delta_b(\nu) \tag{3.19}$$

satisfies

$$\int_{t_c^1}^{t^*+\nu} b(s,\nu)ds = 0.$$

Now we construct a lower solution in the form

$$\underline{U}_2(t,\varepsilon) = \eta\varepsilon^2 \exp\left\{\frac{1}{\varepsilon}\int_{t_c^1}^{t} b(s,\nu)ds\right\}, \tag{3.20}$$

where η is the same constant as in (3.8). For the sequel we assume

$$0 < \eta < \eta_0 = min(\gamma_1, \varphi(t^*)). \tag{3.21}$$

From (3.8) and (3.20) we get

$$\underline{U}_1(t_c^1,\varepsilon) = \underline{U}_2(t_c^1,\varepsilon) = \eta\varepsilon^2,$$

that is, the inequality for the initial condition is fulfilled.
In a similar way as we established the constant $\lambda_a(\nu,\varepsilon)$ we can conclude from (3.20) that there is a positive constant $\lambda_b(\nu,\varepsilon)$ such that

$$\underline{U}_2(t^* + \nu + \lambda_b(\nu,\varepsilon)) = \eta \tag{3.22}$$

and $\lambda_b(\nu,\varepsilon) \to 0$ as $\varepsilon \to 0$.
It is obvious that $\underline{U}_2(t,\varepsilon)$ satisfies the differential equation

$$\varepsilon\frac{d\underline{U}_2}{dt} = b(t,\nu)\underline{U}_2.$$

Using this equation we have

$$\varepsilon \frac{d\underline{U}_2}{dt} - g(\underline{U}_2, t, \varepsilon) = (b(t,\nu) - g_u(0,t,\varepsilon))\underline{U}_2 + g_u(0,t,\varepsilon)\underline{U}_2 - g(\underline{U}_2, t, \varepsilon). \quad (3.23)$$

From (3.19) it follows that for sufficiently small ε

$$b(t,\nu) - g_u(0,t,\varepsilon) \le -\frac{\delta_b(\nu)}{2}. \quad (3.24)$$

By the hypotheses (A_3) and (A_0) there is a constant $\kappa > 0$ such that for $(u,t,\varepsilon) \in D$

$$\begin{aligned} g_u(0,t,\varepsilon)u - g_u(u,t,\varepsilon) &= g_u(0,t,\varepsilon)u - (g(u,t,\varepsilon) - g(0,t,\varepsilon)) \\ &= (g_u(0,t,\varepsilon) - g_u(u_*,t,\varepsilon))u \le \kappa u^2, \end{aligned} \quad (3.25)$$

where $0 < u_* < u$. Thus, it follows from (3.23) – (3.25)

$$\varepsilon \frac{d\underline{U}_2}{dt} - g(\underline{U}_2, t, \varepsilon) \le \underline{U}_2 \left(-\frac{\delta_b(\nu)}{2} + \kappa \underline{U}_2 \right). \quad (3.26)$$

If we choose η such that

$$\eta \le min(\eta_0, \delta_b(\nu)/(2\kappa)),$$

then we get from (3.26)

$$\varepsilon \frac{d\underline{U}_2}{dt} - g(\underline{U}_2, t, \varepsilon) \le 0.$$

Therefore $\underline{U}_2(t,\varepsilon)$ is a nontrivial lower solution of (2.1), (2.2) for $t \in [t_c^1, t^* + \nu + \lambda_b(\nu,\varepsilon)]$. By (3.22) we can conclude

$$u(t^* + \nu + \lambda_b(\nu,\varepsilon), \varepsilon) = \tilde{u} \ge \eta, \quad (3.27)$$

where η does not depend on ε. If we consider the initial value problem (2.1), $u(t^* + \nu + \lambda_b(\nu,\varepsilon)) = \tilde{u}$, where we emphasize that $\tilde{u}$ does not depend on ε and belongs to the basin of attraction of the asymptotically stable equilibrium $\varphi(t^* + \nu + \lambda_b(\nu,\varepsilon))$ of the associated equation (2.4), then we can apply the standard theory of singularly perturbed initial value problems and get the relation

$$\lim_{\varepsilon \to 0} u(t,\varepsilon) = \varphi(t) \quad \text{for } t \in (\, t^* + \nu, t_0 + T\ ]. \quad (3.28)$$

As ν does not depend on ε and can be chosen arbitrary small, the proof of relation (3.1), and consequently of Theorem 3.1 is complete.

4 Periodic forced canards

4.1 Formulation of the Problem. Assumptions

In this section we consider the singularly perturbed scalar differential equation

$$\varepsilon \frac{du}{dt} = g(u,t,\varepsilon), \quad t \in R \tag{4.1}$$

in the case of exchange of stabilities under the additional assumption that the function g is T-periodic in t. Our goal is to study the existence of harmonic solutions of (4.1), i.e. we are looking for solutions satisfying

$$u(t,\varepsilon) = u(t+T,\varepsilon) \quad \forall\, t \in R \tag{4.2}$$

and which exhibit the phenomenon of delayed exchange of stabilities. Such solutions are referred to as periodic forced canards.

Our approach to prove the existence of periodic forced canards is based on the method of lower and upper solutions.

Definition 4.1 *Let the functions $\underline{U}(t,\varepsilon)$ and $\overline{U}(t,\varepsilon)$ be mappings of the domain $D := I_{\varepsilon^*} \times [t_c^1 - \nu, t_c^1 - \nu + T]$ into R. The functions $\underline{U}(t,\varepsilon)$ and $\overline{U}(t,\varepsilon)$ are called ordered lower and upper solutions of the boundary value problem* (4.1), (4.2) *on the interval $[t_c^1 - \nu, t_c^1 - \nu + \omega]$ for $\varepsilon \in I_{\varepsilon^*}$, if they are piecewise continuous and piecewise continuously differentiable with respect to t on D and if they satisfy for $(t,\varepsilon) \in D$ the following relations*

(i)

$$\underline{U}(t,\varepsilon) \le \overline{U}(t,\varepsilon) \tag{4.3}$$

in all points of continuity,

(ii)

$$\underline{U}(\hat{t}+0,\varepsilon) \le \underline{U}(\hat{t}-0,\varepsilon), \;\; \overline{U}(\hat{t}+0,\varepsilon) \ge \overline{U}(\hat{t}-0,\varepsilon) \tag{4.4}$$

in any point $t = \hat{t}$ of discontinuity,

(iii)

$$\varepsilon \frac{d\underline{U}}{dt} - g(\underline{U},t,\varepsilon) \le 0, \tag{4.5}$$

$$\varepsilon \frac{d\overline{U}}{dt} - g(\overline{U},t,\varepsilon) \ge 0, \tag{4.6}$$

where in all points, where $\frac{d\underline{U}}{dt}$ and $\frac{d\overline{U}}{dt}$ have a jump, the derivatives have to be understood in the sense of right and left derivatives.

(iv)

$$\underline{U}(t_c^1 - \nu, \varepsilon) \leq \underline{U}(t_c^1 - \nu + \omega, \varepsilon), \ \overline{U}(t_c^1 - \nu, \varepsilon) \geq \overline{U}(t_c^1 - \nu + \omega, \varepsilon). \tag{4.7}$$

In the sequel, we need the same assumptions as introduced in section 1, additionally we suppose that g is T-periodic in t.

(P_1). *There exists a number $t_0 \in R$ such that the assumptions $(A_1) - (A_6)$ of section 1 are satisfied.*

(P_2). *g is T-periodic in t, i.e.*

$$g(u, t, \varepsilon) = g(u, t + T, \varepsilon) \quad \forall t \in R, \forall u \in I_u, \forall \varepsilon \in I_{\varepsilon_0}. \tag{4.8}$$

4.2 Existence and stability of periodic forced canards

Our first result states the existence of periodic forced canards.

Theorem 4.2 *Assume the hypotheses (P_1) and (P_2) to be valid. Then, for sufficiently small ε, there exists a positive T-periodic solution $u_p(t, \varepsilon)$ of (4.1) satisfying for $n = 0, \pm 1, \ldots$*

$$\lim_{\varepsilon \to 0} u_p(t, \varepsilon) = 0 \quad \textit{for } t \in (t_c^1 \pm nT, t^* \pm nT), \tag{4.9}$$

$$\lim_{\varepsilon \to 0} u_p(t, \varepsilon) = \varphi(t) \quad \textit{for } t \in (t^* \pm nT, t_c^1 + (1 \pm n)T). \tag{4.10}$$

Remark 4.1 The periodic solution $u_p(t, \varepsilon)$ represents a relaxation oscillation. Since it stays for the intervals $(t_c^2 \pm nT, t^* \pm nT)$ near the repelling part of the solution root $u = 0$, it is referred to as periodic forced canard.

Remark 4.2 *If we consider $\lim_{\varepsilon \to 0} u_p(t, \varepsilon)$ for $t \in [t_0, t_0 + T]$ except for $t = t^*$, then we obtain a discontinuous function $u_0(t)$ defined by*

$$u_0(t) := \begin{cases} \varphi(t) & for \quad t_0 \leq t \leq t_c^1, \\ 0 & for \quad t_c^1 \leq t < t^* \\ \varphi(t) & for \quad t^* < t < t_0 + T \end{cases} \tag{4.11}$$

Proof of Theorem 4.2. Under the periodicity condition (4.8), the phase space of the differential equation (2.1) can be considered as the part $S^T \times I_u$ of the cylinder $\mathcal{Z} := S^T \times R$. To each $\varepsilon \in I_{\varepsilon^*}$, where ε^* is a sufficiently small positive number, we construct two curves $\overline{\mathcal{U}_\varepsilon}$ and $\underline{\mathcal{U}_\varepsilon}$ which starts at some straight line $t = \hat{t}$ on the cylinder $\mathcal{Z}$, where their initial points $\overline{P}_\varepsilon$ and $\underline{P}_\varepsilon$ bound some interval I_P^ε. They surround the cylinder $\mathcal{Z}$ without intersecting each other and arrive at the same straight line, where their endpoints $\overline{E}_\varepsilon$ and $\underline{E}_\varepsilon$ bound an interval I_E^ε which is a a subinterval of I_P^ε. We denote by G_ε the region on the cylinder $\mathcal{Z}$ bounded by the curves $\overline{\mathcal{U}_\varepsilon}$ and $\underline{\mathcal{U}_\varepsilon}$ and by I_P^ε and I_E^ε. The curves $\overline{\mathcal{U}_\varepsilon}$ and $\underline{\mathcal{U}_\varepsilon}$

have the important property that a solution of (4.1) starting on I_P^ε will never leave the region G_ε. Hence, the interval I_P^ε will be mapped into itself by the solutions of (4.1) starting on I_P^ε. Consequently, by applying Schauder's fixed point theorem, we get the existence of at least one periodic solution of (4.1) located in G_ε.
The curves $\overline{\mathcal{U}_\varepsilon}$ and $\underline{\mathcal{U}_\varepsilon}$ are determined by the construction of upper and lower solutions for the boundary value problem (4.1), (4.2). For this purpose we use the results of section 3. We start from $t = t_c^1 - \nu$.
As we mentioned in section 3, the solution of the initial value problem (2.1), (2.2) takes for $t = t_c^1 - \nu$ the value u^1 which can be estimated from above by $\overline{U}_1(t_c^1 - \nu, \varepsilon)$, where the function

$$\overline{U}_1(t, \varepsilon) = \hat{u}(t) + \gamma\sqrt{\varepsilon} \tag{4.12}$$

is an upper solution of (2.1) on the interval $[t_c^1 - \nu, t_c^1 + \nu]$. We also recall that the function

$$\overline{U}_2(t, \varepsilon) = \gamma\sqrt{\varepsilon}\exp\Big\{\frac{1}{\varepsilon}\int_{t_c^1+\nu}^{t} a(\rho, \nu)d\rho\Big\}$$

is an upper solution of (2.1) on the interval $[t_c^1 + \nu, t_1]$, where $t_1 := t^* - \omega(\nu) + \lambda_a(\nu, \varepsilon)$, satisfying

$$\overline{U}_2(t_1, \varepsilon) = \gamma_1, \tag{4.13}$$

where γ_1 does not depend on ε (see (3.18)).
A lower solution of (2.1) for the interval $[t_c^1 - \nu, t_c^1]$ is given by

$$\underline{U}_1(t, \varepsilon) = \varphi(t) + \eta\varepsilon^2, \tag{4.14}$$

and for the interval $[t_c^1, t_2]$, where $t_2 := t^* + \nu + \lambda_b(\nu, \varepsilon)$, by

$$\underline{U}_2(t, \varepsilon) = \eta\varepsilon^2\exp\Big\{\frac{1}{\varepsilon}\int_{t_c^1}^{t} b(\rho, \nu)d\rho\Big\},$$

satisfying

$$\underline{U}_2(t_2, \varepsilon) = \eta, \tag{4.15}$$

where η does not depend on ε (see (3.19), (3.20)). Fig. 4.1 contains a schematic representation of the constructed lower and upper solutions.
In the next step we construct an upper solution for the interval $[t_1, t_c^1 - \nu + T]$, and a lower solution for the interval $[t_2, t_c^1 - \nu + T]$.
The idea to do this is as follows. First we note that in the intervals under consideration there is no exchange of stabilities. Next we observe that for $t = t_1$ the upper solution takes the value γ_1 which is independent of ε, and that for $t = t_2$ the lower solution takes the value η which is also independent of ε. Both points γ_1 and η are located in the region of attraction of the equilibria $u = \varphi(t_1)$ and of $u = \varphi(t_2)$, respectively of the associated

equation (2.4).

Taking into account this observations we construct lower and upper solutions in the form

$$\begin{aligned}
\overline{U}_3(t,\varepsilon) &= \varphi(t) + \varepsilon(u_1(t)+\alpha) + \Pi_0^1(\tau_1) + \varepsilon\tilde{\Pi}_1^1(\tau_1),\\
\underline{U}_3(t,\varepsilon) &= \varphi(t) + \varepsilon(u_1(t)-\alpha) + \Pi_0^2(\tau_2) + \varepsilon\tilde{\Pi}_1^2(\tau_2).
\end{aligned} \tag{4.16}$$

Here, $u_1(t)$ is the first order term in the regular expansion of the solution $u(t,\varepsilon)$ of the differential equation (4.1) which is defined by (3.4), $\tau_1 = (t-t_1)/\varepsilon$, and $\tau_2 = (t-t_2)/\varepsilon$, α is some appropriate positive constant to be chosen later. The zeroth order boundary layer functions Π_0^1 and Π_0^2 are defined by the following initial value problems (see, e.g. [30])

$$\frac{d\Pi_0^1}{d\tau_1} = g(\varphi(t_1)+\Pi_0^1, t_1, 0),\ \tau_1 > 0,\ \Pi_0^1(0) = \gamma_1 - \varphi(t_1) \tag{4.17}$$

and

$$\frac{d\Pi_0^2}{d\tau_2} = g(\varphi(t_2)+\Pi_0^2, t_2, 0),\ \tau_2 > 0,\ \Pi_0^2(0) = \eta - \varphi(t_2). \tag{4.18}$$

The first order boundary layer function Π_1^1 is defined by the following initial value problem (see [30])

$$\frac{d\Pi_1^1}{d\tau_1} = g_u(\varphi(t_1)+\Pi_0^1(\tau_1), t_1, 0)\Pi_1^1 + g_1(\tau_1),\ \tau_1 > 0,\ \Pi_1^1(0) = -u_1(t_1), \tag{4.19}$$

where $g_1(\tau_1)$ is defined by

$$\begin{aligned}
g_1(\tau_1) \ := \ & (g_u(\varphi(t_1)+\Pi_0^1(\tau_1), t_1, 0) - g_u(\varphi(t_1), t_1, 0))\varphi'(t_1)\tau_1\\
& +(g_t(\varphi(t_1)+\Pi_0^1(\tau_1), t_1, 0) - g_t(\varphi(t_1), t_1, 0))\tau_1\\
& +g_\varepsilon(\varphi(t_1)+\Pi_0^1(\tau_1), t_1, 0) - g_\varepsilon(\varphi(t_1), t_1, 0)\\
& +(g_u(\varphi(t_1)+\Pi_0^1(\tau_1), t_1, 0) - g_u(\varphi(t_1), t_1, 0))u_1(t_1).
\end{aligned} \tag{4.20}$$

Π_1^2 is the solution of the initial value problem

$$\frac{d\Pi_1^2}{d\tau_2} = g_u(\varphi(t_2)+\Pi_0^2(\tau_2), t_2, 0)\Pi_1^2 + g_2(\tau_2),\ \tau_2 > 0,\ \Pi_1^2(0) = -u_1(t_2), \tag{4.21}$$

where $g_2(\tau_2)$ is defined analogously to (4.20).

The boundary layer corrections $\tilde{\Pi}_1^1(\tau_1)$ and $\tilde{\Pi}_1^2(\tau_2)$ are slight modifications of the first order boundary layer functions $\Pi_1^1(\tau_1)$ and $\Pi_1^2(\tau_2)$, respectively, and are defined by

$$\begin{aligned}
\frac{d\tilde{\Pi}_1^1}{d\tau_1} \ = \ & g_u(\varphi(t_1)+\Pi_0^1(\tau_1), t_1, 0)\tilde{\Pi}_1^1 + g_1(\tau_1)\\
& +(g_u(\varphi(t_1)+\Pi_0(\tau_1), t_1, 0) - g_u(\varphi(t_1), t_1, 0))\alpha,\\
& \tau_1 > 0,\ \tilde{\Pi}_1^1(0) = -u_1(t_1),
\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}
\frac{d\tilde{\Pi}_1^2}{d\tau_2} &= g_u(\varphi(t_2)+\Pi_0^2(\tau_2),t_2,0)\tilde{\Pi}_1^2+g_2(\tau_2) \\
&\quad +(g_u(\varphi(t_2)+\Pi_0^2(\tau_2),t_2,0)-g_u(\varphi(t_2),t_2,0))\alpha, \\
&\quad \tau_2>0,\ \tilde{\Pi}_1^2(0)=-u_1(t_2).
\end{aligned} \tag{4.23}$$

It is known that the solutions of the problems (4.17) – (4.19) and (4.21) –(4.23) exist and exponentially decay to zero (see [30]).
From our construction it follows that $\underline{U}_3(t,\varepsilon)$ and $\overline{U}_3(t,\varepsilon)$ defined in (4.16) are formal asymptotic solutions of (4.1) (see also [18]) and satisfy

$$\varepsilon\frac{d\underline{U}_3}{dt}-g(\underline{U}_3,t,\varepsilon)=\varepsilon\alpha g_u(\varphi(t),t,0)+o(\varepsilon),$$

$$\varepsilon\frac{d\overline{U}_3}{dt}-g(\overline{U}_3,t,\varepsilon)=-\varepsilon\alpha g_u(\varphi(t),t,0)+o(\varepsilon).$$

By assumption (A_2) we have $g_u(\varphi(t),t,0)<0$ for $t>t_c^2$. Therefore, conditions (4.5) and (4.6) in the definition of lower and upper solutions are fulfilled. Since the boundary layer corrections are exponentially decaying functions and the estimate (3.21) is valid, the condition (4.3) can be satisfied for any positive α and sufficiently small ε.
If we construct lower and upper solutions $\underline{U}(t,\varepsilon)$ and $\overline{U}(t,\varepsilon)$ for the boundary value problem (4.1), (4.2) by composing the functions $\underline{U}_i$ and $\overline{U}_i$, $i=1,2,3$, then the functions $\underline{U}$ and $\overline{U}$ have a discontinuity for $t=t_1$ and $t=t_2$, respectively. Now we will check that the conditions (4.4) are fulfilled at these points. From (4.16),(4.17),(4.19), and (4.13) we get

$$\begin{aligned}
\overline{U}_3(t_1,\varepsilon)&=\varphi(t_1)+\varepsilon(u_1(t_1)+\alpha)+\gamma_1-\varphi(t_1)-\varepsilon u_1(t_1)+o(\varepsilon)= \\
&\qquad \gamma_1+\varepsilon\alpha+o(\varepsilon)>\gamma_1=\overline{U}_2(t_1,\varepsilon)>\eta.
\end{aligned}$$

Analogously we obtain from (4.16),(4.18), (4.23), and (4.15)

$$\begin{aligned}
\underline{U}_3(t_2,\varepsilon)&=\varphi(t_2)+\varepsilon(u_1(t_2)-\alpha)+\eta-\varphi(t_2)-\varepsilon u_1(t_2)+o(\varepsilon)= \\
&\qquad \eta-\varepsilon\alpha+o(\varepsilon)<\eta=\underline{U}_2(t_2,\varepsilon)<\gamma_1.
\end{aligned}$$

Therefore, we can conclude that $\underline{U}(t,\varepsilon)$ and $\overline{U}(t,\varepsilon)$ are upper and lower solutions of (4.1), (4.2).

Finally we have to ensure that the relations

$$\underline{U}_3(t_c^1-\nu+T,\varepsilon)>\underline{U}_1(t_c^1-\nu),\quad \overline{U}_3(t_c^1-\nu+T,\varepsilon)<\overline{U}_1(t_c^1-\nu)$$

hold. By (4.16) we have

$$\begin{aligned}
\underline{U}_3(t_c^1-\nu+T,\varepsilon) &= \varphi(t_c^1-\nu+T)+\varepsilon(u_1(t_c^1-\nu+T)-\alpha)+o(\varepsilon) \\
&= \varphi(t_c^1-\nu)+\varepsilon(u_1(t_c^1-\nu)-\alpha)+o(\varepsilon).
\end{aligned}$$

By (3.6), $u_1(t_1-\nu)$ is positive. If we require $\alpha < u_1(t_c^1-\nu)$, then we get by (4.14)

$$\underline{U}_3(t_c^1-\nu+T, \varepsilon > \varphi(t_c^1-\nu)+\eta\varepsilon^2 = \underline{U}_1(t_c^1-\nu)$$

for sufficiently small ε. By (4.16) and (4.12) we have

$$\begin{aligned}\overline{U}_3(t_c^1-\nu+T,\varepsilon) &= \varphi(t_c^1-\nu+T)+\varepsilon(u_1(t_c^1-\nu+T)+\alpha)+o(\varepsilon)\\ &= \varphi(t_c^1-\nu)+\varepsilon(u_1(t_c^1-\nu)+\alpha)+o(\varepsilon) < \varphi(t_c^1-\nu)+\gamma\sqrt{\varepsilon}\\ &= \underline{U}_1(t_c^1-\nu).\end{aligned}$$

This condition holds for sufficiently small ε. Therefore, we can conclude that the boundary value problem (4.1), (4.2) has at least one solution, q.e.d.

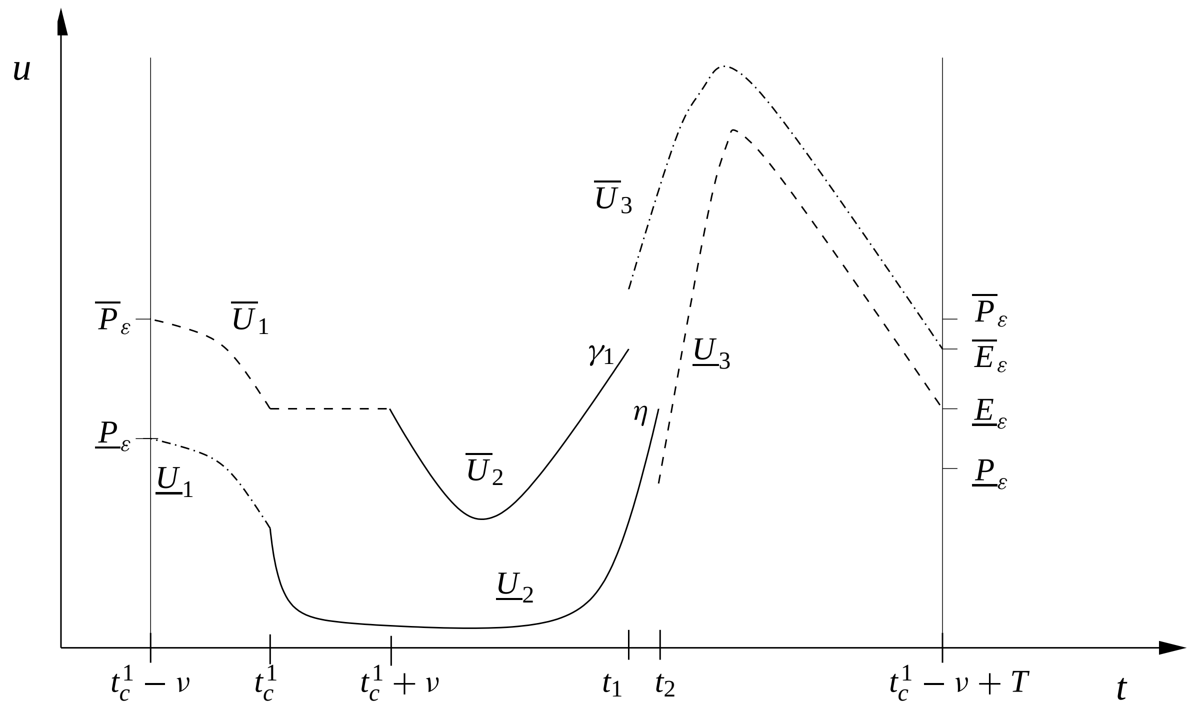

Fig. 4.1. Schematic representation of constructed lower and upper solutions.

We denote by G_ε the bounded region on the cylinder $\mathcal{Z}$ bounded by the upper and lower solution $\underline{U}(t,\varepsilon)$ and $\overline{U}(t,\varepsilon)$.

Theorem 4.3 *Assume the assumptions of Theorem* 4.2 *hold. Then there is a sufficiently small positive* ε_1, $\varepsilon_1 < \varepsilon_0$ *such that any* T*-periodic solution of* (4.1) *located in* G_ε *satisfies*

$$\int_{t_0}^{t_0+T} g_u(u_p(t,\varepsilon),t,\varepsilon)dt < 0.$$

Proof. We consider the integral

$$\int_{t_0}^{t_0+T} g_u(u_0(t),t,0)dt,$$

where the function u_0 has been introduced in (4.11) as the pointwise limit of any T-periodic solution of (4.1) located in G_ε as ε tends to zero. We have

$$\int_{t_0}^{t_0+T} g_u(u_0(t), t, 0)dt = \int_{t_0}^{t_c^1} g_u(\varphi(t), t, 0)dt + \int_{t_c^1}^{t^*} g_u(0, t, 0)dt + \int_{t^*}^{t_0+T} g_u(\varphi(t), t, 0)dt.$$

By assumption (A_2) there are positive constants α_1 and α_2 such that

$$\int_{t_0}^{t_c^1} g_u(\varphi(t), t, 0)dt = -\alpha_1 < 0,$$

$$\int_{t^*}^{t_0+T} g_u(\varphi(t), t, 0)dt = -\alpha_2 < 0,$$

by hypothesis (A_4) we have

$$\int_{t_c^1}^{t^*} g_u(0, t, 0)dt = 0.$$

Hence it holds

$$\int_{t_0}^{t_0+T} g_u(u_0(t), t, 0)dt = -\alpha_1 - \alpha_2 < 0.$$

From our smoothness assumption (A_0) we get that $u_p(t, \varepsilon)$ and $g_u(u_p(t, \varepsilon), t, \varepsilon)$ depend continuously on ε. Hence, we can conclude that there is a sufficiently small positive ε_1, $\varepsilon_1 < \varepsilon_0$, such that for $0 < \varepsilon \leq \varepsilon_1$ any T-periodic solution $u_p(t, \varepsilon)$ of (2.1) located in G_ε fulfills

$$\int_{t_0}^{t_0+T} g_u(u_p(t, \varepsilon), t, \varepsilon)dt < 0.$$

Theorem 4.4 *Assume the hypotheses (P_1) and (P_2) to be valid. Then, for sufficiently small ε system* (4.1) *has a unique asymptotically stable periodic solution in G_ε.*

Proof. It is well-known that a T-periodic solution $u_p(t, \varepsilon)$ of the T-periodic scalar differential equation (4.1) is asymptotically stable if

$$\int_{t_0}^{t_0+T} g_u(u_p(t, \varepsilon), t, \varepsilon)dt < 0.$$

By Theorem 4.2 this relation is satisfied for any T-periodic solution $u_p(t, \varepsilon)$ of (4.1) located in G_ε for $0 < \varepsilon \leq \varepsilon_1$. That means, any such solution is asymptotically stable. Therefore, (4.1) has exactly one T-periodic solution in G_ε.

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