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Approximation of Wiener integrals with respect to the Brownian bridge by simulation of SDEs

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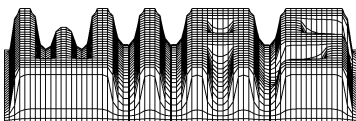   Leicester LE1 7RH, UK
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ABSTRACT. Numerical integration of stochastic differential equations together with the Monte Carlo technique is used to evaluate conditional Wiener integrals of exponential-type functionals. An explicit Runge-Kutta method of order four and implicit Runge-Kutta methods of order two are constructed. The corresponding convergence theorems are proved. To reduce the Monte Carlo error, a variance reduction technique is considered. Results of numerical experiments are presented.

1. Introduction

We consider Wiener integrals

$$(1.1) \qquad \mathcal{J} = \int_{C_{0,a;T,b}^d} F(x(\cdot)) \, d\mu_{0,a}^{T,b}(x)$$

of the exponential-type functionals

$$(1.2) \qquad F(x(\cdot)) = \exp \left[\int_0^T f(t, x(t)) \, dt \right].$$

Here $\mu_{0,a}^{T,b}(x)$ is a conditional Wiener measure which corresponds to the Brownian paths $X_{0,a}^{T,b}(t)$ with fixed initial and final points, i.e., it corresponds to the d-dimensional Brownian bridge from a at the time $t = 0$ into b at the time $t = T$. The integral (1.1) is understood in the sense of Lebesgue integral with respect to the measure $\mu_{0,a}^{T,b}(x)$ and is taken over the set $C_{0,a;T,b}^d$ of all d-dimensional continuous vector-functions $x(t)$ satisfying the conditions $x(0) = a$, $x(T) = b$ (see, e.g. [4]).

A relation of such integrals with quantum physics and some equations of mathematical physics can be found, e.g., in [4, 3, 2, 13, 9]. In particular, the Feynman path integral of the form

$$\mathcal{J} = \int \exp \left(\int_0^T \left[\frac{m \dot{x}^2(t)}{2} - V(x(t)) \right] dt \right) \mathcal{D}x(t)$$

is another writing of the integral (1.1)-(1.2) with $f = -V$.

Numerical evaluation of Wiener integrals is an important and difficult task. Many approaches are proposed for solving this problem (see, e.g. [2, 15, 1] and references therein). As a rule, the known numerical methods reduce a path integral to a high dimensional integral which is then approximated using either classical or Monte Carlo methods. The high order of these integrals makes calculation of the Wiener integrals extremely difficult.

In [5, 14, 10], the approach using numerical integration (in the weak sense) of stochastic differential equations with application of the Monte Carlo technique is proposed for computation of Wiener integrals of the form

$$(1.3) \qquad \mathcal{I} = \int_{C_{0,0}^d} F(x(\cdot)) \, d\mu_{0,0}(x),$$

where $\mu_{0,0}(x)$ is a Wiener measure corresponding to Brownian paths with the fixed initial point $(0, 0)$ and $F(x(\cdot)) = \varphi(x(T), \int_0^T f(t, x(t)) \, dt)$. The approach is based on the following

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probabilistic representation of the integral (1.3):

$$(1.4) \qquad \mathcal{I} = E\varphi(X_{0,0}(T), Z_{0,0,0}(T)),$$

where $X_{0,0}(t)$, $Z_{0,0,0}(t)$, $0 \le t \le T$, is the solution of the $(d+1)$-dimensional system of stochastic differential equations (SDEs)

$$(1.5) \qquad dX = dw(t), \ X_{0,0}(0) = 0,$$
$$dZ = f(t, X^1, \ldots, X^d)dt, \ Z_{0,0,0}(0) = 0,$$

and $w(t) = (w^1(t), \ldots, w^d(t))^\mathsf{T}$ is a d-dimensional standard Wiener process.

An efficiency of this approach is due to the fact that the system (1.5) has the fixed dimension d and the corresponding accuracy is reached by means of a choice of a method and a step of numerical integration and a number M of Monte Carlo simulations. Thus, the problem of calculating the infinite-dimensional Wiener integral $\mathcal{I}$ is reduced to the Cauchy problem (1.5). This problem can naturally be regarded as one-dimensional since it contains one independent variable only. We underline that in other methods the path integral is reduced to a high dimensional Riemann integral and the accuracy is reached on account of increasing its dimension. The approach based on the probabilistic representation (1.4)-(1.5) turned out to be especially effective in evaluating Wiener integrals (1.3) of exponential-type functionals due to the obtained fourth-order Runge-Kutta methods [5, 10].

Here the approach of [5, 14, 10] is developed for evaluating the conditional Wiener integral (1.1)-(1.2). The corresponding probabilistic representation contains a more complicated system than (1.5). The solution of this system gives a Markov representation of the Brownian bridge. The system is singular and this circumstance stipulates a certain complexity of theoretical proofs. Nevertheless the constructed fourth-order Runge-Kutta algorithms are equally simple and effective as in the case of the Wiener integral (1.3). The effectiveness of these algorithms allows us to evaluate integrals (1.1)-(1.2) for large dimension d.

In this paper we restrict ourselves to conditional Wiener integrals of exponential-type functionals although the approach can also be applied to conditional Wiener integrals of functionals of an integral type (cf. [14]).

In Section 2, a fourth-order explicit Runge-Kutta method is constructed. The corresponding convergence theorem is proved in Sections 3 and 4. Implicit methods of order 2 are derived in Section 5. These methods have an implicitness with respect to the linear part only which is easily analytically resolved. In our approach, there are two types of errors: the error of numerical integration and the Monte Carlo error. To reduce the Monte Carlo error, we consider the method of control variates in Section 6. Some results of numerical tests are presented in Section 7.

2. Explicit Runge-Kutta method of order 4

As it is known [7, 8], the d-dimensional Brownian bridge $X(t) = X_{0,a}(t) = X_{0,a}^{T,b}(t)$, $0 \le t \le T$, from a to b can be characterized as the pathwise unique solution of the system

of SDEs

(2.1) $$dX = \frac{b-X}{T-t}dt + dw(t), \ 0 \le t < T, \ X(0) = a,$$

with

(2.2) $$X(T) = b,$$

where $w(t) = (w^1(t), \dots, w^d(t))^\mathsf{T}$ is a d-dimensional standard $\mathcal{F}_t$-Wiener process. The system is considered on a probability space $(\Omega, \mathcal{F}, P)$, and $\mathcal{F}_t$, $0 \le t \le T$, is a non-decreasing family of σ-algebras of $\mathcal{F}$.

Let us also introduce the scalar equation

(2.3) $$dY = f(t, X(t))\, Y\, dt, \ 0 \le t \le T, \ Y(0) = 1,$$

where $X(t)$ is defined by (2.1)-(2.2) and $f(t, x)$ is the same as in (1.2). Then the Wiener integral (1.1)-(1.2) is equal to

(2.4) $$\mathcal{J} = EY(T)\,.$$

Thus, evaluation of the Wiener integral (1.1)-(1.2) is reduced to the problem of numerical integration of the system (2.1)-(2.3).

Introduce a discretization of the time interval $[0, T]$, for definiteness the equidistant one with a time step $h > 0$:

$$t_k = kh, \ \ k = 0, \dots, N, \ \ t_N = T,$$

and let $t_{k+1/2} := t_k + h/2$.

To get a higher order method for (2.1)-(2.3), we need to simulate the solution of (2.1) exactly. The solution of (2.1) is

(2.5) $$X(t) = a\frac{T-t}{T} + b\frac{t}{T} + (T-t)\int_0^t \frac{dw(s)}{T-s}\,.$$

Hence

(2.6) $$X(t+h) = X(t) + h\frac{b - X(t)}{T-t} + (T - t - h)\int_t^{t+h} \frac{dw(s)}{T-s}\,.$$

We have

(2.7) $$E\left[(T-t-h)\int_t^{t+h}\frac{dw(s)}{T-s}\, \middle|\, X(t)\right] = 0,$$

$$E\left[(T-t-h)\int_t^{t+h}\frac{dw(s)}{T-s}\, \middle|\, X(t)\right]^2 = \left(1 - \frac{h}{T-t}\right)h.$$

We can exactly simulate the solution of (2.1) by a simple recurrent procedure based on the formula

(2.8) $$X(t+h) = X(t) + h\frac{b - X(t)}{T-t} + h^{1/2}\sqrt{\frac{T-t-h}{T-t}}\,\xi,$$

where ξ is a random vector which components are Gaussian random variables with zero mean and unit variance and they are independent of $X(t)$.

Now let us formally apply a standard deterministic explicit fourth-order Runge-Kutta method to the equation (2.3) assuming that $X(t)$ is a known function. Then, taking into account (2.8), we obtain the following algorithm for integrating the system (2.1)-(2.3):

$$(2.9) \qquad X(0) = a \,,$$

$$X(t_{k+1/2}) = X(t_k) + \frac{h}{2} \frac{b - X(t_k)}{T - t_k} + \frac{h^{1/2}}{\sqrt{2}} \sqrt{\frac{T - t_{k+1/2}}{T - t_k}} \xi_{k+1/2} \,,$$

$$k = 0, \ldots, N - 1 \,,$$

$$X(t_{k+1}) = X(t_{k+1/2}) + \frac{h}{2} \frac{b - X(t_{k+1/2})}{T - t_{k+1/2}} + \frac{h^{1/2}}{\sqrt{2}} \sqrt{\frac{T - t_{k+1}}{T - t_{k+1/2}}} \xi_{k+1} \,,$$

$$k = 0, \ldots, N - 2 \,, \quad X(t_N) = b \,,$$

$$(2.10) \qquad Y_0 = 1 \,,$$

$$k_1 = f(t_k, X(t_k)) Y_k, \quad k_2 = f(t_{k+1/2}, X(t_{k+1/2})) \left[Y_k + h k_1 / 2 \right] \,,$$

$$k_3 = f(t_{k+1/2}, X(t_{k+1/2})) \left[Y_k + h k_2 / 2 \right] \,, \quad k_4 = f(t_{k+1}, X(t_{k+1})) \left[Y_k + h k_3 \right] \,,$$

$$Y_{k+1} = Y_k + \frac{h}{6} \left(k_1 + 2 k_2 + 2 k_3 + k_4 \right) \,, \quad k = 0, \ldots, N - 1 \,,$$

where $\xi_{k+1/2}$, ξ_{k+1} are d-dimensional random vectors which components are mutually independent random variables with standard normal distribution $\mathcal{N}(0, 1)$.

Since the function $X(t)$ is non-smooth, the deterministic result on the accuracy order of the involved Runge-Kutta method is not applicable here and a separate convergence theorem is needed. In the next two sections the following theorem is proved under some assumptions on the function $f(t, x)$ (see them after (3.4)).

Theorem 2.1. *The method (2.9)-(2.10) applied to evaluation of the Wiener integral (2.4) is of fourth order of accuracy, i.e.,*

$$(2.11) \qquad |\mathcal{J} - EY_N| = |EY(T) - EY_N| \leq K h^4,$$

where the constant K is independent of h.

3. Theorem on one-step error

In this section we consider a one-step error of the method (2.9)-(2.10). We use this analysis in the next section to prove Theorem 2.1 on global error of the method (2.9)-(2.10).

We say that a function $g(s, x)$, $s \in [0, T]$, $x \in \mathbf{R}^d$, belongs to the class $\mathbf{F}$ (with respect to the variable x), written as $g \in \mathbf{F}$, if there are constants $K > 0$ and $\kappa > 0$ such that for all $x \in \mathbf{R}^d$ the following inequality holds uniformly in $s \in [0, T]$:

$$(3.1) \qquad |g(s, x)| \leq K (1 + |x|^\kappa) \,.$$

Introduce the operator

$$(3.2) \qquad L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} \frac{b^i - x^i}{T - t} \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2}{(\partial x^i)^2} , \quad 0 \le t < T .$$

We observe that this operator contains singularity since the denominator $T - t$ tends to zero as t goes to T.

Consider the function

$$(3.3) \qquad u(t, x) = EY_{t,x,1}(T) .$$

It satisfies the Cauchy problem

$$(3.4) \qquad Lu + fu = 0 , \quad 0 \le t < T , \ x \in \mathbf{R}^d ,$$
$$u(T, x) = 1 .$$

We assume that the function $f(t, x)$ is sufficiently smooth, belongs to the class $\mathbf{F}$ together with its partial derivatives of a sufficiently high order and is such that the problem (3.4) has a unique solution which is sufficiently smooth and belongs to the class $\mathbf{F}$ together with its partial derivatives of a sufficiently high order. In addition, we suppose that $EY^2(t)$ exists and bounded on $[0, T]$ and that for all sufficiently small h the second moments EY_k^2 are uniformly bounded with respect to h. For instance, the latter conditions are satisfied when the function $f(t, x)$ is bounded. Therefore, theoretically, we can use Theorem 2.1, approximating $f(t, x)$ (if it is unbounded) by an appropriate bounded function.

Let g be a sufficiently smooth function belonging to the class $\mathbf{F}$ together with its partial derivatives up to a sufficiently high order. Then the expectations for a nonnegative integer m

$$EL^l g(\theta, X(\theta)), \ l = 0, 1, ..., m + 1,$$

$$E \left(\frac{\partial}{\partial x^i} L^l g(\theta, X(\theta)) \right)^2 , \ l = 0, 1, ..., m, \ i = 1, ..., d ,$$

exist and are continuous with respect to $0 \le \theta < T$. And the following formulas are also valid for $t \le s \le t + h < T$:

$$(3.5) \qquad E(g(t + h, X_{t,x}(t + h))|\mathcal{F}_s) = g(s, X_{t,x}(s))$$

$$+ (t + h - s)Lg(s, X_{t,x}(s)) + \cdots + \frac{(t + h - s)^m}{m!} L^m g(s, X_{t,x}(s))$$

$$+ \int_s^{t+h} \frac{(t + h - \theta)^m}{m!} E(L^{m+1} g(\theta, X_{t,x}(\theta))|\mathcal{F}_s) d\theta ,$$

$$(3.6) \qquad Eg(t + h, X_{t,x}(t + h)) = g(t, x) + hLg(t, x) + \cdots + \frac{h^m}{m!} L^m g(t, x)$$

$$+ \int_t^{t+h} \frac{(t + h - \theta)^m}{m!} EL^{m+1} g(\theta, X_{t,x}(\theta)) \, d\theta .$$

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The expansions (3.5) and (3.6) are analogous to an expansion of semigroups. Their proof is available in [10, p. 137].

It is convenient to introduce the additional notation for the approximation defined by (2.10): $\bar{Y}_{0,a,1}(t_k) = Y_k$ and also $\bar{Y}_{s,x,y}(t)$, $t \geq s$, by which we mean the approximation of (2.3) started from y at $t = s$ with $X(s) = x$.

It is not difficult to see that

(3.7)
$$Y_{t,x,y}(t + t') = yY_{t,x,1}(t + t'), \quad \bar{Y}_{t_k,x,y}(t_{k+k'}) = y\bar{Y}_{t_k,x,1}(t_{k+k'}),$$
$$EY_{t,x,y}(T) = yEY_{t,x,1}(T) = yu(t, x),$$

where $u(t, x)$ is the solution of the problem (3.4).

Recall that $t_0 = 0$, $X_0 = a$, $Y_0 = 1$. Using (3.3) and (3.7) and the fact that we simulate $X_k = X(t_k)$ exactly, we can represent the global error of the method (2.9)-(2.10) (cf. (2.11)) in the form

(3.8)
$$\left|EY_{0,a,1}(T) - E\bar{Y}_{0,a,1}(T)\right| = \left|EY_{t_0,X_0,Y_0}(T) - EY_N\right|$$
$$= \left|u(t_0, X_0)Y_0 - Eu(t_N, X_N)Y_N\right|$$
$$= \left|\sum_{k=0}^{N-1} \left[Eu(t_k, X(t_k))Y_k - Eu(t_{k+1}, X(t_{k+1}))\bar{Y}_{t_k,X_k,Y_k}(t_{k+1})\right]\right|$$
$$= \left|\sum_{k=0}^{N-1} EY_k\left[u(t_k, X(t_k)) - u(t_{k+1}, X(t_{k+1}))\bar{Y}_{t_k,X_k,1}(t_{k+1})\right]\right|$$
$$\leq \sum_{k=0}^{N-1} \left|EY_k\left[u(t_k, X(t_k)) - u(t_{k+1}, X_{t_k,X_k}(t_{k+1}))\bar{Y}_{t_k,X_k,1}(t_{k+1})\right]\right|.$$

We have

(3.9)
$$R_k := \left|EY_k\left[u(t_k, X(t_k)) - u(t_{k+1}, X_{t_k,X_k}(t_{k+1}))\bar{Y}_{t_k,X_k,1}(t_{k+1})\right]\right|$$
$$= \left|EY_kE\left[u(t_k, X_k) - u(t_{k+1}, X_{t_k,X_k}(t_{k+1}))\bar{Y}_{t_k,X_k,1}(t_{k+1})|\mathcal{F}_{t_k}\right]\right|.$$

First, we analyze R_k for $k = 0, \ldots, N - 2$. To this end, we consider the one-step error for $0 \leq t < T - h$:

(3.10)
$$r(t, x) := Eu(t + h, X_{t,x}(t + h))\bar{Y}_{t,x,1}(t + h) - u(t, x).$$

We rewrite (2.10) on a single step in the form:

(3.11)
$$\bar{Y}_{t,x,1}(t + h) = 1 + \frac{h}{6}\left(f_0 + 4f_{1/2} + f_1\right) + \frac{h^2}{6}\left(f_0 f_{1/2} + f_{1/2}^2 + f_{1/2}f_1\right)$$
$$+ \frac{h^3}{12}\left(f_0 f_{1/2}^2 + f_{1/2}^2 f_1\right) + \frac{h^4}{24}f_0 f_{1/2}^2 f_1,$$

where $f_0 := f(t, x)$, $f_{1/2} := f(t + h/2, X_{t,x}(t + h/2))$, and $f_1 := f(t + h, X_{t,x}(t + h))$.

Using (3.6), we get

(3.12) $Eu(t + h, X_{t,x}(t + h)) = u(t, x) + hLu(t, x) + \dfrac{h^2}{2}L^2 u(t, x) + \dfrac{h^3}{6}L^3 u(t, x)$

$$+ \dfrac{h^4}{24}L^4 u(t, x) + \int_t^{t+h} \dfrac{(t + h - \theta)^4}{24} EL^5 u(\theta, X_{t,x}(\theta))\, d\theta\,,$$

$$Ef_0 u(t + h, X_{t,x}(t + h)) = f_0 Eu(t + h, X_{t,x}(t + h))$$

$$= f_0 \left[u(t, x) + hLu(t, x) + \dfrac{h^2}{2}L^2 u(t, x) + \dfrac{h^3}{6}L^3 u(t, x) \right.$$

$$\left. + \int_t^{t+h} \dfrac{(t + h - \theta)^3}{6} EL^4 u(\theta, X_{t,x}(\theta))\, d\theta \right]\,,$$

$$Ef_1 u(t + h, X_{t,x}(t + h)) = f_0 u(t, x) + hL\,(fu)\,(t, x) + \dfrac{h^2}{2}L^2\,(fu)\,(t, x)$$

$$+ \dfrac{h^3}{6}L^3\,(fu)\,(t, x) + \int_t^{t+h} \dfrac{(t + h - \theta)^3}{6} EL^4\,(fu)\,(\theta, X_{t,x}(\theta))\, d\theta.$$

Further,

$$Ef_{1/2} u(t + h, X_{t,x}(t + h)) = E\left(f_{1/2} E\left[u(t + h, X_{t,x}(t + h))|\mathcal{F}_{t+h/2} \right] \right)\,,$$

and by (3.5) we obtain

$$E\left[u(t + h, X_{t,x}(t + h))|\mathcal{F}_{t+h/2} \right] = u(t + h/2, X_{t,x}(t + h/2)) + \dfrac{h}{2}Lu(t + h/2, X_{t,x}(t + h/2))$$

$$+ \dfrac{h^2}{8}L^2 u(t + h/2, X_{t,x}(t + h/2)) + \dfrac{h^3}{48}L^3 u(t + h/2, X_{t,x}(t + h/2))$$

$$+ \int_{t+h/2}^{t+h} \dfrac{(t + h - \theta)^3}{6} E\left[L^4 u(\theta, X_{t,x}(\theta))|\mathcal{F}_{t+h/2} \right] d\theta\,,$$

then

(3.13) $Ef_{1/2} u(t + h, X_{t,x}(t + h)) = f_0 u(t, x) + \dfrac{h}{2}L\,(fu)\,(t, x) + \dfrac{h^2}{8}L^2\,(fu)\,(t, x)$

$$+ \dfrac{h^3}{48}L^3\,(fu)\,(t, x) + \int_t^{t+h/2} \dfrac{(t + h/2 - \theta)^3}{6} EL^4\,(fu)\,(\theta, X_{t,x}(\theta))d\theta + \dfrac{h}{2}f_0 Lu(t, x)$$

$$+ \dfrac{h^2}{4}L\,(fLu)\,(t, x) + \dfrac{h^3}{16}L^2\,(fLu)\,(t, x) + \dfrac{h}{2}\int_t^{t+h/2} \dfrac{(t + h/2 - \theta)^2}{2} EL^3\,(fLu)\,(\theta, X_{t,x}(\theta))d\theta$$

$$+ \dfrac{h^2}{8}f_0 L^2 u(t, x) + \dfrac{h^3}{16}L\,(fL^2 u)\,(t, x) + \dfrac{h^2}{8}\int_t^{t+h/2} (t + h/2 - \theta)EL^2\,(fL^2 u)\,(\theta, X_{t,x}(\theta))d\theta$$

$$+ \dfrac{h^3}{48}f_0 L^3 u(t, x) + \dfrac{h^3}{48}\int_t^{t+h/2} EL\,(fL^3 u)\,(\theta, X_{t,x}(\theta))d\theta$$

$$+ Ef_{1/2}\int_{t+h/2}^{t+h} \dfrac{(t + h - \theta)^3}{6} L^4 u(\theta, X_{t,x}(\theta))\, d\theta\,.$$

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Analogously, we get

$$(3.14) \quad E f_0 f_{1/2} u(t+h, X_{t,x}(t+h)) = f_0^2 u(t,x) + \frac{h}{2} f_0 L \left(f u \right)(t,x) + \frac{h^2}{8} f_0 L^2 \left(f u \right)(t,x)$$

$$+ f_0 \int_t^{t+h/2} \frac{(t+h/2-\theta)^2}{2} E L^3 \left(f u \right)(\theta, X_{t,x}(\theta)) d\theta + \frac{h}{2} f_0^2 L u(t,x) + \frac{h^2}{4} f_0 L \left(f L u \right)(t,x)$$

$$+ \frac{h}{2} f_0 \int_t^{t+h/2} (t+h/2-\theta) E L^2 \left(f L u \right)(\theta, X_{t,x}(\theta)) d\theta + \frac{h^2}{8} f_0^2 L^2 u(t,x)$$

$$+ \frac{h^2}{8} f_0 \int_t^{t+h/2} E L \left(f L^2 u \right)(\theta, X_{t,x}(\theta)) d\theta + f_0 E f_{1/2} \int_{t+h/2}^{t+h} \frac{(t+h-\theta)^2}{2} L^3 u(\theta, X_{t,x}(\theta)) \, d\theta \,,$$

$$E f_{1/2}^2 u(t+h, X_{t,x}(t+h)) = f_0^2 u(t,x) + \frac{h}{2} L \left(f^2 u \right)(t,x) + \frac{h^2}{8} L^2 \left(f^2 u \right)(t,x)$$

$$+ \int_t^{t+h/2} \frac{(t+h/2-\theta)^2}{2} E L^3 \left(f^2 u \right)(\theta, X_{t,x}(\theta)) d\theta + \frac{h}{2} f_0^2 L u(t,x) + \frac{h^2}{4} L \left(f^2 L u \right)(t,x)$$

$$+ \frac{h}{2} \int_t^{t+h/2} (t+h/2-\theta) E L^2 \left(f^2 L u \right)(\theta, X_{t,x}(\theta)) d\theta + \frac{h^2}{8} f_0^2 L^2 u(t,x)$$

$$+ \frac{h^2}{8} \int_t^{t+h/2} E L \left(f^2 L^2 u \right)(\theta, X_{t,x}(\theta)) d\theta + E f_{1/2}^2 \int_{t+h/2}^{t+h} \frac{(t+h-\theta)^2}{2} L^3 u(\theta, X_{t,x}(\theta)) \, d\theta \,,$$

$$E f_{1/2} f_1 u(t+h, X_{t,x}(t+h)) = f_0^2 u(t,x) + \frac{h}{2} L \left(f^2 u \right)(t,x) + \frac{h^2}{8} L^2 \left(f^2 u \right)(t,x)$$

$$+ \int_t^{t+h/2} \frac{(t+h/2-\theta)^2}{2} E L^3 \left(f^2 u \right)(\theta, X_{t,x}(\theta)) d\theta + \frac{h}{2} f_0 L \left(f u \right)(t,x) + \frac{h^2}{4} L \left(f L \left(f u \right) \right)(t,x)$$

$$+ \frac{h}{2} \int_t^{t+h/2} (t+h/2-\theta) E L^2 \left(f L \left(f u \right) \right)(\theta, X_{t,x}(\theta)) d\theta + \frac{h^2}{8} f_0 L^2 \left(f u \right)(t,x)$$

$$+ \frac{h^2}{8} \int_t^{t+h/2} E L \left(f L^2 \left(f u \right) \right)(\theta, X_{t,x}(\theta)) d\theta + E f_{1/2} \int_{t+h/2}^{t+h} \frac{(t+h-\theta)^2}{2} L^3 \left(f u \right)(\theta, X_{t,x}(\theta)) \, d\theta \,,$$

$$E f_0 f_{1/2}^2 u(t+h, X_{t,x}(t+h)) = f_0^3 u(t,x) + \frac{h}{2} f_0 L \left(f^2 u \right)(t,x)$$

$$+ f_0 \int_t^{t+h/2} (t+h/2-\theta) E L^2 \left(f^2 u \right)(\theta, X_{t,x}(\theta)) d\theta + \frac{h}{2} f_0^3 L u(t,x)$$

$$+ \frac{h}{2} f_0 \int_t^{t+h/2} E L \left(f^2 L u \right)(\theta, X_{t,x}(\theta)) d\theta + f_0 E f_{1/2}^2 \int_{t+h/2}^{t+h} (t+h-\theta) L^2 u(\theta, X_{t,x}(\theta)) \, d\theta \,,$$

$$Ef_{1/2}^2 f_1 u(t+h, X_{t,x}(t+h)) = f_0^3 u(t,x) + \frac{h}{2} L \left(f^3 u \right)(t,x)$$

$$+ \int_t^{t+h/2} (t+h/2-\theta)EL^2 \left(f^3 u \right)(\theta, X_{t,x}(\theta))d\theta + \frac{h}{2} f_0^2 L(fu)(t,x)$$

$$+ \frac{h}{2} \int_t^{t+h/2} EL \left(f^2 L(fu) \right)(\theta, X_{t,x}(\theta))d\theta + Ef_{1/2}^2 \int_{t+h/2}^{t+h} (t+h-\theta)L^2 \left(fu \right)(\theta, X_{t,x}(\theta)) \, d\theta,$$

$$Ef_0 f_{1/2}^2 f_1 u(t+h, X_{t,x}(t+h)) = f_0^4 u(t,x) + f_0 \int_t^{t+h/2} EL \left(f^3 u \right)(\theta, X_{t,x}(\theta))d\theta$$

$$+ f_0 Ef_{1/2}^2 \int_{t+h/2}^{t+h} L \left(fu \right)(\theta, X_{t,x}(\theta)) \, d\theta.$$

Substituting (3.11)-(3.14) in (3.10), we obtain

$$r = h \left[Lu + fu \right] + \frac{h^2}{2} \left[L^2 u + L(fu) + fLu + f^2 u \right]$$

$$+ \frac{h^3}{6} \left[L^3 u + L^2(fu) + fL^2 u + fL \left(fu \right) + L \left(fLu \right) + L \left(f^2 u \right) + f^2 Lu + f^3 u \right]$$

$$+ \frac{h^4}{24} [L^4 u + L^3(fu) + fL^3 u + fL^2(fu) + f^2 L^2 u + f^2 L \left(fu \right) + f^3 Lu + f^4 u$$

$$+ L^2(fLu) + L^2(f^2 u) + fL(fLu) + fL(f^2 u) + L(fL^2 u) + L(fL(fu))$$

$$+ L(f^2 Lu) + L(f^3 u)] + \tilde{r},$$

where all the operators and functions are evaluated at the point (t,x) and

(3.15)
$$\tilde{r}(t,x) = \int_t^{t+h} \frac{(t+h-\theta)^4}{24} EL^5 u(\theta, X_{t,x}(\theta)) \, d\theta$$

$$+ \frac{h}{6} \left[f_0 \int_t^{t+h} \frac{(t+h-\theta)^3}{6} EL^4 u(\theta, X_{t,x}(\theta)) \, d\theta\theta \right.$$

$$+ 4 \int_t^{t+h/2} \frac{(t+h/2-\theta)^3}{6} EL^4 \left(fu \right)(\theta, X_{t,x}(\theta)) d$$

$$+ 2h \int_t^{t+h/2} \frac{(t+h/2-\theta)^2}{2} EL^3 \left(fLu \right)(\theta, X_{t,x}(\theta)) d\theta$$

$$+ \frac{h^2}{2} \int_t^{t+h/2} (t+h/2-\theta)EL^2 \left(fL^2 u \right)(\theta, X_{t,x}(\theta))d\theta + \frac{h^3}{12} \int_t^{t+h/2} EL \left(fL^3 u \right)(\theta, X_{t,x}(\theta))d\theta$$

$$+ 4Ef_{1/2} \int_{t+h/2}^{t+h} \frac{(t+h-\theta)^3}{6} L^4 u(\theta, X_{t,x}(\theta)) \, d\theta + \int_t^{t+h} \frac{(t+h-\theta)^3}{6} EL^4 \left(fu \right)(\theta, X_{t,x}(\theta)) \, d\theta \right]$$

$$+ \frac{h^2}{6} \left[f_0 \int_t^{t+h/2} \frac{(t+h/2-\theta)^2}{2} EL^3\left(fu\right)(\theta, X_{t,x}(\theta))d\theta \right.$$

$$+ \frac{h}{2} f_0 \int_t^{t+h/2} (t+h/2-\theta) EL^2\left(fLu\right)(\theta, X_{t,x}(\theta))d\theta$$

$$+ \frac{h^2}{8} f_0 \int_t^{t+h/2} EL\left(fL^2u\right)(\theta, X_{t,x}(\theta))d\theta$$

$$+ f_0 E f_{1/2} \int_{t+h/2}^{t+h} \frac{(t+h-\theta)^2}{2} L^3 u(\theta, X_{t,x}(\theta))\, d\theta$$

$$+ \int_t^{t+h/2} \frac{(t+h/2-\theta)^2}{2} EL^3\left(f^2u\right)(\theta, X_{t,x}(\theta))d\theta$$

$$+ \frac{h}{2} \int_t^{t+h/2} (t+h/2-\theta) EL^2\left(f^2Lu\right)(\theta, X_{t,x}(\theta))d\theta$$

$$+ \frac{h^2}{8} \int_t^{t+h/2} EL\left(f^2L^2u\right)(\theta, X_{t,x}(\theta))d\theta$$

$$+ E f_{1/2}^2 \int_{t+h/2}^{t+h} \frac{(t+h-\theta)^2}{2} L^3 u(\theta, X_{t,x}(\theta))\, d\theta$$

$$+ \int_t^{t+h/2} \frac{(t+h/2-\theta)^2}{2} EL^3\left(f^2u\right)(\theta, X_{t,x}(\theta))d\theta$$

$$+ \frac{h}{2} \int_t^{t+h/2} (t+h/2-\theta) EL^2\left(fL\left(fu\right)\right)(\theta, X_{t,x}(\theta))d\theta$$

$$+ \frac{h^2}{8} \int_t^{t+h/2} EL\left(fL^2\left(fu\right)\right)(\theta, X_{t,x}(\theta))d\theta$$

$$\left. + E f_{1/2} \int_{t+h/2}^{t+h} \frac{(t+h-\theta)^2}{2} L^3\left(fu\right)(\theta, X_{t,x}(\theta))\, d\theta \right]$$

$$+ \frac{h^3}{12} \left[f_0 \int_t^{t+h/2} (t+h/2-\theta) EL^2\left(f^2u\right)(\theta, X_{t,x}(\theta))d\theta \right.$$

$$+ \frac{h}{2} f_0 \int_t^{t+h/2} EL\left(f^2Lu\right)(\theta, X_{t,x}(\theta))d\theta$$

$$+ f_0 E f_{1/2}^2 \int_{t+h/2}^{t+h} (t+h-\theta) L^2 u(\theta, X_{t,x}(\theta))\, d\theta$$

$$+ \int_t^{t+h/2} (t+h/2-\theta) EL^2\left(f^3u\right)(\theta, X_{t,x}(\theta))d\theta$$

$$+\frac{h}{2}\int_t^{t+h/2} EL\left(f^2L(fu)\right)(\theta, X_{t,x}(\theta))d\theta + Ef_{1/2}^2\int_{t+h/2}^{t+h}(t+h-\theta)L^2\left(fu\right)(\theta, X_{t,x}(\theta))\,d\theta\Bigg]$$

$$+\frac{h^4}{24}\left[f_0\int_t^{t+h/2} EL\left(f^3u\right)(\theta, X_{t,x}(\theta))d\theta + f_0Ef_{1/2}^2\int_{t+h/2}^{t+h} L\left(fu\right)(\theta, X_{t,x}(\theta))\,d\theta\right].$$

Taking into account that $u(t,x)$ satisfies the equation from (3.4), we get

$$(3.16) \qquad\qquad\qquad r(t,x) = \tilde{r}(t,x)\,.$$

If the terms in the one-step error $r(t,x)$ of the method (2.9)-(2.10) (i.e., the terms in (3.15)) were bounded by $K(x)h^5$, $K(x) \in \mathbf{F}$, for all $t \leq T - h$, the relations (3.8)-(3.10) would imply that $\sum_{k=0}^{N-2} R_k \leq Ch^4$, where C is independent of h. But we see that the one-step error consists of integrals with integrands containing terms of the form $A(t,x) = L^n\left(q_1 L^l q_2\right)(t,x)$, where $q_1(t,x)$ and $q_2(t,x)$ are some functions from the class $\mathbf{F}$. The functions $A(t,x)$ belong to the class $\mathbf{F}$ for $t \in [0, T_*]$, where $T_* < T$ is a fixed (independent of h) time moment. Then $|r(t,x)| \leq K(x)h^5$, $K(x) \in \mathbf{F}$, $t \in [0, T_*]$, with $K(x)$ depending on T_*. However, the functions $A(t,x)$ do not belong to the class $\mathbf{F}$ for $t \in [0, T)$ due to the singularity in L (see (3.2)). Consequently, $r(t,x)$ can not be bounded by $K(x)h^5$, $K(x) \in \mathbf{F}$, for all $t < T$, and a more detailed analysis of the one-step error is required to prove the convergence theorem. In particular, we need to consider the structure of the functions $A(t,x)$ in detail. We always assume that L^0 is an identity operator.

Lemma 3.1. *Let $q_1(t,x)$ and $q_2(t,x)$ be sufficiently smooth functions belonging to the class $\mathbf{F}$ together with their partial derivatives of a sufficiently high order. Then for $0 \leq t < T$:*

$$(3.17) \qquad L^n\left(q_1 L^l q_2\right)(t,x) = g_0(t,x) + \sum_{j=1}^m \sum_{\alpha_j} g_{\alpha_j}(t,x)\psi^{\alpha_j}(t,x)\,,$$

$$l, n = 0, 1, \ldots,\ m = l + n,$$

where α_j is a multiindex such that $\alpha_j = (i_1, \ldots, i_j)$ and each i_k is from $\{1, \ldots, d\}$, the summation in (3.17) is over all possible values of α_j, g_0 and g_{α_j} are some functions from the class $\mathbf{F}$, and

$$(3.18) \qquad\qquad\qquad \psi^r = \frac{b^r - x^r}{T - t},\ r = 1, \ldots, d,$$

$$\psi^{\alpha_{j+1}} = \frac{b^{i_{j+1}} - x^{i_{j+1}}}{T - t}\psi^{\alpha_j} + \frac{\partial}{\partial x^r}\psi^{\alpha_j},\ \alpha_j = (i_1, \ldots, i_j),$$

$$\alpha_{j+1} = (i_1, \ldots, i_j, i_{j+1}),\ j = 1, 2, \ldots,$$

and for all α_j

$$(3.19) \qquad\qquad\qquad L\psi^{\alpha_j} = 0\,.$$

Proof. We prove the lemma by induction and start with (3.19). For $j = 1$ the (3.19) is easily verified. Suppose (3.19) is true for all j and prove it for $j + 1$. We have

$$L\psi^{\alpha_{j+1}} = L\left(\frac{b^{i_{j+1}} - x^{i_{j+1}}}{T-t}\psi^{\alpha_j} + \frac{\partial}{\partial x^r}\psi^{\alpha_j}\right)$$

$$= \psi^{\alpha_j} L\frac{b^{i_{j+1}} - x^{i_{j+1}}}{T-t} + \frac{b^{i_{j+1}} - x^{i_{j+1}}}{T-t}L\psi^{\alpha_j} - \frac{1}{T-t}\frac{\partial}{\partial x^{i_{j+1}}}\psi^{\alpha_j}$$

$$+ \frac{\partial}{\partial x^{i_{j+1}}}L\psi^{\alpha_j} + \frac{1}{T-t}\frac{\partial}{\partial x^{i_{j+1}}}\psi^{\alpha_j} = 0.$$

Now we prove (3.17). For $m = 1$ the relations (3.17) are evident. Assume they to hold for all $m = n + l$; we will prove them for $m + 1$. If $l \le m$, we have $n \ge 0$ and

$$L^{n+1}(q_1 L^l q_2) = L\left(g_0 + \sum_{j=1}^{m}\sum_{\alpha_j} g_{\alpha_j}\psi^{\alpha_j}\right)$$

$$= Lg_0 + \sum_{j=1}^{m}\sum_{\alpha_j}\left(\psi^{\alpha_j} Lg_{\alpha_j} + \sum_{r=1}^{d}\frac{\partial g_{\alpha_j}}{\partial x^r}\frac{\partial}{\partial x^r}\psi^{\alpha_j}\right)$$

$$= \tilde{g}_0 + \sum_{j=1}^{m}\sum_{\alpha_j}\tilde{g}_{\alpha_j}\psi^{\alpha_j} + \sum_{j=1}^{m}\sum_{\alpha_j}\sum_{r=1}^{d}\left(\psi^{\alpha_j}\frac{b^r - x^r}{T-t} + \frac{\partial}{\partial x^r}\psi^{\alpha_j}\right)\frac{\partial g_{\alpha_j}}{\partial x^r}$$

$$= \tilde{g}_0 + \sum_{j=1}^{m}\sum_{\alpha_j}\tilde{g}_{\alpha_j}\psi^{\alpha_j} + \sum_{j=1}^{m}\sum_{\alpha_{j+1}}\psi^{\alpha_{j+1}}\breve{g}_{\alpha_{j+1}},$$

i.e., we get (3.17) with the new functions g_0, g_{α_j} from the class **F**. If $l = m + 1$ then we deal with $q_1 L^{m+1} q_2$ which also has the form (3.17) due to the just proved result. $\square$

Corollary 3.1. We have for $t \le \theta < T$:

$$E\psi^{\alpha_j}(\theta, X_{t,x}(\theta)) = \psi^{\alpha_j}(t, x),$$

i.e., $\psi^{\alpha_j}(\theta, X_{0,a}(\theta))$ is a martingale.

Let us now consider some other properties of the functions $\psi^{\alpha_j}(t, x)$. We note that $\psi^{\alpha_j}(t, x)$ does not depend on the order of $i_1, \ldots, i_j$ in α_j (to see this it is enough to show that $\psi^{(i_1,\ldots,i_{j-2},l,r)}(t, x) = \psi^{(i_1,\ldots,i_{j-2},r,l)}(t, x)$). Introduce the function $\chi(r, \alpha_j)$, $\alpha_j = (i_1, \ldots, i_j)$, which is equal to the number of appearances of r in the set $\{i_1, \ldots, i_j\}$. In what follows we will sometimes denote by the same α_j different multiindices having the length j, and therefore functions ψ^{α_j} may differ although they have the same notation.

Lemma 3.2. *We have for $t < T$:*

$$(3.20) \qquad \frac{\partial}{\partial x^r}\psi^{\alpha_j}(t, x) = -\frac{\chi(r, \alpha_j)}{T-t}\psi^{\alpha_{j-1}}(t, x), \quad j = 2, 3, \ldots.$$

Proof. Let $j = 2$ and $\alpha_2 = (i_1, i_2)$. Then (see (3.18)):

$$\frac{\partial}{\partial x^r} \psi^{(i_1, i_2)}(t, x) = \frac{\partial}{\partial x^r} \left(\psi^{(i_2)} \psi^{(i_1)} + \frac{\partial}{\partial x^{i_2}} \psi^{(i_1)} \right) = \frac{\partial}{\partial x^r} \left(\psi^{(i_2)} \psi^{(i_1)} \right)$$

$$= -\frac{1}{T-t} (\delta_{r,i_2} \psi^{(i_1)} + \delta_{r,i_1} \psi^{(i_2)}) = -\frac{\chi(r, \alpha_2)}{T-t} \psi^{\alpha_1},$$

where $\delta_{r,i}$ is the Kronecker symbol. Thus, (3.20) holds for $j = 2$. Note that here α_1 is either (i_1) or (i_2).

Suppose that (3.20) is valid for all $2 \leq j \leq l$. We have

$$\frac{\partial}{\partial x^r} \psi^{(i_1, \ldots, i_{l+1})}(t, x) = \frac{\partial}{\partial x^r} \left(\psi^{(i_{l+1})} \psi^{(i_1, \ldots, i_l)} + \frac{\partial}{\partial x^{i_{l+1}}} \psi^{(i_1, \ldots, i_l)} \right)$$

$$= -\frac{\delta_{r,i_{l+1}}}{T-t} \psi^{\alpha_l} - \frac{b^{i_{l+1}} - x^{i_{l+1}}}{T-t} \frac{\chi(r, \alpha_l)}{T-t} \psi^{\alpha_{l-1}} - \frac{\chi(r, \alpha_l)}{T-t} \frac{\partial}{\partial x^{i_{l+1}}} \psi^{\alpha_{l-1}}$$

$$= -\frac{\delta_{r,i_{l+1}}}{T-t} \psi^{\alpha_l} - \frac{\chi(r, \alpha_l)}{T-t} \psi^{\alpha_l} = -\frac{\chi(r, \alpha_{l+1})}{T-t} \psi^{\alpha_l} .$$

$\square$

Corollary 3.2. We have for $t < T$:

(3.21) $$\psi^{\alpha_{j+1}}(t, x) = \frac{b^r - x^r}{T-t} \psi^{\alpha_j}(t, x) - \frac{\chi(r, \alpha_j)}{T-t} \psi^{\alpha_{j-1}}(t, x), \; j > 1,$$

(3.22) $$(b^r - x^r) \psi^{\alpha_j}(t, x) = (T-t) \psi^{\alpha_{j+1}}(t, x) + \chi(r, \alpha_j) \psi^{\alpha_{j-1}}(t, x), \; j > 1,$$

(3.23) $$\left(b^l - x^l \right) (b^r - x^r) \psi^{\alpha_j}(t, x) = (T-t)^2 \, \psi^{\alpha_{j+2}}(t, x) + \chi(l, \alpha_{j+1}) (T-t) \psi^{\alpha_j}(t, x)$$
$$+ \chi(r, \alpha_j) (T-t) \psi^{\alpha_j}(t, x) + \chi(r, \alpha_j) \chi(l, \alpha_{j-1}) \psi^{\alpha_{j-2}}(t, x), \;\; j > 2.$$

Note that ψ^{α_j} in (3.23) are, in general, different. We do not distinguish them because in the following analysis we will concern with the length of multiindices only.

Lemma 3.3. *We have for* $\theta < T$:

(3.24)
$$\left(E \left[\left(b^{r_1} - X_{0,a}^{r_1}(\theta) \right) \times \cdots \times \left(b^{r_l} - X_{0,a}^{r_l}(\theta) \right) \times \psi^{\alpha_j}(\theta, X_{0,a}(\theta)) \right]^{2n} \right)^{1/(2n)} \leq C \cdot (T - \theta)^{(l-j)/2},$$

$$j = 1, 2, \ldots, \;\; l = 0, 1, \ldots, \;\; n = 1, 2, \ldots,$$

where the constant $C > 0$ *is independent of* θ *(of course, it depends on* n*).*

Proof. Using (2.5), it is not difficult to show that

$$E(X_{0,a}^i(\theta) - b^i)^{2n} \leq C \cdot (T - \theta)^n ,$$

where the constant C is independent of θ. Then, taking into account that $X_{0,a}^{r_i}(\theta)$ and $X_{0,a}^{r_j}(\theta)$ are independent for $r_i \neq r_j$, we get

(3.25) $$E \left[\left(b^{r_1} - X_{0,a}^{r_1}(\theta) \right) \times \cdots \times \left(b^{r_l} - X_{0,a}^{r_l}(\theta) \right) \right]^{2n} \leq C \cdot (T - \theta)^{nl} ,$$

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even if some of r_i in (3.25) coincide.

We have

$$(3.26) \qquad E\left[\psi^{\alpha_1}(\theta, X_{0,a}(\theta))\right]^{2n} = \frac{E(b^i - X_{0,a}^i(\theta))^{2n}}{(T - \theta)^{2n}} \leq C \cdot (T - \theta)^{-n} ,$$

and it is also easy to verify that

$$(3.27) \qquad E\left[\psi^{\alpha_2}(\theta, X_{0,a}(\theta))\right]^{2n} \leq C \cdot (T - \theta)^{-2n} .$$

By (3.23), (3.26)-(3.27), and the Cauchy-Bunyakovskii inequality, we prove by induction that

$$(3.28) \qquad E\left[\psi^{\alpha_j}(\theta, X_{0,a}(\theta))\right]^{2n} \leq C \cdot (T - \theta)^{-nj} .$$

Finally, applying the Cauchy-Bunyakovskii inequality to the left-hand side of (3.24) and using (3.25) and (3.28), we obtain the estimate (3.24). $\square$

Now we are in position to make a detailed analysis of the remainder $r(t, x) = \tilde{r}(t, x)$ from (3.15). Let us recall that the one-step error consists of integrals with integrands containing terms of the form $A(t, x) = L^n\left(q_1 L^l q_2\right)(t, x)$, where $q_1(t, x)$ and $q_2(t, x)$ are some functions from the class $\mathbf{F}$. Since $L^5 u = -L^4(fu)$ (cf. (3.4)), the number $m = l + n$ for all the terms $A(t, x)$ participating in the remainder is less than or equal to 4. Using Lemma 3.1, we can represent the term $A = L^n(q_1 L^l q_2)(\theta, X_{t,x}(\theta))$ as

$$(3.29) \qquad A(\theta, X_{t,x}(\theta)) = g_0(\theta, X_{t,x}(\theta)) + \sum_{j=1}^{m} \sum_{\alpha_j} g_{\alpha_j}(\theta, X_{t,x}(\theta))\psi^{\alpha_j}(\theta, X_{t,x}(\theta)) .$$

By Lemma 3.3 (see (3.28)), we get that

$$\left(E\left[A(\theta, X_{0,a}(\theta))\right]^{2n}\right)^{1/(2n)} \leq \frac{C}{(T - \theta)^{m/2}} ,$$

where the constant C is independent of θ.

Consequently (recall that $m \leq 4$), we obtain the following estimate

$$(3.30) \qquad \left(E\left[r(t, X_{0,a}(t); h)\right]^{2n}\right)^{1/2n} \leq \frac{Ch^5}{(T - t - h)^2} .$$

Using this rough estimate, we can show (see further discussion in Remark 4.2) that the method (2.9)-(2.10) is at least of order 3. To prove the fourth-order of its convergence, a more sophisticated analysis based on extraction of singularity is needed.

To clarify the matter, we consider, for example, the following integral from the remainder $r(t, x)$ (see (3.15)):

(3.31)
$$\int_t^{t+h} \frac{(t+h-\theta)^4}{24} EL^5 u(\theta, X_{t,x}(\theta)) \, d\theta = -\int_t^{t+h} \frac{(t+h-\theta)^4}{24} EL^4 (fu)(\theta, X_{t,x}(\theta)) \, d\theta$$

$$= -\int_t^{t+h} \frac{(t+h-\theta)^4}{24} \left[Eg_0(\theta, X_{t,x}(\theta)) + \sum_{j=1}^4 \sum_{\alpha_j} Eg_{\alpha_j}(\theta, X_{t,x}(\theta)) \psi^{\alpha_j}(\theta, X_{t,x}(\theta)) \right] d\theta .$$

We will demonstrate the extraction of singularity analyzing the term with the highest singularity $g_{\alpha_4}(\theta, X_{t,x}(\theta)) \psi^{\alpha_4}(\theta, X_{t,x}(\theta))$. The singularity of $\psi^{\alpha_4}(\theta, X_{t,x}(\theta))$ is of order 2 (see (3.28)), i.e.,

$$\left(E[\psi^{\alpha_4}(\theta, X_{0,a}(\theta))]^2 \right)^{1/2} \le \frac{C}{(T-\theta)^2} .$$

At the same time, $\psi^{\alpha_4}(\theta, X_{0,a}(\theta))$ is a martingale (see Corollary 3.1) and $E\psi^{\alpha_4}(\theta, X_{0,a}(\theta)) = \psi^{\alpha_4}(0, a)$, which is a constant independent of θ. To exploit this property of ψ^{α_4} in further analysis, we expand $g(\theta, X_{t,x}(\theta)) := g_{\alpha_4}(\theta, X_{t,x}(\theta))$ at (T, b) :

(3.32) $\quad g(\theta, X_{t,x}(\theta)) = g(T, b) + \frac{\partial g}{\partial t}(T, b)(\theta - T) + \sum_{r=1}^d \frac{\partial g}{\partial x^r}(T, b)\left(X_{t,x}^r(\theta) - b^r \right)$

$$+\frac{1}{2} \sum_{r_1, r_2=1}^d \frac{\partial^2 g}{\partial x^{r_1} \partial x^{r_2}}(T, b)\left(X_{t,x}^{r_1}(\theta) - b^{r_1} \right)\left(X_{t,x}^{r_2}(\theta) - b^{r_2} \right)$$

$$+\frac{1}{2} \frac{\partial^2 g}{\partial t^2}(\vartheta, X_{t,x}(\theta))(\theta - T)^2 + \sum_{r=1}^d \frac{\partial^2 g}{\partial t \partial x^r}(T, \eta_1)(\theta - T)\left(X_{t,x}^r(\theta) - b^r \right)$$

$$+\frac{1}{6} \sum_{r_1, r_2, r_3=1}^d \frac{\partial^3 g}{\partial x^{r_1} \partial x^{r_2} \partial x^{r_3}}(T, \eta_2)\left(X_{t,x}^{r_1}(\theta) - b^{r_1} \right)\left(X_{t,x}^{r_2}(\theta) - b^{r_2} \right)\left(X_{t,x}^{r_3}(\theta) - b^{r_3} \right) ,$$

where ϑ is a time between θ and T and η_1 and η_2 are points between $X_{t,x}(\theta)$ and b.

Then, using Corollaries 3.1 and 3.2, we obtain

(3.33) $\quad \int_t^{t+h} \frac{(t+h-\theta)^4}{24} Eg_{\alpha_4}(\theta, X_{t,x}(\theta)) \psi^{\alpha_4}(\theta, X_{t,x}(\theta)) \, d\theta$

$$= \int_t^{t+h} \left[\frac{(t+h-\theta)^4}{24} g(T, b)\psi^{\alpha_4}(t, x) + \frac{\partial g}{\partial t}(T, b)(\theta - T)\psi^{\alpha_4}(t, x) \right.$$

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$$-\sum_{r=1}^{d} \frac{\partial g}{\partial x^r}(T,b)\left\{(T-\theta)\,\psi^{\alpha_5}(t,x) + \chi(r,\alpha_4)\,\psi^{\alpha_3}(t,x)\right\}$$

$$+\frac{1}{2}\sum_{r_1,r_2=1}^{d} \frac{\partial^2 g}{\partial x^{r_1}\partial x^{r_2}}(T,b)\left\{(T-\theta)^2\,\psi^{\alpha_6}(t,x) + \chi(r_1,\alpha_5)\,(T-\theta)\,\psi^{\alpha_4}(t,x)\right.$$

$$\left.+\chi(r_2,\alpha_4)\,(T-\theta)\,\psi^{\alpha_4}(t,x) + \chi(r_2,\alpha_4)\chi(r_1,\alpha_3)\,\psi^{\alpha_2}(t,x)\right\}\Big]\,d\theta + E\rho'(t,x;h)$$

$$= h^5 S'(t,x) + E\rho'(t,x;h)\,,$$

where $S'(t,x)$ is a linear combination of the functions $\psi^{\alpha_2}(t,x)$, $\psi^{\alpha_3}(t,x)$, $\psi^{\alpha_4}(t,x)$, $(T-t)\psi^{\alpha_4}(t,x)$, $h\psi^{\alpha_4}(t,x)$, $(T-t)\psi^{\alpha_5}(t,x)$, $h\psi^{\alpha_5}(t,x)$, $(T-t)^2\psi^{\alpha_6}(t,x)$, $(T-t)h\psi^{\alpha_6}(t,x)$, $h^2\psi^{\alpha_6}(t,x)$, coefficients in this linear combination are independent of t, x, and h;

$$E\rho'(t,x;h) = E\int_t^{t+h} \frac{(t+h-\theta)^4}{24}\psi^{\alpha_4}(\theta,X_{t,x}(\theta)) \times \left[\frac{1}{2}\frac{\partial^2 g}{\partial t^2}(\vartheta, X_{t,x}(\theta))\,(\theta-T)^2\right.$$

$$+\sum_{r=1}^{d} \frac{\partial^2 g}{\partial t\partial x^r}(T,\eta_1)\,(\theta-T)\,\left(X_{t,x}^r(\theta) - b^r\right)$$

$$+\frac{1}{6}\sum_{r_1,r_2,r_3=1}^{d} \frac{\partial^3 g}{\partial x^{r_1}\partial x^{r_2}\partial x^{r_3}}(T,\eta_2)\,\left(X_{t,x}^{r_1}(\theta) - b^{r_1}\right)\left(X_{t,x}^{r_2}(\theta) - b^{r_2}\right)\left(X_{t,x}^{r_3}(\theta) - b^{r_3}\right)\right]d\theta\,.$$

Using the Cauchy-Bunyakovskii inequality and Lemma 3.3, we obtain that

$$(3.34) \qquad \left(E\left[\rho'(t,X_{0,a}(t);h)\right]^{2n}\right)^{1/2n} \leq \frac{Ch^5}{\sqrt{T-t-h}}\,.$$

Thus, we extract the singularity by presenting the integral (3.33) as the sum of the singular part $S'(t,x)$ and the remainder. The singular part contains singularities of order from 1 to 4, while the remainder has non-singular terms and terms with singularity of order $1/2$. By further expansion of $g(\theta,X_{t,x}(\theta))$ (cf. (3.32)), we could also include the singularity of order $1/2$ in the singular part making the remainder singular-free. But for our purposes (i.e., for proving Theorem 2.1) the obtained expression (3.33) is sufficient.

We similarly analyze the other terms in the integral (3.31). Note that ψ^{α_1} has singularity of order $1/2$ and we include it in the remainder. So, S' does not contain any ψ^{α_1}. Analogously, we consider all the integrals of (3.15). In the case of an integral from $t+h/2$ to $t+h$, we first take the conditional expectation of the term like A with respect to $\mathcal{F}_{t+h/2}$ in a similar way as above and then we repeat the procedure once again taking the expectation of the product of the conditional expectation and $f_{1/2}$ (or $f_{1/2}^2$). As a result, we obtain an expression like the right-hand side of (3.33). Thus, we have proved the following theorem on one-step error.

Theorem 3.3. *The one-step error of the method (2.9)-(2.10) can be written in the form*

$$(3.35) \qquad r(t,x) = h^5 S(t,x) + E\rho(t,x;h)\,,$$

where $S(t,x)$ is a linear combination of the functions $\psi^{\alpha_2}(t,x)$, $\psi^{\alpha_3}(t,x)$, $\psi^{\alpha_4}(t,x)$, $(T-t)\psi^{\alpha_4}(t,x)$, $h\psi^{\alpha_4}(t,x)$, $(T-t)\psi^{\alpha_5}(t,x)$, $h\psi^{\alpha_5}(t,x)$, $(T-t)^2\psi^{\alpha_6}(t,x)$, $(T-t)h\psi^{\alpha_6}(t,x)$,

$h^2 \psi^{\alpha_6}(t,x)$, *coefficients in this linear combination are independent of t, x, and h; $\rho(t,x;h)$ is such that*

$$\left(E\left[\rho(t,X_{0,a}(t);h)\right]^{2n}\right)^{1/2n} \leq \frac{Ch^5}{\sqrt{T-t-h}}$$

with a constant C independent of t and h.

4. Proof of the convergence theorem

According to (3.9) and (3.10), we have

$$R_k = |EY_k r(t_k, X_k)|$$

with $r(t,x)$ from (3.35).

Using the rough estimate (3.30) and the Cauchy-Bunyakovskii inequality, we get straight-forward that (recall that we assume uniform boundedness of the moments EY_k^2) :

(4.1) $$R_k \leq \frac{Kh^5}{(T-t_{k+1})^2}\,, \quad k=0,1,\ldots,N-2\,.$$

But to prove (2.11), a more accurate estimate of R_k is needed. We obtain such an estimate using Theorem 3.3.

Lemma 4.1. *We have*

(4.2) $$R_k \leq \frac{Kh^5}{\sqrt{T-t_{k+1}}}, \quad k=0,\ldots,N-2\,,$$

where K is independent of k and h.

Proof. Note that we use the same letters C and K for different constants which are independent of h and k.

By Theorem 3.3 and the Cauchy-Bunyakovskii inequality, we obtain

(4.3) $$R_k = |EY_k r(t_k, X_k)| = \left|EY_k\left[h^5 S_k + \rho(t_k, X_k; h)\right]\right|$$

$$\leq h^5 |EY_k S_k| + \frac{Ch^5}{\sqrt{T-t_{k+1}}}\,,$$

where $S_k := S(t_k, X_k)$, $S(t,x)$ and $\rho(t,x;h)$ are from (3.35). Recall that S_k has singularity of order 2, more precisely

$$\left(E\left[S_k\right]^2\right)^{1/2} \leq \frac{C}{(T-t_k)^2}\,.$$

Let $F_i := F\left(t_{i-1}, t_{i-1/2}, t_i, X_{i-1}, X_{i-1/2}, X_i\right)$ be the function defined by the method (2.10) (see also (3.11)), i.e., the last line of (2.10) for $k=i-1$ can be written as

$$Y_i = Y_{i-1} + hF_i\,.$$

Introduce $S_{k,i} := E\left(S_k|\mathcal{F}_{t_i}\right)$, $i < k$. Due to Theorem 3.3 and Corollary 3.1, $S_{k,i}$ is a linear combination of $\psi^{\alpha_2}(t_i, X_i)$, $\psi^{\alpha_3}(t_i, X_i)$, $\psi^{\alpha_4}(t_i, X_i)$, $(T-t_k)\psi^{\alpha_4}(t_i, X_i)$, $h\psi^{\alpha_4}(t_i, X_i)$, $(T-t_k)\psi^{\alpha_5}(t_i, X_i)$, $h\psi^{\alpha_5}(t_i, X_i)$, $(T-t_k)^2\psi^{\alpha_6}(t_i, X_i)$, $(T-t_k)h\psi^{\alpha_6}(t_i, X_i)$, $h^2\psi^{\alpha_6}(t_i, X_i)$,

coefficients in this linear combination are independent of t_k, t_i, x, and h. Consequently (cf. (3.28))

$$\left(E \left[S_{k,i} \right]^2 \right)^{1/2} \leq \frac{C}{\left(T - t_i \right)^2} \,.$$

We see that though $S_{k,i}$ has the same order of singularity as S_k, the singularity is shifted. Roughly speaking, $S_{k,i}$ is less singular than $S_{k,i+1}$. Also note that $E \left(S_{k,i} | \mathcal{F}_{t_{i-1}} \right) = S_{k,i-1}$ since ψ^{α_j} are martingales (see Corollary 3.1).

We fix $k > 0$ and consider $B_i := \left| EY_i S_{k,i} \right|$, $i = k, k-1, \ldots, 1$:

(4.4) $\qquad B_i = \left| EY_i S_{k,i} \right| = \left| EY_{i-1} \left[1 + hF_i \right] S_{k,i} \right| \leq \left| EY_{i-1} S_{k,i-1} \right| + h \left| EY_{i-1} F_i \, S_{k,i} \right| \,.$

We expand the terms, which form F_i, at (T, b) up to terms of first order, i.e., we write F_i as a constant plus a remainder consisting of products of $f(t, x)$, some its derivatives and $X_j^r - b^r$ or $t_j - T$ with $j = i$, $i - 1/2$, or $i - 1$. Then, using the Cauchy-Bunyakovskii inequality and Lemma 3.3, we get:

$$\left| EY_{i-1} F_i \, S_{k,i} \right| \leq K \left| EY_{i-1} S_{k,i} \right| + \frac{C}{\left(T - t_i \right)^{3/2}} = K \left| EY_{i-1} S_{k,i-1} \right| + \frac{C}{\left(T - t_i \right)^{3/2}} \,.$$

Hence, due to (4.4), we obtain

(4.5) $\qquad B_i \leq B_{i-1} + KhB_{i-1} + \frac{Ch}{\left(T - t_i \right)^{3/2}} \,, \quad i = k, k-1, \ldots, 1 \,,$

where B_0 is evidently a constant.

Therefore

(4.6)
$$B_k \leq \left(1 + Kh \right)^k B_0 + \left(1 + Kh \right)^{k-1} \frac{Ch}{\left(T - t_1 \right)^{3/2}}$$

$$+ \left(1 + Kh \right)^{k-2} \frac{Ch}{\left(T - t_2 \right)^{3/2}} + \cdots + \frac{Ch}{\left(T - t_k \right)^{3/2}}$$

$$\leq B_0 \, e^{KT} + C e^{KT} h \sum_{i=1}^{k} \frac{1}{\left(T - t_i \right)^{3/2}} \leq \frac{C}{\sqrt{T - t_{k+1}}} \,,$$

which together with (4.3) implies (4.2). $\quad \square$

Remark 4.1. It is possible to prove that $R_k \leq Kh^5$, $k = 0, \ldots, N - 2$, with the constant K independent of h and k. But we restrict ourselves here to the estimate (4.2) since it is sufficient for proving Theorem 2.1 and is obtained by less efforts than it would be needed for a more accurate estimate.

Since the operator L is not defined at $t = T$, we need a separate analysis of the error on the last step R_{N-1}. We can prove that $R_{N-1} \leq Kh^5$ but we will obtain just the estimate $R_{N-1} \leq Kh^4$ which is enough for our purposes.

Lemma 4.2. *We have*

(4.7) $$R_{N-1} \leq Kh^4,$$

where K is independent of h.

Proof. Since $EY_{N-1}u\left(t_{N-1}, X_{N-1}\right) = EY_{t_{N-1}, X_{N-1}, Y_{N-1}}(T)$ and $u(T, X(T)) = 1$, we can write the error on the last step as (cf. (3.9)):

$$(4.8) \qquad R_{N-1} = \left| EY_{t_{N-1}, X_{N-1}, Y_{N-1}}(T) - E\bar{Y}_{t_{N-1}, X_{N-1}, Y_{N-1}}(T) \right| .$$

Introduce $\varepsilon = h^4$ and write $EY_{t_{N-1}, X_{N-1}, Y_{N-1}}(T)$ as

$$(4.9) \quad EY_{t_{N-1}, X_{N-1}, Y_{N-1}}(T) = EY_{N-1} + E \int_{t_{N-1}}^{T} f\left(\theta, X_{t_{N-1}, X_{N-1}}(\theta)\right) Y_{t_{N-1}, X_{N-1}, Y_{N-1}}(\theta)\, d\theta$$

$$= EY_{N-1} + EY_{N-1} \int_{t_{N-1}}^{T-\varepsilon} E\left[f\left(\theta, X_{t_{N-1}, X_{N-1}}(\theta)\right) Y_{t_{N-1}, X_{N-1}, 1}(\theta) | \mathcal{F}_{t_{N-1}} \right] d\theta$$

$$+ \int_{T-\varepsilon}^{T} E f\left(\theta, X_{t_{N-1}, X_{N-1}}(\theta)\right) Y_{t_{N-1}, X_{N-1}, Y_{N-1}}(\theta)\, d\theta .$$

The last term in the right-hand side of (4.9) is bounded by $K\varepsilon = Kh^4$. By (3.5), we expand the second term in the right-hand side of (4.9):

$$(4.10) \qquad EY_{N-1} \int_{t_{N-1}}^{T-\varepsilon} E\left[f\left(\theta, X_{t_{N-1}, X_{N-1}}(\theta)\right) Y_{t_{N-1}, X_{N-1}, 1}(\theta) | \mathcal{F}_{t_{N-1}} \right] d\theta$$

$$= EY_{N-1} \left[1 + (h - \varepsilon) f(t_{N-1}, X_{N-1}) + \frac{(h-\varepsilon)^2}{2} \left(Lf + f^2 \right) (t_{N-1}, X_{N-1}) \right.$$

$$+ \frac{(h-\varepsilon)^3}{6} \left(L^2 f + Lf^2 + fLf + f^3 \right) (t_{N-1}, X_{N-1})$$

$$+ \frac{(h-\varepsilon)^4}{24} \left(L^3 f + L^2 f^2 + LfLf + Lf^3 L^2 f \right.$$

$$\left. + fLf^2 + f^2 Lf + f^4 + Lf^3 \right) (t_{N-1}, X_{N-1}) \Bigg]$$

$$+ EY_{N-1} r_1(X_{N-1}),$$

where $r_1(x)$ consists of integrals from $T - h$ to $T - \varepsilon$ which form the remainder of the expansion.

Consider the integrals from r_1. For definiteness, we take the following one

$$(4.11) \qquad \int_{t_{N-1}}^{T-\varepsilon} \frac{(T - \varepsilon - \theta)^4}{24} E\left[Y_{t_{N-1}, X_{N-1}, 1}(\theta) \left(L^4 f \right) \left(\theta, X_{t_{N-1}, X_{N-1}}(\theta)\right) | \mathcal{F}_{t_{N-1}} \right] d\theta .$$

Due to Lemma 3.1, we write

$$(4.12) \qquad \left(L^4 f \right) \left(\theta, X_{t_{N-1}, X_{N-1}}(\theta)\right) = g_0\left(\theta, X_{t_{N-1}, X_{N-1}}(\theta)\right)$$

$$+ \sum_{j=1}^{4} \sum_{\alpha_j} g_{\alpha_j}\left(\theta, X_{t_{N-1}, X_{N-1}}(\theta)\right) \psi^{\alpha_j}\left(\theta, X_{t_{N-1}, X_{N-1}}(\theta)\right) .$$

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Let us analyze the term with ψ^{α_4} which singularity is of order 2. Expand $g_{\alpha_4}(\theta, X_{t_{N-1},X_{N-1}}(\theta))$ at (T, b) (cf. (3.32)):

$$g_{\alpha_4}(\theta, X_{t_{N-1},X_{N-1}}(\theta)) =: g(\theta, X_{t_{N-1},X_{N-1}}(\theta))$$

$$= g(T,b) + \sum_{r=1}^{d} \frac{\partial g}{\partial x^r}(T,b)\left(X_{t_{N-1},X_{N-1}}^r(\theta) - b^r\right) + \frac{\partial g}{\partial t}(\vartheta, X_{t_{N-1},X_{N-1}}(\theta))(\theta - T)$$

$$+ \frac{1}{2}\sum_{r_1,r_2=1}^{d} \frac{\partial^2 g}{\partial x^{r_1}\partial x^{r_2}}(T,\eta)\left(X_{t_{N-1},X_{N-1}}^{r_1}(\theta) - b^{r_1}\right)\left(X_{t_{N-1},X_{N-1}}^{r_2}(\theta) - b^{r_2}\right),$$

where ϑ is a time between $\theta \geq t_{N-1}$ and T and η is a point between $X_{t_{N-1},X_{N-1}}(\theta)$ and b.

Further, we expand $Y_{t_{N-1},X_{N-1},1}(\theta)$, $t_{N-1} \leq \theta \leq T - \varepsilon$ as

$$Y_{t_{N-1},X_{N-1},1}(\theta) = 1 + \rho_Y(\theta)$$

with $E\left(\rho_Y(\theta)\right)^2 \leq C \cdot h^2$.

Using these expansions together with Corollaries 3.1 and 3.2 and analyzing the remainder terms by the Cauchy-Bunyakovskii inequality and Lemma 3.3, we obtain

$$\int_{t_{N-1}}^{T-\varepsilon} \frac{(T-\varepsilon-\theta)^4}{24} E\left[Y_{t_{N-1},X_{N-1},1}(\theta) g_{\alpha_4}(\theta, X_{t_{N-1},X_{N-1}}(\theta))\,\psi^{\alpha_4}(\theta, X_{t_{N-1},X_{N-1}}(\theta))\,|\mathcal{F}_{t_{N-1}}\right]$$

$$= h^5 S'(X_{N-1}) + E\left(\rho'|\mathcal{F}_{t_{N-1}}\right),$$

where $S'(x)$ is a linear combination of $\psi^{\alpha_3}(T-h,x)$, $\psi^{\alpha_4}(T-h,x)$, $h\psi^{\alpha_5}(T-h,x)$, coefficients in this linear combination are independent of x and h; ρ' contains terms with ψ^{α_1} and ψ^{α_2} and other terms with singularity of order less than or equal to one. Besides, the remainder ρ' has the terms $\varepsilon^l h^{5-l} S'(X_{N-1})$ with $0 < l \leq 5$ which expectations are bounded by Ch^4 due to the choice of ε. The terms with ψ^{α_1} and ψ^{α_2} are estimated in the following way by the Cauchy-Bunyakovskii inequality and Lemma 3.3:

$$E\left[\int_{T-h}^{T-\varepsilon} \frac{(T-\varepsilon-\theta)^4}{4!} g_{\alpha_2}(\theta, X(\theta))\psi^{\alpha_2}(\theta, X(\theta))d\theta\right]^2$$

$$\leq Ch \int_{T-h}^{T-\varepsilon} (T-\varepsilon-\theta)^8 E\left[g_{\alpha_2}(\theta, X(\theta))\psi^{\alpha_2}(\theta, X(\theta))\right]^2 d\theta$$

$$\leq Ch \int_{T-h}^{T-\varepsilon} (T-\varepsilon-\theta)^8 \sqrt{E\left[g_{\alpha_2}(\theta, X(\theta))\right]^4}\sqrt{E\left[\psi^{\alpha_2}(\theta, X(\theta))\right]^4}d\theta$$

$$\leq Kh \int_{T-h}^{T-\varepsilon} \frac{(T-\varepsilon-\theta)^8}{(T-\theta)^2}d\theta \leq Kh \int_{T-h}^{T-\varepsilon} (T-\theta)^6 d\theta \leq Kh^8.$$

As a result, we obtain

$$\left(E\left[\rho'\right]^2\right)^{1/2} \leq Ch^4.$$

Analyzing analogously the other terms of the integral (4.11) and also in a similar way all the integrals from r_1, we get

(4.13) $$r_1(X_{N-1}) = h^5 S(X_{N-1}) + E\left(\rho|\mathcal{F}_{t_{N-1}}\right),$$

where $S(x)$ is a linear combination of $\psi^{\alpha_3}(T-h,x)$, $\psi^{\alpha_4}(T-h,x)$, $h\psi^{\alpha_5}(T-h,x)$, and ρ is such that

$$(4.14) \qquad\qquad \left(E\rho^2\right)^{1/2} \le Ch^4 .$$

Consider $EY_{N-1}r_1(X_{N-1})$. By (4.13)-(4.14), we obtain (cf. (4.3)):

$$(4.15) \qquad\qquad |EY_{N-1}r_1(X_{N-1})| = \left|EY_{N-1}\left[h^5 S_{N-1} + \rho\right]\right|$$
$$\le h^5 \left|EY_{N-1}S_{N-1}\right| + Ch^4,$$

where $S_{N-1} := S(X_{N-1})$. Using arguments similar to those used in the proof of Lemma 4.1, we get that $|EY_{N-1}S_{N-1}|$ is bounded by $C/\sqrt{h}$ (cf. (4.5)-(4.6)). Hence we obtain

$$|EY_{N-1}r_1(X_{N-1})| \le Ch^4.$$

Further, we write h instead of $h-\varepsilon$, h^2 instead of $(h-\varepsilon)^2$, etc. in (4.10). Due to (4.10), (4.15), and the choice of $\varepsilon = h^4$, we arrive at

(4.16)

$$EY_{t_{N-1},X_{N-1},Y_{N-1}}(T) = EY_{N-1}\left[1 + hf(t_{N-1},X_{N-1}) + \frac{h^2}{2}\left(Lf + f^2\right)(t_{N-1},X_{N-1})\right.$$

$$+\frac{h^3}{6}\left(L^2 f + Lf^2 + fLf + f^3\right)(t_{N-1},X_{N-1})$$

$$\left.+\frac{h^4}{24}\left(L^3 f + L^2 f^2 + LfLf + Lf^3 L^2 f + fLf^2 + f^2 Lf + f^4 + Lf^3\right)(t_{N-1},X_{N-1})\right] + E\hat{r},$$

with $|E\hat{r}|$ bounded by Ch^4.

Now consider the approximation on the last step (cf. (3.11)):

$$E\bar{Y}_{t_{N-1},X_{N-1},Y_{N-1}}(T) = EY_{N-1}E\left[\bar{Y}_{t_{N-1},X_{N-1},1}(T)|\mathcal{F}_{t_{N-1}}\right]$$

$$= EY_{N-1}E\left[1 + \frac{h}{6}\left(f_0 + 4f_{1/2} + f_1\right) + \frac{h^2}{6}\left(f_0 f_{1/2} + f_{1/2}^2 + f_{1/2}f_1\right) + \frac{h^3}{12}\left(f_0 f_{1/2}^2 + f_{1/2}^2 f_1\right)\right.$$

$$\left.+\frac{h^4}{24}f_0\, f_{1/2}^2\, f_1\,\right|\mathcal{F}_{t_{N-1}}\right] ,$$

where $f_0 := f(t_{N-1},X_{N-1})$, $f_{1/2} := f(t_{N-1/2},X_{N-1/2})$, and $f_1 := f(T,b)$.

Since we cannot expand f_1 using (3.5) directly, we represent it as

$$(4.17) \quad f_1 \equiv f(T,b) = f(T-\varepsilon^2, X_{0,a}(T-\varepsilon^2)) + \left[f(T,b) - f(T-\varepsilon^2, X_{0,a}(T-\varepsilon^2))\right] .$$

Due to $E(b - X_{0,a}(T-\varepsilon^2))^2 \le C\varepsilon$, the last term in (4.17) is estimated as

$$E\left[f(T,b) - f(T-\varepsilon^2, X_{0,a}(T-\varepsilon^2))\right]^2 \le C\varepsilon.$$

The expectation of the first term in (4.17) can already be expanded by (3.5). Using arguments similar to those used in the proof of Theorem 3.3, we get

(4.18)

$$E\bar{Y}_{t_{N-1},X_{N-1},Y_{N-1}}(T) = EY_{N-1}\left[1 + hf(t_{N-1}, X_{N-1}) + \frac{h^2}{2}\left(Lf + f^2\right)(t_{N-1}, X_{N-1})\right.$$

$$+\frac{h^3}{6}\left(L^2 f + Lf^2 + fLf + f^3\right)(t_{N-1}, X_{N-1})$$

$$+\frac{h^4}{24}\left(L^3 f + L^2 f^2 + LfLf + Lf^3 L^2 f + fLf^2 + f^2 Lf + f^4 + Lf^3\right)(t_{N-1}, X_{N-1})\right]$$

$$+EY_{N-1}r_2(X_{N-1}),$$

where

$$r_2(X_{N-1}) = h^5 \tilde{S}(X_{N-1}) + E\left(\tilde{\rho}|\mathcal{F}_{t_{N-1}}\right).$$

Here $\tilde{S}(x)$ is a linear combination of the same functions as in the case of $S(x)$ and $\tilde{\rho}$ obeys the inequality of the form (4.14). Then, by arguments similar to those used in the proof of Lemma 4.1 and in estimating $|EY_{N-1}r_1(X_{N-1})|$ above, we get that $|EY_{N-1}r_2(X_{N-1})|$ is bounded by Ch^4.

Finally, substituting (4.16) and (4.18) in (4.8), we arrive at (4.7). $\square$

Now we are in position to prove the convergence theorem.

Proof of Theorem 2.1. Lemmas 4.1 and 4.2 and the relations (3.8)-(3.9) imply

$$\left|EY_{0,a,1}(T) - E\bar{Y}_{0,a,1}(T)\right| \le \sum_{k=0}^{N-2}\frac{Kh^5}{\sqrt{T - t_{k+1}}} + Kh^4.$$

Since $\sum_{k=0}^{N-2}\frac{h}{\sqrt{T - t_{k+1}}} \le C$, we get

$$\left|EY_{0,a,1}(T) - E\bar{Y}_{0,a,1}(T)\right| \le Kh^4,$$

i.e., we have proved that the method (2.9)-(2.10) is of order 4. $\square$

Remark 4.2. A rough estimation of the remainders r_1 and r_2 arising in the proof of Lemma 4.2 gives us that

(4.19) $$\left(Er_i^2\right)^{1/2} \le Ch^3, \quad i = 1, 2.$$

Indeed, consider, for instance, the integral (4.11) from r_1. The expectation in this integral is estimated by the Cauchy-Bunyakovskii inequality and Lemma 3.3 (see (3.28)) as $C/(T - \theta)^2$ (recall that the term $L^4 f$ has singularity of order 2). Then the integral (4.11) is estimated as Ch^3. Note that to get (4.19) for $i = 1$ we do not need the expansion (4.12) and the detailed analysis after it. To get (4.19) for $i = 2$ is also simple.

Using (4.19), we clearly arrive at

$$R_{N-1} \le Kh^3.$$

This and the estimate (4.1) imply that the global error of the method (2.9)-(2.10) is bounded as

$$\left| EY_{0,a,1}(T) - E\bar{Y}_{0,a,1}(T) \right| \le Ch^5 \sum_{k=0}^{N-2} \frac{1}{(T-t_k)^2} + Ch^3 \,.$$

Since the sum $\displaystyle\sum_{k=0}^{N-1} \frac{h^2}{(T-t_k)^2} \le C$, we get that the method (2.9)-(2.10) is at least of order 3. To obtain this result, we do not need, for instance, Lemmas 3.2 and 4.1 and we prove analogues of Theorem 3.3 and Lemma 4.2 much simpler. A significant part of the previous and current sections dealt with the extraction of singularities that was needed in order to get the fourth order of convergence.

5. Implicit Runge-Kutta methods

From the point of view of possible applications, the most interesting case is when the function f is bounded from above, for example, when f is negative. In this case the explicit Runge-Kutta method from Section 2 may cause some computational problems since, for instance, Y_{k+1} in (2.10) can become a large negative number while the exact $Y(t)$ is always positive. Apparently, this may occasionally lead to some instabilities and require a very small time step to achieve a reasonable accuracy. In such a situation an implicit method can behave better.

Let us formally apply the deterministic midpoint method to (2.3) provided $X(t)$ is a known function. As a result, we obtain

$$(5.1) \qquad X(h/2) = a + \frac{h}{2}\frac{b-a}{T} + \sqrt{\frac{h}{2}}\sqrt{\frac{T-h/2}{T}}\,\xi_{1/2}\,,$$

$$X(t_{k+1/2}) = X(t_{k-1/2}) + h\frac{b - X(t_{k-1/2})}{T - t_{k-1/2}} + \sqrt{h}\sqrt{\frac{T - t_{k+1/2}}{T - t_{k-1/2}}}\,\xi_{k+1/2}\,,$$

$$k = 1,\ldots,N-1\,,$$

$$(5.2) \qquad Y_0 = 1\,,$$

$$Y_{k+1} = Y_k + hf(t_{k+1/2}, X(t_{k+1/2}))\frac{Y_k + Y_{k+1}}{2}\,,\quad k = 0,\ldots,N-1\,,$$

where $\xi_{k+1/2}$, $k = 0,\ldots,N-1$, are d-dimensional random vectors which components are mutually independent random variables with standard normal distribution $\mathcal{N}(0,1)$.

Resolving the implicitness in (5.2), we get

$$(5.3) \qquad Y_{k+1} = Y_k \frac{1 + \dfrac{h}{2}f(t_{k+1/2}, X(t_{k+1/2}))}{1 - \dfrac{h}{2}f(t_{k+1/2}, X(t_{k+1/2}))}\,.$$

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To ensure that the denominator in (5.3) does not vanish for all sufficiently small h, we should require that the function $f(t,x)$ is bounded from above, i.e., that $f(t,x) \leq c$ for all (t,x), c is a constant. In this case for all sufficiently small h the denominator in (5.3) is positive. If $f(t,x) \leq 0$, then $-1 \leq Y_k \leq 1$ for all k.

We prove the convergence theorem for the method (5.1)-(5.2) under the same assumptions as in Section 3 (see p. 5). Note that in the case of $f(t,x) \leq 0$, the condition $EY_k^2 \leq C$ is satisfied due to the uniform boundedness of the random variables Y_k.

Theorem 5.1. *The method* (5.1)-(5.2) *applied to evaluation of the Wiener integral* (2.4) *is of second accuracy order, i.e.,*

$$(5.4) \qquad |\mathcal{J} - EY_N| = |EY(T) - EY_N| \leq Kh^2,$$

where the constant K is independent of h.

Proof. In the method (5.1)-(5.2), the approximation $\bar{Y}(t)$ is evaluated at $t = t_k$, $k = 1, \ldots, N$, while $X(t)$ is simulated at $t = t_{k+1/2}$ and $X(t_k)$ is not used in the algorithm. Due to this reason, we cannot directly make use of relations like (3.8)-(3.9) to prove convergence of the method (5.1)-(5.2). To overcome this difficulty, we consider the other algorithm:

$$(5.5) \quad X(t_{k+1/2}) = X(t_k) + \frac{h}{2}\frac{b - X(t_k)}{T - t_k} + \sqrt{\frac{h}{2}}\sqrt{\frac{T - t_{k+1/2}}{T - t_k}}\,\xi_{k+1/2}\,, \quad k = 0, \ldots, N-1\,,$$

$$X(t_{k+1}) = X(t_{k+1/2}) + \frac{h}{2}\frac{b - X(t_{k+1/2})}{T - t_{k+1/2}} + \sqrt{\frac{h}{2}}\sqrt{\frac{T - t_{k+1}}{T - t_{k+1/2}}}\,\xi_{k+1}\,,$$

$$k = 0, \ldots, N-2\,, \quad X(t_N) = b\,,$$

and Y_k, $k = 0, \ldots, N-1$, are simulated by the same formulas as in (5.2) (or, what is the same, (5.3)). In (5.5), $\xi_{k+1/2}$ and ξ_{k+1} are d-dimensional random vectors which components are mutually independent random variables with standard normal distribution $\mathcal{N}(0,1)$.

Since $X(t)$ is simulated exactly both in (5.1) and (5.5) and, in particular, $X(t_{k+1/2})$ from (5.1) have the same distributions as their counterparts in (5.5), it is clear that the estimate (5.4) for the algorithm (5.5), (5.2) implies this estimate for (5.1)-(5.2). At the same time, due to the presence of $X(t_{k+1})$ in (5.5), we can make use of relations like (3.8)-(3.9) to estimate the error of the algorithm (5.5), (5.2). In what follows, we prove (5.4) for (5.5), (5.2).

We write the global error of (5.5), (5.2) in the form (3.8)-(3.9) and introduce the one-step error of (5.5), (5.2) as in (3.10):

$$(5.6) \qquad r(t,x) := Eu(t+h, X_{t,x}(t+h))\bar{Y}_{t,x,1}(t+h) - u(t,x)\,.$$

We rewrite (5.3) on a single step and expand it as

$$(5.7) \qquad \bar{Y}_{t,x,1}(t+h) = 1 + hf_{1/2} + \frac{h^2}{2}f_{1/2}^2 + \rho,$$

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where $f_{1/2} := f(t + h/2, X_{t,x}(t + h/2))$ and the random variable ρ is such that

(5.8)
$$\left(E\rho^2\right)^{1/2} \leq Ch^3 .$$

We substitute (5.7) in (5.6) and then expand the terms in the obtained relation using (3.5), (3.6) as we did in the proof of Theorem 3.3 (see pp. 6-9). In fact, the expansions are simpler here since we are proving the second order of convergence only. We have (cf. (3.12)-(3.14)):

(5.9)
$$Eu(t + h, X_{t,x}(t + h)) = u(t, x) + hLu(t, x) + \frac{h^2}{2}L^2 u(t, x)$$
$$+ \int_t^{t+h} \frac{(t + h - \theta)^2}{2} EL^3 u(\theta, X_{t,x}(\theta))\, d\theta ,$$

(5.10)
$$Ef_{1/2}u(t + h, X_{t,x}(t + h)) = f(t, x)u(t, x) + \frac{h}{2}L\left(fu\right)(t, x)$$
$$+ \int_t^{t+h/2} (t + h/2 - \theta)\, EL^2\left(fu\right)(\theta, X_{t,x}(\theta))d\theta + \frac{h}{2}f(t, x)Lu(t, x)$$
$$+ \frac{h}{2}\int_t^{t+h/2} EL\left(fLu\right)(\theta, X_{t,x}(\theta))d\theta + Ef_{1/2}\int_{t+h/2}^{t+h} (t + h - \theta)\, L^2 u(\theta, X_{t,x}(\theta))\, d\theta ,$$

(5.11)
$$Ef_{1/2}^2 u(t + h, X_{t,x}(t + h)) = f^2(t, x)u(t, x) + \int_t^{t+h/2} EL\left(f^2 u\right)(\theta, X_{t,x}(\theta))d\theta$$
$$+ Ef_{1/2}^2 \int_{t+h/2}^{t+h} Lu(\theta, X_{t,x}(\theta))\, d\theta .$$

Substituting (5.7), (5.9)-(5.11) in (5.6) and taking into account that $u(t, x)$ is a solution of (3.4), we arrive at

(5.12)
$$r(t, x) = -\int_t^{t+h} \frac{(t + h - \theta)^2}{2} EL^2(fu)(\theta, X_{t,x}(\theta))\, d\theta$$
$$+ h\int_t^{t+h/2} (t + h/2 - \theta)\, EL^2\left(fu\right)(\theta, X_{t,x}(\theta))d\theta$$
$$+ hEf_{1/2}\int_{t+h/2}^{t+h} (t + h - \theta)\, L^2 u(\theta, X_{t,x}(\theta))\, d\theta$$
$$+ \frac{h^2}{2}Ef_{1/2}^2 \int_{t+h/2}^{t+h} Lu(\theta, X_{t,x}(\theta))\, d\theta + Eu(t + h, X_{t,x}(t + h))\, \rho .$$

Then, using Lemmas 3.1 and 3.3 (cf. the proof of Theorem 3.3), we obtain that the one-step error of the method (5.1)-(5.2) can be written in the form

(5.13)
$$r(t, x) = h^3 S(t, x) + E\tilde{\rho}(t, x; h) ,$$

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where $S(t, x)$ is a linear combination of the functions $\psi^{\alpha_2}(t, x)$, coefficients in this linear combination are independent of t, x, and h; $\tilde{\rho}(t, x; h)$ is such that

$$\left(E\left[\tilde{\rho}(t, X_{0,a}(t); h)\right]^{2n}\right)^{1/2n} \leq \frac{Ch^3}{\sqrt{T - t - h}}$$

with a constant C independent of t and h. We see that $S(t, x)$ in (5.13) and, consequently, the one-step error $r(t, x)$, has singularity of order one.

Further, using arguments similar to those in the proofs of Lemmas 4.1 and 4.2 (in fact, due to the lower order of convergence and lower order of singularity, much simpler calculations are needed here), we obtain (5.4). $\square$

If we formally apply the deterministic Gauss method of order 4 (see, e.g., [6, p. 71]) to (2.3), assuming that $X(t)$ is a known function, we obtain

$$(5.14) \qquad X(\gamma h) = a + \gamma h \frac{b - a}{T} + \sqrt{\gamma h} \sqrt{\frac{T - \gamma h}{T}} \, \xi_\gamma \,,$$

$$X((1 - \gamma)h) = X(\gamma h) + (1 - 2\gamma)h \frac{b - X(\gamma h)}{T - \gamma h} + \sqrt{(1 - 2\gamma)h} \sqrt{\frac{T - (1 - \gamma)h}{T - \gamma h}} \, \xi_{1-\gamma} \,,$$

$$X(t_k + \gamma h) = X(t_{k-1} + (1 - \gamma)h) + 2\gamma h \frac{b - X(t_{k-1} + (1 - \gamma)h)}{T - t_k + \gamma h}$$
$$+ \sqrt{2\gamma h} \sqrt{\frac{T - t_k - \gamma h}{T - t_k + \gamma h}} \, \xi_{k+\gamma} \,,$$

$$X(t_k + (1 - \gamma)h) = X(t_k + \gamma h) + (1 - 2\gamma)h \frac{b - X(t_k + \gamma h)}{T - t_k - \gamma h}$$
$$+ \sqrt{(1 - 2\gamma)h} \sqrt{\frac{T - t_{k+1} + \gamma h}{T - t_k - \gamma h}} \, \xi_{k+1-\gamma} \,, \qquad k = 1, \ldots, N - 1 \,,$$

$$(5.15) \qquad Y_0 = 1 \,,$$

$$k_1 = f(t_k + \gamma h, X(t_k + \gamma h)) \left[Y_k + \frac{h}{4} k_1 + \left(\frac{1}{4} - \frac{\sqrt{3}}{6} \right) h k_2 \right] \,,$$

$$k_2 = f(t_k + (1 - \gamma)h, X(t_k + (1 - \gamma)h)) \left[Y_k + \left(\frac{1}{4} + \frac{\sqrt{3}}{6} \right) h k_1 + \frac{h}{4} k_2 \right] \,,$$

$$Y_{k+1} = Y_k + \frac{h}{2} (k_1 + k_2) \,, \quad k = 0, \ldots, N - 1 \,,$$

where $\gamma = \dfrac{1}{2} - \dfrac{\sqrt{3}}{6}$ and $\xi_{k+\gamma}$, $\xi_{k+1-\gamma}$, $k = 0, \ldots, N - 1$, are d-dimensional random vectors which components are mutually independent random variables with standard normal distribution $\mathcal{N}(0, 1)$.

Resolving (5.15) with respect to k_1 and k_2, we get

$$(5.16) \qquad Y_{k+1} = Y_k \frac{1 + \dfrac{h}{4}(f_1 + f_2) + \dfrac{h^2}{12}f_1 f_2}{1 - \dfrac{h}{4}(f_1 + f_2) + \dfrac{h^2}{12}f_1 f_2},$$

where $f_1 := f(t_k + \gamma h, X(t_k + \gamma h))$ and $f_2 := f(t_k + (1-\gamma)h, X(t_k + (1-\gamma)h))$.

The denominator in (5.16) does not vanish for all sufficiently small h for functions $f(t,x)$ being bounded from above. And if $f(t,x) \le 0$, then $-1 \le Y_k \le 1$ for all k.

The intuition built on the previous analysis of the methods (2.9)-(2.10) and (5.1)-(5.2) tells us that the method (5.14)-(5.15) should be of order 4. But this assertion turned out to be wrong, the method is of order 2 only just as the method (5.1)-(5.2). Analogously to Theorems 2.1 and 5.1, we prove the convergence theorem.

Theorem 5.2. *The method* (5.14)-(5.15) *applied to evaluation of the Wiener integral* (2.4) *is of second order of accuracy, i.e.,*

$$(5.17) \qquad |\mathcal{J} - EY_N| = |EY(T) - EY_N| \le Kh^2,$$

where the constant K is independent of h.

Although the methods (5.1)-(5.2) and (5.14)-(5.15) are of the same order of convergence, in our numerical tests (see Section 7) the method (5.14)-(5.15) gives more accurate results. Apparently, this is due to the fact that the constant K in (5.17) is, in general, less than its counterpart in (5.4).

6. Variance reduction

To evaluate $E\bar{Y}(T)$ in practice, we need to apply the Monte Carlo technique. As a result, in addition to the error of numerical integration considered in the previous sections, there is also the Monte Carlo error:

$$(6.1) \qquad E\bar{Y}(T) = \frac{1}{M}\sum_{m=1}^{M} \bar{Y}^{(m)}(T) \pm R_{mc},$$

where M is the number of independent realizations $\bar{Y}^{(m)}(T)$ of $\bar{Y}(T)$. The Monte Carlo error R_{mc} is estimated as

$$(6.2) \qquad R_{mc} = \frac{c}{\sqrt{M}}\sqrt{Var\,\bar{Y}(T)} \doteq \frac{c}{\sqrt{M}}\sqrt{Var\,Y(T)}$$

with, for example, the fiducial probability 0.997 for $c = 3$ and 0.95 for $c = 2$.

Thus, if the variance $Var\,Y(T)$ is big, a large number of trajectories M has to be simulated in order to reach a satisfactory accuracy. To reduce the Monte Carlo error, a variance reduction technique can be used. The basic idea of variance reduction techniques (see [5, 10, 12, 11]) is to substitute $Y(T)$ by another random variable which has the same expectation as $Y(T)$ but a smaller variance. Two variance reduction methods are known: the method of important sampling [5, 10, 11] and the method of control variates [12, 11].

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A combining method is given in [11]. The method of important sampling is based on Girsanov's transformation. In our case its application changes the linear system (2.1) for X to a system with, in general, a nonlinear drift. As a result, we lose the advantage of simulating $X(t)$ exactly and of approximating the conditional Wiener integral by higher-order numerical integrators from Sections 2 and 5. This shortcoming does not arise in the case of the method of control variates. That is why, we restrict ourselves here to this method only.

In connection with the evaluation of the Wiener integral (1.1)-(1.2) consider the following system of Ito SDEs (cf. (2.1)-(2.3)):

$$(6.3) \qquad dX = \frac{b - X}{T - t} dt + dw(t), \; X(s) = x,$$

$$(6.4) \qquad dY = f(t, X(t)) \, Y \, dt, \; Y(s) = y,$$

$$(6.5) \qquad dZ = G^{\top}(t, X) Y \, dw(t), \; Z(s) = z \, .$$

Here Z is scalar and $G(t, x)$ is a column-vector of dimension d with good analytical properties, the other notation is the same as before.

It is clear that
$$u(s, x) = EY_{s,x,1}(T) = E\left[Y_{s,x,1}(T) + Z_{s,x,1,0}(T)\right] \, .$$

As it is known [11]

$$(6.6) \qquad Var\left[Y_{s,x,1}(T) + Z_{s,x,1,0}(T)\right] = E \int_s^T Y_{s,x,1}^2(t) \sum_{i=1}^d \left(\frac{\partial u}{\partial x^i} + G^i\right)^2 dt \, ,$$

where $u(t, x)$ is the solution of (3.4). Then by choosing $G(t, x)$ as

$$(6.7) \qquad G^i = -\frac{\partial u}{\partial x^i}, \quad j = 1, \ldots, d \, ,$$

we obtain that the variance of $Y_{s,x,1}(T) + Z_{s,x,1,0}(T)$ is equal to zero.

Applying a numerical method to (6.3)-(6.5), we get the approximate $\bar{Y}_{s,x,1}(T)$ and $\bar{Z}_{s,x,1,0}(T)$. The variance $Var\left[\bar{Y}_{s,x,1}(T) + \bar{Z}_{s,x,1,0}(T)\right]$ is close to $Var\left[Y_{s,x,1}(T) + Z_{s,x,1,0}(T)\right]$, i.e., it is small in the case of G from (6.7), and, consequently, less number of independent realizations M is needed to have a satisfactory accuracy.

Of course, in practice the solution $u(t, x)$ is not known. However, an approximate solution $\tilde{u}$ to the problem (3.4) can be known. In this case we can take $G(t, x)$ in the form of (6.7) with $\tilde{u}$ instead of u and we may expect a variance reduction. This is demonstrated in numerical examples (see the next section).

7. Numerical tests

1. We take $f(t, x)$ in the form

$$(7.1) \qquad f(t, x) = (A(t) x, x) + (a_1(t), x) + a_0(t) \, ,$$

where $A(t)$ is a $d \times d$ symmetric matrix, $a_1(t)$ is a d-dimensional vector, and $a_0(t)$ is a scalar function.

Let $u(t, x)$ be the solution of (3.4) with f from (7.1). Introduce the function $P(t, x)$:

(7.2)
$$u(t, x) = \exp(P(t, x)) \,.$$

This function satisfies the problem

(7.3)
$$LP + (A(t)\,x, x) + (a_1(t), x) + a_0(t) + \frac{1}{2} \sum_{i=1}^{d} \left(\frac{\partial P}{\partial x^i} \right)^2 = 0 \,, \quad x \in \mathbf{R}^d, \ t < T,$$
$$P(T, x) = 0 \,.$$

We look for a solution of (7.3) in the form

(7.4)
$$P(t, x) = \frac{1}{2} (P(t)x, x) + (p(t), x) + q(t) \,,$$

where $P(t)$ is a $d \times d$ symmetric matrix, $p(t)$ is a d-dimensional vector, and $q(t)$ is a scalar function.

Substituting (7.4) in (7.3) and collecting terms $(\cdot\, x, x)$, $(\cdot\, , x)$ and terms independent of x separately, we arrive at the system for $P(t)$, $p(t)$, and $q(t)$:

(7.5)
$$P'(t) - \frac{2}{T - t} P + 2A(t) + P^2(t) = 0 \,, \quad P(T) = 0 \,,$$

(7.6)
$$p'(t) - \frac{1}{T - t} p + \frac{1}{T - t} P(t)b + P(t)p + a_1(t) = 0 \,, \quad p(T) = 0 \,,$$

(7.7)
$$q'(t) + \frac{1}{T - t} (p(t), b) + \frac{1}{2} tr\, P(t) + \frac{1}{2} (p(t), p(t)) + a_0(t) = 0 \,, \quad q(T) = 0 \,.$$

Note that if $a_1(t) \equiv 0$ and $b = 0$, then $p(t) \equiv 0$. And if in addition $a_0(t) \equiv 0$, then

$$q(t) = \frac{1}{2} \int_t^T tr\, P(s)\, ds \,.$$

The solution of (7.5) can be expanded in (positive) powers of $T - t$. If $A(t)$ is a constant matrix A, then this formal expansion starts with the terms:

$$P(t) = \frac{2}{3} A \cdot (T - t) + \frac{4}{45} A^2 \cdot (T - t)^3 + \cdots \,.$$

For test purposes, it is convenient to have an exact solution of (7.5)-(7.7) in a closed analytical form. To this end, we choose a variable matrix $A(t)$ such that

(7.8)
$$A(t) = A - \frac{2}{9} A^2 \cdot (T - t)^2,$$

where A is a constant symmetric matrix. Then the exact solution of the system (7.5)-(7.7) with $b = 0$, $a_0(t) \equiv 0$, and $a_1(t) \equiv 0$ has the form

(7.9)
$$P(t) = \frac{2}{3}(T - t)A, \quad p(t) = 0, \quad q(t) = \frac{(T - t)^2}{6} tr\, A \,.$$

TABLE 1. The results of simulaton of the conditional Wiener integral (1.1)-(1.2) for f from (7.1) with $a_0 = 0$, $a_1 = 0$, $A(t)$ from (7.8), (7.11) and for $a = b = 0$, $T = 1$ by the explicit Runge-Kutta method (2.9)-(2.10) and the implicit Runge-Kutta methods (5.1), (5.3) and (5.14), (5.16). The exact soultion is 1.

h	M	(2.9)-(2.10)	(5.1), (5.3)	(5.14), (5.16)
0.2	10^6	0.9994 ± 0.0013	1.0176 ± 0.0044	1.0040 ± 0.0013
0.1	10^8	1.00002 ± 0.00013	1.00361 ± 0.00015	1.00093 ± 0.00013
0.05	10^8	0.99996 ± 0.00013	1.00089 ± 0.00013	1.00019 ± 0.00013

Consequently, the solution of (7.4) is

$$(7.10) \qquad P(t,x) = \frac{T-t}{3}(Ax, x) + \frac{(T-t)^2}{6} tr A \ .$$

Then the conditional Wiener integral (1.1)–(1.2) for f from (7.1) with $a_0 = 0$, $a_1 = 0$, $A(t)$ from (7.8) and for $a = b = 0$ is equal to

$$\mathcal{J} = u(0,0) = \exp\left(\frac{T^2}{6} tr A\right) \ .$$

In our experiments we take the dimension $d = 4$ and the following matrix A :

$$(7.11) \qquad A = \begin{bmatrix} -1 & -0.5 & 0 & 0 \\ -0.5 & 2 & -0.5 & 0 \\ 0 & -0.5 & -2 & -0.5 \\ 0 & 0 & -0.5 & 1 \end{bmatrix} ,$$

for which $tr A = 0$.

In Table 1 we give results of simulation of the conditional Wiener integral (1.1)-(1.2) for f from (7.1) with $a_0 = 0$, $a_1 = 0$, $A(t)$ from (7.8), (7.11) and for $a = b = 0$, $T = 1$ by the explicit Runge-Kutta method (2.9)-(2.10) and the implicit Runge-Kutta methods (5.1), (5.3) and (5.14), (5.16). As it was mentioned in Section 6, we have two types of errors in numerical simulations here: the error of a method used and the Monte Carlo error. The results in the table are approximations of $E\bar{Y}(1)$ calculated as in (6.1)-(6.2) with $c = 2$. Note that the "$\pm$" reflects the Monte Carlo error only and it does not reflect the error of a method. The results obtained are in agreement with the proved convergence theorems (see also Table 2). Recall that the implicit methods (5.1)-(5.2) and (5.14)-(5.15) are both of order two. In our tests the method (5.14)-(5.15) performs better. Apparently, this is due to the fact that the constant K in (5.17) is, in general, less than its counterpart in (5.4).

2. To reduce the Monte Carlo error in simulation of the above test problem, we can use the variance reduction technique from Section 6. For f from (7.1) with $a_0 = 0$, $a_1 = 0$, $A(t)$ from (7.8), (7.11) and for $b = 0$, the solution $u(t,x)$ of (3.4) has the form (7.2),

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TABLE 2. The results of simulaton of the conditional Wiener integral (1.1)-(1.2) for f from (7.1) with $a_0 = 0$, $a_1 = 0$, $A(t)$ from (7.8), (7.11) and for $a = b = 0$, $T = 1$ by the explicit Runge-Kutta method (2.9)-(2.10) and the implicit Runge-Kutta methods (5.1), (5.3) and (5.14), (5.16) using the variance reduction technique. The exact soultion is 1.

h	M	(2.9)-(2.10)	(5.1), (5.3)	(5.14), (5.16)
0.1	10^7	0.99977 ± 0.00024	1.00396 ± 0.00050	1.00103 ± 0.00023
0.05	10^7	0.99992 ± 0.00017	1.00098 ± 0.00017	1.00023 ± 0.00016
0.05	10^8	0.99999 ± 0.00005	1.00088 ± 0.00005	1.00027 ± 0.00005
0.01	10^7	1.00003 ± 0.00007	1.00001 ± 0.00007	1.00003 ± 0.00007

(7.10). Therefore, in this case the vector function G defined in (6.7) is equal to

$$(7.12) \qquad G^i(t, x) = -\frac{2}{3}(T - t) \, \exp(P(t, x)) \sum_{j=1}^{d} A^{ij} x^j \,, \quad i = 1, \ldots, d,$$

where $P(t, x)$ is from (7.10) and A is from (7.11).

Applying the Euler method to the equation (6.5), we get

$$(7.13) \qquad\qquad Z_0 = 0,$$

$$Z_{k+1} = Z_k + G^\top(t_k, X) Y_k \, \Delta w_k, \quad k = 1, \ldots, N - 1.$$

If we approximate (6.3)-(6.4) using the explicit fourth-order Runge-Kutta method (2.9)-(2.10), then Y_k in (7.13) is from (2.10) and the Wiener increment is

$$\Delta w_k := w(t_{k+1}) - w(t_k) = \frac{h^{1/2}}{\sqrt{2}} \left(\xi_{k+1/2} + \xi_{k+1} \right) \,,$$

where $\xi_{k+1/2}$ and ξ_{k+1} are the same as in (2.9)-(2.10).

It is clear that $EZ_{k+1} = 0$. This implies that the method (2.9)-(2.10), (7.13) applying to (6.3)-(6.5) to approximate the Wiener integral $\mathcal{J} = EY(T)$ is of order four, i.e., the above realization of the variance reduction technique does not affect the accuracy of the numerical method. The variance $Var\, Y(T)$ is approximated with accuracy $O(h)$. Consequently, for a fixed number of realizations M the Monte Carlo error in simulations using the variance reduction technique is $\sim 1/\sqrt{h}$ times less than in simulations without variance reduction. In other words, in the case of variance reduction the Monte Carlo error is proportional to $\sqrt{h}/\sqrt{M}$. This is illustrated in Table 2. In particular, we see for $h = 0.05$ that to produce results of the same quality we need $M = 10^8$ independent trajectories without variance reduction and $M = 10^7$ independent realizations in the variance reduction case (compare Tables 1 and 2).

Remark 7.1. Recall that the implicit methods (5.1), (5.3) and (5.14), (5.16) do not contain simulation of $X(t_{k+1})$, and the random variables involved in these methods are not enough to evaluate the Wiener increments Δw_k on the intervals $[t_k, t_{k+1}]$. At the same time, these Wiener increments are needed to realize (7.13). Thus, to use the variance reduction technique in connection with the implicit methods (5.1), (5.3) and (5.14), (5.16),

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we introduce additional random variables and simulation of $X(t_{k+1})$ in the corresponding algorithms (see (5.5) in the case of the method (5.1),(5.3)).

3. Now we illustrate the assertion made at the end of Section 6. To this end we take the function $f(t,x)$ in the form (7.1) with the constant matrix $A(t) \equiv A$ from (7.11) and $a_0 = 0$, $a_1 = 0$. We also put $b = 0$. In this case we do not know the exact solution $u(t,x)$ of (3.4). But for the variance reduction we can use an approximation $\tilde{u}(t,x)$ of the solution based on the formal expansion (7.9):

$$(7.14) \qquad \tilde{u}(t,x) = \exp\left(\frac{1}{2}\left(\tilde{P}(t)x, x\right)\right),$$

where

$$\tilde{P}(t) = \frac{2}{3}A \cdot (T - t).$$

Deriving (7.14), we take into account that $tr\tilde{P}(t) = 0$ because of the specific choice of the matrix A which is from (7.11).

Then we take the function G in (7.13) of the form

$$G^i(t,x) = -\frac{\partial \tilde{u}}{\partial x^i}, \quad j = 1, \ldots, d.$$

Putting $a = 0$ and $T = 1$, we evaluate the corresponding conditional Wiener integral (1.1)-(1.2) by the fourth-order explicit Runge-Kutta method (2.9)-(2.10) with time step $h = 0.01$ and we simulate $M = 10^5$ independent realizations. Without variance reduction, we get: $\mathcal{J} \doteq 1.1536 \pm 0.0093$, while applying the variance reduction technique (i.e., using the method (2.9)-(2.10), (7.13) for (6.3)-(6.5)) we obtain $\mathcal{J} \doteq 1.1482 \pm 0.0018$. We see that the Monte Carlo error is 5 times less when we use the variance reduction technique.

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