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Heteroclinic orbits between rotating waves of semilinear parabolic equations on the circle

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1 Introduction

One of the simplest reaction-advection-diffusion equations is the scalar partial differential equation

$$u_t = u_{xx} + f(u, u_x) \tag{1.1}$$

for one real variable $u = u(t, x)$ and in one space dimension x. We mainly consider periodic boundary conditions

$$u(t, 0) = u(t, 2\pi), \qquad u_x(t, 0) = u_x(t, 2\pi), \tag{1.2}$$

alias $x \in S^1 = \mathbb{R}/2\pi\,\mathbb{Z}$. The case of Neumann boundary conditions

$$u_x(t, 0) = u_x(t, 2\pi) = 0 \tag{1.3}$$

has been studied in considerable detail in the literature for more or less general nonlinearities $f = f(x, u, u_x)$; [FR91], [Fie94], [FR96], [Wol02a], [Wol02b], the survey [FS02], and the references there. In the case $(\mathcal{N})$ of (1.1) with Neumann boundary conditions (1.3), bounded solutions $u(t, x)$ tend to time independent equilibrium solutions $v(x)$ for $t \to \infty$. The case $(\mathcal{P})$ of (1.1) with periodic boundary conditions (1.2), in contrast, features time periodic solutions which turn to be *rotating waves*

$$u(t, x) = v(x - ct), \tag{1.4}$$

rotating at constant wave speed $c \neq 0$ around the circle $x \in S^1$; [AF88], [Mat88], [MN97]. *Heteroclinic orbits* are solutions $u(t, x)$ which converge to different limiting objects, here equilibria and rotating waves, for $t \to -\infty$ and $t \to +\infty$, respectively. In the Neumann case $(\mathcal{N})$, heteroclinic orbits can only connect equilibria, due to a variational structure; see [Zel68], [Mat78], [Mat88], [FR96]. In the periodic case $(\mathcal{P})$, heteroclinic orbits between rotating waves can arise. The central question of the present paper, answered in theorems 1.3 and 1.4 below, is therefore the following: given two rotating waves v, w, does there exist a heteroclinic orbit from v to w?

To answer this question, we fix the following technical setting and notation. We consider nonlinearities $f \in C^2$, and solutions $u(t, \cdot) \in X = H^s(S^1)$; the fractional Sobolev space with exponent $s > \frac{3}{2}$, so that X embeds into $C^1(S^1)$. By standard semigroup theory, (1.1), (1.2) defines a solution semiflow on X. This settle existence and uniqueness questions for the Cauchy problem with given initial data $u(0, x) = u_0(x)$ in X. See [Hen81] or [Paz83] for details.

We also assume f is *dissipative* in the sense of [Hal88], [BV92]: there exists a large absorbing ball in X which any solution $u(t, \cdot)$ eventually enters, after some time which may depend on the initial condition $u_0 \in X$. For specific sufficient conditions on f, which entail dissipativeness, see for example [FR96]. As an important consequence, we obtain a nonempty compact *global attractor* $\mathcal{A}$ of (1.1), (1.2). By definition, $\mathcal{A}$ is the smallest set which attracts any bounded set of initial conditions, for $t \to +\infty$. Equivalently, $\mathcal{A}$ is the maximal compact invariant set, or the set of all globally defined bounded solutions $u(t, \cdot) \in X, t \in \mathbb{R}$.

An explicit condition on f which ensures dissipativeness is the following:

$$\begin{array}{ll} | f(v,p) | \leq C_0, & \text{for all} \quad v,p \in \mathbb{R}, \text{ and} \\ f(v,0) \cdot v < 0 & \text{for} \qquad \text{large } |v|. \end{array} \tag{1.5}$$

See for example [MN97] for less restrictive growth conditions on f which, following [Am76], ensure dissipativeness. In the present paper we are only interested in the dynamics on the global attractor $\mathcal{A}$. Because $\mathcal{A}$ is bounded in C^1-norm, we may then modify f outside the values which $(v,p) = (u, u_x)$ attain on $\mathcal{A}$. We may thus assume (1.5) to hold, without loss of generality.

Homogeneous equilibria, rotating and frozen (alias stationary) waves are examples of elements of the global attractor. Define the set $\mathcal{E} \subset X$ of *homogeneous equilibria* of (1.1) to be the set of those $e \in \mathbb{R}$, for which

$$f(e,0) = 0. \tag{1.6}$$

Clearly $u(t,x) \equiv e$ is then a spatially homogeneous equilibrium solution. *Rotating waves* $u = v(x - ct), c \neq 0$, have been defined in (1.4) above, and constitute the set $\mathcal{R} \subset X$. Here we assume v to be nonconstant and include $v(\cdot)$ in $\mathcal{R}$ together with its shifted copies $v(\cdot + \vartheta), \vartheta \in \mathbb{R}$. Clearly, a rotating wave is a 2π-periodic solution of the ordinary differential equation

$$0 = v_{xx} + f(v, v_x) + cv_x, \tag{1.7}$$

with $c \neq 0$. If v happens to solve (1.7) with rotation speed $c = 0$, then we call v a *frozen wave.* The set $\mathcal{F}$ thus consists of all spatially nonhomogeneous equilibrium solutions of (1.1), (1.2). Note that

$$\mathcal{E} \cup \mathcal{F} \cup \mathcal{R} \subset \mathcal{A}. \tag{1.8}$$

Indeed, equilibria and periodic solutions are globally bounded, for all $t \in \mathbb{R}$, and hence belong to the global attractor $\mathcal{A}$.

Let $\mathcal{H}$ denote the set of heteroclinic orbits between different elements of $\mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$. Clearly $\mathcal{H} \subset \mathcal{A}$. In theorem 1.2, we will see that

$$\mathcal{E} \cup \mathcal{F} \cup \mathcal{R} \cup \mathcal{H} = \mathcal{A}. \tag{1.9}$$

To describe all heteroclinic orbits is therefore the central, and most difficult, ingredient to a full description of the global attractor. Note that $\mathcal{E}, \mathcal{F}, \mathcal{R}$ require only ODE information, as is involved in equations (1.6), (1.7) above. The heteroclinic set $\mathcal{H}$, in contrast, involves information on the PDE (1.1), (1.2).

To describe the heteroclinic set, we use two structural ingredients. The first ingredient, which in fact holds for x-dependent nonlinearities $f = f(x, u, u_x)$, is the *Sturm property.* This property originated with Sturm [Stu36]; see [Ang88] for a more recent account. Matano [Mat82] revived the relevance of this structure for the global analysis of reaction-advection-diffusion equations. For any function $\varphi \in C^1(S^1)$;

let the zero number $z(\varphi)$ count the (even) number of strict sign changes of φ. Let $u^1(t,x), u^2(t,x)$ denote any two solutions of (1.1) with periodic or Neumann boundary conditions (1.2), (1.3). Then

$$z\left(u^1(t,\cdot) - u^2(t,\cdot)\right) \tag{1.10}$$

is finite, for any $t > 0$, and nonincreasing with t. Moreover z drops strictly, at any multiple zero of $x \mapsto u^1(t_0,x) - u^2(t_0,x)$. In other words, z drops strictly, at $t = t_0$, if and only if there exists $x_0 \in S^1$ such that

$$\begin{array}{rcl} u^1(t_0,x_0) & = & u^2(t_0,x_0) \\ u^1_x(t_0,x_0) & = & u^2_x(t_0,x_0). \end{array} \tag{1.11}$$

We briefly digress to give a typical argument involving zero numbers. Let $v^1, v^2 \in \mathcal{R}$ be different rotating waves. Then [MN97] have observed that $v^1(\cdot+\vartheta_1) - v^2(\cdot+\vartheta_2)$ possesses only simple zeros, and in particular

$$(\vartheta_1,\vartheta_2) \mapsto z\left(v^1(\cdot+\vartheta_1) - v^2(\cdot+\vartheta_2)\right) \equiv k \tag{1.12}$$

does not depend on ϑ_1, ϑ_2. To see this fundamental fact, we argue indirectly. Let

$$\varphi(t) := v^1(\cdot - c_1 t + \vartheta_1) - v^2(\cdot - c_2 t + \vartheta_2) \in X, \tag{1.13}$$

where c_1, c_2 denote the nonzero rotation speeds of the rotating waves v^1, v^2. With any multiple zero of $\varphi(0)$, the zero number

$$t \mapsto z(\varphi(t)) \tag{1.14}$$

would drop strictly, at $t = 0$. If c_1/c_2 is rational, then (1.14) is periodic. This is a contradiction because, once dropped, z cannot ever recover its initial value at any later time. Next suppose c_1/c_2 is irrational. Choose $t_1 < 0$ such that $\varphi(t_1) \in X$ possesses only simple zeros. Note that $z(\varphi)$ is then constant, near $\varphi = \varphi(t_1)$. By density of $\varphi(t)$ on the torus of pairs $(v^1(\cdot+\tilde{\vartheta}_1), (v^2(\cdot+\tilde{\vartheta}_2)) \in X \times X$, we then find some possibly large time $t_2 > 0$ such that

$$z(\varphi(t_1)) = z(\varphi(t_2)) < z(\varphi(t_1)). \tag{1.15}$$

The equality holds for $\varphi(t_2)$ near $\varphi(t_1)$, and the inequality follows from the Sturm property (1.10). Indeed $z(\varphi(t))$ strictly drops, at $t = 0$, and $t_1 < 0 < t_2$. This contradiction proves claim (1.12).

The second structural ingredient, which only holds for x-independent nonlinearities $f = f(u,u_x)$, is S^1-*equivariance.* This property simply states that $u(t,x+\vartheta)$ is a solution of (1.1) with periodic boundary conditions (1.2), for any fixed $\vartheta \in S^1$, whenever $u(t,x)$ itself is a solution. Rotating waves $u(t,x) = v(x-ct)$, in this setting, are relative equilibria to the group action: the time orbit $u(t,x)$ coincides with the initial condition $u(0,x) = v(x)$, shifted by $-ct$, and thus remains in a single group orbit. We also note the invariance of the zero number $z(\varphi)$ under shifts of $\varphi \in C^1(S^1)$.

Hyperbolicity of equilibria and periodic orbits is the only remaining assumption for formulating the main results, theorems 1.3 and 1.4 below. For a homogeneous equilibrium v, we consider eigenvalues μ of the linearization of (1.1) with periodic or Neumann boundary conditions. We call v hyperbolic, if all eigenvalues μ possess nonzero real part. The number of eigenvalues μ with strictly positive real part, counting algebraic multiplicities, is called the *unstable dimension* or *Morse index* $i(v)$. For periodic orbits $u(t,x)$, we require the trivial Floquet multiplier $\mu = 1$, with eigenfunction u_t, to be simple and to be the only Floquet multiplier on the complex unit circle. The (strong) *unstable dimension* $i(u)$ then counts the total algebraic multiplicity of Floquet multipliers with modulus $|\mu| > 1$. For frozen wave equilibria, we require *normal hyperbolicity:* all eigenvalues μ possess nonzero real part, except for a simple trivial eigenvalue $\mu = 0$ with eigenfunction given by v_x. The Morse index $i(v)$ is then counted as for homogeneous equilibria above.

We recall that hyperbolic equilibria and periodic orbits v come along with their stable and unstable manifolds $W^u(v)$ and $W^s(v)$. Note that

$$\dim W^u(v) = i(v) + 1, \quad \operatorname{codim} W^s(v) = i(v) \tag{1.16}$$

for periodic orbits v, by standard terminology. In case of rotating waves v we obtain v as a hyperbolic frozen wave, in corotating coordinates. For consistency, we therefore use the same notation $W^u(v), W^s(v)$, for frozen wave equilibria v, even though – as a tribute to conventional terminology – these manifolds ought to be called the center unstable, and center stable manifold of v, respectively. In particular, (1.16) then holds for $v \in \mathcal{F} \cup \mathcal{R}$.

As Brunovský has observed, [Bru00], the above hyperbolicity assumptions hold true for a generic set of nonlinearities $f(u, u_x)$ in $C^k, k \geq 3$. As we will see in lemmas 4.4 and 5.3 below, hyperbolicity is related to the Sard property of a suitably defined time map $T(\alpha)$.

Heteroclinic orbits in the Neumann case $(\mathcal{N})$ of (1.1), (1.3) have been studied by [FR96]. We present this result in the significantly refined and simplified formulation of [Wol02a], [Wol02b]. Let $v^1, v^2 \in \mathcal{E} \cup \mathcal{F}$ be two different equilibria. We call v^1, v^2 *k-adjacent*, or k-$(\mathcal{N})$-adjacent, if the following holds:

$$z(v^1 - v^2) = k, \tag{1.17}$$

and there does not exist an equilibrium $w \in \mathcal{E} \cup \mathcal{F}$ such that

$$\begin{array}{c} z(v^1 - w) = z(v^2 - w) = k, \text{ and} \\ w(0) \text{ is between } v^1(0) \text{ and } v^2(0) \end{array} \tag{1.18}$$

Theorem 1.1 *Consider the Neumann case $(\mathcal{N})$ of (1.1), (1.3). Assume the nonlinearity $f = f(x, u, u_x)$ is C^2 dissipative, and all equilibria $v \in \mathcal{E} \cup \mathcal{F}$ are hyperbolic. Then the global attractor $\mathcal{A}$ decomposes into equilibria and heteroclinic orbits:*

$$\mathcal{A} = \mathcal{E} \cup \mathcal{F} \cup \mathcal{H}. \tag{1.19}$$

In particular, any orbit in $\mathcal{A} \setminus (\mathcal{E} \cup \mathcal{F})$ is heteroclinic, connecting two different equilibria.

Consider any two different equilibria $v^1, v^2 \in \mathcal{E} \cup \mathcal{F}$. Let $k := z(v^1 - v^2)$. Then there exists a heteroclinic orbit between v^1 and v^2 if, and only if, v^1 and v^2 are k-$(\mathcal{N})$-adjacent. Moreover, there is a unique heteroclinic orbit $u(t, x)$, such that

$$z(v^1 - u(t, \cdot)) = z(v^2 - u(t, \cdot)) = k \tag{1.20}$$

for all $t \in \mathrm{IR}$. In particular $u(t, 0)$ moves monotonically between $v^1(0)$ and $v^2(0)$. The heteroclinic orbit runs in the direction of decreasing Morse index.

We now consider the case $(\mathcal{P})$ of periodic boundary conditions. The analogue of the first part of theorem 1.1 has been established by [AF88], as follows.

Theorem 1.2 *Consider the S^1-equivariant case $(\mathcal{P})$ of (1.1), (1.2) with dissipative C^2-nonlinearity $f = f(u, u_x)$ and with hyperbolic homogeneous equilibria, frozen and rotating waves. Then the global attractor $\mathcal{A}$ decomposes into homogeneous equilibria $\mathcal{E}$, frozen wave equilibria $\mathcal{F}$, rotating waves $\mathcal{R}$ and their heteroclinic orbits $\mathcal{H}$:*

$$\mathcal{A} = \mathcal{E} \cup \mathcal{F} \cup \mathcal{R} \cup \mathcal{H}. \tag{1.21}$$

In particular, any periodic orbit is a rotating wave, and any orbit in $\mathcal{A} \setminus (\mathcal{E} \cup \mathcal{F} \cup \mathcal{R})$ is heteroclinic, connecting two different elements v^1, v^2 of $\mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$.

We emphasize that (1.21) in particular excludes the possibility of homoclinic orbits $u(t, x)$, asymptotic to the same orbit $v \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$ for $t \to \pm\infty$. This was observed by Matano and Nakamura, [MN97].

The main result of the present paper, concerns heteroclinic connectivity, that is, which pairs $v^1, v^2 \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$ possess a heteroclinic orbit with v^1, v^2 as limits for $t \to \pm\infty$. To formulate this result, we have to slightly modify our notion of k-adjacency. Similarly to (1.17), (1.18) above, two different $v^1, v^2 \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$ are called *k-adjacent* in $(\mathcal{P})$, or simply k-$(\mathcal{P})$-adjacent, if the following holds:

$$z(v^1 - v^2) = k \tag{1.22}$$

and there does not exist an element $w \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$ such that

$$\begin{array}{c} z(v^1 - w) = z(v^2 - w) = k, \text{ and} \\ \max_{x \in S^1} w(x) \text{ is strictly between } \max_{S^1} v^1 \text{ and } \max_{S^1} v^2. \end{array} \tag{1.23}$$

We note that $z(v^1 - v^2)$ is well-defined and time-independent, for any two different elements v^1, v^2. For rotating waves $v^1, v^2 \in \mathcal{R}$ this follows from the Sturm property; see (1.12) above. The other cases, involving frozen waves or homogeneous equilibria, are even easier and are left to the reader as a warm-up. In particular, we observe that k-$(\mathcal{P})$-adjacency of $v^1, v^2 \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$ persists under shifts $v^j(x) \mapsto v^j(x + \vartheta^j)$.

For $v^1, v^2 \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$ we denote stable and unstable manifolds $W^s(v^j), W^u(v^j)$ as explained above. We call v^1 and v^2 *connected*, if either $W^u(v^1) \cap W^s(v^2)$, or $W^u(v^2) \cap W^s(v^1)$, is not empty. Note that in the case of frozen waves, this establishes the existence of a heteroclinic orbit $u(t,x)$ converging to suitably shifted copies $v^j(x+\vartheta^j)$, only. For orbits $u(t,x)$ converging to rotating waves, i.e. to time periodic orbits, likewise the asymptotic phases for $t \to \pm\infty$ are not prescribed.

Theorem 1.3 *Let the assumptions of theorem 1.2 hold, for the S^1-equivariant periodic case $(\mathcal{P})$. Let v^1, v^2 be two different elements of $\mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$, and let $k := z(v^1 - v^2)$. Then v^1 and v^2 are connected if, and only if, they are k-$(\mathcal{P})$-adjacent. In particular, there is a heteroclinic connecting orbit $u(t,x)$, satisfying*

$$z(v^1 - u(t,\cdot)) = z(v^2 - u(t,\cdot)) = k \quad \textit{for all} \quad t \in \mathbb{R}. \tag{1.24}$$

We repeat that some care is needed in the precise interpretation of this theorem. We consider phase-shifted copies $v^2(x) = v^1(x+\vartheta)$ of frozen waves $\mathcal{F}$ as representing the same element of $\mathcal{F}$, rather than different ones. Similarly, consider v^1, v^2 representing different frozen waves. We then call v^1, v^2 connected by a heteroclinic orbit $u(t,\cdot)$, if this orbit converges to suitably phase-shifted copies $v^j(x+\vartheta_j)$. This interpretation is common use for rotating waves v^1, v^2, that is, for time periodic orbits. Indeed, the above theorem then does not make any claims about the asymptotic phases of the heteroclinic orbit, for $t \to \pm\infty$.

The above result fails to determine the time direction of heteroclinic orbits between k-$(\mathcal{P})$-adjacent elements $v^1, v^2 \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$. To close this gap we restate the following result of [MN97]:

Theorem 1.4 *Under the assumptions of theorem 1.2, let $u(t,x)$ be an orbit of $(\mathcal{P})$ from v^1 for $t \to -\infty$, to v^2, for $t \to +\infty$, with both v^1 and v^2 in $\mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$. Then u runs in the direction of decreasing (strong) unstable dimension, that is*

$$i(v^1) > i(v^2). \tag{1.25}$$

We now outline the proof of theorems 1.3 and 1.4 which occupies the remaining sections of the paper. In section 2 we prove the easy "only if" part of theorem 1.3. This will be an easy consequence of the Sturm property (1.10), (1.11). Under the name of "blocking principle", a very similar argument has already been used in [BF88] to exclude heteroclinic connections. In section 3 we review the strong Morse-Smale property, which is absolutely central to our proof of the "if" part of theorem 1.3. For the Neumann case $(\mathcal{N})$, this property has been discovered independently by [Hen85], [Ang86]. For the S^1-equivariant periodic case $(\mathcal{P})$, the Morse-Smale property is established in section 3 following [FuOl90]. Loosely speaking it states that hyperbolicity of $\mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$ alone is sufficient to imply transverse intersections of stable and unstable manifolds. This automatic transversality between stable and unstable manifolds of hyperbolic periodic orbits has been shown to hold in some special classes of finite-dimensional systems, [Pal78], [FuOl90]. The infinite-dimensional case has recently been considered in [Oli02], where a class of retarded functional differential equations is studied. Following work of [PS70], [Pal69], in the finite-dimensional case, and of [Oli02] in the infinite-dimensional case, the strong Morse-Smale property implies local C^0-orbit conjugacy of the flows on the global attractors, under C^2-small perturbations of the nonlinearity f. Remarkably, the strong Morse-Smale property fails for x-dependent nonlinearities $f = f(x, u, u_x)$. We will mainly use the strong Morse-Smale property to show that nonempty heteroclinic connections between $v^1, v^2 \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$, cannot disappear under any global, dissipative homotopies of the nonlinearity $f = f(u, u_x)$, as long as v^1, v^2 remain hyperbolic and k-adjacent. We also observe that theorem 1.4 is a simple consequence of theorem 1.3 and the strong Morse-Smale property.

The first such homotopy is introduced in section 4: freezing of rotating waves. For example we will show that, in absence of homogeneous equilibria of saddle type, any disc bounded by a rotating wave v^1 and containing a second wave v^2, in the (v, v_x)-plane, is filled with rotating waves – albeit of different wave speeds and of inappropriate spatial period. Passing to coordinates which corotate at this nonuniform wave speed $c(v, v_x)$, to be constant along each of these intermediate rotating waves, all these rotating wave speeds freeze to zero. In addition the phase portrait (1.7) of these frozen waves, with $c = c(v, v_x)$, becomes integrable in the disk. Moreover, hyperbolicity of the rotating waves v^1 and v^2, as well as their spatial periods, are preserved during the homotopy. By Morse-Smale transversality, this preserves their connectivity by heteroclinic orbits.

In section 5, by a second homotopy, we further modify the nonlinearity $f = f(u, u_x)$ in the disc to become even in u_x. Again the strong Morse-Smale property preserves connectivity. Moreover, the frozen rotating wave equation (1.7), already equipped with wave speeds $c \equiv 0$, now becomes x-reversible, in addition. From the equivariance point of view, the symmetry group $S^1 = SO(2)$ of x-shifts is now enhanced to $O(2)$, by the additional reflection $x \mapsto 2\pi - x$.

As a first benefit, we can determine the unstable dimensions of rotating (and frozen) waves v from their lap number $\ell(v) = z(v_x)$ by a time map argument reminiscent of the Neumann case; see lemma 5.3. Together with theorem 1.4 this determines, in particular, the time direction of the heteroclinic orbit between v^1 and v^2, in theorem 1.3.

After freezing and the above symmetrization, we will choose suitably shifted copies of the already frozen waves such that they attain their maximum at $x = 0$. In particular, the hyperbolic frozen waves v^1, v^2 now satisfy Neumann boundary conditions (1.3), in addition to the periodic boundary conditions (1.2). Moreover, both hyperbolicity and k-adjacency is preserved, under this Neumann interpretation. Cutting S^1 at $x = 2\pi$, in section 6, theorem 1.1 provides us with a Neumann heteroclinic orbit $u(t, x)$ from v^1 to v^2, satisfying (1.20). In general, however the boundary values $u(t, x)$ might differ, at $x = 0$ and $x = 2\pi$.

This is where the reflection symmetry $x \mapsto 2\pi - x$ strikes in section 7: by uniqueness of u and reflection symmetry of v^1, v^2, the heteroclinic orbit $u(t, x)$ will be 2π-

periodic in x. Indeed

$$u(t, 2\pi - x) = u(t, x),$$

also holds at $x = 0$. Because also

$$u_x(t, 2\pi) = u_x(t, 0) = 0,$$

by the Neumann boundary condition (1.3), the periodic boundary conditions (1.2) are automatically satisfied by the Neumann-heteroclinic orbit $u(t, x)$. Thus we have found a heteroclinic orbit u between v^1 and v^2, for the periodic problem $(\mathcal{P})$, and after two homotopies. Since $(u, u_x)(t, x)$ attains only values in the disc, Morse-Smale transversality provides survival of the heteroclinic orbit $u(t, x)$ under both homotopies, and proves theorem 1.3.

We conclude the paper, in section 8, with a retrospective of the main lines of proof, and with a few remarks on remaining cases. We also recall some of the difficulties, and chances, concerning the intriguing open case $f = f(x, u, u_x)$ of not S^1-equivariant, x-dependent nonlinearities with x on the circle.

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2 Blocking and k-adjacency

In this short section, we prove the easy "only if" part of theorem 1.3.

Assume $v^1, v^2 \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$ possess a heteroclinic orbit $u(t, x)$ from v^1 to v^2. Let $k := z(v^1 - v^2)$. We show, indirectly, that v^1, v^2 must then be k-adjacent under periodic boundary conditions $(\mathcal{P})$ as given in (1.2). Suppose they are not. Then there exists $w \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$ satisfying (1.23) above, that is

$$\begin{gathered} z(v^1 - w) = z(v^2 - w) = k, \text{ and} \\ \max w \text{ is between } \max v^1 \text{ and } \max v^2. \end{gathered} \tag{2.1}$$

Note that (2.1) also holds, if we replace v^1, w, v^2 by arbitrarily shifted copies, or by any small perturbation in $X \supseteq C^1(S^1)$. The heteroclinic orbit $u(t, \cdot)$ which converges to v^1, v^2, (or a shifted copy) for $t \to -\infty, t \to +\infty$, respectively, therefore satisfies

$$z\left(u(\pm t, \cdot) - w\right) = k, \tag{2.2}$$

for all sufficiently large positive t. Again we emphasize that w in (2.2) may also be replaced by any spatially shifted copies.

Continuity of $\max_{S^1}$, on the other hand, implies that

$$\max_{x\in S^1} u(t_0, x) = \max_{x\in S^1} w(x) \tag{2.3}$$

holds, for some $t_0 \in \mathbb{R}$. Shifting w, if necessary, we may assume that the maximum is attained at the same point $x = x_0 \in S^1$, for both sides of (2.3). Therefore

$$u(t_0, x) - w(x) \tag{2.4}$$

possesses a multiple zero, at $x = x_0$. By the Sturm property the zero number of the difference (2.4) must drop *strictly*, as time evolves. Therefore

$$z(u(+t, \cdot) - w) < z(u(-t, \cdot) - w), \tag{2.5}$$

for sufficiently large $t > 0$. Since (2.5) contradicts (2.2), this proves the "only if" part of theorem 1.3.

3 Hyperbolicity and the strong Morse-Smale property

In this section we review, and slightly adapt the strong Morse-Smale property. We consider the Neumann case $(\mathcal{N})$ of (1.1), (1.3), as well as the case $(\mathcal{P})$ of (1.1) with periodic boundary conditions (1.2); see proposition 3.1. Along the way, we collect some facts concerning unstable dimensions, zero numbers and lap numbers of equilibria and rotating waves.

Consider the Neumann case $(\mathcal{N})$ first, and let v^1, v^2 denote two hyperbolic equilibria. Let $W^u(v^1)$ denote the unstable manifold of v^1, of dimension given by the Morse index $i(v^1)$. Similarly, let $W^s(v^2)$ denote the stable manifold of v^2, of codimension $i(v^2)$. Based on the Sturm property (1.10), (1.11), [Hen85] and [Ang86] have concluded that transversality of these manifolds must hold *automatically*:

$$W^u(v^1) \pitchfork W^s(v^2). \tag{3.1}$$

Note that the intersection may be empty: this is the case where there does not exist a heteroclinic orbit running from v^1 to v^2. If the intersection is nonempty, that is, if a heteroclinic orbit from v^1 to v^2 exists, then the set of all heteroclinic orbits is an embedded submanifold of the underlying Sobolev space X, of dimension

$$0 < \dim \left(W^u(v^1) \cap W^s(v^2)\right) = i(v^1) - i(v^2). \tag{3.2}$$

In particular, this excludes homoclinic orbits which possess the same equilibrium as α- and ω-limit set. Moreover, (3.2) excludes the possibility of heteroclinic cycles which describe a closed oriented path of heteroclinics between hyperbolic equilibria. Indeed, the Morse index drops strictly along every heteroclinic orbit.

We also remark that the strong Morse-Smale property (3.1) holds true, more generally, for x-dependent nonlinearities $f = f(x, u, u_x)$, in the Neumann case $(\mathcal{N})$.

This is not the case, for $f = f(x, u, u_x)$, under periodic boundary conditions $(\mathcal{P})$. In fact, [SF92] have shown that any planar autonomous vector field can then be realized, for suitable $f = f(x, u, u_x)$, in an invariant linear subspace of $X = H^s(S^1)$. In particular homoclinic orbits to hyperbolic equilibria become possible.

This failure of the strong Morse-Smale property is the deeper technical reason why we have restricted our analysis of heteroclinic connectivity to the S^1-equivariant case of x-independent nonlinearities $f = f(u, u_x)$, in the present paper. The following proposition 3.2 is a slight adaptation of work by Fusco and Oliva on the strong Morse-Smale property for finite-dimensional, spatially discrete cyclic Jacobi systems, which share the Sturm property with our periodic case $(\mathcal{P})$. We prepare with some facts on the zero numbers on stable and unstable manifolds.

Proposition 3.1 *Consider the periodic case $(\mathcal{P})$ of (1.1), (1.2).*

(a) *Let $e \in \mathcal{E}$ be a homogeneous, hyperbolic, unstable equilibrium, with unstable manifold $W^u(e)$ and stable manifold $W^s(e)$. Then $i(e) = \dim W^u(e)$ is odd, and*

$$z(u_0 - e) < i(e) < z(\tilde{u}_0 - e), \tag{3.3}$$

for any $u_0 \in W^u(e) \setminus \{e\}$, $\tilde{u}_0 \in W^s(e) \setminus \{e\}$.

(b) *Let $v \in \mathcal{F} \cup \mathcal{R}$ be a hyperbolic rotating or frozen wave, with unstable manifold $W^u(v)$ and stable manifold $W^s(v)$. Then*

$$\dim W^u(v) = i(v) + 1 \ , \ \operatorname{codim} W^s(v) = i(v) \tag{3.4}$$

For any given lap number $\ell(v) := z(v_x) \geq 2$, *moreover,*

$$i(v) \in \{\ell(v) - 1, \ell(v)\}. \tag{3.5}$$

More specifically, only the following two cases occur.

In the more stable case, $i(v) = \ell(v) - 1$, we have

$$z(u_0 - v) < \ell(v) \leq z(\tilde{u}_0 - v), \tag{3.6}$$

for any $u_0 \in W^u(v), \tilde{u}_0 \in W^s(v)$, which do not lie on the rotating wave orbit v.

In the more unstable case, $i(v) = \ell(v)$, we have

$$z(u_0 - v) \leq \ell(v) < z(\tilde{u}_0 - v), \tag{3.7}$$

for any $u_0 \in W^u(v), \tilde{u}_0 \in W^s(v)$ orbit v. In either case, we have

$$z(u_0 - v) \leq i(v) < z(\tilde{u}_0 - v). \tag{3.8}$$

Proposition 3.2 *Consider the periodic case* $(\mathcal{P})$ *of (1.1), (1.2). Let* $v^1, v^2 \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$ *be hyperbolic homogeneous equilibria, or hyperbolic frozen or rotating waves. Let* $W^u(v^1)$ *denote the unstable manifold of* v^1 *and* $W^s(v^2)$ *the stable manifold of* v^2*. Then transversality holds:*

$$W^u(v^1) \pitchfork W^s(v^2). \tag{3.9}$$

Moreover, if $W^u(v^1) \cap W^s(v^2) \neq \emptyset$*, then*

$$i(v^1) > i(v^2). \tag{3.10}$$

Proof of proposition 3.1: Claims (a) are immediate, because the eigenfunctions of the linearization at a homogeneous S^1-invariant equilibrium e are the representation subspaces

$$\langle \cos kx, \ \sin kx \rangle \tag{3.11}$$

$k = 0, 1, 2, \cdots$ where $z(\cos kx) = z(\sin kx) = 2k$. Indeed $(u(t) - e)/|u(t) - e|$ approaches an eigenspace $\langle \cos kx, \ \sin kx \rangle$, for $t \to \pm\infty$ and $u(0)$ in the appropriate stable or unstable manifold of e. The Sturm property (1.10) then implies claim (3.3). See also [BF86], [AF88], for similar arguments.

Claim (b), (3.4) follows by definition; see (1.16). Claim (b), (3.5) involves a decomposition of the Floquet eigenspaces, according to zero number $z = 2k$, very similar to – but less explicit than – (3.11) above. See [AF88] and also [FuOl90] for details. The lap number $\ell(v) = z(v_x)$ enters diacritically, because $v_t = cv_x$ is the Floquet eigenfunction of the trivial Floquet multiplier 1, and thus separates the strongly stable from the strongly unstable Floquet eigenspace. This sketches the proof of proposition 3.1. ⋈

Proof of proposition 3.2: We first observe that claim (3.10) is an immediate consequence of proposition 3.1(b). Indeed, choose a heteroclinic orbit $u(t) \in W^u(v^1) \cap W^s(v^2) \neq \emptyset$. Then (3.8) implies

$$i(v^2) < z(u(-t) - v^2) = z(v^1 - v^2) = z(v^2 - v^1) = z(u(t) - v^1) \leq i(v^1) \ , \tag{3.12}$$

provided $t > 0$ is chosen large enough. This proves (3.10).

To show transversality, (3.9), we invoke [FuOl90], theorem 4. In the technically completely analogous finite-dimensional setting of Jacobi systems, transversality was shown there to hold, unless v^1 and v^2 are both equilibria. Either by repeating their arguments, verbatim, or else by symmetric finite difference discretization of (1.1), their results extend to the present case.

To deal with equilibria, we first eliminate all frozen waves by passing to a new coordinate system which rotates at a suitable nonzero speed c_0 :

$$\tilde{u}_t = \tilde{u}_{xx} + f(\tilde{u}, \tilde{u}_x) + c_0 \tilde{u}_x \ . \tag{3.13}$$

Due to S^1-equivariance of the original system (1.1), on S^1, the transversality property (3.9) is not affected by this adjustment. By hyperbolicity, all rotating waves of

(1.1) are isolated, hence finite in number, with a finite number of associated wave speeds. These can easily be avoided by c_0, so that all frozen waves of (1.1) become rotating waves, in (3.13), and none of the rotating waves of (1.1) is frozen.

With all frozen wave equilibria thus eliminated, we only have to deal with the possibility that v^1 and v^2 in (3.9) are both in fact homogeneous equilibria. The spectral analysis of proposition 3.1(a) then precisely provides the sufficient (and necessary, in view of the example [SF92]) condition, under which [FuOl90], theorem 5, asserts transversality (3.9) to hold. This completes the proof of proposition 3.2. $\bowtie$

Proof of theorem 1.4: Let there exist an orbit $u(t,x)$ from v^1 to v^2, both in $\mathcal{E}\cup\mathcal{F}\cup\mathcal{R}$. Then $W^u(v^1)\cap W^s(v^2)\neq\emptyset$, and theorem 1.4 follows because proposition 3.2, (3.10) asserts $i(v^1)>i(v^2)$. Note in particular that $i(v^1)>i(v^2)$ implies $v^1\neq v^2$. Therefore $u(t,x)$ must be heteroclinic, indeed. This proves theorem 1.4. $\bowtie$

We can now prove the "if" part of our main result, theorem 1.3, in case $k=z(v^1-v^2)=0$. First suppose at least one of v^1, v^2 is a rotating or frozen wave, say v^1. Moreover v^2-v^1 is of constant sign, say positive. Then $\ell(v^1)=z(v^1_x)\geq 2$, because $x\in S^1$ and v^1 is not identically constant. Therefore, proposition 3.1(b) implies instability, $i(v^1)\geq 1$. This instability was already observed, in the more general context of strongly monotone dynamical systems, by [Hir85]. By positivity of the first eigenfunction, we obtain a trajectory

$$u(t,x)>v^1(t,x) \tag{3.14}$$

in $W^u(v^1)$. Let $w\in\mathcal{E}\cup\mathcal{F}\cup\mathcal{R}$ be the ω-limit set of u. If $w\not\equiv v^2$, then w prevents v^1, v^2 from being 0-adjacent. Therefore $w\equiv v^2$. Finally, suppose both v^1 and $v^2\in\mathcal{E}\subset\mathbb{R}$ are homogeneous equilibria and therefore $z(v^1-v^2)=0$. Note that $f(v,0)$ changes sign at v if, and only if, $v\in\mathcal{E}$. Moreover, the one-dimensional flow

$$\dot{u}=f(u,0),\quad u\in\mathbb{R} \tag{3.15}$$

embeds into (1.1), (1.2). Therefore 0-adjacent elements v^1, v^2 of $\mathcal{E}$ possess an obvious x-homogeneous heteroclinic orbit (3.15). Together with section 2, this proves theorem 1.3 for $k=0$.

To prove theorem 1.3 for k-adjacent $v^1, v^2\in\mathcal{E}\cup\mathcal{F}\cup\mathcal{R}$ with $k>0$ we construct several homotopies, in the following chapters, which preserve both k-adjacency and hyperbolicity of v^1 and v^2. The following lemma ensures that possible heteroclinic orbits between v^1 and v^2 are neither created nor destroyed by such homotopies.

Lemma 3.3 *Let $v^1, v^2\in\mathcal{E}\cup\mathcal{F}\cup\mathcal{R}$ be hyperbolic and k-adjacent for a C^1-family f^τ of C^2-nonlinearities, $0\leq\tau\leq 1$. For $\tau=0$, assume that there exists a heteroclinic orbit $u(t,\cdot)$, $u(t,\cdot)\to v^{1,2}$ for $t\to\mp\infty$, such that*

$$z(u(t,\cdot)-v^j)=k. \tag{3.16}$$

for $j=1,2$ and all real t. Then v^1, v^2, or suitably shifted copies of v^1, v^2, possess a heteroclinic orbit with this property, for all $0\leq\tau\leq 1$.

Proof: Let $T \subseteq [0,1]$ denote the set of τ for which there exists a connecting orbit $u(t,\cdot) \in W^u(v^1) \cap W^s(v^2)$ satisfying (3.16). Note that $0 \in T$, by assumption. By the assumed hyperbolicity of v^1, v^2, and the Morse-Smale property $W^u(v^1) \pitchfork W^s(v^2)$, the set T is open in $[0,1]$. The lemma is proved, once we show that T is also closed.

Consider therefore any convergent sequence $T \ni \tau_n \to \tau \in [0,1]$ with associated heteroclinics $u_n(t,\cdot)$. By compactness of the semiflow we may assume convergence of $u_n(0,\cdot)$ to some initial condition $u_0 \in X$ at parameter τ. Consider the bounded global solution $u(t,\cdot)$ through $u(0,\cdot) = u_0$. We now show that v^j, $j = 1,2$, or a phase shifted copy of them, is in the ω-limit set $\omega(u_0)$. The analogous argument for $\alpha(u_0)$, which will complete the proof, will be omitted.

Let $w \in \omega(u_0) \subset \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$. Suppose indirectly, that w is neither a shifted copy of v^1, nor of v^2. Then $w - v^j$ possesses only simple zeros. By continuity of the zero number, (3.16) therefore implies

$$z(w - v^j) = z(u(t,\cdot) - v^j) = z(u_n(t,\cdot) - v^j) = k \qquad (3.17)$$

for sufficiently large t, n and for $j = 1,2$. This takes care of one of the conditions (1.23) involved in violations of k-adjacency. Because (3.16) also implies that $\max_{S^1} u_n(t,x)$ is strictly between $\max_{S^1} v^1(x)$ and $\max_{S^1} v^2(x)$, the same holds for $\max_{S^1} w(x)$. Here we use $w \neq v^1, v^2$, also after phase-shifts. This takes care of the other condition in (1.23) and shows that w prevents k-adjacency of v^1, v^2. Our indirect argument therefore shows that w coincides with a shifted copy of either v^1 or v^2.

Similarly we prove that $\alpha(u_0)$ coincides with a shifted copy of v^1 or v^2. Because homoclinic orbits do not exist here, this proves that $u(t,\cdot)$ is an orbit, connecting v^1 and v^2. A forteriori, $u(t,\cdot)$ is a transverse intersection of the respective stable and unstable manifolds and hence also satisfies (3.16), by (3.17). This proves $\tau \in T$. Hence T is also closed and the lemma is proved. ⋈

4 Nesting, freezing, and hyperbolicity

At the end of the previous section we have proved our main result, theorem 1.3, in case $k = z(v^1 - v^2) = 0$. From now on, we therefore assume

$$k = z(v^1 - v^2) \geq 2 \qquad (4.1)$$

for the even zero number k on the circle. Moreover, we fix the numbering v^1, v^2 such that

$$\max_{S^1} v^2 > \max_{S^1} v^1. \qquad (4.2)$$

In proposition 4.1 below, we show that v^1 must then be contained in the interior of the closed orbit of v^2, in the phase plane (v, v_x). We call this fact *"nesting"*. In lemma 4.2, central to the entire paper, we show how all relevant rotating waves can be frozen to zero wave speed, simultaneously, even if the individual wave speeds are different.

Lemma 4.3 then shows how a freezing homotopy affects neither hyperbolicity nor, due to section 3, transversality and heteroclinic connectivity.

The following proposition is essentially due to Matano and Nakamura, [MN97]. We include a proof, for the convenience of the reader.

Proposition 4.1 *Let assumptions (4.1), (4.2) above hold for* $v^1, v^2 \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$. *Then*

(i) the 2π*-periodic, possibly constant* x*-orbit* $(v^1(x), v^1_x(x))$ *of* v^1 *lies inside the* 2π*-periodic, non-constant* x*-orbit* $(v^2(x), v^2_x(x))$ *of* v^2

(ii) $k = z(v^1 - v^2) = \ell(v^2) = z(v^2_x)$.

Proof: As we have noted in the introduction, the x-orbits of v^1 and v^2 cannot intersect, or else $z(v^1 - v^2)$ would have to drop. If $\min v^2 > \max v^2$, then $k = z(v^1 - v^2) = 0$, contrary to assumption (4.1). Therefore the Jordan curve theorem implies

$$\min v^2 < \min v^1 \leq \max v^1 < \max v^2. \tag{4.3}$$

This establishes our nesting claim (i).

We prove claim (ii), $z(v^1 - v^2) = \ell(v^2)$, next. By (4.3), $v^2 - v^1$ changes sign at least once between any two adjacent extrema of $v^2(x), x \in S^1$. Therefore $z(v^1 - v^2) \geq \ell(v^2)$. Because the x-orbit of v^1 is nested strictly inside the x-orbit of v^2, we also observe

$$\mathrm{sign}(v^2_x - v^1_x) = \mathrm{sign}\, v^2_x \neq 0 \tag{4.4}$$

at any zero of $v^2 - v^1$. Therefore $v^2 - v^1$ changes sign at most once between two adjacent extrema of $v^2(x)$, $x \in S^1$, and hence $z(v^1 - v^2) \leq \ell(v^2)$. Together this proves $z(v^1 - v^2) = \ell(v^2)$, and the proposition. $\bowtie$

As a preparation for the freezing homotopy, we study the rotating wave equation

$$0 = v_{xx} + f(v, v_x) + cv_x \ , \tag{4.5}$$

with the wave speed $c \in \mathbb{R}$ as a parameter. Recall that nonconstant periodic solutions $v(x)$ of (4.5) with minimal period $T > 0$ indeed give rise to rotating wave solutions, for $c \neq 0$, or frozen wave solutions, for $c = 0$, of (1.1), (1.2) on the circle $x \in S^1 = \mathbb{R}/nT\mathbb{Z}$, for any $n \in \mathbb{N}$. We can also view (4.5) as a rather simple problem of global Hopf bifurcation for the planar system

$$\begin{aligned} v_x &= p \\ p_x &= -f(v,p) - cp \end{aligned} \tag{4.6}$$

with parameter c. Note that equilibria $(e, 0) \in \mathcal{E}$ of (4.6) undergo a single Hopf bifurcation, as c increases, when

$$f_v(e, 0) > 0. \tag{4.7}$$

We then call e a *center*, even though $(e,0)$ will actually be a saddle of (4.6) for large $|\ c\ |$. We call e a *saddle*, if $f_v(e,0) < 0$, that is, if $(e,0)$ is a saddle of (4.6) for all values of $c \in \mathbb{R}$. We now investigate the properties of the set $\mathcal{C} \subset \mathbb{R}^2$, which consists of all centers $(e,0)$ and of all points (v,p), for which there exists a parameter value c such that the orbit of (4.6) through (v,p) is periodic. We call $\mathcal{C}$ the *cyclicity set.* Lemma 4.2 below is inspired by [MN97].

Lemma 4.2 *Assume the C^2-nonlinearity f satisfies the dissipativeness condition (1.5). Moreover suppose all zeros e of $v \mapsto f(v,0)$ are simple. Then the cyclicity set $\mathcal{C}$ defined above is bounded, and open. Furthermore, if $\mathcal{C}$ is nonempty, there exist C^2-functions*

$$c, T : \mathcal{C} \to \mathbb{R} \tag{4.8}$$

with the following properties.

(i) For each nonstationary point $(v,p) \in \mathcal{C}$, the value $c(v,p)$ indicates the unique *wave speed c for which (v,p) lies on a periodic orbit of system (4.6). Similarly, $T(v,p)$ indicates the minimal period.*

(ii) The wave speeds c are uniformly bounded on $\mathcal{C}$.

(iii) The minimal periods T tend to infinity, at the boundary $\partial\mathcal{C}$ of $\mathcal{C}$.

(iv) $\partial\mathcal{C}$ consists of saddles, and of points which are homoclinic or heteroclinic to saddles, for some parameter c.

Proof: By dissipativeness condition (1.5), we have $f(v,0)v < 0$ for large $|v|$. Therefore $v \mapsto f(v,0)$ possesses at least one zero which is a saddle. If this saddle is the only zero of $f(\cdot,0)$, then $\mathcal{C}$ is empty. If, however, $f(\cdot,0)$ possesses additional zeros, then one of them must be a center and $\mathcal{C}$ is nonempty. In this case, the same sign constraint ensures uniform bounds, on the maximum and minimum of v, on any bounded solution of (4.6). Together with the bounds (1.5) on f this shows boundedness of $\mathcal{C}$.

We now prove openness of $\mathcal{C}$. By standard Hopf bifurcation of (4.6) at centers $(e,0)$ with parameter $c = -f_p(e,0)$ and limiting period $T = 2\pi/\sqrt{f_v(e,0)}$, the cyclicity set $\mathcal{C}$ contains neighborhoods of centers. Next consider some nonstationary $(v_0,p_0) \in \mathcal{C}$, at parameter c_0 and with minimal period $T_0 > 0$. Let $\varphi(v,p,T,c)$ denote the flow (4.6) with "time" $x = T$. Then

$$\varphi(v,p,T,c) - (v,p) = 0 \tag{4.9}$$

at (v_0,p_0,T_0,c_0). By the implicit function theorem, openness of $\mathcal{C}$ follows if we can show that

$$\det(\varphi_T, \varphi_c) \neq 0, \tag{4.10}$$

for the 2×2 matrix of partial derivatives φ_T, φ_c at (v_0,p_0,T_0,c_0).

To show (4.10), we abbreviate $\mathbf{v} = (v, p)$, and

$$\begin{array}{rcl l} \mathbf{v} & = & \mathbf{v}(x) & := \varphi(v_0, p_0, x, c_0) \\ \boldsymbol{\eta} & = & \boldsymbol{\eta}(x) & := \varphi_c(v_0, p_0, x, c_0) \\ \mathbf{f} & = & \mathbf{f}(\mathbf{v}, c) & := (p, -f(v,p) - cp) \ . \end{array} \tag{4.11}$$

With the Jacobian $\mathbf{f_v}$ along $\mathbf{v}(x)$ and the partial derivative $\mathbf{f}_c$ we observe

$$\begin{array}{rcl} \mathbf{v}_x & = & \mathbf{f}(\mathbf{v}, c) \\ (\mathbf{v}_x)_x & = & \mathbf{f_v} \cdot \mathbf{v}_x \\ \boldsymbol{\eta}_x & = & \mathbf{f_v} \cdot \boldsymbol{\eta} + \mathbf{f}_c \ . \end{array} \tag{4.12}$$

Denoting transpose by $'$ and letting $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we also define a solution $\mathbf{v}_*(x)$ by

$$\begin{array}{rcl} (\mathbf{v}_*)_x & = & -\mathbf{f_v}' \cdot \mathbf{v}_* \\ \mathbf{v}_*(0) & = & J\mathbf{v}_x(0) \neq 0 \ . \end{array} \tag{4.13}$$

Then $\mathbf{v}_*(x) \neq 0$, for all x, and

$$\frac{d}{dx}(\mathbf{v}_*(x)'\mathbf{v}_x(x)) = 0 \ . \tag{4.14}$$

This invariance of the scalar product, together with the initial condition for $\mathbf{v}_*(0)$, implies

$$\mathbf{v}_*(x) = \lambda(x) J \mathbf{v}_x(x) \tag{4.15}$$

for some strictly positive scalar function $\lambda(x)$. Similarly to (4.14) we also observe that

$$\begin{array}{rcl} \frac{d}{dx}(\mathbf{v}_*(x)'\boldsymbol{\eta}(x)) & = & (\mathbf{v}_*)_x'\boldsymbol{\eta} + \mathbf{v}_*'\boldsymbol{\eta}_x \\ & = & (-\mathbf{f_v}' \cdot \mathbf{v}_*)'\boldsymbol{\eta} + \mathbf{v}_*'(\mathbf{f_v} \cdot \boldsymbol{\eta} + \mathbf{f}_c) \\ & = & \mathbf{v}_*'\mathbf{f}_c = \lambda(J\mathbf{v}_x)'\mathbf{f}_c \\ & = & \lambda(J\mathbf{f})'\mathbf{f}_c = -\lambda p^2 < 0 \ , \end{array} \tag{4.16}$$

except at the discrete points x where $p(x) = 0$. Here we have used (4.12), (4.13), and (4.15). Because $\boldsymbol{\eta}(0) = 0$, by (4.11), we can now conclude

$$\begin{array}{rcl} \lambda(T)\det(\varphi_T, \varphi_c) & = & \lambda(J\mathbf{v}_x)'\boldsymbol{\eta}\,|_{x=T} \;=\; \mathbf{v}_*'\boldsymbol{\eta}\,|_{x=T} \\ & = & \int_0^T \frac{d}{dx}(\mathbf{v}_*'\boldsymbol{\eta})dx \;=\; -\int_0^T \lambda p^2 dx < 0 \ . \end{array} \tag{4.17}$$

This proves claim (4.10) and openness of the cyclicity set $\mathcal{C}$.

We now show that any nonstationary $\mathbf{v}_0 \in \mathcal{C}$ can be periodic for only one unique value of the wave speed c. Suppose, indirectly, that there exist two solutions $\mathbf{v}^1(x), \mathbf{v}^2(x)$ of (4.6) with wave speeds c_1, c_2, respectively. Without loss of generality, assume $c_1 < c_2$. If $p_0 \neq 0$, then it is easy to see that $\mathbf{v}^2(x)$ must transversely enter the interior of the closed curve $\mathbf{v}^1(x)$, and cannot leave it. For $p_0 = 0$, the same statement holds true, but with a cubic tangency at $x = 0$. This contradicts the assumption that $\mathbf{v}^1(x)$ is a closed curve, and proves existence of the wave speed function $c = c(v_0, p_0)$. Existence of the minimal period function $T = T(v_0, p_0)$ is then obvious. Differentiability of the function c follows from the implicit function theorem applied to (4.9). Differentiability of the minimal period T follows, similarly, because period doubling bifurcations and the like are impossible in planar systems. This proves part (i) of the lemma.

A singular perturbation argument for $c \to \pm\infty$, together with dissipativeness of f shows that (4.6) cannot possess periodic solutions for $|c|$ above a certain threshold C. This proves boundedness of c, (ii). For an alternative argument involving integration by parts see also [MN97].

To prove (iii) suppose, indirectly, that the minimal periods T_n remain bounded as points $(v_n, p_n) \in \mathcal{C}$ with bounded wave speeds c_n approach a point $(v_0, p_0) \in \partial\mathcal{C}$. Without loss of generality assume $T_n \to T_0, c_n \to c_0$. Because minimal periods are uniformly bounded from below [AbRo67], [Yor69], we have $T_0 > 0$ and $(v_0, p_0) \in \mathcal{C}$. Because $\mathcal{C}$ is open, this contradicts $(v_0, p_0) \in \partial\mathcal{C}$, and proves (iii).

To prove (iv), consider $(v_0, p_0) \in \partial\mathcal{C}$. If (v_0, p_0) is an equilibrium, then $p_0 = 0$ and $v_0 = e$ is a saddle, because centers belong to the open set $\mathcal{C}$.

Next suppose $\mathbf{v}_0 = (v_0, p_0) \in \partial\mathcal{C}$ is not an equilibrium. Choose

$$\mathcal{C} \ni \mathbf{v}_n \to \mathbf{v}_0 \ . \tag{4.18}$$

Without loss of generality assume convergence of the associated wave speeds

$$c_n = c(\mathbf{v}_n) \to c_0 \ , \tag{4.19}$$

and consider (4.6) with wave speed $c = c_0$. Note that the trajectory $\mathbf{v}(x)$ of $\mathbf{v}_0$ is bounded, by uniform boundedness of the trajectories $\mathbf{v}_n(x) \in \mathcal{C}$. To prove (iv) we have to show that the ω-limit set and the α-limit set of $\mathbf{v}_0$ consist of equilibria $(e, 0)$, only.

Suppose, indirectly, that there exists a non-equilibrium $\mathbf{w}_0 \in \omega(\mathbf{v}_0)$. Because $\mathbf{v}_0 \notin \mathcal{C}$ cannot be periodic, the trajectory $\mathbf{v}(x)$ of $\mathbf{v}_0$ must intersect a Poincaré section to the flow at $\mathbf{w}_0$ infinitely often, and at infinitely many different intersection points. Already two distinct intersection points, however, contradict the periodicity of $\mathbf{v}_n(x)$, for some large n. This contradiction proves claim (iv), and the lemma. $\bowtie$

With the wave speed function $c = c(v, p)$ of lemma 4.2 at hand, we now consider the *frozen system*

$$0 = v_{xx} + f(v, v_x) + c(v, v_x)v_x \ , \tag{4.20}$$

defined for $(v, v_x) \in \mathcal{C}$, for the moment. By construction, the function $c(v, v_x)$ is in fact a first integral of (4.20). Indeed, $c(\mathbf{v}_0)$ is the unique wave speed parameter c of (4.6), for which the orbit $\mathbf{v}(x)$ through $\mathbf{v}_0 = (v_0, p_0)$ becomes periodic. Therefore all points on the periodic orbit $\mathbf{v}(x)$ share the same value $c = c(\mathbf{v}(x))$.

As an aside, we mention that the condition for $c(v, p)$ to be a first integral is equivalent to the nonlinear hyperbolic conservation law

$$pc_v - (pc + f)c_p = 0 \ . \tag{4.21}$$

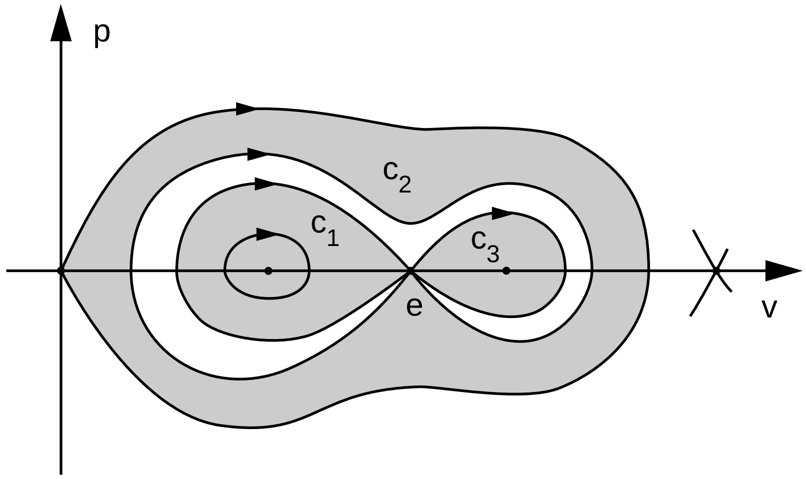

Figure 4.1: Gaps in the cyclicity domain $\mathcal{C}$ (hashed). The limiting wave speeds for the three homoclinic orbits to the saddle e satisfy $c_1 > c_2 > c_3$, in this example.

Note the degeneracy at $p = f = 0$. It is therefore not surprising that c cannot be defined everywhere, as a first integral. See figure 4.1 for an example, where the gap of $\mathcal{C}$ between the inner homoclinic lobes and the outer lobe would have to be filled by rarefaction fans.

For the purposes of our boundary value problem $(\mathcal{P})$ on $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, however, rotating waves (4.6), (4.20) with minimal periods $T > 2\pi$ are irrelevant. We therefore modify and extend $c(\mathbf{v})$ now, by an arbitrary, globally defined, bounded C^2-function outside the set $\mathcal{C}'$ of $\mathbf{v} \in \mathcal{C}$ with $T(\mathbf{v}) \leq 3\pi$. With this extension of $c(\mathbf{v})$ we now consider the *freezing homotopy*

$$\begin{gathered} u_t = u_{xx} + f^\tau(u, u_x), \quad x \in S^1, \\ \text{where } f^\tau(v,p) := f(v,p) + \tau c(v,p)p \ , \end{gathered} \tag{4.22}$$

for $0 \leq \tau \leq 1$. Clearly (4.22) preserves all rotating wave solutions (v, v_x) of (1.1), (1.2), reducing their wave speeds

$$c^\tau = (1-\tau)c^0 \tag{4.23}$$

to zero, by $\tau = 1$. Indeed

$$0 = v_{xx} + f^\tau(v, v_x) + (1-\tau)c(v, v_x)v_x \ . \tag{4.24}$$

Note that (4.22) may, in general, introduce additional spurious rotating or frozen waves, for $\tau > 0$, which are not present in the original problem for $\tau = 0$. However, a careful choice of the extension of $c(\mathbf{v})$ outside the set $\mathcal{C}'$ avoids this nuisance in the following way. Consider the gap associated to the saddle point $(v,p) = (e, 0)$ and its three homoclinic orbits corresponding to the limiting wave speeds $c_1 > c_2 > c_3$, (see again Figure 4.1 for this generic situation). Let $B_\varepsilon(e)$ denote an open ball of radius

$\varepsilon > 0$ centered at the saddle point $(e, 0)$. For small $\varepsilon > 0$ the set $\mathcal{C} \cap B_\varepsilon(e)$ consists of three open disjoint sectors determined by the three homoclinic orbits. In $\mathcal{C} \setminus B_\varepsilon(e)$ we leave $c(\mathbf{v})$ unchanged, as given by lemma 4.2. In $B_\varepsilon(e)$ we modify $c(\mathbf{v})$ only in the sectors corresponding to the wave speeds c_1 and c_3, adjusting C^2-smoothly so that $c(e, 0) = c_2$. For f^τ defined in this way, any spurious periodic orbit is confined to the annular regions outside $\mathcal{C}'$, but inside the homoclinic orbits corresponding to c_1 or c_3. Indeed, the annular region corresponding to c_1 bounds a subset of $\mathcal{C}'$ which, for wave speeds $c \neq c_1$, is either positively or negatively invariant. Taking ε sufficiently small we conclude that in this annular region a spurious periodic orbit occurs only for wave speeds ε-close to c_1. Continuous dependence of solutions of (4.22) with respect to parameters finally implies that the period T of this orbit is very large, say $T > 3\pi$. The same reasoning applies to the region corresponding to c_3 completing the argument.

Lemma 4.3 *Consider a periodic solution* $\mathbf{v} = (v, v_x)$ *of the freezing homotopy (4.24), with x-period* 2π *and wave speed* $c^\tau = (1-\tau)c^0$*. Then the corresponding rotating or frozen wave solution*

$$u(t, x) := v(x - c^\tau t) \tag{4.25}$$

of (4.22), which freezes for $\tau = 1$*, is hyperbolic for all* $0 \leq \tau \leq 1$*, if it is hyperbolic for* $\tau = 0$.

To prove lemma 4.3, we first relate the two cases $i(v) \in \{\ell(v) - 1, \ell(v)\}$ of the unstable dimension $i(v)$ of hyperbolic rotating or frozen waves, as established in proposition 3.1(b), to a monotonicity property of the x-period T, in lemma 4.2. To this end, let $\mathbf{v}^0 \in \mathcal{C}$ be a rotating or frozen wave solution of (4.6) with wave speed c^0. By lemma 4.2, $\mathbf{v}^0(x)$ lies in a local family $\mathbf{v}^\alpha(x)$ of rotating or frozen waves. Without loss of generality, we parametrize this family by its maxima:

$$\begin{array}{rcl} \mathbf{v}^\alpha(0) & = & (\alpha, 0) \;, \\ v^\alpha_{xx}(0) & < & 0 \;. \end{array} \tag{4.26}$$

By lemma 4.2 we have the corresponding wave speeds and minimal periods

$$c^\alpha := c(\alpha, 0); \quad T^\alpha := T(\alpha, 0) \;. \tag{4.27}$$

The following lemma is reminiscent of stability results involving the time map, in the Neumann case ($\mathcal{N}$); see [STW80], [Smo83], [BC84].

Lemma 4.4 *In the above setting, the rotating or frozen wave solution* $u(t, x) := v^0(x - c^0 t)$ *is hyperbolic if, and only if,*

$$\dot{T} := \partial_\alpha T(\alpha, 0) \mid_{\alpha=0} \neq 0 \;. \tag{4.28}$$

Proof: Let $k = \ell(v^0) = z(v^0_x) \geq 2$. Then $T^0 = 4\pi/k$. We only consider the case $k = 2$ here, where $2\pi = T^0$ is the minimal period. The more general case $k > 2$ is precisely analogous, and is omitted. Passing to a corotating coordinate system (t, x), we may also assume $c^0 = 0$, so that $\mathbf{v}^0$ is a frozen wave.

Denoting $\dot{} = \partial_\alpha \mid_{\alpha=0}$ we may differentiate the rotating wave family

$$\begin{aligned} 0 &= v^\alpha_{xx} + f(v^\alpha, v^\alpha_x) + c^\alpha v^\alpha_x \\ \alpha &= v^\alpha(0) = v^\alpha(T^\alpha) \\ 0 &= v^\alpha_x(0) = v^\alpha_x(T^\alpha) \end{aligned} \tag{4.29}$$

with respect to α, at $\alpha = 0$, to obtain

$$\begin{aligned} 0 &= \mathcal{L}\dot{v} + \dot{c}v^0_x \\ 1 &= \dot{v}(0) = \dot{v}(2\pi) \\ 0 &= \dot{v}_x(0) = \dot{v}_x(2\pi) + v^0_{xx}(2\pi) \cdot \dot{T}\ . \end{aligned} \tag{4.30}$$

Here we have abbreviated $\mathcal{L}$ to denote the linearization

$$\mathcal{L}w = w_{xx} + f_p w_x + f_v w \tag{4.31}$$

with f_v, f_p evaluated at $\mathbf{v}^0(x)$.

Note that μ is an eigenvalue of the frozen wave $\mathbf{v}^0(x)$, if and only if we find a nontrivial eigenfunction $w \in X = H^s(S^1)$ such that

$$\mu w = \mathcal{L}w\ . \tag{4.32}$$

The trivial eigenvalue $\mu = 0$ comes with the kernel eigenfunction $w = v^0_x$ of zero number $z(v^0_x) = k = 2$. The two instability cases of proposition 3.1(b), as enumerated in (3.5), are decided on by sign μ for the second eigenfunction w with $z(w) = k = 2$. We now construct this second eigenfunction, based on $\dot{v}$.

First suppose $\dot{T} = 0$. Then $\dot{v} \in X$ is 2π-periodic, by (4.30), and

$$\mathcal{L}\dot{v} = -\dot{c}v^0_x\ . \tag{4.33}$$

Moreover $\dot{v}$ is linearly independent of v^0_x, because $\dot{v}(0) \neq 0 = v^0_x(0)$. Therefore the frozen wave $\mathbf{v}^0$ is not hyperbolic, if $\dot{T} = 0$.

Conversely, suppose $\mathbf{v}^0$ is not hyperbolic. Then we can solve the initial value problem

$$\begin{aligned} 0 &= \mathcal{L}\dot{v} + \dot{c}v^0_x \\ 1 &= \dot{v}(0) \\ 0 &= \dot{v}_x(0) \end{aligned} \tag{4.34}$$

uniquely for $\dot{v}$, and obtain $\dot{v}$ as a unique linear combination of v^0_x and the second generalized eigenfunction w. Periodicity of both v^0_x and w implies 2π-periodicity of $\dot{v}$, and hence

$$v^0_{xx}(2\pi) \cdot \dot{T} = 0\ , \tag{4.35}$$

by (4.30). Because

$$v^0_{xx}(2\pi) < 0 \tag{4.36}$$

at the maximum of $v^0(x)$, this implies $\dot{T} = 0$, as claimed in lemma 4.4. ⋈

Proof of lemma 4.3: We embed the solution $\mathbf{v} := \mathbf{v}^0 \in \mathcal{C}$ of lemma 4.3 into a family $\mathbf{v}^\alpha \in \mathcal{C}$ as in (4.26), (4.27). Then lemma 4.4 implies that $\dot{T} \neq 0$, because u is hyperbolic for $\tau = 0$. Since the freezing homotopy neither changes any of the solutions $\mathbf{v}^\alpha$, nor any of the x-periods T^α, lemma 4.4 also implies the hyperbolicity of u, for any $0 \leq \tau \leq 1$. This proves lemma 4.3. ⋈

5 Reflection symmetry

In this section we provide the second crucial homotopy which, together with the freezing homotopy (4.22), will help to prove theorem 1.3 in section 7. This second homotopy

$$u_t = u_{xx} + f^\tau(u, u_x) \tag{5.1}$$

connects a nonlinearity $f^0 = f(v,p)$ with a *reversible nonlinearity* $f^1 = f^1(v,p)$ which is even in p:

$$f^1(v, -p) = f^1(v, p). \tag{5.2}$$

The homotopy will be defined by a family of symmetrizing transformations $\mathcal{C}^\tau$ of the cyclicity set $\mathcal{C}$ introduced in section 4; see lemma 4.2. The transformations will have the form

$$Q^\tau : (v, p) \mapsto (v, q^\tau(v, p)) \; . \tag{5.3}$$

For $\tau = 1$, all periodic orbits (v, v_x) in the cyclicity set $\mathcal{C}^1 = Q^1(\mathcal{C})$ will be axisymmetric with respect to the v-axis. Working with frozen waves, throughout this section, the second order frozen wave equation

$$0 = v_{xx} + f^\tau(v, v_x) \; , \tag{5.4}$$

which results from the symmetrizing transformation Q^τ, will then be x-reversible on $\mathcal{C}^1$, for $\tau = 1$. This establishes the symmetry (5.2).

To define the symmetrizing transformation Q^τ, consider any $(v_0, p_0) \in \mathcal{C}^0$ on an x-periodic orbit $\mathbf{v}(x) = (v(x), v_x(x))$ of (5.4), for $\tau = 0$. If $p_0 \neq 0$, that is if v_0 is neither the maximum nor the minimum of $v(x)$, then there exists a unique second point (v_0, p_0') on the same orbit. Indeed $v_x(x) \neq 0$ between extrema of $v(x)$. Note that p_0 and its *mirror point* p_0' have opposite sign. We define

$$(q^\tau(v_0, p_o))^{-1} := (1 - \tau)\frac{1}{p_0} + \frac{1}{2}\tau\left(\frac{1}{p_0} - \frac{1}{p_0'}\right) \; . \tag{5.5}$$

For $\tau = 1$, we obtain the harmonic mean of p_0 and $-p_0'$. Our definition of q^τ, Q^τ is motivated by the period invariance property (5.9) of the following lemma. A straightforward calculation shows that the homotopy f^τ in the nonlinearity of (5.4) is then given explicitly by

$$f^\tau \circ Q^\tau = \left(q^\tau_p f - q^\tau_v p\right) \cdot q^\tau / p \ . \tag{5.6}$$

Lemma 5.1 *Let $f \in C^5$. The transformation $q^\tau(v,p)$ defined in (5.5) is of class C^2 on the open cyclicity set $\mathcal{C} = \mathcal{C}^0$, with the properties*

$$q^\tau(v,0) := 0 \tag{5.7}$$

$$q^\tau_p(v,p) > 0. \tag{5.8}$$

In particular $Q^\tau(v,p) = (v, q^\tau(v,p))$ defines a continuous family of C^2-diffeomorphisms from $\mathcal{C}$ onto its image $\mathcal{C}^\tau = Q^\tau(\mathcal{C})$.

The period function $T(v,p)$, defined in lemma 4.2, is invariant under the homotopy Q^τ :

$$T^\tau(Q^\tau(v,p)) = T(v,p) \ , \tag{5.9}$$

where T^τ refers to the transformed nonlinearity f^τ.

Corollary 5.2 *Frozen waves and homogeneous equilibria remain hyperbolic under the symmetrizing transformations Q^τ.*

Proof of corollary 5.2: For frozen waves this follows from hyperbolicity lemma 4.4, and the period invariance (5.9) of lemma 5.1, under the symmetrizing transformations Q^τ. For homogeneous equilibria, the claim is obvious from the construction of f^τ. $\bowtie$

Proof of lemma 5.1: We prove differentiability and monotonicity (5.8) of $q^\tau(v,\cdot)$ first. For $(v,p) \in \mathcal{C}, p \neq 0$, these claims are obvious, because $\{v\} \times \mathbb{R}$ defines a local Poincaré section to

$$\begin{aligned} v_x &= p \\ p_x &= -f(v,p) \ . \end{aligned} \tag{5.10}$$

Indeed the map

$$p' = \varphi(v,p), \tag{5.11}$$

is C^5 and $\varphi_p \neq 0$, for $p \neq 0$. By the nesting property of proposition 4.1(i), we observe

$$\varphi_p(v,p) < 0. \tag{5.12}$$

This proves $q^\tau \in C^5$ and (5.8), for $p \neq 0$.

We now consider the case $p = 0$ and $v = v_0$ not a center. For v near v_0 and small $|p|$ let

$$\alpha = \psi(v,p) \tag{5.13}$$

near v_0 be such that $(\alpha, 0)$ lies on the same orbit of (5.9) as (v,p). By the flow box theorem,

$$(v,p) \mapsto (\psi(v,p), 0) \tag{5.14}$$

is a local C^5-diffeormorphism. Moreover (5.10) and $v_{xx} = f(v,p) \neq 0$ imply

$$\begin{aligned} \psi(v,0) &= v \\ \psi_p(v,0) &= 0 \\ \psi_{pp}(v,0) &\neq 0 \ . \end{aligned} \tag{5.15}$$

By the one-dimensional Morse Lemma, there exists an α-dependent local diffeomorphism

$$\begin{aligned} h(v,p) &= \tilde{p} \\ h(v,0) &= 0 \\ h_p(v,0) &\neq 0 \end{aligned} \tag{5.16}$$

such that the transformed ψ, given by $\tilde{\psi}(v,\tilde{p}) := \psi(v,p)$, simply satisfies

$$\tilde{\psi}(v,\tilde{p}) = \tilde{p}^2 + v \ . \tag{5.17}$$

See for example ([Arn71], sec. 12). Note that v just enters as a parameter. The transformation h is only of class C^3, in general. In $\tilde{p}$-coordinates (5.17) we then simply have $\tilde{p}' = -\tilde{p}$. This proves our differentiability and monotonicity claims for φ and q^τ, except at centers.

Let $(v_0, 0)$ be a center, without loss of generality with $v_0 = 0$. Similarly to (5.13), we then define ψ locally by

$$\alpha^2 = \psi(v,p), \tag{5.18}$$

where $(0,\alpha)$, with $\alpha \geq 0$, lies on the same orbit of (5.9) as (v,p). By the Hopf bifurcation theorem, ψ is in fact of class C^4. See for example [CR77], [Van89]. Again we lose one derivative here. Moreover

$$\begin{aligned} \psi(0,p) &= p^2 \\ \psi_p(v,0) &= 0 \\ \psi_{pp}(v,0) &\neq 0 \ . \end{aligned} \tag{5.19}$$

Applying the one-dimensional Morse Lemma with parameter v, as in (5.15), (5.17) above, we obtain

$$\tilde{\psi}(v,\tilde{p}) = \tilde{p}^2 + \psi(v,0) \tag{5.20}$$

at the expense of another two derivatives. Again $\tilde{p}' = -\tilde{p}$ and all differentiability and monotonicity claims for φ and q^τ are proved.

It only remains to check the period invariance property (5.9) which will be crucial to the preservation of hyperbolicity throughout our symmetrizing homotopy (5.4). Let $\mathbf{v}^0(x) = (v^0(x), p^0(x)) \in \mathcal{C}^0$ be any periodic orbit of the symmetrization homotopy (5.4), for $\tau = 0$. Let $a = \min v^0$, $b = \max v^0$. We can then write

$$v_x^0(x) = p^0(x) = P_\pm^0(v^0(x)) \ , \tag{5.21}$$

separately for the bottom part of $\mathbf{v}^0(x)$, where $p^0(x) \leq 0$, and for the top part, where $p^0(x) \geq 0$. The minimal period $T^0(\mathbf{v}^0(x)) = T(\mathbf{v}^0(x))$ is then given by

$$T^0(\mathbf{v}^0) = \int_0^{T^0} dx = \int_a^b \left(\frac{1}{P_+^0(v)} - \frac{1}{P_-^0(v)} \right) dv \ , \tag{5.22}$$

simply by substituting $v^0(x)$ for x, separately on either part of $\mathbf{v}^0(x)$. Note convergence of the integrals, by existence of T^0.

The same derivation (5.21), (5.22) applies throughout the symmetrization homotopy by $Q^\tau(v,p) = (v, q^\tau(v,p))$. We may replace the superscript "0" by "τ", if we define

$$P_\pm^\tau(v) := q^\tau(v, P_\pm^0(v)) \ . \tag{5.23}$$

Therefore (5.5) implies

$$T^\tau(\mathbf{v}^\tau) = \int_a^b \left(\frac{1}{P_+^\tau(v)} - \frac{1}{P_-^\tau(v)} \right) dv = (1-\tau)T^0(\mathbf{v}^0) + \tau T^1(\mathbf{v}^1) \ . \tag{5.24}$$

By definition of the mirror point p_0' to p_0, the map $p_0 \mapsto p_0'$ just interchanges $P_+^0(v)$ and $P_-^0(v)$, for any v. Therefore (5.5) implies

$$\begin{array}{rcl} T^1(\mathbf{v}^1) &=& \int_a^b \left(\frac{1}{P_+^1(v)} - \frac{1}{P_-^1(v)} \right) dv \\ &=& \frac{1}{2} \int_a^b \left(\left(\frac{1}{P_+^0(v)} - \frac{1}{P_-^0(v)} \right) - \left(\frac{1}{P_-^0(v)} - \frac{1}{P_+^0(v)} \right) \right) dv \\ &=& T^0(\mathbf{v}^0) \ . \end{array} \tag{5.25}$$

Together, (5.24) and (5.25) prove the period invariance property (5.9), and the lemma. ⋈

In the reversible frozen form

$$\begin{array}{rcl} v_t &=& v_{xx} + f(v, v_x) \\ f(v,-p) &=& f(v,p) \end{array} \tag{5.26}$$

it is now easy to refine lemma 4.4 and determine the unstable dimension $i(v^0) \in \{\ell(v^0)-1, \ell(v^0)\}$ of a rotating wave v^0 by its lap number $\ell(v^0) = z(v_x^0)$, explicitly. As in lemma 4.4, we consider v^0 as embedded into a rotating wave family v^α of minimal period T^α, see (4.26)–(4.29).

Lemma 5.3 *In the setting of lemma 4.4, the sign of*

$$\dot{T} = \partial_\alpha T(\alpha, 0) \mid_{\alpha=0} \neq 0 \tag{5.27}$$

decides the unstable dimension of a hyperbolic rotating or frozen wave v^0 to be given by

$$\begin{array}{rclcl} i(v^0) &=& \ell(v^0) - 1 & \Leftrightarrow & \dot{T} > 0 \\ i(v^0) &=& \ell(v^0) & \Leftrightarrow & \dot{T} < 0 \end{array} \tag{5.28}$$

Proof: By lemmas 4.3, 4.4 and corollary 5.2 above, we may first freeze and symmetrize the rotating wave family $\mathbf{v}^\alpha(x)$, without loss of generality. In particular $\dot{c} = 0$ in (4.29). Again we restrict to $k = \ell(v^0) = 2, T^0 = 2\pi$, for simplicity. From (4.29), (4.31) we recall that $\dot{v} = \partial_\alpha v \mid_{\alpha=0}$ satisfies

$$\begin{array}{rclcl} 0 &=& \mathcal{L}\dot{v} && \\ 1 &=& \dot{v}(0) &=& \dot{v}(2\pi) \\ 0 &=& \dot{v}_x(0) &=& \dot{v}_x(2\pi) + v_{xx}^0(2\pi)\dot{T} \end{array} \tag{5.29}$$

with $v^0_{xx}(2\pi) = v^0_{xx}(0) < 0$ at $v^0(0) = \max v^0$. Here $\mathcal{L}$ denotes the linearization at $\mathbf{v}^0(x)$, by now frozen. In comparison, let w satisfy

$$\mu w = \mathcal{L} w \ . \tag{5.30}$$

If w happens to be 2π-periodic, and if moreover $z(w) = 2$, then w is an eigenfunction of μ and

$$\begin{array}{rcll} i(v^0) &=& \ell(v^0) - 1, & \text{if} \quad \mu < 0 \\ i(v^0) &=& \ell(v^0), & \text{if} \quad \mu > 0 \ . \end{array} \tag{5.31}$$

We therefore construct μ, first, and then define w with the above properties.

To construct μ consider the initial value problem

$$\begin{array}{rcl} 0 &=& \mathcal{L}\tilde{w} - \mu\tilde{w} \\ 1 &=& \tilde{w}(0) \\ 0 &=& \tilde{w}_x(0) \ . \end{array} \tag{5.32}$$

Note that $\tilde{w} = \dot{v}$, for $\mu = 0$, by (5.29). Let $\tilde{\mathbf{w}} := (\tilde{w}, \tilde{w}_x)$. For positive μ, the nonzero vector $\tilde{\mathbf{w}}$ rotates slower than $\dot{\mathbf{v}} = (\dot{v}, \dot{v}_x)$, whereas for negative μ it rotates faster. See for example [CL55], [Har64]. We can in fact choose μ of the appropriate sign such that $\tilde{\mathbf{w}}(x)/|\tilde{\mathbf{w}}(x)|$ becomes 2π-periodic, and $z(\tilde{w}) = 2$. Comparing $\tilde{\mathbf{w}}(x)$ with the reference solution $\dot{\mathbf{v}}(x)$, for $\mu = 0$, we conclude from (5.29) that

$$\begin{array}{ll} \mu < 0, & \text{if} \quad -v^0_{xx}(2\pi)\dot{T} > 0 \\ \mu > 0, & \text{if} \quad -v^0_{xx}(2\pi)\dot{T} < 0 \ . \end{array} \tag{5.33}$$

Here we have used that the orbits $\mathbf{v}^\alpha(x)$ are nested, by proposition 4.1(i), and $\dot{\mathbf{v}}(x)/|\dot{\mathbf{v}}(x)|$ describes the limiting direction of the difference vector $\mathbf{v}^\alpha(x) - \mathbf{v}^0(x)$, for $\alpha \searrow 0$. The 2π-periodicity of $\tilde{\mathbf{w}}/|\tilde{\mathbf{w}}|$ implies

$$\tilde{w}(2\pi) = \beta, \quad \tilde{w}_x(2\pi) = 0, \tag{5.34}$$

for some positive β.

We now invoke reversibility (5.26) of f. Because $v^0(2\pi) = v^0(0), v^0_x(2\pi) = v^0_x(0) = 0$, reversibility implies

$$v^0(2\pi - x) = v^0(x). \tag{5.35}$$

Similarly, (5.32), (5.33) imply

$$\tilde{w}(2\pi - x) = \beta\tilde{w}(x) \ , \tag{5.36}$$

simply because $\tilde{w}(2\pi - x)$ solves the reversible linear equation $0 = \mathcal{L}\tilde{w} - \mu\tilde{w}$ if, and only if, $\tilde{w}(x)$ does. Relation (5.36) then follows because $\tilde{w}(2\pi - x)$ and $\beta\tilde{w}(x)$ satisfy the same initial conditions at $x = 0$. Inserting $x = 2\pi$ into (5.36), however, implies

$$1 = \tilde{w}(0) = \beta\tilde{w}(2\pi) = \beta^2 \ , \tag{5.37}$$

and therefore $\beta = \pm 1$. Because β was positive, we conclude 2π-periodicity of $(\tilde{w}, \tilde{w}_x)$. With $z(\tilde{w}) = 2$, we have thus constructed an eigenfunction $w = \tilde{w}$. Combining (5.31), (5.33) therefore proves the lemma. $\bowtie$

At this stage, we have first frozen all rotating waves and second symmetrize the resulting nonlinearity, at least as far as periodic orbits of

$$0 = v_{xx} + f(v, v_x) \tag{5.38}$$

of x-period $T \leq 3\pi$ in $\mathcal{C}$ are concerned. We remark that, conversely, any rotating wave solution of

$$0 = v_{xx} + f(v, v_x) + cv_x \tag{5.39}$$

is necessarily frozen, $c = 0$, provided that $f(v, -p) = f(v, p)$ is reversible along the rotating wave profile $(v, p) = \mathbf{v}(x)$. Without loss of generality, suppose $v_x(0) = 0$. Indeed, reversibility implies that $\tilde{\mathbf{v}}(x) := \mathbf{v}(-x)$ also solves (5.39), albeit with wave speed $-c$, along with $\mathbf{v}(x)$ itself. By (1.12)–(1.14), different rotating waves possess disjoint orbits, in the (v, v_x)-plane. Since $\tilde{\mathbf{v}}(0) = \mathbf{v}(0) = (v(0), 0)$, the two rotating wave orbits $\tilde{\mathbf{v}}$ and $\mathbf{v}$ must therefore coincide. Therefore their wave speeds coincide, $-c = c$, which implies $c = 0$.

6 Cut and paste: Neumann versus S^1

In this section we compare frozen waves, their hyperbolicity and heteroclinic orbit for (1.1), in the case $(\mathcal{P})$ of periodic boundary conditions (1.2), $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, with the case $(\mathcal{N})$ of Neumann boundary conditions (1.3), at $x = 0, 2\pi$. Throughout this section, we assume reversibility of the nonlinearity f,

$$f(v, -p) = f(v, p) \ . \tag{6.1}$$

We recall that any rotating or frozen wave solution v of $(\mathcal{P})$ is then frozen, automatically, and hence satisfies

$$0 = v_{xx} + f(v, v_x) \ ; \tag{6.2}$$

see (5.39) above.

The "cut" transition passes from $(\mathcal{P})$ to $(\mathcal{N})$, by cutting the periodic circle S^1 at $x = 2\pi$, to become the Neumann interval $0 \leq x \leq 2\pi$. The paste" transition, conversely, passes from $(\mathcal{P})$ to $(\mathcal{N})$, by pasting solutions of the Neumann problem together at $x = 0, 2\pi$. Clearly, successful pasting requires identical boundary values of the solution, at $x = 0, 2\pi$. The following lemma collects the various aspects of our cut-and-paste procedure.

Lemma 6.1 *(i) Let $f(e, 0) = 0$ define a homogeneous equilibrium of both $(\mathcal{P})$ and $(\mathcal{N})$. Then e is hyperbolic for $(\mathcal{P})$ if and only if $f_v(e, 0) \neq k^2, k \in \mathbb{Z}$, while e is hyperbolic for $(\mathcal{N})$ if and only if $f_v(e, 0) \neq (k/2)^2, k \in \mathbb{Z}$. The respective Morse indices, $i^{\mathcal{P}}$ and $i^{\mathcal{N}}$, satisfy*

$$\begin{aligned} i^{\mathcal{P}}(e) &= 1 + 2\left[\sqrt{f_v(e,0)}\right] && \text{if } f_v(e,0) > 0 \\ i^{\mathcal{N}}(e) &= 1 + \left[2\sqrt{f_v(e,0)}\right] && \text{if } f_v(e,0) > 0 \\ i^{\mathcal{P}}(e) &= i^{\mathcal{N}}(e) = 0 && \text{if } f_v(e,0) < 0 \end{aligned} \tag{6.3}$$

where $[\cdot]$ *denotes the* integer part.

(ii) *Let* $v(x)$ *be a frozen wave solution of (6.2), positioned such that*

$$v_x(0) = 0 \ . \tag{6.4}$$

Then $v(x)$ *is an equilibrium of both* $(\mathcal{P})$ *and* $(\mathcal{N})$. *Moreover* v *is hyperbolic for* $(\mathcal{P})$ *if and only if it is hyperbolic for* $(\mathcal{N})$ *and we have*

$$i^{\mathcal{P}}(v) = i^{\mathcal{N}}(v) - 1 \ . \tag{6.5}$$

(iii) *Conversely, let* $v(x)$ *be an equilibrium of* $(\mathcal{N})$ *with even lap number* $\ell(v) = z(v_x) \geq 2$. *Then* v *is a frozen wave solution of* $(\mathcal{P})$, *positioned such that (6.4) holds.*

(iv) *Let* v^1, v^2 *be k-adjacent equilibria of* $(\mathcal{P})$, *homogeneous or nonhomogeneous, positioned such that*

$$v^j(0) = \max_{x \in S^1} v^j(x) \qquad j = 1, 2 \ . \tag{6.6}$$

Then v^1, v^2 *are k-adjacent equilibria for* $(\mathcal{N})$.

(v) *Let* $u(t, x)$ *be any solution of* $(\mathcal{N})$ *in* $W^{1,\infty}$, *such that*

$$u(t, 0) = u(t, 2\pi) \tag{6.7}$$

for all t *in an open interval. Then* $u(t, x)$ *also solves* $(\mathcal{P})$, *for those* t.

Proof: The proof of (i) is an elementary computation, involving sines and cosines.

To prove (ii), note that also $v_x(2\pi) = 0$, by 2π-periodicity of the frozen wave $\mathbf{v}(x)$. Hyperbolicity of v is equivalent, for $(\mathcal{P})$, to $\dot{T} \neq 0$ for the associated time map. See lemma 4.4. Because reversibility (6.1) freezes all rotating waves, anyways, the time map $T^\alpha = T^\alpha_{\mathcal{P}}$ near $\mathbf{v}$ in fact coincides with the usual time map $T^\alpha_{\mathcal{N}}$ of the Neumann problem, up to a factor 2:

$$T^\alpha_{\mathcal{P}} = 2T^\alpha_{\mathcal{N}} \ . \tag{6.8}$$

Indeed, by reversibility, the Neumann time map $T^\alpha_{\mathcal{N}}$ is given by half the minimal periods of the associated orbits of (6.2). Therefore

$$\dot{T}_{\mathcal{P}} = 2\dot{T}_{\mathcal{N}} \ , \tag{6.9}$$

in obvious notation. Hyperbolicity for $(\mathcal{N})$ is equivalent to $\dot{T}_{\mathcal{N}} \neq 0$; see for example the proof in [BC84]. The hyperbolicity claim of (ii) therefore follows from lemma 4.4.

To prove the relation (6.5) of the unstable dimensions, we recall from lemma 5.3 that

$$i^{\mathcal{P}}(v) \in \{\ell(v), \ell(v) - 1\}, \tag{6.10}$$

depending on the sign of $\dot{T}_{\mathcal{P}}$. Similarly,

$$i^{\mathcal{N}}(v) \in \{\ell(v)+1, \ell(v)\}, \tag{6.11}$$

with the analogous dependence on the sign of $\dot{T}_{\mathcal{N}}$. This proves claim (6.5), and (ii).
Claim (iii) is an obvious special case of (v). Indeed

$$v(2\pi) = v(0) \tag{6.12}$$

for even ℓ. Together with the Neumann boundary condition $v_x(2\pi) = v_x(0) = 0$, this shows that v is a frozen wave of $(\mathcal{P})$.

Consider claim (iv) next, with $k \geq 2$. The simple case $k = 0$ is omitted. Let v^1, v^2 be k-adjacent in $(\mathcal{P})$ with maxima attained at $x = 0$. To prove k-adjacency in $(\mathcal{N})$, suppose w is an equilibrium of $(\mathcal{N})$, such that

$$\begin{array}{ccccc} z(v^1 - w) & = & z(v^2 - w) & = & k \, , \\ v^2(0) & > & w(0) & > & v^1(0) \, . \end{array} \tag{6.13}$$

By nesting proposition 4.1 and non-intersection of frozen waves, $\mathbf{w}(x)$ is then trapped in the annulus between $\mathbf{v}^2(x)$ and $\mathbf{v}^1(x)$. In particular, we may also assume $w(0) = \max w$. Nesting, again, then implies

$$k = z(v^1 - w) = \ell(w) = z(w_x) \, . \tag{6.14}$$

Note that $k = z(v^1 - v^2)$ is even here. Therefore (iii) exhibits w as a frozen wave of $(\mathcal{P})$. Because $w(0) = \max w, v^j(0) = \max v^j$, this contradicts the k-adjacency of v^1, v^2 in $(\mathcal{P})$, and thus proves (iv).

To prove (v), simply observe that the solution $u(t,\cdot) \in W^{1,\infty}([0,2\pi], \mathbb{R})$ of $(\mathcal{N})$ also defines a weak solution $u(t,\cdot) \in W^{1,\infty}(S^1, \mathbb{R})$, simply by pasting $x = 0$ and $x = 2\pi$. This follows from assumption (6.7) and the Neumann boundary condition $u_x(t,0) = 0 = u_x(t,2\pi)$. By standard regularity theory, this solution coincides with a solution in $X = H^s(S^1), s > \frac{3}{2}$, as considered here. This completes the proof of the lemma. ⋈

7 Proof of theorem 1.3

We first recall that the easy "only if" part of theorem 1.3 has already been proved in section 2. The "if" part has been proved at the end of section 3, for $k = 0$. We therefore consider k-adjacent hyperbolic $v^1, v^2 \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$ for the periodic boundary value problem $(\mathcal{P})$ of (1.1), (1.2), with $k \geq 2$. We assume that

$$\max_{x \in S^1} v^1(x) < \max_{x \in S^1} v^2(x) \tag{7.1}$$

so that $\mathbf{v}^1(x) = (v^1(x), v^1_x(x))$ is contained inside $\mathbf{v}^2(x) = (v^2(x), v^2_x(x))$, by nesting proposition 4.1. We have to establish the existence of a connecting orbit $u(t,x)$ between v^1 and v^2, which satisfies (1.24).

Suppose first that the closed disc D bounded by $\mathbf{v}^2(x)$ belongs to the cyclicity set $\mathcal{C}$. Since $\mathcal{C}$ does not contain saddles, this disc contains exactly one homogeneous equilibrium e which is a center, that is, $f_v(e,0) > 0$. We remark that if $v^1 \notin \mathcal{E}$ we have a closed annulus between $\mathbf{v}^1(x)$ and $\mathbf{v}^2(x)$ and e is contained in the disc inside $\mathbf{v}^1(x)$. Of course the annulus degenerates to a disk if $v^1(x) \equiv e$.

In the previous situation, we invoke freezing lemma 4.2 and the freezing homotopy f^τ, (4.22). Because periodic solutions in $\mathcal{C}$ are unaffected by this homotopy – only their wave speeds are – v^1 and v^2 remain k-adjacent. By lemma 4.3, v^1 and v^2 also remain hyperbolic. Due to lemma 3.3, a heteroclinic connection, satisfying (1.24), persists under this process. We may therefore assume f frozen to start with, without loss of generality.

Once f is frozen in the closed disk bounded by $\mathbf{v}^2(x)$, we apply the symmetrizing homotopy (5.6) induced by the symmetrizing transformations q^τ, Q^τ; see (5.3), (5.5), and lemma 5.1. By corollary 5.2, the frozen waves or homogeneous equilibria $Q^\tau(\mathbf{v}^1), Q^\tau(\mathbf{v}^2)$ remain hyperbolic throughout the homotopy. Moreover, k-adjacency is again preserved. As before, from lemma 3.3 we obtain the persistence of connectivity of v^1, v^2 throughout this second homotopy. We may therefore assume f in D to be frozen and reversible to start with, without loss of generality.

We now invoke the cut-and-paste lemma 6.1 twice. By (ii) we may consider v^1, v^2 as hyperbolic equilibrium solutions of the Neumann problem $(\mathcal{N})$ on $0 \le x \le 2\pi$. To this end, we use the possibility of phase-shifts. Indeed, we may assume that v^1, v^2 attain their maxima at $x = 0$, by simply shifting v^1, v^2, separately, in $x \in S^1$. Recall that k-adjacency in $(\mathcal{P})$ persists under phase-shifts and, by lemma 6.1 (iv), implies k-adjacency in $(\mathcal{N})$.

By theorem 1.1, there exists a heteroclinic orbit $u(t,x)$ of $(\mathcal{N})$, between v^1 and v^2. Suppose we can show that

$$u(t,0) = u(t,2\pi) \tag{7.2}$$

holds for all real t. Then cut-and-paste lemma 6.1(v) implies that $u(t,x)$ is also a heteroclinic orbit between v^1 and v^2 for the original problem $(\mathcal{P})$ with periodic boundary conditions.

To prove (7.2) for the Neumann heteroclinic orbit u, we recall from theorem 1.1 that for k-adjacent v^1, v^2 there exists a special heteroclinic orbit u with constant zero numbers

$$z(u(t,\cdot) - v^1) = z(u(t,\cdot) - v^2) = z(u(t_1,\cdot) - u(t_2,\cdot)) = z(v^1 - v^2) = k \tag{7.3}$$

for all real t and all real $t_1 \neq t_2$. Moreover the heteroclinic orbit with these properties is unique in $(\mathcal{N})$, up to time shift. Note that $\mathbf{u}(t,x) = (u(t,x), u_x(t,x))$ has to be contained in D for all $t \in \mathbb{R}$. Indeed, any intersection of the two curves $\mathbf{v}^2(x)$ and $\mathbf{u}(t,x)$, $x \in S^1$, for some $t \in \mathbb{R}$, leads to a dropping of $z(u(\cdot,t) - v^2(\cdot + \vartheta))$ for some $\vartheta \in S^1$. This obviously contradicts (7.3), if $\vartheta = 0$. For $\vartheta \neq 0$, we obtain a contradiction to the dropping of the zero number from proposition 4.1 as follows.

For the limits v^1, v^2 of $u(t,x)$ for $t \to \pm\infty$, we have

$$z(v^1(\cdot) - v^2(\cdot + \vartheta)) = l(v^2) = k \tag{7.4}$$

and

$$z(v^2(\cdot) - v^2(\cdot + \vartheta)) = l(v_x^2) = k. \tag{7.5}$$

By reversibility symmetry of f in D, we in fact have now constructed two heteroclinic orbits $u(t,x)$ and $u(t, 2\pi - x)$ between v^1 and v^2, which must coincide up to a time shift, by uniqueness:

$$u(t + \vartheta, x) = u(t, 2\pi - x). \tag{7.6}$$

Applying (7.6) twice, we have

$$u(t + 2\vartheta, x) = u(t, x) \tag{7.7}$$

for $x = 0, 2\pi$ and all t. On the other hand, $t \mapsto u(t,x)$ is strictly monotone at $x = 0$, by theorem 1.1. Therefore $\vartheta = 0$ in (7.7), (7.6) and claim (7.2) is proved. Moreover, (7.3) implies (1.24), and theorem 1.3 is now proved under the assumption that the closed disk D bounded by $\mathbf{v}^2$ belongs to the cyclicity set $\mathcal{C}$. Indeed, we have shown that

$$\mathbf{u}(t,x) \in D \tag{7.8}$$

for all t, x. Moreover, k-adjacency is defined via the non-existence of certain $w(x) \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$, and the imposed conditions (1.22), (1.23) enforce $\mathbf{w}(x) \in D$, for all x. Therefore the f-homotopies and transformations can be defined arbitrarily outside D, without affecting the constructed heteroclinic orbit $u(t,x)$ between v^1 and v^2. This was the case under our initial assumption that D was contained in the cyclicity set $\mathcal{C}$.

We now drop this assumption, finally, and deal with cyclicity sets $\mathcal{C}$ which exhibit gaps in the disk D. Such gaps can be caused, only, by saddle points $\mathbf{v}^- = (v^-, 0)$ in the disk D, for which

$$f_v(\mathbf{v}^-) < 0 \ ; \tag{7.9}$$

see lemma 4.2 (iv). To complete the proof of theorem 1.3 it is therefore sufficient to construct an additional homotopy, again denoted $f^\tau, 0 \le \tau \le 1$, which removes all saddles in D together with some rotating or frozen waves. Due to lemma 3.3 this will not interfere with the existence of connecting orbits between v^1 and v^2, as long as hyperbolicity and k-adjacency of v^1 and v^2 are preserved.

Clearly, $\mathbf{v}^-$ either belongs to the disk inside $\mathbf{v}^1$, in which case we have

$$v^- < \max v^1 \ , \tag{7.10}$$

or lies in the annulus between $\mathbf{v}^1$ and $\mathbf{v}^2$. In this situation we assume

$$\max v^1 < v^- < \max v^2 \ , \tag{7.11}$$

without loss of generality. Otherwise reflect $u \mapsto -u$.

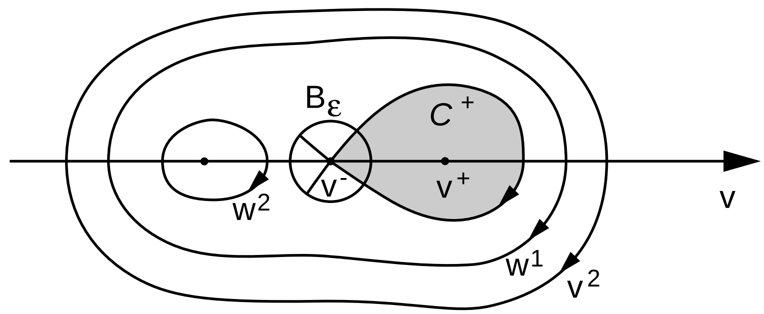

Figure 7.1: The cyclicity subset $\mathcal{C}^+$. For $\mathbf{v}^1 = \mathbf{w}^1$, the saddle $\mathbf{v}^-$ belongs to the disk inside $\mathbf{v}^1$. For $\mathbf{v}^1 = \mathbf{w}^2$, the saddle $\mathbf{v}^-$ belongs to the annulus between $\mathbf{v}^1$ and $\mathbf{v}^2$.

To prepare for the *saddle removal* f^τ, we assume the saddle v^- is chosen to be the maximal one in D. Because $f(\max v^2, 0) > 0$, there also exists an equilibrium $v^+ \in (v^-, \max v^2)$ such that

$$f_v(\mathbf{v}^+) > 0 \tag{7.12}$$

holds for $\mathbf{v}^+ = (v^+, 0)$. Note that $\mathbf{v}^+$ is a center, and therefore $\mathbf{v}^+ \in \mathcal{C}$. Because $\mathbf{v}^-$ is a saddle, not in $\mathcal{C}$, the boundary of the $\mathcal{C}$-component $\mathcal{C}^+$ of the center $\mathbf{v}^+$ also contains a saddle $(v_-, 0)$. By maximality of v^- in D, we therefore have $v_- = v^-$. Moreover, $\partial\mathcal{C}^+$ is homoclinic to v^-, and $\mathcal{C}^+$ itself is the bubble-shaped interior of the homoclinic boundary. See Figure 7.1.

It is important to observe that $\partial\mathcal{C}^+$ indeed becomes a homoclinic orbit for some wave speed c^+. This follows by choosing $\mathcal{C}^+ \ni \mathbf{v}_n \to \partial\mathcal{C}^+$ as in the proof of lemma 4.2, and choosing a convergent subsequence of the bounded sequence c_n of associated wave speeds. Arguing as for periodic orbits, the homoclinic bubble

$$\mathcal{C}^+ \text{ is } \left\{ \begin{array}{l} \text{positively} \\ \text{positively and negatively} \\ \text{negatively} \end{array} \right\} \text{invariant for } \left\{ \begin{array}{l} c > c^+ \\ c = c^+ \\ c < c^+ \end{array} \right. \tag{7.13}$$

See also Fig. 4.1 and the arguments there.

With $B_\varepsilon(v^-)$ an ε-ball around $\mathbf{v}^-$ we define

$$\mathcal{C}^+_\varepsilon := B_\varepsilon(v^-) \cup \mathcal{C}^+. \tag{7.14}$$

To construct the final part of the saddle removal homotopy f^τ, we choose $\varepsilon > 0$ small, below, and let $f^\tau \equiv f$ outside $\mathcal{C}^+_\varepsilon$, for $0 \le \tau \le 1$. Inside, we choose any smooth homotopy, for $0 \le \tau \le \delta < 1$, such that

$$f^\delta(v, 0) \left\{ \begin{array}{l} = 0 \\ > 0 \end{array} \right. \quad \text{for} \quad \begin{array}{l} v^- \le v \le v^+ \\ v < v^- \text{ or } v > v^+ \end{array} \tag{7.15}$$

$$f^\tau(v, 0) \left\{ \begin{array}{l} < 0 \\ > 0 \end{array} \right. \quad \text{for} \quad \begin{array}{l} v^- < v < v^+ \\ v < v^- \text{ or } v > v^+ \end{array}, \quad 0 \le \tau < \delta \tag{7.16}$$

$$f^\tau(\mathbf{v}^-) = f^\tau(\mathbf{v}^+) = 0\ ,\ |f^\tau| \leq C_0\ ,\ Lip(f^\tau) \leq L_0 + 1\ . \tag{7.17}$$

Here C_0 is a bound for f^0 which was introduced in the explicit dissipativeness condition (1.5) above. Lip denotes the Lipschitz constant and $L_0 := Lip(f^0)$ on D. We claim, and prove below, that conditions (7.15)–(7.17) ensure that hyperbolicity and k-adjacency of $\mathbf{v}^1, \mathbf{v}^2$ are preserved throughout the homotopy f^τ, for $0 \leq \tau \leq \delta$, provided that ε is chosen small enough. A final homotopy segment

$$f^\tau := f^\delta + \tau - \delta \tag{7.18}$$

for $\delta \leq \tau \leq 1$ then removes the interval $[v^-, v^+]$ of equilibria which was produced at $\tau = \delta$. For δ sufficiently close to 1, k-adjacency of $\mathbf{v}^1, \mathbf{v}^2$ must persist during this final step, being an open property. Likewise, hyperbolicity remains unaffected.

We now prove that the homotopy $f^\tau, 0 \leq \tau \leq \delta$, with (7.15)–(7.17) indeed preserves hyperbolicity and k-adjacency of $\mathbf{v}^1, \mathbf{v}^2$. Hyperbolicity is trivial, because $\mathbf{v}^1$ and $\mathbf{v}^2$ are contained outside $\mathcal{C}^+_\varepsilon$ and hence $f^\tau \equiv f$ in a neighborhood of $\mathbf{v}^1, \mathbf{v}^2$.

To prove the persistence of k-adjacency, we have to exclude the possibility of introducing any new solution $w \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$, for some τ, which satisfies (1.23) and hence would destroy the k-adjacency of v^1, v^2. We argue indirectly: suppose $w \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$, does destroy the k-adjacency of v^1, v^2, for some τ. In particular $z(v^j - w) = k$ for both $j = 1, 2$. Proposition 4.1 implies that $\mathbf{w}(x) = (w(x), w_x(x))$ is contained in the annulus between $\mathbf{v}^1(x)$ and $\mathbf{v}^2(x)$. Since $f^\tau \equiv f^0$ outside $\mathcal{C}^+_\varepsilon$, we must have $\mathbf{w}(x) \in \mathcal{C}^+_\varepsilon$, for some x, say for $x = 0$. Also $\mathbf{w}(x_1) = (w(x_1), 0)$ with $w(x_1) < \min v^1$, for some x_1, because $z(v^1 - w) = k \geq 2$. In particular $\mathbf{w}(x_1) \notin (\mathcal{C}^+ \cup \partial\mathcal{C}^+)$. Independently of the wave speed c_w of w, flow invariance (7.13) of $\mathcal{C}^+ \cup \partial\mathcal{C}^+$ (which is either positively or negatively invariant) now implies $\mathbf{w}(x_0) \in B_\varepsilon(v^-)$, for some x_0. Indeed (7.13) prevents $\mathbf{w}(x)$ from entering, or leaving, $\mathcal{C}^+$ through $\partial\mathcal{C}^+ \setminus B_\varepsilon(v^-) \subseteq \partial\mathcal{C}^+_\varepsilon$, because $f^\tau \equiv f$ there.

For small $\varepsilon > 0$, we now reach a contradiction between $\mathbf{w}(x_0) \in B_\varepsilon(v^-)$, near $\mathbf{v}^-$, and the positioning of $\mathbf{w}(x_1)$. Because $Lip(f^\tau) \leq L_0 + 1$ on D, $f^\tau(\mathbf{v}^-) = 0$, and $|\mathbf{w}(x_0) - \mathbf{v}^-| < \varepsilon$, the Gronwall Lemma implies

$$|\mathbf{w}(x_1) - \mathbf{v}^-| \leq \varepsilon e^{L|x_1 - x_0|} \tag{7.19}$$

for $L := L_0 + 2 + c_0$. Here c_0 is an a priori bound for the wave speed $|c|$, depending on the bound C_0 for $|f^\tau|$. Indeed the rotating wave term cp is then Lipschitz in $p = u_x$ with Lipschitz constant c_0.

The positioning of $\mathbf{w}(x_1)$ and 2π-periodicity of $\mathbf{w}$ now imply

$$0 < v^- - \min v^1 < |\mathbf{w}(x_1) - \mathbf{v}^-| \leq \varepsilon e^{2\pi L}. \tag{7.20}$$

For small ε, this is a contradiction, which proves that our saddle removal homotopy f^τ indeed preserves hyperbolicity and k-adjacency of v^1, v^2.

Finally, because the number of saddles is finite, we can remove all saddles in the disc D bounded by $\mathbf{v}^2$, successively, by finitely many of the above homotopies.

Together with the saddles, we have removed all gaps of the cyclicity set inside D. Hence $D \subset \mathcal{C}$. The arguments at the beginning of this section then prove the existence of a heteroclinic orbit $u(t,x)$ between v^1 and v^2 at the end of the final homotopy. Moreover this heteroclinic orbit satisfies assumption (3.16) of lemma 3.3. Therefore, backtracking all homotopies, any two k-adjacent and hyperbolic elements $v^1, v^2 \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$, or suitably x-shifted copies, possess a heteroclinic connection with property (3.16), alias (1.24). This completes the proof of theorem 1.3.

8 Discussion

In the present paper we have established heteroclinic orbits for one-dimensional parabolic reaction-advection-diffusion equations, in the S^1-equivariant case of nonlinearities $f = f(u, u_x)$ which do not depend on $x \in S^1$ explicitly. More precisely, for any two hyperbolic homogeneous equilibria, frozen or rotating waves $v^1, v^2 \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$, we have given a necessary and sufficient condition for a heteroclinic orbit to connect them. Heteroclinicity was understood up to phase shifts of v^1, v^2, only. See theorems 1.3 and 1.4 for complete technical details.

Putting our result in perspective, we conclude by outlining several related problems.

Motivated by the S^1-equivariance generated by x-shifts we can attempt to characterize unique heteroclinic orbits, which represent all heteroclinics between v^1 and v^2, much in the spirit of the results for the Neumann case in [Wol02a]. Our set-up, in contrast, has applied different phase shifts ϑ_1, ϑ_2 to v^1, v^2, separately, to enforce a reduction to Neumann boundary conditions after freezing.

A geometric setting for this problem would be the 2-torus $(\vartheta_1, \vartheta_2) \in T^2$ generated by the translates $(v^1(\cdot + \vartheta_1), v^2(\cdot + \vartheta_2))$. We can then distinguish the heteroclinic set $\mathcal{H} \subseteq T^2$, defined by the asymptotic phases ϑ_1, ϑ_2 of any heteroclinic orbit. Clearly $\mathcal{H}$ is invariant under the diagonal action $\vartheta \mapsto (\vartheta_1 + \vartheta, \vartheta_2 + \vartheta)$ of S^1-equivariance on T^2. By transversality and k-($\mathcal{P}$)-adjacency, $\mathcal{H}$ is a closed submanifold of T^2. If $i(v^1) - i(v^2) \geq 2$, then $\mathcal{H}$ is also open, for dimensional reasons. Then $\mathcal{H} = T^2$ and all asymptotic phases are realized by heteroclinic orbits. If $i(v^1) - i(v^2) = 1$, in contrast, then $\mathcal{H}$ consists of one or several S^1 group orbits on T^2. Uniqueness, however, has not been proved.

A completely different description of orbits $u(t, \cdot)$ in unstable manifolds $W^u(v^1)$ has been pursued in [AF88]. For arbitrary non-increasing time tracks $t \mapsto \zeta(t)$ compatible with the constraints of zero numbers, the existence of orbits $u(t, \cdot) \in W^u(v^1)$ has been shown, which realize the prescribed time track ζ:

$$z(u(t, \cdot) - v^1(t, \cdot)) = \zeta(t). \tag{8.1}$$

In addition, certain phase conditions could be prescribed.

Our present result, in contrast, has singled out a heteroclinic orbits with $\zeta \equiv k =$

const.. Phase constrains, however, have been lost. It has not been attempted, but seems desirable to reconcile both approaches in a geometric way.

Our result is centrally based on a reduction from the case $(\mathcal{P})$ of periodic boundary conditions to the case $(\mathcal{N})$ of separated boundary conditions: freezing, symmetrization, and the cut-paste process are crucial steps of this reduction. This immediately raises the question of the precise geometric relationship between the class $\mathcal{A}_{\mathcal{N}}$ of all Neumann attractors and the class $\mathcal{A}_{\mathcal{P}}$ of all S^1-equivariant attractors under periodic boundary conditions – of course under the respective generic hyperbolicity assumptions. Our cut-paste process clearly indicates that these two classes are closely related.

Very detailed descriptions of the Neumann class have been given by Brunovský, Fiedler, Rocha and Wolfrum; see [BF86, BF88, FR91, Fie94, Fie96, FR96, FR99, FR00, Wol02a, Wol02b]. Although these descriptions coincide, in principle, the details of presentation differ substantially. For elegance of formulation, we have favoured the Wolfrum approach here. Alternatively, it seems feasible, to develop an analog of the permutation approach of Fusco, Fiedler, and Rocha to explicitly and computationally encode all possible constellations of zero numbers $z(v^i - v^j)$ and Morse indices $i(v^j)$ which can arise in the S^1-context. In particular the braid type of $\mathcal{E} \cup \mathcal{F} \cup \mathcal{R}$, seen as curves $(x, v(x), v_x(x)) \in S^1 \times \mathbb{R}^2$, comes to mind here.

The Poincaré–Bendixson result of Fiedler, Mallet-Paret [FMP89], and, independently, Nadirashvili [Nad90] indicates, that general x-dependent nonlinearities $f = f(x, u, u_x)$ lead to global attractors $\mathcal{A}$ which consist of equilibria, periodic and heteroclinic or homoclinic orbits. Again periodic boundary conditions $x \in S^1$ are assumed, but S^1-equivariance – a central technical ingredient to our result – is lost. A simple flow embedding due to [SF92] realizes any planar flow in global attractors $\mathcal{A}$. In particular, homoclinic orbits to hyperbolic equilibria do occur. As indicated above, a braid description of periodic orbits and equilibria seems possible. The homoclinic lack of an automatic Morse-Smale property, however, in the sense of Henry, Angenent, and Oliva [Hen85, FuOl90, Ang86] still seems to pose a severe obstacle to this problem.

Returning to our original S^1-equivariant problem, we conclude with the problem of viscous profile solutions in the singular limit

$$u_t = \varepsilon^2 u_{xx} + f(u, u_x), \qquad x \in S^1 \tag{8.2}$$

for $\varepsilon \searrow 0$. The case of separated boundary conditions has been studied successfully by Härterich; see [Här98, Här99]. In particular, for nonlinearities of balance law type

$$f(u, u_x) = a(u)_x + b(x), \tag{8.3}$$

sufficient conditions have been derived such that the global attractor $\mathcal{A}_\varepsilon$ possesses a finite-dimensional limit for $\varepsilon \searrow 0$. The separated boundary conditions, however, force boundary layers to occur which are of course pinned to the boundary of x. In the periodic case $x \in S^1$ of a single conservation law, $b(x) \equiv 0$, a celebrated result by Dafermos shows convergence of $u(t, \cdot)$ to its spatial average for $t \to \infty$, provided

$a(u)$ possesses isolated inflection points; see [Daf85]. Hopefully, our results for $\varepsilon > 0$ may provide an access to this interesting problem, for more general nonlinearities $f(u, u_x)$.

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