Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Numerical analysis of Monte Carlo finite difference evaluation of Greeks

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submitted: 20th January 2003

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No. 808
Berlin 2003

2000 *Mathematics Subject Classification.* 60H30, 65C30, 91B28.

Key words and phrases. Derivative pricing and hedging, probabilistic representations, weak approximation of solutions of stochastic differential equations, Monte Carlo simulation.

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Abstract. An error analysis of approximation of derivatives of the solution to the Cauchy problem for parabolic equations by finite differences is given taking into account that the solution itself is evaluated using weak-sense numerical integration of the corresponding system of stochastic differential equations together with the Monte Carlo technique. It is shown that finite differences are effective when the method of dependent realizations is exploited in the Monte Carlo simulations. This technique is applicable to evaluation of Greeks. In particular, it turns out that it is possible to evaluate both the option price and deltas by a single simulation run that reduces the computational costs. Results of some numerical experiments are presented.

1. Introduction

To evaluate a hedging strategy for a European claim depending on values of stocks (risky assets) at maturity time, we have to know at every moment the solution of the Cauchy problem for a parabolic equation (the value of the hedging portfolio) and its derivatives (deltas). Other derivatives (price sensitivities known as Greeks) are an important measure of risk.

Consider the Cauchy problem for the equation of parabolic type:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(t,x)\frac{\partial u}{\partial x^i} + c(t,x)u + g(t,x) = 0\,, \tag{1.1}$$

$$t_0 \leq \ t < T,\ x \in \mathbf{R}^d,$$

with the initial condition

$$u(T,x) = f(x)\,. \tag{1.2}$$

The solution of the problem (1.1)-(1.2) has various probabilistic representations:

$$u(t,x) = E(f(X_{t,x}(T))Y_{t,x,1}(T) + Z_{t,x,1,0}(T))\,, \tag{1.3}$$

where $X_{t,x}(s)$, $Y_{t,x,y}(s)$, $Z_{t,x,y,z}(s)$, $s \geq t$, is the solution of the Cauchy problem for the system

$$\begin{aligned} dX &= b(s,X)ds - \sigma(s,X)\mu(s,X)ds + \sigma(s,X)\,dw(s),\ X(t) = x\,, \\ dY &= c(s,X)Y ds + \mu^\top(s,X)Y\,dw(s),\ Y(t) = y\,, \\ dZ &= g(s,X)Y ds + F^\top(s,X)Y\,dw(s),\ Z(t) = z\,. \end{aligned} \tag{1.4}$$

In (1.4) $w(s) = (w^1(s), \ldots, w^d(s))^\top$ is a d-dimensional standard Wiener process, $b(s,x) = \{b^i(s,x)\}$ is a column-vector of dimension d, $\sigma(s,x)$ is a matrix of dimension $d \times d$ such that $\sigma(s,x)\sigma^\top(s,x) = a(s,x) = \{a^{ij}(s,x)\}$, Y and Z are scalars, $\mu(s,x)$ and $F(s,x)$ are column-vectors of dimension d with good analytical properties (for example, they have bounded derivatives up to some order) but arbitrary otherwise.

Let $F = 0$. Then the usual representation can be seen if $\mu = 0$, the others rest on Girsanov's theorem. For $F \neq 0$, the representation (1.3) is evidently true as well. We see that the expectation in (1.3) does not depend on a choice of both μ and F. At the same time, $Var(f(X_{t,x}(T))Y_{t,x,1}(T) + Z_{t,x,1,0}(T))$ does depend on them. A suitable choice of μ

and F allows us to reduce the variance (see, e.g., the method of important sampling in [12, 15, 17], the method of control variates in [15], the combining method in [13, 14]).

We should note that stochastic models in financial mathematics are usually considered in $\mathbf{R}^d_+$ (not in $\mathbf{R}^d$). However, the results obtained here can be carried over to the case of $\mathbf{R}^d_+$. Besides, we may also avoid these difficulties by using a suitable transformation of $\mathbf{R}^d_+$ into $\mathbf{R}^d$.

In the case of $c=0,\ g=0,\ \mu=0,\ F=0$ we have

$$u(t,x) = Ef(X_{t,x}(T)) \,, \tag{1.5}$$

where $X_{t,x}(s)$ is the solution of the problem

$$dX = b(s,X)ds + \sigma(s,X)\,dw(s)\,, \quad X(t)=x\,. \tag{1.6}$$

The coefficients b and σ are assumed to have bounded derivatives up to some order. In particular, this ensures the existence and uniqueness of the continuous solution to (1.6) on $[t,T]$. The uniform ellipticity condition is imposed on the matrix σ :

$$\exists\ \lambda>0,\ y^\intercal\sigma(s,x)\sigma^\intercal(s,x)y \geq \lambda|y|^2, \text{ for any } s\in[t_0,T],\ x,y\in\mathbf{R}^d.$$

The function $f(x)$ and its derivatives up to some order are assumed to satisfy inequalities of the form

$$|f(x)| \leq K\cdot(1+|x|^\varkappa)\,, \tag{1.7}$$

where K and $\varkappa$ are positive constants. Under these conditions, the solution $u(t,x)$ and its derivatives satisfy inequalities of the type (1.7) [7].

For differentiable functions f, a probabilistic representation for $\dfrac{\partial u}{\partial x^k}(t,x)$ can be obtained by straightforward differentiating (1.5) (see [2, 13]):

$$\frac{\partial u}{\partial x^k}(t,x) = E\sum_{i=1}^d \frac{\partial f}{\partial x^i}(X_{t,x}(T))\,\delta_k X^i(T)\,, \tag{1.8}$$

where $\delta_k X^i(s) := \partial X^i_{t,x}(s)/\partial x^k,\ t\leq s\leq T$, satisfies the following system of variational equations associated with (1.6):

$$\begin{aligned} d\delta_k X &= \sum_{i=1}^d \frac{\partial b(s,X)}{\partial x^i}\delta_k X^i ds + \sum_{i=1}^d \frac{\partial\sigma(s,X)}{\partial x^i}\delta_k X^i\ dw(s)\,, \\ \delta_k X^i(t) &= 0\,,\ \text{if } i\neq k,\ \text{and } \delta_k X^k(t)=1\,. \end{aligned} \tag{1.9}$$

When $f(x)$ is nondifferentiable or when a European claim is specified by a more complicated payoff functional than the payoff function $f(X(T))$ at maturity time T, the authors of [6] propose to use Malliavin calculus for numerical computation of Greeks. In particular, their approach is based on the integration-by-parts formula which gives

$$(T-t)\frac{\partial u}{\partial x^k}(t,x) = Ef(X_{t,x}(T))\int_t^T [\sigma^{-1}(s,X_{t,x}(s))\delta_k X(s)]^\intercal\ dw(s)\ . \tag{1.10}$$

Let us note that if the problem under consideration depends on some parameter α, i.e., $X = X_{t,x}(s;\alpha)$, $u = u(t,x,\alpha)$, then it is possible to find $\partial u(t,x,\alpha)/\partial\alpha$, in the same way. Of course, we need $\delta_\alpha X(s) := \partial X_{t,x}(s;\alpha)/\partial\alpha$. However, if this derivative does not exist, we cannot use, for example, the formula (1.10). We are faced with such a situation in the case of finding theta $\partial u(t,x)/\partial t$ (e.g., the problem $dX = dw(s)$, $X(t) = x$, $s \geq t$, has the solution $X_{t,x}(s) = x + w(s) - w(t)$ which is evidently nondifferentiable with respect to t). In [13], to evaluate theta, a system of linear parabolic equations is derived which belongs to a class of systems admitting probabilistic representations for their solutions [11]. It is shown in [13] that the computational costs are comparable with those of the straightforward differentiation method (1.8)-(1.9).

Both formulas (1.8) and (1.10) require computation of $\delta_k X(s)$, i.e., to evaluate deltas by these methods one has to integrate not only the d-dimensional system (1.6) but also d additional systems, each of dimension d. This can present severe computational difficulties. At the same time, there is a very simple method which makes use of evaluating the values of u only to evaluate deltas. This method rests on the finite difference formula

$$(1.11)\quad \frac{\partial u}{\partial x^i} = \frac{u(t,x^1,\ldots,x^i+\Delta x,\ldots,x^d) - u(t,x^1,\ldots,x^i-\Delta x,\ldots,x^d)}{2\Delta x} + O\left((\Delta x)^2\right) .$$

Of course, in (1.11) we are forced to use the approximate values $\hat{u}(t,x^1,\ldots,x^i \pm \Delta x,\ldots,x^d)$ instead of $u(t,x^1,\ldots,x^i \pm \Delta x,\ldots,x^d)$:

$$(1.12)\qquad \hat{u}(t,x^1,\ldots,x^i+\Delta x,\ldots,x^d) = \frac{1}{M}\sum_{m=1}^{M} f(\bar{X}^{(m)}_{t,x^1,\ldots,x^i+\Delta x,\ldots,x^d}(T))$$
$$\doteq \bar{u} = Ef(\bar{X}_{t,x^1,\ldots,x^i+\Delta x,\ldots,x^d}(T)) \doteq u = Ef(X_{t,x^1,\ldots,x^i+\Delta x,\ldots,x^d}(T)) .$$

The same can be written for $\hat{u}(t,x^1,\ldots,x^i-\Delta x,\ldots,x^d)$. In (1.12), $\bar{X}(s)$, $t \leq s \leq T$, is an approximate solution of (1.6) obtained by a scheme of numerical integration, $\bar{X}^{(m)}(T)$, $m = 1,\ldots,M$, are independent realizations of $\bar{X}(T)$. There are two errors in (1.12): the error of numerical integration, say $O(h^p)$, and the statistical error of the Monte Carlo method which is estimated as $O(1/\sqrt{M})$. Therefore, the error R of the approximation

$$\frac{\partial u}{\partial x^i} = \frac{\hat{u}(t,x^1,\ldots,x^i+\Delta x,\ldots,x^d) - \hat{u}(t,x^1,\ldots,x^i-\Delta x,\ldots,x^d)}{2\Delta x} + R$$

is equal, in general, to

$$R = O\left((\Delta x)^2\right) + O\left(\frac{h^p}{\Delta x}\right) + O\left(\frac{1}{\Delta x\sqrt{M}}\right) .$$

Due to the presence of small Δx in the denominators, the difference approach seems to be not admissible. Fortunately, the more accurate arguments and the employment of the dependent realizations in simulation of $\hat{u}(t,x^1,\ldots,x^i+\Delta x,\ldots,x^d)$ and $\hat{u}(t,x^1,\ldots,x^i-\Delta x,\ldots,x^d)$ rehabilitate the difference approach. In Section 2, we prove that the error of numerical integration by the weak Euler method ($p = 1$) contributes $O(h) + O(h^2/\Delta x)$ (not $O(h/\Delta x)$) to the total error of evaluation of the derivative. This result is due to the Talay-Tubaro expansion of the error of numerical integration [16]. In Section 3, we prove that the method of dependent realizations, which is close to using common random numbers for Monte Carlo estimators (see [1, 4, 5, 10]), contributes just $O(1/\sqrt{M})$ to the total error. Thus

$$R = O\left((\Delta x)^2\right) + O\left(\frac{h^2}{\Delta x}\right) + O\left(\frac{1}{\sqrt{M}}\right) .$$

If we put $\Delta x = \alpha h^\beta,\ \alpha > 0,\ 1/2 \leq \beta \leq 1$, then

$$R = O(h) + O(1/\sqrt{M}) .$$

Hence we get the same convergence rate in evaluating deltas as in evaluation of option prices. We should note that this inference is rigorously obtained here in the case of simple payoff functions which depend on the underlying asset process X at maturity time T only: $f = f(X(T))$. Most likely, the results obtained here can be justified in the case when a contingent claim is defined as a functional of the asset process, for example, $f = f(X(t_1), \ldots, X(t_m)),\ t_0 < \cdots < t_m = T$.

It turns out (see Section 4) that the calculations for evaluating the option price $u(t,x)$ can be used for evaluation of deltas, i.e., we obtain both u and its first derivatives by a single simulation run. This allows us to reduce the computational costs. In Section 5, evaluation of other Greeks is discussed. Results of numerical experiments are presented in Section 6.

2. Influence of the error of numerical integration on evaluating deltas by finite differences

For simplicity, we first consider the one-dimensional Cauchy problem, more precisely the problem (1.1)-(1.2) with $d = 1$, $c = g = 0$:

$$Lu := \frac{\partial u}{\partial t} + b(t,x)\frac{\partial u}{\partial x} + \frac{\sigma^2(t,x)}{2}\frac{\partial^2 u}{\partial x^2} = 0 \tag{2.1}$$

$$u(T,x) = f(x). \tag{2.2}$$

Then

$$u(t,x) = Ef(X_{t,x}(T))\,, \tag{2.3}$$

where $X_{t,x}(s)$, $s \geq t$, is the solution of the Cauchy problem (1.6).

We approximate the solution of (1.6) by a weak method, for instance by the weak Euler method:

$$\bar{X}_{t,x}(s_{k+1}) := X_{k+1} = X_k + hb(s_k, X_k) + h^{1/2}\sigma(s_k, x)\xi_k, \quad k = 0, \ldots, N-1, \tag{2.4}$$

where h is a step of discretization of the time interval $[t, T]$:

$$s_k = t + kh, \quad k = 0, \ldots, N, \quad s_N = T\,;$$

ξ_k are independent random variables taking the values $+1$ and -1 with probability $1/2$, and $X_0 = x$. The scheme (2.4) has the first order of accuracy in the sense of weak approximation (see, e.g. [12]).

Using (2.3) and (2.4), we can evaluate the solution $u(t,x)$ of (2.1)-(2.2) as follows

$$u(t,x) \doteq \bar{u}(t,x) = Ef(\bar{X}_{t,x}(T))\,. \tag{2.5}$$

As is known [16], the error of this approximation can be expanded in powers of h :

$$\rho(t,x) := \bar{u}(t,x) - u(t,x) = hE\int_t^T B(\vartheta, X_{t,x}(\vartheta))\ d\vartheta\ + O(h^2)\,, \tag{2.6}$$

where $B(t,x)$ is determined by the coefficients of the problem (2.1)-(2.2) and does not depend on h :

$$B(t,x) = (L^2u(t,x) - A(t,x))/2$$

with

$$A(t,x) = \frac{\partial^2 u}{\partial t^2} + 2b\frac{\partial^2 u}{\partial t\partial x} + \sigma^2\frac{\partial^3 u}{\partial t\partial x^2} + b^2\frac{\partial^2 u}{\partial x^2} + b\sigma^2\frac{\partial^3 u}{\partial x^3} + \frac{1}{12}\sigma^4\frac{\partial^4 u}{\partial x^4}\,.$$

We approximate the derivative $\frac{\partial u}{\partial x}(t,x)$ by the central finite difference:

$$\frac{\partial u}{\partial x}(t,x) = \frac{u(t,x+\Delta x) - u(t,x-\Delta x)}{2\Delta x} + O((\Delta x)^2)\,. \tag{2.7}$$

Our aim in this section is to answer on the question: if we replace u in (2.7) by $\bar{u}$ from (2.5), how the error ρ influences the error of the evaluation of the derivative. We have

$$\begin{aligned}\frac{\partial u}{\partial x}(t,x) &= \frac{u(t,x+\Delta x) - u(t,x-\Delta x)}{2\Delta x} + O((\Delta x)^2)\\ &= \frac{\bar{u}(t,x+\Delta x) - \bar{u}(t,x-\Delta x)}{2\Delta x} + \frac{h}{2\Delta x}\left(v(t,x+\Delta x) - v(t,x-\Delta x)\right)\\ &+ O\left(\frac{h^2}{\Delta x}\right) + O((\Delta x)^2)\,,\end{aligned}$$

where

$$v(t,x) = E\int_t^T B(\vartheta, X_{t,x}(\vartheta))\ d\vartheta \tag{2.8}$$

which is a smooth function due to the assumptions on the coefficients made in Introduction. Then, expanding $v(t,x\pm\Delta x)$ around (t,x), we obtain

$$\frac{h}{2\Delta x}\left(v(t,x+\Delta x) - v(t,x-\Delta x)\right) = O(h)\ .$$

Therefore

$$\frac{\partial u}{\partial x}(t,x) = \frac{\bar{u}(t,x+\Delta x) - \bar{u}(t,x-\Delta x)}{2\Delta x} + O\left(h + \frac{h^2}{\Delta x} + (\Delta x)^2\right),$$

and if we select $\Delta x = \alpha h^\beta$, $\alpha > 0$, $1/2 \le \beta \le 1$, then

$$\frac{\partial u}{\partial x}(t,x) = \frac{\bar{u}(t,x+\alpha h^\beta) - \bar{u}(t,x-\alpha h^\beta)}{2\alpha h^\beta} + O(h).$$

Thus, we have proved the theorem.

Theorem 2.1. *Let $\bar{u}(t,x)$ be evaluated according to (2.5) with $\bar{X}_{t,x}(T)$ obtained by the Euler scheme (2.4), α be a positive number, and $1/2 \le \beta \le 1$. Then*

$$\frac{\bar{u}(t,x+\alpha h^\beta) - \bar{u}(t,x-\alpha h^\beta)}{2\alpha h^\beta} - \frac{\partial u}{\partial x}(t,x) = O(h)\,. \tag{2.9}$$

Remark 2.1. If the coefficients of (2.1)-(2.2) have a sufficient number of bounded derivatives, the error $\rho(t,x)$ can be expanded as [16]:

$$\rho(t,x) = \bar{u}(t,x) - u(t,x) = h \sum_{i=0}^{n-1} h^i \, E \int_t^T B_i(\vartheta, X_{t,x}(\vartheta)) \; d\vartheta + O(h^{n+1}) \, .$$

Then

$$\frac{\partial u}{\partial x}(t,x) = \frac{\bar{u}(t,x+\Delta x) - \bar{u}(t,x-\Delta x)}{2\Delta x} + O\left(h + \frac{h^{n+1}}{\Delta x} + (\Delta x)^2\right),$$

and we achieve the accuracy $O(h)$ by choosing $\Delta x = \alpha h^\beta$, $\alpha > 0$, $1/2 \le \beta \le n$.

Remark 2.2. Let $\bar{X}_{t,x}(T)$ be obtained by a second-order weak scheme [12]. Then we have the expansion

$$\bar{u}(t,x) - u(t,x) = h^2 E \int_t^T B(\vartheta, X_{t,x}(\vartheta)) \; d\vartheta \; + O(h^3) \, ,$$

(with another function $B(t,x)$, of course) and

$$\frac{\partial u}{\partial x}(t,x) = \frac{\bar{u}(t,x+\Delta x) - \bar{u}(t,x-\Delta x)}{2\Delta x} + O\left(h^2 + \frac{h^3}{\Delta x} + (\Delta x)^2\right) \, .$$

Taking $\Delta x = \alpha h$, $\alpha > 0$, we get

$$\frac{\partial u}{\partial x}(t,x) = \frac{\bar{u}(t,x+\alpha h) - \bar{u}(t,x-\alpha h)}{2\alpha h} + O(h^2) \, .$$

Theorem 2.1 can easily be generalized to the general problem (1.1)-(1.2). Indeed, we can approximate the solution of (1.4) by, e.g., the weak Euler method and evaluate $\bar{u}$ by the formula

$$\bar{u}(t,x) = E\left[f(\bar{X}_{t,x}(T))\bar{Y}_{t,x,1}(T) + \bar{Z}_{t,x,1,0}(T)\right] \, . \tag{2.10}$$

Then, by the same arguments as above, we obtain the theorem.

Theorem 2.2. *Let $\bar{u}(t,x)$ be evaluated according to (2.10) with $\bar{X}_{t,x}(T)$, $\bar{Y}_{t,x,1}(T)$, $\bar{Z}_{t,x,1,0}(T)$ obtained by the weak Euler method applied to (1.4), α be a positive number, and $1/2 \le \beta \le 1$. Then*

$$\frac{\bar{u}(t,x+\alpha h^\beta \nu) - \bar{u}(t,x-\alpha h^\beta \nu)}{2\alpha h^\beta} - \frac{\partial u(t,x)}{\partial \nu} = O(h) \, , \tag{2.11}$$

where ν is a unit vector and

$$\frac{\partial u(t,x)}{\partial \nu} = \left.\frac{\partial u(t,x+r\nu)}{\partial r}\right|_{r=0}$$

is the derivative in the direction of ν.

3. Influence of the Monte Carlo error on evaluating deltas by finite differences

For simplicity, we again first consider the one-dimensional case (2.1)-(2.2). To realize the formula (2.5) (or (2.10)) in practice, we need to apply the Monte Carlo technique. As a result, in addition to the error of numerical integration considered in the previous section, there is also the Monte Carlo error:

$$\bar{u}(t,x) = Ef(\bar{X}_{t,x}(T)) = \frac{1}{M}\sum_{m=1}^{M} f(\bar{X}_{t,x}^{(m)}(T)) \pm r_u, \tag{3.1}$$

where M is the number of independent realizations $\bar{X}_{t,x}^{(m)}(T)$ of $\bar{X}_{t,x}(T)$. The Monte Carlo error r_u is estimated as

$$r_u = \frac{c}{\sqrt{M}}\sqrt{Var\, f(\bar{X}_{s,x}(T))} = \frac{c}{\sqrt{M}}\left(\sqrt{Var\, f(X_{s,x}(T))} + O(h)\right)$$

with, for example, the fiducial probability 0.997 for $c = 3$ and 0.95 for $c = 2$. Below, for definiteness, we take $c = 3$. Thus, the total error R_u in the evaluation of u is estimated as

$$R_u = O(h) + O\left(\frac{1}{\sqrt{M}}\right),$$

which is of order $O(h)$ if we choose $M \sim 1/h^2$.

We have

$$\frac{\partial u}{\partial x}(t,x) = \frac{1}{2\alpha h^\beta}\frac{1}{M}\left[\sum_{m=1}^{M} f(\bar{X}_{t,x+\alpha h^\beta}^{(m)}(T)) - \sum_{m=1}^{M} f(\bar{X}_{t,x-\alpha h^\beta}^{(m)}(T))\right] + O(h) \pm \frac{1}{2\alpha h^\beta} r_{u'}, \tag{3.2}$$

where the Monte Carlo error $r_{u'}$ is estimated for sufficiently large M as

$$\begin{aligned} r_{u'} &= \frac{3}{\sqrt{M}}\sqrt{Var\, f(\bar{X}_{t,x+\alpha h^\beta}(T)) + Var\, f(\bar{X}_{t,x-\alpha h^\beta}(T))} \\ &= \frac{3}{\sqrt{M}}\left(\sqrt{2\, Var\, f(X_{t,x}(T))} + O(h)\right) \end{aligned} \tag{3.3}$$

with the fiducial probability 0.997. We have assumed here that all the realizations $\bar{X}_{t,x+\alpha h^\beta}^{(m)}(T)$ and $\bar{X}_{t,x-\alpha h^\beta}^{(m)}(T)$ are independent. The second relation in (3.3) is obtained by the following arguments: the values $Var\, f(\bar{X}_{t,x\pm\alpha h^\beta}(T))$ differ from $Var\, f(X_{t,x\pm\alpha h^\beta}(T))$ by $O(h)$ and $Var\, f(X_{t,x\pm\alpha h^\beta}(T))$ differ from $Var\, f(X_{t,x}(T))$ by $O(h^{2\beta})$.

Thus, the total error $R_{u'}$ in the evaluation of $\partial u/\partial x$ due to (3.2) is estimated as

$$R_{u'} = O(h) + O\left(\frac{1}{\sqrt{Mh^{2\beta}}}\right), \tag{3.4}$$

which is of order $O(h)$ if we choose $\beta = 1/2$ and $M \sim 1/h^3$ that is $1/h$ times larger than it is required in the evaluation of the solution u itself.

Now, instead of simulating the independent trajectories, let us simulate them in a pairwise dependent way (that is similar to the use of common random numbers [1, 4, 5, 10]). More precisely, we now simulate M pairs of trajectories, each pair consists of a trajectory starting from $x+\alpha h^\beta$ and a trajectory starting from $x-\alpha h^\beta$. The pairs are independent, but the two trajectories of the same pair are dependent: they correspond to the same realization of the Wiener process. Then we have

$$\begin{aligned}(3.5)\qquad \frac{\partial u}{\partial x}(t,x) = \frac{1}{2\alpha h^\beta}\frac{1}{M}\sum_{m=1}^{M}\left[f(\bar{X}^{(m)}_{t,x+\alpha h^\beta}(T)) - f(\bar{X}^{(m)}_{t,x-\alpha h^\beta}(T))\right] \\ + O(h) \pm \frac{1}{2\alpha h^\beta} r^{new}_{u'} \,,\end{aligned}$$

where the Monte Carlo error $r^{new}_{u'}$ is estimated for sufficiently large M as

$$(3.6)\qquad r^{new}_{u'} = \frac{3}{\sqrt{M}}\sqrt{Var\left[f(\bar{X}_{t,x+\alpha h^\beta}(T)) - f(\bar{X}_{t,x-\alpha h^\beta}(T))\right]} \,.$$

Since $\bar{X}_{t,x}(T)$ approximates $X_{t,x}(T)$ with the first weak order, we get

$$\begin{aligned}(3.7)\qquad & Var\left[f(\bar{X}_{t,x+\alpha h^\beta}(T)) - f(\bar{X}_{t,x-\alpha h^\beta}(T))\right] \\ & = Var\left[f(X_{t,x+\alpha h^\beta}(T)) - f(X_{t,x-\alpha h^\beta}(T))\right] + O(h) \,.\end{aligned}$$

We have

$$\begin{aligned} & Var\left[f(X_{t,x+\alpha h^\beta}(T)) - f(X_{t,x-\alpha h^\beta}(T))\right] \\ & = E\left[f(X_{t,x+\alpha h^\beta}(T)) - f(X_{t,x-\alpha h^\beta}(T))\right]^2 - \left[E\left(f(X_{t,x+\alpha h^\beta}(T)) - f(X_{t,x-\alpha h^\beta}(T))\right)\right]^2 \,.\end{aligned}$$

Due to the conditions imposed on the coefficients of the problem (2.1)-(2.2) and continuous dependence of solutions to SDEs on the initial data (see [3, Section 8]), we obtain

$$E\left[f(X_{t,x+\alpha h^\beta}(T)) - f(X_{t,x-\alpha h^\beta}(T))\right]^2 \le K\cdot 4\alpha^2 h^{2\beta}$$

and

$$\left[E\left(f(X_{t,x+\alpha h^\beta}(T)) - f(X_{t,x-\alpha h^\beta}(T))\right)\right]^2 \le K\cdot 4\alpha^2 h^{2\beta}$$

(here and in what follows we denote by the same letter K various positive constants independent of h).

Hence

$$(3.8)\qquad Var\left[f(X_{t,x+\alpha h^\beta}(T)) - f(X_{t,x-\alpha h^\beta}(T))\right] \le K\cdot 4\alpha^2 h^{2\beta}$$

and

$$(3.9)\qquad r^{new}_{u'} \le \frac{3}{\sqrt{M}}(K\cdot 4\alpha^2 h^{2\beta} + O(h))^{1/2}.$$

By a more accurate analysis, it is possible to prove that

$$(3.10)\qquad r^{new}_{u'} = \frac{3C\cdot 2\alpha h^\beta}{\sqrt{M}}\left(1+O(h)\right) \,,$$

where the constant C can be estimated in the Monte Carlo simulation in the usual way. Indeed, let $\bar{\varphi}(x) := f(\bar{X}_{t,x}(T))$. It follows from the proof of Lemma 4.2 that the derivatives of $\bar{\varphi}(x)$ exist and some their moments are bounded uniformly with respect to h. We have

$$\bar{\varphi}(x+\alpha h^\beta) - \bar{\varphi}(x-\alpha h^\beta) = 2\alpha h^\beta \bar{\varphi}'(x) + \frac{\left(\alpha h^\beta\right)^3}{6}\left(\bar{\varphi}'''(\eta_1) + \bar{\varphi}'''(\eta_2)\right),$$

where η_1 and η_2 are some intermediate points. Then

$$V := Var\left[\bar{\varphi}(x+\alpha h^\beta) - \bar{\varphi}(x-\alpha h^\beta)\right] = 4\alpha^2 h^{2\beta} \cdot Var\left[\bar{\varphi}'(x) + \frac{\left(\alpha h^\beta\right)^2}{12}\left(\bar{\varphi}'''(\eta_1) + \bar{\varphi}'''(\eta_2)\right)\right],$$

and, using Lemma 4.2, we get

$$V = 4\alpha^2 h^{2\beta} \cdot \left[Var\,\bar{\varphi}'(x) + O(h^{2\beta})\right].$$

The joint system of difference equations for $\bar{X}_{t,x}(s_k)$, $\dfrac{\partial \bar{X}_{t,x}}{\partial x}(s_k)$ (cf. (2.4) and (4.6)-(4.7)) is the weak Euler approximation for the joint system for $X_{t,x}(s)$, $\dfrac{\partial X_{t,x}}{\partial x}(s)$, $s \geq t$ (cf. (1.6) and (1.9)). Then

$$Var\,\bar{\varphi}'(x) = Var\,\varphi'(x) + O(h),$$

where $\varphi(x) := f(X_{t,x}(T))$. Finally, we obtain (recall that $1/2 \leq \beta \leq 1$)

$$V = C^2 \cdot 4\alpha^2 h^{2\beta}\left(1 + O(h)\right)$$

with

$$C = \sqrt{Var\,\varphi'(x)}\ ,$$

whence (3.10) follows.

The total error $R_{u'}^{new}$ in the evaluation of $\partial u/\partial x$ due to (3.5) is estimated as

$$R_{u'}^{new} = O\left(h + \frac{1}{\sqrt{M}}\right). \tag{3.11}$$

Note that for $\beta = 1/2$ the estimate (3.11) follows from (3.5) and (3.9), i.e., (3.10) is not needed. We see that it is sufficient to take $M \sim 1/h^2$ like in the evaluation of the solution u itself. Thus, the following theorem is proved.

Theorem 3.1. *Let $\bar{X}_{t,x}(T)$ be obtained by the Euler scheme (2.4), the pairs $(\bar{X}_{t,x+\alpha h^\beta}^{(m)}(T)$, $\bar{X}_{t,x-\alpha h^\beta}^{(m)}(T))$, $m = 1, \ldots, M$, $\alpha > 0$, $1/2 \leq \beta \leq 1$, be independent but the two realizations from the same pair be dependent (i.e., they are supposed to correspond to the same realization of the Wiener process). Then*

$$\frac{\partial u}{\partial x}(t,x) = \frac{1}{2\alpha h^\beta}\frac{1}{M}\sum_{m=1}^{M}\left[f(\bar{X}_{t,x+\alpha h^\beta}^{(m)}(T)) - f(\bar{X}_{t,x-\alpha h^\beta}^{(m)}(T))\right] + O(h) \pm \frac{3C}{\sqrt{M}}. \tag{3.12}$$

Remark 3.1. In the case of one-sided finite difference we can obtain the similar result:

$$\frac{\partial u}{\partial x}(t,x) = \frac{1}{\alpha h}\frac{1}{M}\sum_{m=1}^{M}\left[f(\bar{X}_{t,x+\alpha h}^{(m)}(T)) - f(\bar{X}_{t,x}^{(m)}(T))\right] + O(h) \pm \frac{3C}{\sqrt{M}}, \ \alpha > 0, \tag{3.13}$$

where the pairs $(\bar{X}_{t,x+\alpha h}^{(m)}(T),\ \bar{X}_{t,x}^{(m)}(T))$, $m = 1,\dots,M$, are independent but the two realizations from the same pair are dependent. It is clear that in practice the formula (3.12) with $\beta = 1/2$ is usually preferable to (3.13) due to its better stability with respect to inherent errors.

In the case of the general problem (1.1)-(1.2) we have with the fiducial probability 0.997 :

$$\bar{u}(t,x) = Ef(\bar{X}_{t,x}(T)) = \frac{1}{M}\sum_{m=1}^{M}\left[f(\bar{X}_{t,x}^{(m)}(T))\,\bar{Y}_{t,x,1}^{(m)}(T) + \bar{Z}_{t,x,1,0}^{(m)}(T)\right] \pm \frac{3}{\sqrt{M}}\left(\sqrt{Var\,\left[f(X_{t,x}(T))Y_{t,x,1}(T) + Z_{t,x,1,0}(T)\right]} + O\left(h\right)\right), \tag{3.14}$$

where M is the number of independent realizations $\bar{\Lambda}_{t,x}^{(m)}(T) := (\bar{X}_{t,x}^{(m)}(T), \bar{Y}_{t,x,1}^{(m)}(T), \bar{Z}_{t,x,1,0}^{(m)}(T))$ of $\bar{\Lambda}_{t,x}(T) := (\bar{X}_{t,x}(T), \bar{Y}_{t,x,1}(T), \bar{Z}_{t,x,1,0}(T))$. It is not difficult to generalize Theorem 3.1 and obtain the following assertion.

Theorem 3.2. *Let $\bar{\Lambda}_{t,x}(T) := (\bar{X}_{t,x}(T),\ \bar{Y}_{t,x,1}(T),\ \bar{Z}_{t,x,1,0}(T))$ be obtained by the weak Euler method applied to* (1.4)*, the pairs $\left(\bar{\Lambda}_{t,x+\alpha h^{\beta}\nu}^{(m)}(T), \bar{\Lambda}_{t,x-\alpha h^{\beta}\nu}^{(m)}(T)\right)$, $m = 1,\dots,M$, $\alpha > 0$, $1/2 \le \beta \le 1$, be independent but the two realizations from the same pair correspond to the same realization of the Wiener process. Then for a unit vector ν*

$$\frac{\partial u(t,x)}{\partial \nu} = \frac{1}{2\alpha h^{\beta}}\frac{1}{M}\sum_{m=1}^{M}\Big[f(\bar{X}_{t,x+\alpha h^{\beta}\nu}^{(m)}(T))\bar{Y}_{t,x+\alpha h^{\beta}\nu,1}^{(m)}(T) + \bar{Z}_{t,x+\alpha h^{\beta}\nu,1,0}^{(m)}(T) - f(\bar{X}_{t,x-\alpha h^{\beta}\nu}^{(m)}(T))\bar{Y}_{t,x-\alpha h^{\beta}\nu,1}^{(m)}(T) - \bar{Z}_{t,x-\alpha h^{\beta}\nu,1,0}^{(m)}(T)\Big] + O(h) \pm \frac{3C}{\sqrt{M}}. \tag{3.15}$$

4. The use of intermediate values from simulation of the option price for evaluating deltas

The realization of the approach to evaluating derivatives explained in the previous two sections requires many simulation runs. More precisely, to find the option price and all the deltas in the general case (1.1)-(1.2), we need $2d+1$ values of the solution $u(t,x)$ at different points, i.e., we have to simulate the system (1.4) with $2d+1$ different initial values using the corresponding Monte Carlo technique. However, it turns out that it is possible to evaluate derivatives using just a single simulation run if we use intermediate values from simulation of the solution itself. This allows us to reduce computational costs essentially.

In this section we put $\mu = 0$ and $F = 0$ in (1.4) in order to simplify the exposition only. All the results are easily carried over to the case of arbitrary μ and F that does not require any new ideas.

Let $X_{t,x}(s)$, $Y_{t,x,y}(s)$, $Z_{t,x,y,z}(s)$, $s \geq t$, be a solution of the system of SDEs (1.4) with $\mu = 0$ and $F = 0$ and $X_k = \bar{X}_{t,x}(s_k)$, $Y_k = \bar{Y}_{t,x}(s_k)$, $Z_k = \bar{Z}_{t,x}(s_k)$, $k = 0, \ldots, N$, $s_0 = t$, $s_N = T$, be its approximation obtained by the weak Euler method:

$$
\begin{aligned}
X_{k+1} &= X_k + hb(s_k, X_k) + h^{1/2}\sigma(s_k, x)\xi_k \\
Y_{k+1} &= Y_k + hc(s_k, X_k)\, Y_k \\
Z_{k+1} &= Z_k + hg(s_k, X_k)\, Y_k, \quad k = 0, \ldots, N-1,
\end{aligned} \tag{4.1}
$$

where $\xi_k = (\xi_k^1, \ldots, \xi_k^d)^\intercal$ are d-dimensional vectors which components are i.i.d. random variables with the law $P(\xi^i = \pm 1) = 1/2$.

In addition, denote by $\bar{X}$ the one-step approximation

$$
\bar{X}_{t,x}(t+h) = \bar{X} = x + hb(t,x) + h^{1/2}\sigma(t,x)\,\xi\,. \tag{4.2}
$$

Lemma 4.1. *Let $U(t,x)$ be a sufficiently smooth function. Then*

$$
\sigma^\intercal \bigtriangledown U(t,x) = \frac{1}{\sqrt{h}} E\left[U(t+h, \bar{X})\,\xi\right] + O(h)\,, \tag{4.3}
$$

where $\bigtriangledown U(t,x) = grad\, U = \left(\frac{\partial U}{\partial x^1}, \ldots, \frac{\partial U}{\partial x^d}\right)^\intercal$ *and $\bar{X}$ is from* (4.2).

Proof. We have

$$
\begin{aligned}
E\left[U(t+h,\bar{X})\,\xi\sqrt{h}\right] &= h^{1/2} E\left[U(t+h, x + hb(t,x) + h^{1/2}\sigma(t,x)\xi)\,\xi\right] \\
&= h^{1/2} E\left[U(t+h,x)\,\xi\right] + hE\left[\sum_{i=1}^{d} \frac{\partial U}{\partial x^i}(t+h,x)\,\left(h^{1/2}b^i + (\sigma\xi)^i\right)\,\xi\right] \\
&\quad + h^{3/2} E\left[\frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 U}{\partial x^i \partial x^j}(t+h,x)\,(\sigma\xi)^i\,(\sigma\xi)^i\,\xi\right] + O(h^2) \\
&= hE\left[\sum_{i=1}^{d} \frac{\partial U}{\partial x^i}(t,x)\,(\sigma\xi)^i\,\xi\right] + O(h^2) = h\sigma^\intercal \bigtriangledown U(t,x) + O(h^2)\,.
\end{aligned}
$$

□

Now let $u(t,x)$ be the solution of the problem (1.1)-(1.2). Then, due to Lemma 4.1, we get

$$
E\left[u(t+h,\bar{X})\,\xi\sqrt{h}\right] = h\sigma^\intercal \bigtriangledown u(t,x) + O(h^2), \tag{4.4}
$$

where $\bar{X}$ is from (4.2).

Lemma 4.2. *The function $\bar{u}(t,x)$ defined by* (2.10) *has derivatives up to some order which are uniformly bounded with respect to h.*

Proof. Recall that we assumed (see Introduction) that the coefficients of the problem (1.1)-(1.2) have bounded derivatives up to some order. Let us prove, for example, that $\bar{u}(t,x)$ given by (2.10) has the continuous first derivatives with respect to x. We can formally write

$$\begin{aligned}
\frac{\partial}{\partial x^i}\bar{u}(t,x) &= \frac{\partial}{\partial x^i}E\left[f(\bar{X}_{t,x}(T))\bar{Y}_{t,x,1}(T)+\bar{Z}_{t,x,1,0}(T)\right] \\
&= E\left[\sum_{j=1}^d \frac{\partial}{\partial x^j}f(\bar{X}_{t,x}(T))\ \frac{\partial}{\partial x^i}\bar{X}^j_{t,x}(T)\ \bar{Y}_{t,x,1}(T)\right. \\
&\left. +f(\bar{X}_{t,x}(T))\frac{\partial}{\partial x^i}\bar{Y}_{t,x,1}(T)+\frac{\partial}{\partial x^i}\bar{Z}_{t,x,1,0}(T)\right].
\end{aligned} \tag{4.5}$$

We have (cf. (4.1)):

$$\bar{X}^j_{t,x}(s_{k+1}) = \bar{X}^j_{t,x}(s_k)+hb^j(s_k,\bar{X}_{t,x}(s_k))+h^{1/2}\sum_{l=1}^d \sigma^{jl}(s_k,\bar{X}_{t,x}(s_k))\ \xi^l_k\ ,$$
$$j=1,\dots,d,\quad k=0,\dots,N-1,$$

and $s_0=t,\quad s_N=T,\ \bar{X}_{t,x}(t)=x.$

We get

$$\frac{\partial}{\partial x^i}\bar{X}^j_{t,x}(s_1)=\delta_{ij}+h\frac{\partial}{\partial x^i}b^j(t,x)+h^{1/2}\sum_{l=1}^d\frac{\partial}{\partial x^i}\sigma^{jl}(t,x)\ \xi^l_1\ ,$$

(here δ_{ij} is the Kronecker symbol). Hence the derivative $\frac{\partial}{\partial x^i}\bar{X}^j_{t,x}(s_1)$ evidently exists due to the assumptions on the coefficients. By induction we obtain that $\frac{\partial}{\partial x^i}\bar{X}^j_{t,x}(s_k)$, $k=0,\dots,N$, $j=1,\dots,d$, exist and satisfy the linear system of stochastic difference equations

$$\begin{aligned}
\frac{\partial}{\partial x^i}\bar{X}^j_{t,x}(s_{k+1}) &= \frac{\partial}{\partial x^i}\bar{X}^j_{t,x}(s_k)+h\sum_{l=1}^d\frac{\partial}{\partial x^l}b^j(s_k,\bar{X}_{t,x}(s_k))\ \frac{\partial}{\partial x^i}\bar{X}^l_{t,x}(s_k) \\
&+h^{1/2}\sum_{l,m=1}^d\frac{\partial}{\partial x^m}\sigma^{jl}(s_1,\bar{X}_{t,x}(s_k))\ \xi^l_k\ \frac{\partial}{\partial x^i}\bar{X}^m_{t,x}(s_k), \\
& j=1,\dots,d,\quad k=0,\dots,N-1,
\end{aligned} \tag{4.6}$$

with the initial condition

$$\frac{\partial}{\partial x^i}\bar{X}^j_{t,x}(s_0)=\delta_{ij}\ . \tag{4.7}$$

It is not difficult to show that the second moments of the derivatives $\frac{\partial}{\partial x^i}\bar{X}^j_{t,x}(T)$ are uniformly bounded with respect to h and that the derivatives $\frac{\partial}{\partial x^i}\bar{Y}_{t,x}(T)$ and $\frac{\partial}{\partial x^i}\bar{Z}^j_{t,x}(T)$

possess the same properties. Then (4.5) implies the existence of the first derivatives uniformly bounded with respect to h. Differentiating (4.5) further, we arrive at the assertion of the lemma. □

By Lemmas 4.1 and 4.2, we obtain

$$E\left[\bar{u}(t+h,\bar{X})\,\xi\sqrt{h}\right] = h\sigma^{\mathsf{T}} \bigtriangledown \bar{u}(t,x) + O(h^2) \tag{4.8}$$

with $\bar{X}$ from (4.2).

Theorem 4.1. *We have*

$$\sigma^{\mathsf{T}} \bigtriangledown u(t,x) = \frac{1}{\sqrt{h}} E\left[\bar{u}(t+h,\bar{X})\,\xi\right] + O(h)\,. \tag{4.9}$$

Proof. The relations (4.4) and (4.8) imply

$$\sigma^{\mathsf{T}}\left(\bigtriangledown u(t,x) - \bigtriangledown \bar{u}(t,x)\right) = \frac{1}{\sqrt{h}} E\left[u(t+h,\bar{X}) - \bar{u}(t+h,\bar{X})\right]\xi + O(h)\,. \tag{4.10}$$

Let us estimate $E\left[u(t+h,\bar{X}) - \bar{u}(t+h,\bar{X})\right]\xi$. For simplicity, consider the problem (2.1)-(2.2) (note that the generalization of the proof to the problem (1.1)-(1.2) does not require any additional ideas). We have (cf. (2.5) and (2.6)):

$$\begin{aligned}\bar{u}(t+h,\bar{X}) &= E\left(f(\bar{X}_{t+h,\bar{X}}(T))\,|\,\bar{X}\right)\\ &= u(t+h,\bar{X}) + hE\Big(\int_{t+h}^{T} B(\vartheta, X_{t+h,\bar{X}}(\vartheta))\,d\vartheta\,|\,\bar{X}\Big) + O(h^2)\,.\end{aligned}$$

Introducing $v(t,x)$ as in (2.8) and recalling the relation (4.2) for $\bar{X}$, we obtain

$$\begin{aligned}\bar{u}(t+h,\bar{X}) &= u(t+h,\bar{X}) + hv(t+h,\bar{X}) + O(h^2)\\ &= u(t+h,\bar{X}) + hv(t+h,x+hb+h^{1/2}\sigma\,\xi) + O(h^2)\\ &= u(t+h,\bar{X}) + hv(t+h,x+hb) + h^{3/2}\frac{\partial v}{\partial x}(t+h,x+hb)\sigma\,\xi + O(h^2)\,.\end{aligned}$$

Then

$$\begin{aligned}&E\left[u(t+h,\bar{X}) - \bar{u}(t+h,\bar{X})\right]\xi\\ &= -E\left[hv(t+h,x+hb) + h^{3/2}\frac{\partial v}{\partial x}(t+h,x+hb)\sigma\,\xi + O(h^2)\right]\xi\\ &= -h^{3/2}E\left[\frac{\partial v}{\partial x}(t+h,x+hb)\sigma\,\xi^2\right] + O(h^2) = O(h^{3/2})\,.\end{aligned} \tag{4.11}$$

Substituting (4.11) in (4.10) and taking into account (4.8), we get (4.9). Theorem 4.1 is proved. □

Remark 4.1. If we evaluate $\bar{u}(t+h,\bar{X})$ using (2.10) with $\bar{X}_{t,x}(T)$, $\bar{Y}_{t,x,1}(T)$, $\bar{Z}_{t,x,1,0}(T)$ obtained by a second-order weak scheme with the time step $\sqrt{h}$ applied to (1.4), then

$$\sigma^{\mathsf{T}} \bigtriangledown u(t,x) = \frac{1}{\sqrt{h}} E\left[\bar{u}(t+h,\bar{X})\,\xi\right] + O(h)\,,$$

i.e., we get the same accuracy as before but for lower computational costs.

To approximately find the derivatives of the solution $u(t,x)$ to (1.1)-(1.2) by (4.9), we need to evaluate the expectation $E\left[\bar{u}(t+h,\bar{X})\,\xi\right]$. Let us denote by $\xi_{(\gamma)} = \left(\xi^1_{(\gamma)},\ldots,\xi^d_{(\gamma)}\right)^\mathsf{T}$, $\gamma = 1,\ldots,2^d$, all the different values of the vector ξ from (4.2) and assign indices to these values so that $\xi_{(2^{d-1}+\nu)} = -\xi_{(\nu)}$, $\nu = 1,\ldots,2^{d-1}$. We denote by $\bar{X}_{(\gamma)}$ the value of the vector $\bar{X}$ from (4.2) corresponding to $\xi_{(\gamma)}$.

We have

$$E\left[\bar{u}(t+h,\bar{X})\,\xi\right] = \frac{1}{2^d}\sum_{\gamma=1}^{2^d} \bar{u}(t+h,\bar{X}_{(\gamma)})\,\xi_{(\gamma)}\,. \tag{4.12}$$

Note that $\bar{X}_{(\gamma)}$ and $\xi_{(\gamma)}$ are deterministic (not random) vectors. The approximate solution $\bar{u}(t,x)$ of (1.1)-(1.2) can be represented in the form (cf. (2.10)):

$$\begin{aligned}\bar{u}(t,x) &= E\left[f(\bar{X}_{t,x}(T))\bar{Y}_{t,x,1}(T) + \bar{Z}_{t,x,1,0}(T)\right]\\ &= E\left[f(\bar{X}_{t+h,\bar{X}_{t,x}(t+h)}(T))\bar{Y}_{t+h,\bar{X}_{t,x}(t+h),\bar{Y}_{t,x,1}(t+h)}(T)\right.\\ &\quad\left.+\bar{Z}_{t+h,\bar{X}_{t,x}(t+h),\bar{Y}_{t,x,1}(t+h),\bar{Z}_{t,x,1,0}(t+h)}(T)\right]\,.\end{aligned}$$

Taking into account that (see (4.1))

$$\bar{Y}_{t+h,\bar{X}_{t,x}(t+h),\bar{Y}_{t,x,1}(t+h)}(T) = \bar{Y}_{t+h,\bar{X}_{t,x}(t+h),1}(T)\cdot\bar{Y}_{t,x,1}(t+h),$$

$$\begin{aligned}\bar{Z}_{t+h,\bar{X}_{t,x}(t+h),\bar{Y}_{t,x,1}(t+h),\bar{Z}_{t,x,1,0}(t+h)}(T) &= \bar{Z}_{t,x,1,0}(t+h) + h\sum_{k=1}^{N-1} g(s_k,\bar{X}_{t,x}(s_k))\,\bar{Y}_{t,x,1}(s_k)\\ &= \bar{Z}_{t,x,1,0}(t+h) + \bar{Z}_{t+h,\bar{X}_{t,x}(t+h),1,0}(T)\cdot\bar{Y}_{t,x,1}(t+h)\,,\end{aligned}$$

we get (recall that we have put $\mu = 0$, $F = 0$)

$$\bar{u}(t,x) = \frac{1}{2^d}\sum_{\gamma=1}^{2^d}\bar{u}(t+h,\bar{X}_{(\gamma)})\,[1+hc(t,x)] + hg(t,x)\,, \tag{4.13}$$

i.e., we can simulate both the approximate solution and derivatives using the same intermediate values $\bar{u}(t+h,\bar{X}_{(\gamma)})$.

We find the approximations $\bar{u}(t+h,\bar{X}_{(\gamma)})$ due to (2.10), which is realized by the Monte Carlo technique. If for this purpose we used (3.1) (i.e., if we simulated $M = 2^d\times L$ independent realizations $\bar{\Lambda}^{(l)}_{t+h,\bar{X}_{(\gamma)}}(T) = (\bar{X}^{(l)}_{t+h,\bar{X}_{(\gamma)}}(T),\bar{Y}^{(l)}_{t+h,\bar{X}_{(\gamma)},1}(T),\bar{Z}^{(l)}_{t+h,\bar{X}_{(\gamma)},1,0}(T))$, $\gamma = 1,\ldots,2^d$, $l = 1,\ldots,L$), the error in evaluation of $u(t+h,\bar{X}_{(\gamma)})$ would be $O(h)+O(1/\sqrt{M})$ but the error in evaluation of $\sigma^\mathsf{T}\bigtriangledown u(t,x)$ would be $O(h)+O(1/\sqrt{hM})$ (cf. (3.4)) that is not acceptable. To decrease the Monte Carlo error, we use dependent realizations again. Let us simulate $M/2 = 2^{d-1}\times L$ pairs $(\bar{\Lambda}^{(l)}_{t+h,\bar{X}_{(\nu)}}(T),\bar{\Lambda}^{(l)}_{t+h,\bar{X}_{(2^{d-1}+\nu)}}(T))$, $\nu = 1,\ldots,2^{d-1}$, $l = 1,\ldots,L$, so that the pairs are independent, but the two realizations of the same pair are dependent. We mean by "dependence" the same as in the previous section: the dependent trajectories correspond to the same realization of the Wiener process. Our aim now is to estimate the Monte Carlo errors arising in simulation of $E\left[\bar{u}(t+h,\bar{X})\,\xi\right]$ due to (4.12) and $\bar{u}(t,x)$ due to (4.13) in the case of dependent realizations.

Theorem 4.2. *Let* $\bar{\Lambda}_{t,x}(T) = (\bar{X}_{t,x}(T), \bar{Y}_{t,x,1}(T), \bar{Z}_{t,x,1,0}(T))$ *be obtained by the weak Euler method* (4.1), *the pairs* $(\bar{\Lambda}^{(l)}_{t+h,\bar{X}_{(\nu)}}(T), \bar{\Lambda}^{(l)}_{t+h,\bar{X}_{(2^{d-1}+\nu)}}(T))$, $\nu = 1,\dots,2^{d-1}$, $l = 1,\dots,L$, *be independent but the two realizations from the same pair be dependent. Then*

$$\bar{u}(t,x) = \frac{1}{M}\sum_{\gamma=1}^{2^d}\sum_{l=1}^{L}\left[f(\bar{X}^{(l)}_{t+h,\bar{X}_{(\gamma)}}(T))\bar{Y}^{(l)}_{t+h,\bar{X}_{(\gamma)},1}(T) + \bar{Z}^{(l)}_{t+h,\bar{X}_{(\gamma)},1,0}(T)\right] \times [1 + hc(t,x)] + hg(t,x)$$
$$\pm\frac{3\sqrt{2}}{\sqrt{M}}\left(\sqrt{Var\left(f(X_{t+h,x}(T))Y_{t+h,x,1}(T) + Z_{t+h,x,1,0}(T)\right)} + O(h)\right),$$
$$M = 2^d \times L\,, \tag{4.14}$$

with the fiducial probability 0.997.

Proof. Denote by $\bar{\eta}_{(\gamma)}$ the random variable $f(\bar{X}_{t+h,\bar{X}_{(\gamma)}}(T))\,\bar{Y}_{t+h,\bar{X}_{(\gamma)},1}(T) + \bar{Z}_{t+h,\bar{X}_{(\gamma)},1,0}(T)$. We have for sufficiently large L :

$$V := Var\left[\frac{1}{2^d}\frac{1}{L}\sum_{\gamma=1}^{2^d}\sum_{l=1}^{L}\bar{\eta}^{(l)}_{(\gamma)}\right] = \frac{1}{2^{2d}}\frac{1}{L}\sum_{\nu=1}^{2^{d-1}} Var\left[\bar{\eta}_{(\nu)} + \bar{\eta}_{(2^{d-1}+\nu)}\right]$$
$$= \frac{1}{2^{2d}}\frac{1}{L}\sum_{\nu=1}^{2^{d-1}} Var\left[\eta_{(\nu)} + \eta_{(2^{d-1}+\nu)}\right] + O\left(\frac{h}{M}\right),$$

where $\eta_{(\gamma)} := f(X_{t+h,\bar{X}_{(\gamma)}}(T))\,Y_{t+h,\bar{X}_{(\gamma)},1}(T) + Z_{t+h,\bar{X}_{(\gamma)},1,0}(T)$ and $\eta_{(\nu)}$ and $\eta_{(2^{d-1}+\nu)}$ are dependent.

Using the closeness of $\eta_{(\nu)} + \eta_{(2^{d-1}+\nu)}$ to $2\left[f(X_{t+h,x}(T))\,Y_{t+h,x,1}(T) + Z_{t+h,x,1,0}(T)\right]$, we obtain

$$Var\left[\eta_{(\nu)} + \eta_{(2^{d-1}+\nu)}\right] = 4Var\left[f(X_{t+h,x}(T))\,Y_{t+h,x,1}(T) + Z_{t+h,x,1,0}(T)\right] + O(h)\,.$$

Hence,

$$V = \frac{2}{M}Var\left[f(X_{t+h,x}(T))\,Y_{t+h,x,1}(T) + Z_{t+h,x,1,0}(T)\right] + O\left(\frac{h}{M}\right),$$

whence the Monte Carlo error in (4.14) follows. □

Remark 4.2. If all the M realizations in (4.14) are independent, then the Monte Carlo error is $\sqrt{2}$ times smaller and equal to the Monte Carlo error in the case of evaluating $\bar{u}(t,x)$ by the usual formula (3.14). Thus, (4.14) is slightly less effective for evaluation of the approximate solution than the formula (3.14). But the use of dependent realizations allows simultaneous simulation of $u(t,x)$ and its first derivatives in x that reduces the computational costs in around $2d$ times when we need both the option price and deltas.

Theorem 4.3. *Let* $\bar{\Lambda}_{t,x}(T) = (\bar{X}_{t,x}(T), \bar{Y}_{t,x,1}(T), \bar{Z}_{t,x,1,0}(T))$ *be obtained by the weak Euler method* (4.1), *the pairs* $\left(\bar{\Lambda}^{(l)}_{t+h,\bar{X}_{(\nu)}}(T), \bar{\Lambda}^{(l)}_{t+h,\bar{X}_{(2^{d-1}+\nu)}}(T)\right)$, $\nu = 1,\dots,2^{d-1}$, $l = 1,\dots,L$, *be* *independent but the two realizations from the same pair be dependent. Then*

$$E\left[\bar{u}(t+h,\bar{X})\,\xi^i\right] = \frac{1}{2^d}\sum_{\nu=1}^{2^{d-1}}\left[\hat{u}(t+h,\bar{X}_{(\nu)}) - \hat{u}(t+h,\bar{X}_{(2^{d-1}+\nu)})\right]\,\xi^i_{(\nu)} \pm \frac{3\,C\sqrt{h}}{\sqrt{M}}\,(1+O(h))\,,\; i=1,\ldots,d,\; M = 2^d\times L\,, \tag{4.15}$$

where

$$\hat{u}(t+h,\bar{X}_{(\nu)}) - \hat{u}(t+h,\bar{X}_{(2^{d-1}+\nu)}) := \frac{1}{L}\sum_{l=1}^{L}\Big[f(\bar{X}^{(l)}_{t+h,\bar{X}_{(\nu)}}(T))\bar{Y}^{(l)}_{t+h,\bar{X}_{(\nu)},1}(T) + \bar{Z}^{(l)}_{t+h,\bar{X}_{(\nu)},1,0}(T) - f(\bar{X}^{(l)}_{t+h,\bar{X}_{(2^{d-1}+\nu)}}(T))\bar{Y}^{(l)}_{t+h,\bar{X}_{(2^{d-1}+\nu)},1}(T) - \bar{Z}^{(l)}_{t+h,\bar{X}_{(2^{d-1}+\nu)},1,0}(T)\Big]\,, \quad \nu = 1,\ldots,2^{d-1}\,. \tag{4.16}$$

Proof. We have

$$V := Var\left[\frac{1}{2^d}\sum_{\nu=1}^{2^{d-1}}\left[\hat{u}(t+h,\bar{X}_{(\nu)}) - \hat{u}(t+h,\bar{X}_{(2^{d-1}+\nu)})\right]\,\xi_{(\nu)}\right] = \frac{1}{2^{2d}}\sum_{\nu=1}^{2^{d-1}}\xi_{(\nu)}\,Var\left[\hat{u}(t+h,\bar{X}_{(\nu)}) - \hat{u}(t+h,\bar{X}_{(2^{d-1}+\nu)})\right]\,.$$

By the arguments similar to those in the proof of Theorem 3.1, we get

$$Var\left[\hat{u}(t+h,\bar{X}_{(\nu)}) - \hat{u}(t+h,\bar{X}_{(2^{d-1}+\nu)})\right] \le \frac{K\,h}{L}\,,$$

whence (4.15) follows. □

Theorems 4.1 and 4.3 imply the following theorem.

Theorem 4.4. *Let $\bar{\Lambda}_{t,x}(T) = (\bar{X}_{t,x}(T), \bar{Y}_{t,x,1}(T), \bar{Z}_{t,x,1,0}(T))$ be obtained by the weak Euler method* (4.1)*, the pairs $\left(\bar{\Lambda}^{(l)}_{t+h,\bar{X}_{(\nu)}}(T), \bar{\Lambda}^{(l)}_{t+h,\bar{X}_{(2^{d-1}+\nu)}}(T)\right)$, $\nu = 1,\ldots,2^{d-1}$, $l = 1,\ldots,L$, be independent but the two realizations from the same pair be dependent. Then*

$$(\sigma^{\mathsf{T}}\bigtriangledown u(t,x))^i = \frac{1}{\sqrt{h}}\frac{1}{2^d}\sum_{\nu=1}^{2^{d-1}}\left[\hat{u}(t+h,\bar{X}_{(\nu)}) - \hat{u}(t+h,\bar{X}_{(2^{d-1}+\nu)})\right]\,\xi^i_{(\nu)} + O(h) \pm \frac{3\,C}{\sqrt{M}}\,,\quad i=1,\ldots,d, \tag{4.17}$$

with $\hat{u}$ from (4.16) *and $M = 2^d\times L$.*

5. Evaluation of other Greeks

Finite differences can also effectively be exploited for evaluating gammas, rho, and vega. As an example, here we give error estimates for the evaluation of gammas.

For simplicity, consider the one-dimensional Cauchy problem (2.1)-(2.2). We approximate the second derivative by the central finite difference:

$$\frac{\partial^2 u}{\partial x^2}(t,x) = \frac{u(t,x+\Delta x) - 2u(t,x) + u(t,x-\Delta x)}{(\Delta x)^2} + O((\Delta x)^2)\,. \tag{5.1}$$

By the arguments similar to those in the proof of Theorem 2.1, we obtain the lemma.

Lemma 5.1. *Let $\bar{u}(t,x)$ be evaluated according to* (2.5) *with $\bar{X}_{t,x}(T)$ obtained by the Euler scheme* (2.4)*. Then*

$$\frac{\bar{u}(t,x+\alpha h^{1/2}) - 2\bar{u}(t,x) + \bar{u}(t,x-\alpha h^{1/2})}{\alpha^2 h} - \frac{\partial^2 u}{\partial x^2}(t,x) = O(h)\,. \tag{5.2}$$

Note that the accuracy can be improved by using a second-order weak scheme (cf. Remark 2.2). The use of dependent realizations gives a computationally effective procedure for evaluating the second derivatives.

Theorem 5.1. *Let $\bar{X}_{t,x}(T)$ be obtained by the Euler scheme* (2.4)*, the triples $(\bar{X}^{(m)}_{t,x+\alpha h^{1/2}}(T)$, $\bar{X}^{(m)}_{t,x}(T)$, $\bar{X}^{(m)}_{t,x-\alpha h^{1/2}}(T))$, $m = 1,\ldots,M$, $\alpha > 0$, be independent but the realizations from the same triple be dependent* (*i.e., they are supposed to correspond to the same realization of the Wiener process*)*. Then*

$$\frac{\partial^2 u}{\partial x^2}(t,x) = \frac{1}{\alpha^2 h}\frac{1}{M}\sum_{m=1}^{M}\left[f(\bar{X}^{(m)}_{t,x+\alpha h^{1/2}}(T)) - 2f(\bar{X}^{(m)}_{t,x}(T)) + f(\bar{X}^{(m)}_{t,x-\alpha h^{1/2}}(T))\right] + O(h) \pm \frac{3C}{\sqrt{M}}\,. \tag{5.3}$$

The proof of this theorem is analogous to the proof of Theorem 3.1. A generalization of Theorem 5.1 to the general problem (1.1)-(1.2) is obvious (cf. Theorem 3.2).

Evaluation of theta, i.e. $\partial u/\partial t$, by the direct application of finite differences is not so effective from the computational point of view. Consider the problem (2.1)-(2.2). By arguments similar to those used in the proof of Theorem 2.1, we get

$$\frac{\partial u}{\partial t}(t,x) = \frac{\bar{u}(t+h,x) - \bar{u}(t,x)}{h} + O(h),$$

where $\bar{u}(t,x)$ is from (2.5) with $\bar{X}_{t,x}(T)$ obtained by the Euler scheme (2.4).

It is not difficult to show that (cf. (3.8))

$$Var\ (f(X_{t+h,x}(T)) - f(X_{t,x}(T))) = O(h)$$

and, as a result, we have (cf. (3.12))

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{hM}\sum_{m=1}^{M}\left[f(\bar{X}^{(m)}_{t+h,x}(T)) - f(\bar{X}^{(m)}_{t,x}(T))\right] + O(h) \pm \frac{3C}{\sqrt{Mh}}\,, \tag{5.4}$$

where the pairs $(\bar{X}^{(m)}_{t+h,x}(T), \bar{X}^{(m)}_{t,x}(T))$, $m = 1, \ldots, M$, are independent but the two realizations from the same pair are dependent.

To achieve the accuracy $O(h)$ by the approximation from (5.4), one needs to take $M \sim 1/h^3$ that is $1/h$ times larger than it is required for evaluating the solution u, the deltas (see (3.12)), and the gammas (see (5.3)) with the same accuracy $O(h)$. Thus, the approximation (5.4) is computationally not effective. We note that if the realization from the same pair in (5.4) were independent, the estimate of the Monte Carlo error would be worse: $O(1/(h\sqrt{M}))$.

We can improve the accuracy of (5.4) by using the central finite difference:

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2\sqrt{h}M}\sum_{m=1}^{M}\left[f(\bar{X}^{(m)}_{t+\sqrt{h},x}(T)) - f(\bar{X}^{(m)}_{t-\sqrt{h},x}(T))\right] + O(h) \pm \frac{3C}{h^{1/4}\sqrt{M}}\,,$$

where $(\bar{X}^{(m)}_{t+\sqrt{h},x}(T), \bar{X}^{(m)}_{t-\sqrt{h},x}(T))$, $m = 1, \ldots, M$, are simulated by the Euler scheme with the time step h, these pairs are independent but the two realizations from the same pair are dependent.

Of course, we can evaluate theta due to the original equation (2.1) after finding deltas and gammas according to (3.12) and (5.3), respectively. In this case the total error of approximation of theta is $O(h + 1/\sqrt{M})$. Another way for evaluating theta is considered in [13].

6. Numerical experiment

Example 6.1. We shall compare the approximation of deltas by (4.17) with the one from [13]. Let $u(t,x)$ be the solution of (1.1)-(1.2). Then (see [13]) the functions $v_k(t,x) := \frac{\partial u}{\partial x^k}(t,x)$, $k = 1, \ldots, d$, can be represented as:

$$v_k(t,x) = E\left[\sum_{i=1}^{d}\frac{\partial f}{\partial x^i}(X_{t,x}(T))\cdot\delta_k X^i(T)\cdot Y_{t,x,1}(T) + f(X_{t,x}(T))\cdot\delta_k Y(T) + \delta_k Z(T)\right]\,, \tag{6.1}$$

where

$$\delta_k X^i(s) := \delta_k X^i_{t,x}(s) := \frac{\partial X^i_{t,x}(s)}{\partial x^k}\,, \quad \delta_k Y(s) := \delta_k Y_{t,x,1}(s) := \frac{\partial Y_{t,x,1}(s)}{\partial x^k}\,,$$
$$\delta_k Z(s) := \delta_k Z_{t,x,1,0}(s) := \frac{\partial Z_{t,x,1,0}(s)}{\partial x^k}\,,$$

which satisfy the system of variational equations associated with (1.4):

$$\text{(6.2)}\quad d\delta_k X = \sum_{l=1}^{d} \frac{\partial\left(b(s,X) - \sigma(s,X)\mu(s,X)\right)}{\partial x^l} \cdot \delta_k X^l\, ds + \sum_{l=1}^{d} \frac{\partial \sigma(s,X)}{\partial x^l} \cdot \delta_k X^l\, dw(s)\,,$$

$$\delta_k X^l(t) = \delta_{kl}\,,$$

$$d\delta_k Y = \sum_{l=1}^{d} \frac{\partial c(s,X)}{\partial x^l} \cdot \delta_k X^l \cdot Y\, ds + c(s,X)\cdot \delta_k Y\, ds + \sum_{l=1}^{d} \frac{\partial \mu^\top(s,X)}{\partial x^l} \cdot \delta_k X^l \cdot Y\, dw(s)$$
$$+\mu^\top(s,X)\cdot \delta_k Y\, dw(s)\,, \quad \delta_k Y(t) = 0\,,$$

$$d\delta_k Z = \sum_{l=1}^{d} \frac{\partial g(s,X)}{\partial x^l} \cdot \delta_k X^l \cdot Y\, ds + g(s,X)\cdot \delta_k Y\, ds + \sum_{l=1}^{d} \frac{\partial F^\top(s,X)}{\partial x^l} \cdot \delta_k X^l \cdot Y\, dw(s)$$
$$+F^\top(s,X)\cdot \delta_k Y\, dw(s)\,, \quad \delta_k Z(t) = 0\,.$$

Of course, $v_k(t,x)$ can be found by the probabilistic method together with evaluating $u(t,x)$ by (3.14):

$$\text{(6.3)}\qquad v_k(t,x) = \frac{1}{M}\sum_{m=1}^{M}\left[\sum_{i=1}^{d} \frac{\partial f}{\partial x^i}(\bar{X}_{t,x}^{(m)}(T)) \cdot \delta_k \bar{X}^{i,(m)}(T)\cdot \bar{Y}_{t,x,1}^{(m)}(T)\right.$$
$$\left. + f(\bar{X}_{t,x}^{(m)}(T)) \cdot \delta_k \bar{Y}^{(m)}(T) + \delta_k \bar{Z}^{(m)}(T)\right],$$

where M is the number of independent realizations of the Euler approximation $\bar{X}_{t,x}(T)$, $\bar{Y}_{t,x,1}(T)$, $\bar{Z}_{t,x,1,0}(T)$, $\delta_k\bar{X}(T)$, $\delta_k\bar{Y}(T)$, $\delta_k\bar{Z}(T)$ of the solution to (1.4), (6.2).

Consider the following particular case of the problem (1.1)-(1.2) (cf. Example 3.1 in [13]). Let the coefficients of (1.1) be of the form

$$\text{(6.4)}\qquad a^{ij}(t,x) = \nu^{ij}(t)\, x^i\, x^j,\quad b^i(t,x) = x^i r^i(t),\quad i,j = 1,\ldots,d,$$
$$c(t,x) = c(t),\quad g(t,x) = g(t)$$

and the payoff function be a sum

$$\text{(6.5)}\qquad f(X(T)) = f_1(X^1(T)) + \cdots + f_d(X^d(T))\,.$$

In this case the solution of the problem (1.1)-(1.2) is given by

$$\text{(6.6)}\qquad u(t,x) = \frac{1}{\sqrt{2\pi}}\sum_{i=1}^{d}\int_{-\infty}^{\infty} f_i(x^i\mu^i(t)\exp(\lambda_i(t)y))\cdot \exp(-y^2/2)\,dy$$
$$\times\exp\left(\int_t^T c(s)ds\right) + \int_t^T g(s)\exp\left(\int_t^T c(s')ds'\right)ds\,,$$

where

$$\lambda_i(t) = \left(\int_t^T \sum_{j=1}^{d}\left[\nu^{ij}(s)\right]^2 ds\right)^{1/2}, \qquad \mu^i(t) = \exp\left(\int_t^T r^i(s)ds - \frac{1}{2}\lambda_i^2(t)\right).$$

Obviously, derivatives of $u(t,x)$ can also be calculated explicitly.

TABLE 1. Results of simulation of the solution $u(3,10)$ to (6.7) and its derivative $v(3,10)$ with $\nu = 0.2$, $r = 0.1$, $\alpha = 0.02$. Here $\mu = F = 0$. The results for u and v in the second column are obtained by (4.14) and (4.17), respectively, with $L = 10^6$. The results for u and v in the third column are obtained by (3.14) and (6.3), respectively, with $M = 10^6$. The exact values are $u(3,10) \doteq 3.043923$, $v(3,10) \doteq 0.608785$.

h	(4.14), (4.17)		(3.14), (6.3)	
	$\bar{u}(3,10)$	$\bar{v}(3,10)$	$\bar{u}(3,10)$	$\bar{v}(3,10)$
0.25	2.9920 ± 0.0063	0.5931 ± 0.0013	2.9901 ± 0.0067	0.5980 ± 0.0013
0.01	3.0431 ± 0.0071	0.6084 ± 0.0014	3.0412 ± 0.0071	0.6082 ± 0.0014
0.0025	3.0463 ± 0.0072	0.6092 ± 0.0014	3.0457 ± 0.0072	0.6091 ± 0.0014

We take

$$d = 1, \;\; \nu^{11}(t) = \nu, \;\; r^1(t) = r, \;\; c(t) = -r, \; g(t) = 0, \;\; f(x) = \alpha x^2 .$$

So, in this example we deal with the following Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{\nu^2}{2} x^2 \frac{\partial^2 u}{\partial x^2} + rx \frac{\partial u}{\partial x} - ru = 0, \quad 0 \le t < T, \; x \in \mathbf{R}, \qquad u(T,x) = \alpha x^2 , \tag{6.7}$$

the solution of which is

$$u(t,x) = \alpha\, x^2 \exp\left[\left(r + \nu^2\right)(T-t)\right]$$

and

$$\frac{\partial u}{\partial x}(t,x) = 2\alpha\, x \exp\left[\left(r + \nu^2\right)(T-t)\right] .$$

The results of Table 1 are obtained by (4.14), (4.17) and by (3.14), (6.3) for the same computational cost. And both approaches produce the results of the same quality. The experiment approves, in particular, the order of convergence of the procedures and the proportionality of the Monte Carlo error to $1/\sqrt{M}$ (not to $1/\sqrt{hM}$).

To reduce the Monte Carlo error in evaluation of the option price $u(t,x)$ by (3.14) or (4.14) and the deltas $v_k(t,x)$ by (4.17) or (6.3), a variance reduction technique can be used. Here we restrict ourselves to the method of control variates [15, 13], i.e., we put $\mu(t,x) = 0$ in (1.4) and (6.2) and reduce the variances of random variables at the expectations in (1.3) and (6.1) by choosing an appropriate $F(t,x)$. It is known [13] that for

$$F^j = -\sum_{i=1}^{d} \sigma^{ij} \frac{\partial u}{\partial x^i}, \quad j = 1, \ldots, d, \tag{6.8}$$

these variances are equal to zero. It is not difficult to see that in this case the corresponding Monte Carlo errors become $O(\sqrt{h/M})$ (they are not zero due to the error of numerical integration). The results of simulating (6.7) by (4.14), (4.17) with the optimal F from (6.8) are given in Table 2.

TABLE 2. Results of simulation of the solution $u(3,10)$ to (6.7) and its derivative $v(3,10)$ by (4.14) and (4.17), respectively. Here $\mu = 0$, F is from (6.8), $L = 10$, and the other parameters are the same as in Table 1. The exact values are $u(3,10) \doteq 3.043923$, $v(3,10) \doteq 0.608785$.

h	$\bar{u}(3,10)$	$\bar{v}(3,10)$
0.25	2.9752 ± 0.0655	0.5898 ± 0.0013
0.01	3.0431 ± 0.0019	0.60838 ± 0.00038
0.0025	3.04382 ± 0.00044	0.60870 ± 0.00009
0.0001	3.04390 ± 0.00003	0.608778 ± 0.000007

TABLE 3. Results of simulation of the solution $u(3,10)$ to (6.9) and its derivative $v(3,10)$ by (4.14) and (4.17), respectively. The parameters are $\nu = 0.2$, $r = 0.1$, $\alpha = 0.02$, $\varepsilon = 0.2$, $\mu = 0$, F is from (6.10), and $L = 10$.

h	$\bar{u}(3,10)$	$\bar{v}(3,10)$
0.25	2.9858 ± 0.0654	0.0591 ± 0.0130
0.01	3.0548 ± 0.0011	0.6096 ± 0.0003
0.0025	3.0552 ± 0.0014	0.6098 ± 0.0001
0.0001	3.0557 ± 0.0014	0.60996 ± 0.00013

Example 6.2. In the multi-dimensional case, especially when the coefficients of the considered problem (1.1)-(1.2) are complicated, the variational system (6.2) becomes too complex and the approach of [13] to evaluation of Greeks is not easy for practical realization and rather costly (see also the discussion in the Introduction). The same comment is true for the approach based on Malliavin calculus [6]. But realization of the finite-difference evaluation of Greeks remains very simple and quite cheap (at least, in the case of smooth payoff functions).

In this example we consider a one-dimensional problem but with a more complicated diffusion coefficient than in (6.7):

$$(6.9) \qquad \frac{\partial u}{\partial t} + \frac{\nu^2}{2}x^2\left(1 + \frac{\varepsilon}{\sqrt{1+x^2}}\right)\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0, \quad 0 \le t < T,\ x \in \mathbf{R},$$
$$u(T,x) = \alpha x^2,$$

$\varepsilon > 0$ is a small parameter here.

To reduce the Monte Carlo error in evaluating of the solution $u(t,x)$ to (6.9) and its derivative $v(t,x)$ by (4.14) and (4.17), we take

$$(6.10) \qquad \mu = 0, \quad F = -2\nu\alpha\, x^2 \exp\left[\left(r + \nu^2\right)(T-t)\right],$$

which are optimal in the case of the problem (6.7) (cf. (6.8)).

A very small modification of the computer program used in Example 6.1 for realization of the finite difference evaluation of deltas is needed here, while the required modification of the program used in Example 6.1 for realization of the approach from [13] is more complicated. The results of experiments are presented in Table 3. We see that in this

TABLE 4. Results of simulation of the solution $u(3,10)$ to (6.11) and its derivative $v(3,10)$ by (4.14) and (4.17), respectively. The parameters are $\nu = 0.2$, $r = 0.1$, $K = 10$, $\mu = F = 0$, and $L = 10^6$. The exact values are $u(3,10) \doteq 2.907491$, $v(3,10) \doteq 0.850651$.

h	$\bar{u}(3,10)$	$\bar{v}(3,10)$
0.25	2.8512 ± 0.0089	0.8453 ± 0.0016
0.01	2.9061 ± 0.0096	0.8506 ± 0.0017
0.0025	2.9109 ± 0.0097	0.8510 ± 0.0017

case the finite difference approach also works quite well (cf. the results in Table 3 and $\bar{u}(3,10) = 3.05591 \pm 0.00007$, $\bar{v}(3,10) = 0.609982 \pm 0.000007$ obtained with $h = 0.000025$ and $L = 10000$).

Example 6.3. Consider the Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{\nu^2}{2}x^2\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0, \quad 0 \le t < T, \ x \in \mathbf{R}, \qquad (6.11)$$
$$u(T,x) = \max(0, x - K)\,,$$

the solution of which for $x > 0$ and $K > 0$ is

$$u(t,x) = x\,\Phi(y_* + \nu\sqrt{T-t}) - K\,e^{-r(T-t)}\,\Phi(y_*)$$

and

$$\frac{\partial u}{\partial x}(t,x) = \Phi(y_* + \nu\sqrt{T-t}) + \frac{1}{\nu\sqrt{T-t}}\exp\left(-\frac{\left(y_* + \nu\sqrt{T-t}\right)^2}{2}\right)$$
$$- \frac{K}{\nu x\sqrt{T-t}}\exp\left(-\frac{y_*^2}{2} - r(T-t)\right)\,,$$

where

$$y_* = \frac{1}{\nu\sqrt{T-t}}\left[\ln\frac{x}{K} + (r - \frac{\nu^2}{2})(T-t)\right] \quad \text{and} \quad \Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-y^2/2}dy\,.$$

It is not difficult to show that the estimate (3.9) of the Monte Carlo error in the finite difference evaluation of deltas also holds for non-smooth globally Lipschitz payoff functions (e.g., for $f(x) = \max(0, x-K)$ and $f(x) = \max(0, K-x)$), i.e., in this case the Monte Carlo error is also proportional just to $1/\sqrt{M}$ as for smooth functions. We have not made a complete numerical analysis of the finite-difference approach (we have not proved Theorem 2.2 and 4.4) in the non-smooth case but the numerical experiments for the problem (6.11) demonstrate (see Table 4) that evaluation of deltas by the finite difference can be effective for non-smooth globally Lipschitz payoff functions as well.

Acknowledgment

We acknowledge partial support from the EPSRC Research Grant GR/R90017/01.

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