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Stability of gyroscopic systems under small random excitations

Rafail Khasminskii[1] Grigori N. Milstein[2]

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[1] Wayne State University
Detroit
USA
E-mail: Rafail@math.wayne.edu

[2] Weierstrass Institute for Applied
Analysis and Stochastics
Mohrenstr. 39
D-10117 Berlin, Germany
E-Mail: milstein@wias-berlin.de

and

Ural State University
Lenin Street 51
620083 Ekaterinburg
Russia

No. 883
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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

ABSTRACT. Gyroscopic systems with two degrees of freedom under small random perturbations are investigated by use of the stochastic averaging principle. It is proved that the principal term of the Lyapunov exponent for the original system coincides with the Lyapunov exponent for the averaged system. An explicit formula for the averaged Lyapunov exponent is derived. The averaged moment Lyapunov exponent is considered as well. An example is given in which an unstable gyroscopical system is stabilized by noise of the Sratonovich type.

1. Introduction

Consider the linear autonomous system of stochastic differential equations (SDEs) in the sense of Stratonovich

$$
\begin{aligned}
&a_1\ddot{y}_1 - g\dot{y}_2 + c_1 y_1 + \varepsilon B_{10}^{\intercal}Y + \sqrt{\varepsilon}\sum_{r=1}^{q} B_{1r}^{\intercal}Y \circ \dot{w}_r = 0 \\
&a_2\ddot{y}_2 + g\dot{y}_1 + c_2 y_2 + \varepsilon B_{20}^{\intercal}Y + \sqrt{\varepsilon}\sum_{r=1}^{q} B_{2r}^{\intercal}Y \circ \dot{w}_r = 0\ ,
\end{aligned}
\tag{1.1}
$$

where Y is the column vector with four components $y_1,\ \dot{y}_1,\ y_2,\ \dot{y}_2$, the coefficients $a_1,\ a_2$ are positive, $g,\ c_1,\ c_2$ are constants, $B_{ir}^{\intercal}$ $i = 1, 2,\ r = 0, 1, ..., q$, are row vectors, $w_r(t),\ r = 1, ..., q$, are independent standard Wiener processes, $\varepsilon > 0$ is a small parameter.

For $\varepsilon = 0$, we have the deterministic system with gyroscopic forces

$$
\begin{aligned}
&a_1\ddot{y}_1 - g\dot{y}_2 + c_1 y_1 = 0 \\
&a_2\ddot{y}_2 + g\dot{y}_1 + c_2 y_2 = 0\ .
\end{aligned}
\tag{1.2}
$$

The characteristic equation for (1.2) reads

$$
a_1 a_2 \lambda^4 + (g^2 + a_1 c_2 + a_2 c_1)\lambda^2 + c_1 c_2 = 0. \tag{1.3}
$$

The roots of (1.3) can be found explicitly.

The following facts are well known and can be directly verified. For $c_1 > 0,\ c_2 > 0,\ g = 0$, the system (1.2) is a conservative system of two oscillators which is stable. Adding gyroscopic forces ($g \neq 0$) does not break the stability. For $c_1 = 0,\ c_2 = 0,\ g \neq 0$, the trivial solution of (1.2) is also stable. For $c_1 < 0,\ c_2 < 0,\ g = 0$, it is not stable. It can be stabilized by sufficiently large gyroscopic forces, namely, iff

$$
|g| > \sqrt{a_1|c_2|} + \sqrt{a_2|c_1|}\ . \tag{1.4}
$$

In both cases $c_1 > 0,\ c_2 > 0$ with any g and $c_1 < 0,\ c_2 < 0$ with $|g| > \sqrt{a_1|c_2|} + \sqrt{a_2|c_1|}$ all the eigenvalues of system (1.2) are pure imaginary (denote them by $\pm i\omega_1,\ \pm i\omega_2$), and the system (1.2) can be decomposed into two independent oscillators

$$
\begin{aligned}
&\ddot{z}_1 + \omega_1^2 z_1 = 0 \\
&\ddot{z}_2 + \omega_2^2 z_2 = 0
\end{aligned}
\tag{1.5}
$$

by a linear change of coordinates.

Further, if $c_1 > 0,\ c_2 > 0$, the stable system (1.2) acquires the asymptotically stable trivial solution after adding resistance forces with full dissipation. For example, the system

$$
\begin{aligned}
a_1\ddot{y}_1 + k_1\dot{y}_1 - g\dot{y}_2 + c_1 y_1 &= 0 \\
a_2\ddot{y}_2 + k_2\dot{y}_2 + g\dot{y}_1 + c_2 y_2 &= 0
\end{aligned}
\tag{1.6}
$$

is asymptotically stable under $c_1 > 0,\ c_2 > 0,\ k_1 > 0,\ k_2 > 0$, and any g. But if $c_1 < 0,\ c_2 < 0$, then the dissipative forces with anyhow small $k_1 > 0,\ k_2 > 0$ destroy the achieved stability by gyroscopic forces. If $k_1 > 0$ and $k_2 < 0$ (i.e., if along with a positive damping $k_1\dot{y}_1$ we use a negative damping $k_2\dot{y}_2$) the trivial solution of the system becomes asymptotically stable iff

$$
\begin{aligned}
&\frac{c_2}{c_1}k_1 < |k_2| < k_1, \\
&g^2 > |c_1| + |c_2| + k_1|k_2| + \frac{(|k_2|c_1 - k_1c_2)^2 + c_1c_2(k_1 - |k_2|)^2}{(|k_2|c_1 - k_1c_2)(k_1 - |k_2|)}\ .
\end{aligned}
\tag{1.7}
$$

This follows from the Hurwitz stability criterion.

Clearly, there is a linear transformation which translates the system (1.1) to the linear 4-dimensional SDEs in the sense of Stratonovich

$$
dX^\varepsilon = JX^\varepsilon dt + \varepsilon A_0^{(s)} X^\varepsilon dt + \sqrt{\varepsilon}\sum_{r=1}^{q} A_r X^\varepsilon \circ dw_r(t),\ \ X^\varepsilon(0) = x,
\tag{1.8}
$$

where

$$
J = \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{pmatrix}\ .
\tag{1.9}
$$

Thus, investigation of gyroscopic systems (1.1) is reduced to an analysis of coupled harmonic oscillators under small random excitations. Nevertheless, the benefit of considering gyroscopic systems in the form (1.1) is that the very important influence of their original parameters including a structure of the noise can be taken into account directly.

Coupled oscillators under small random excitations were considered in a number of papers (see [1, 2, 18, 17]). In [1], the stochastic averaging principle is used for analysis of stability properties of some systems (1.8). We also apply the averaging principle however not immediately to the system (1.8) but to the diffusion process

$$
\Lambda_\lambda^\varepsilon(t) := \frac{X_x^\varepsilon(t)}{|X_x^\varepsilon(t)|}\ ,\ \lambda = \frac{x}{|x|}\ ,\ x \neq 0,
$$

defined on the unit sphere $\mathbf{S}^3$ with center at the origin. In the next section we give a rigorous derivation of averaging systems which are connected with the Lyapunov exponent and moment Lyapunov exponent for the general two-degree-of-freedom oscillator system (1.8). Sections 3 and 4 are devoted to explicit formulas for the coefficients of the averaging systems and for the averaged Lyapunov exponent. The averaged system is one-dimensional if $\omega_1 \neq \omega_2,\ \omega_1 \neq 3\omega_2,\ 3\omega_1 \neq \omega_2$. The cases $\omega_1 = \omega_2,\ \omega_1 = 3\omega_2,\ 3\omega_1 = \omega_2$ are resonant [5]. In the presence of resonances a new slow variable arises and the averaged system becomes two-dimensional. As in the deterministic theory, it can be constructed by a procedure called partial averaging [5]. The resonance cases require an additional investigation and they are not considered here. In Section 5 we prove that $\lambda^\varepsilon = \varepsilon\bar{\lambda} + o(\varepsilon)$, where λ^ε is the Lyapunov exponent for (1.8) and $\bar{\lambda}$ is the averaged Lyapunov exponent. Most likely, an analogous fact is valid for the moment Lyapunov exponents too but for now we have not any complete proof of this result. In addition, let us note that for single-degree-of-freedom oscillator systems this fact is valid and it can be verified directly. In Section 6 we first give transformations reducing (1.1) to (1.8) in an explicit form. This allows us to take into account the coefficients of system (1.1) in a constructive manner. The main result of Section 6 is the proof of possibility to stabilize some unstable gyroscopic systems with positive and negative damping by a small noise of the Stratonovich type. We should emphasize that such a stabilization is impossible for single-degree-of-freedom oscillator systems (even for systems with periodic coefficients, see [9]). Finally, in Section 7, we briefly consider another approach to averaging belonging to the authors of [1]. In conclusion we have to note that this paper adjoins a great deal of works devoted to second order conservative systems with small random excitations (see [6, 4, 16, 17, 13, 15, 9, 7], and references therein).

2. **Averaging**

Consider an autonomous linear four-dimensional system of Ito SDEs

$$dX^\varepsilon = JX^\varepsilon dt + \varepsilon A_0 X^\varepsilon dt + \sqrt{\varepsilon}\sum_{r=1}^{q} A_r X^\varepsilon dw_r(t),\ \ X^\varepsilon(0) = x. \tag{2.1}$$

The diffusion process

$$\Lambda^\varepsilon_\lambda(t) := \frac{X^\varepsilon_x(t)}{|X^\varepsilon_x(t)|}\ ,\ \lambda = \frac{x}{|x|}\ ,\ x \neq 0,$$

defined on the unit sphere $\mathbf{S}^3$ with center at the origin, satisfies the system

$$d\Lambda^\varepsilon = J\Lambda^\varepsilon dt + \varepsilon h_0(\Lambda^\varepsilon)dt + \sqrt{\varepsilon}\sum_{r=1}^{q} h_r(\Lambda^\varepsilon)dw_r(t), \tag{2.2}$$

where the vector fields $h_r(\lambda)$, $r = 0, 1, ..., q$, on $\mathbf{S}^3$ are equal to

$$\begin{gathered} h_0(\lambda) = A_0\lambda - (A_0\lambda,\lambda)\lambda \\ -\frac{1}{2}\sum_{r=1}^{q}(A_r\lambda, A_r\lambda)\lambda - \sum_{r=1}^{q}(A_r\lambda,\lambda)A_r\lambda + \frac{3}{2}\sum_{r=1}^{q}(A_r\lambda,\lambda)^2\lambda, \\ h_r(\lambda) = A_r\lambda - (A_r\lambda,\lambda)\lambda,\ \ r = 1, ..., q. \end{gathered} \tag{2.3}$$

Let us underline that $(J\lambda, \lambda) = 0$.

It is assumed that the following condition of nondegeneracy is fulfilled:

$$\dim LA\{\tilde{h}_0, h_1, ..., h_q\} = 3 \text{ for all } \lambda = x/|x| \in \mathbf{S}^3, \tag{2.4}$$

where

$$\tilde{h}_0(\lambda) = \tilde{A}_0\lambda - (\tilde{A}_0\lambda, \lambda)\lambda,\ \tilde{A}_0 = J + \varepsilon A_0 - \frac{1}{2}\varepsilon\sum_{r=1}^{q} A_r^2,$$

$LA\{\}$ denotes the Lie algebra generated by the vector fields which occur in the brackets (see [3]).

It is well known that under (2.4) the Lyapunov exponent

$$\lambda^\varepsilon = \lim_{t\to\infty}\frac{1}{t}E\ln|X_x^\varepsilon(t)|,\ x \neq 0, \tag{2.5}$$

and the moment Lyapunov exponent

$$g^\varepsilon(p) := \lim_{t\to\infty}\frac{1}{t}\ln E|X_x^\varepsilon(t)|^p,\ x\neq 0,\ -\infty < p < \infty, \tag{2.6}$$

exist and are independent of $x \neq 0$.

For $|X_x^\varepsilon(t)|^p,\ -\infty < p < \infty$, we have the following linear equation

$$d|X_x^\varepsilon|^p = \varepsilon(pQ(\Lambda^\varepsilon) + \frac{1}{2}p^2R(\Lambda^\varepsilon))|X_x^\varepsilon|^p dt + \sqrt{\varepsilon}p\sum_{r=1}^{q}(A_r\Lambda^\varepsilon, \Lambda^\varepsilon)|X_x^\varepsilon|^p dw_r(t), \tag{2.7}$$

where Λ^ε satisfies (2.2) with initial data $\Lambda^\varepsilon(0) = x/|x|$, the functions $Q(\lambda)$ and $R(\lambda)$ are equal to

$$Q(\lambda) = (A_0\lambda, \lambda) + \frac{1}{2}\sum_{r=1}^{q}(A_r\lambda, A_r\lambda) - \sum_{r=1}^{q}(A_r\lambda,\lambda)^2, \tag{2.8}$$

$$R(\lambda) = \sum_{r=1}^{q}(A_r\lambda,\lambda)^2.$$

The process $\rho^\varepsilon(t) := \ln|X_x^\varepsilon(t)|$ satisfies

$$d\rho^\varepsilon = \varepsilon Q(\Lambda^\varepsilon)dt + \sqrt{\varepsilon}\sum_{r=1}^{q}(A_r\Lambda^\varepsilon, \Lambda^\varepsilon)dw_r(t). \tag{2.9}$$

Let us note that

$$|X_x^\varepsilon(t)|^p = \exp(p\rho^\varepsilon(t)). \tag{2.10}$$

We see from (2.2), (2.7), (2.9) that $|X_x^\varepsilon(t)|^p$ and $\rho^\varepsilon(t)$ are slow variables and $\Lambda^\varepsilon(t)$ is a fast one.

Introduce the variable

$$\Gamma^\varepsilon(t) = F^{-1}(t)\Lambda^\varepsilon(t) = F(-t)\Lambda^\varepsilon(t), \tag{2.11}$$

where

$$F(t) = \begin{pmatrix} \cos\omega_1 t & \sin\omega_1 t & 0 & 0 \\ -\sin\omega_1 t & \cos\omega_1 t & 0 & 0 \\ 0 & 0 & \cos\omega_2 t & \sin\omega_2 t \\ 0 & 0 & -\sin\omega_2 t & \cos\omega_2 t \end{pmatrix}.$$

Clearly, the transformation $F = F(t)$ is orthogonal and therefore $\Gamma^\varepsilon \in \mathbf{S}^3$. Since (see (1.9))

$$(F^{-1})'F + F^{-1}JF = 0,$$

we get

$$d\Gamma^\varepsilon = \varepsilon v_0(t,\Gamma^\varepsilon)dt + \sqrt{\varepsilon}\sum_{r=1}^{q} v_r(t,\Gamma^\varepsilon)dw_r(t), \tag{2.12}$$

where the vector fields $v_r(t,\gamma)$, $r = 0,1,...,q$, on $\mathbf{S}^3$ are equal to

$$v_0(t,\gamma) = F^{-1}A_0F\gamma - (A_0F\gamma, F\gamma)\gamma \tag{2.13}$$

$$-\frac{1}{2}\sum_{r=1}^{q}(A_rF\gamma, A_rF\gamma)\gamma - \sum_{r=1}^{q}(A_rF\gamma, F\gamma)F^{-1}A_rF\gamma + \frac{3}{2}\sum_{r=1}^{q}(A_rF\gamma, F\gamma)^2\gamma,$$

$$v_r(t,\gamma) = F^{-1}A_rF\gamma - (A_rF\gamma, F\gamma)\gamma, \ \ r = 1,...,q.$$

We see that $\Gamma^\varepsilon(t)$ is a slow variable.

Introduce

$$Y^\varepsilon := (\Gamma_1^\varepsilon)^2 + (\Gamma_2^\varepsilon)^2,$$

which is a slow variable too. Clearly,

$$(\Gamma_3^\varepsilon)^2 + (\Gamma_4^\varepsilon)^2 = 1 - Y^\varepsilon. \tag{2.14}$$

It is not difficult to get

$$dY^\varepsilon = \varepsilon a(t,\Gamma^\varepsilon, Y^\varepsilon)dt + \sqrt{\varepsilon}\sum_{r=1}^{q}\sigma_r(t,\Gamma^\varepsilon, Y^\varepsilon)dw_r(t), \tag{2.15}$$

where

$$a(t,\gamma,y) = 2\sum_{k=1}^{2}(F^{-1}A_0F\gamma)_k\gamma_k - 2(A_0F\gamma, F\gamma)y \tag{2.16}$$

$$-\sum_{r=1}^{q}(A_rF\gamma, A_rF\gamma)y - 4\sum_{r=1}^{q}(A_rF\gamma, F\gamma)\sum_{k=1}^{2}(F^{-1}A_rF\gamma)_k\gamma_k$$

$$+4\sum_{r=1}^{q}(A_rF\gamma, F\gamma)^2y + \sum_{r=1}^{q}\sum_{k=1}^{2}(F^{-1}A_rF\gamma)_k^2,$$

$$\sigma_r(t,\gamma,y) = 2\sum_{k=1}^{2}(F^{-1}A_rF\gamma)_k\gamma_k - 2(A_rF\gamma, F\gamma)y. \tag{2.17}$$

Denote $Z_p^\varepsilon := |X_x^\varepsilon|^p$ and rewrite (2.7) in variables Γ^ε :

$$dZ_p^\varepsilon = \varepsilon b(t,\Gamma^\varepsilon;p)Z_p^\varepsilon dt + \sqrt{\varepsilon}p\sum_{r=1}^{q}\zeta_r(t,\Gamma^\varepsilon)Z^\varepsilon dw_r(t), \tag{2.18}$$

where

$$b(t,\gamma;p) = p(A_0F\gamma, F\gamma) + \frac{1}{2}p\sum_{r=1}^{q}(A_rF\gamma, A_rF\gamma) \tag{2.19}$$

$$+ (\frac{1}{2}p^2 - p)\sum_{r=1}^{q}(A_rF\gamma, F\gamma)^2,$$

$$\zeta_r(t,\gamma) = (A_rF\gamma, F\gamma).$$

Also rewrite (2.9) in variables Γ^ε :

$$d\rho^\varepsilon = \varepsilon b^0(t,\Gamma^\varepsilon)dt + \sqrt{\varepsilon}\sum_{r=1}^{q}\zeta_r(t,\Gamma^\varepsilon)dw_r(t), \tag{2.20}$$

where

$$b^0(t,\gamma) = (A_0F\gamma, F\gamma) + \frac{1}{2}\sum_{r=1}^{q}(A_rF\gamma, A_rF\gamma) - \sum_{r=1}^{q}(A_rF\gamma, F\gamma)^2. \tag{2.21}$$

Introduce the slow time s :

$$s = \varepsilon t$$

and new variables

$$\tilde{Y}^\varepsilon(s) = Y^\varepsilon(s/\varepsilon),\ \tilde{Z}_p^\varepsilon(s) = Z_p^\varepsilon(s/\varepsilon),\ \tilde{\rho}^\varepsilon(s) = \rho^\varepsilon(s/\varepsilon),\ \tilde{\Gamma}^\varepsilon(s) = \Gamma^\varepsilon(s/\varepsilon). \tag{2.22}$$

It is easy to see from (2.15), (2.18), (2.20), (2.12) that

$$d\tilde{Y}^\varepsilon = a(\vartheta,\tilde{\Gamma}^\varepsilon,\tilde{Y}^\varepsilon)ds + \sum_{r=1}^{q}\sigma_r(\vartheta,\tilde{\Gamma}^\varepsilon,\tilde{Y}^\varepsilon)d\tilde{w}_r(s), \tag{2.23}$$

$$d\tilde{Z}^\varepsilon = b(\vartheta,\tilde{\Gamma}^\varepsilon;p)\tilde{Z}^\varepsilon ds + \sum_{r=1}^{q}\zeta_r(\vartheta,\tilde{\Gamma}^\varepsilon)\tilde{Z}^\varepsilon d\tilde{w}_r(s), \tag{2.24}$$

$$d\tilde{\rho}^\varepsilon = b^0(\vartheta,\tilde{\Gamma}^\varepsilon)ds + \sum_{r=1}^{q}\zeta_r(\vartheta,\tilde{\Gamma}^\varepsilon)d\tilde{w}_r(s), \tag{2.25}$$

$$d\tilde{\Gamma}^\varepsilon = v_0(\vartheta,\tilde{\Gamma}^\varepsilon)ds + \sum_{r=1}^{q}v_r(\vartheta,\tilde{\Gamma}^\varepsilon)d\tilde{w}_r(s), \tag{2.26}$$

where $\vartheta = s/\varepsilon$ is a fast variable, $\tilde{Y}^\varepsilon,\ \tilde{Z}_p^\varepsilon,\ \tilde{\rho}^\varepsilon,\ \tilde{\Gamma}^\varepsilon$ are slow variables, and $\tilde{w}_r(t),\ r = 1,...,q$, are the independent standard Wiener processes.

We recall that the equation for Y^ε was obtained as an equation for $(\Gamma_1^\varepsilon)^2 + (\Gamma_2^\varepsilon)^2$. It follows from here that if $y = \gamma_1^2 + \gamma_2^2$, then $\tilde{Y}^\varepsilon = (\tilde{\Gamma}_1^\varepsilon)^2 + (\tilde{\Gamma}_2^\varepsilon)^2$. To be in the position for applying averaging principle, let us for a while consider $\tilde{Y}^\varepsilon$ without such a connection with $\tilde{\Gamma}^\varepsilon$, i.e., we take y to be not obligatory equal to $\gamma_1^2 + \gamma_2^2$. Due to the averaging principle (see [10, 11, 8]), we have to fix any $y,\ z,\ \rho$, and γ (of course, with $|\gamma| = 1$) and to find averaged $\bar{a}(\gamma,y)$, $\overline{(\sigma_r)^2}(\gamma,y)$, $\bar{b}(\gamma;p)$, $\overline{\sigma_r\zeta_r}(\gamma)$, $\overline{(\zeta_r)^2}(\gamma,y)$, $\bar{b}^0(\gamma)$, $\overline{v_0}(\gamma)$, $\overline{\sigma_r v_r^k}(\gamma,y)$, $\overline{\zeta_r v_r^k}(\gamma)$, $\overline{v_r^k v_r^j}(\gamma)$, $k,j = 1,2,3,4$. Then we are able to write the corresponding system of SDEs for averaged $\bar{Y}$, $\bar{Z}_p$, $\bar{\rho}$, $\bar{\Gamma}$. This system can be chosen with a matrix of diffusion coefficients having a triangular form so that there is a single noise in the equation for $\bar{Y}$, two noises in the equation for $\bar{Z}_p$, three noises in the equation for $\bar{\Gamma}_1$, and so on. Due to the averaging principle, for any initial y, z, ρ, γ, the variables $\tilde{Y}^\varepsilon$, $\tilde{Z}^\varepsilon_p$, $\tilde{\rho}^\varepsilon$, $\tilde{\Gamma}^\varepsilon$ weakly converge to $\bar{Y}$, $\bar{Z}_p$, $\bar{\rho}$, $\bar{\Gamma}$ as $\varepsilon \to 0$. In particular, it follows from here that if $y = \gamma_1^2 + \gamma_2^2$, then $\tilde{Y}^\varepsilon$ (being equal to $(\tilde{\Gamma}_1^\varepsilon)^2 + (\tilde{\Gamma}_2^\varepsilon)^2$) tends to $\bar{\Gamma}_1^2 + \bar{\Gamma}_2^2$. Therefore if $y = \gamma_1^2 + \gamma_2^2$ then $\bar{Y} = \bar{\Gamma}_1^2 + \bar{\Gamma}_2^2$ and the averaged characteristics $\bar{a}$, $\overline{(\sigma_r)^2}$, $\bar{b}$, $\overline{\sigma_r\zeta_r}$, $\overline{(\zeta_r)^2}$, $\bar{b}^0$ are equal to $\bar{a}(\bar{\Gamma}, \bar{\Gamma}_1^2 + \bar{\Gamma}_2^2)$, $\overline{(\sigma_r)^2}(\bar{\Gamma}, \bar{\Gamma}_1^2 + \bar{\Gamma}_2^2)$, $\bar{b}(\bar{\Gamma}; p)$, $\overline{\sigma_r\zeta_r}(\bar{\Gamma})$, $\overline{(\zeta_r)^2}(\bar{\Gamma}, \bar{\Gamma}_1^2 + \bar{\Gamma}_2^2)$, $\bar{b}^0(\bar{\Gamma})$. Below we prove (see Section 3) that if $y = \gamma_1^2 + \gamma_2^2$ (and $\omega_1 \neq \omega_2$, $\omega_1 \neq 3\omega_2$, $3\omega_1 \neq \omega_2$), then all the functions $\bar{a}(\gamma, y)$, $\overline{(\sigma_r)^2}(\gamma, y)$, $\bar{b}(\gamma; p)$, $\bar{b}^0(\gamma)$, $\overline{\sigma_r\zeta_r}(\gamma)$, $\overline{(\zeta_r)^2}(\gamma, y)$ depend on y only. Denote them by $\alpha(y)$, $(\sigma\sigma)_r(y)$, $\beta(y; p)$, $\beta^0(y)$, $(\sigma\zeta)_r(y)$, $(\zeta\zeta)_r(y)$. Thus, $\bar{Y}$ with $\bar{Y}(0) = y = \gamma_1^2 + \gamma_2^2$, $\bar{Z}_p$, and $\bar{\rho}$ satisfy the system

$$d\bar{Y} = \alpha(\bar{Y})dt + \sqrt{\sum_{r=1}^{q} (\sigma\sigma)_r(\bar{Y})}\, d\bar{w}_1, \tag{2.27}$$

$$\begin{aligned} d\bar{Z}_p = \beta(\bar{Y}; p)\bar{Z}_p dt + p\frac{\sum_{r=1}^q (\sigma\zeta)_r(\bar{Y})}{\sqrt{\sum_{r=1}^q (\sigma\sigma)_r(\bar{Y})}}\bar{Z}_p d\bar{w}_1 \\ +p\sqrt{\frac{\sum_{r=1}^q (\sigma\sigma)_r(\bar{Y}) \cdot \sum_{r=1}^q (\zeta\zeta)_r(\bar{Y}) - (\sum_{r=1}^q (\sigma\zeta)_r(\bar{Y}))^2}{\sum_{r=1}^q (\sigma\sigma)_r(\bar{Y})}}\bar{Z}_p d\bar{w}_2\ , \end{aligned} \tag{2.28}$$

$$\begin{aligned} d\bar{\rho} = \beta^0(\bar{Y})dt + \frac{\sum_{r=1}^q (\sigma\zeta)_r(\bar{Y})}{\sqrt{\sum_{r=1}^q (\sigma\sigma)_r(\bar{Y})}} d\bar{w}_1 \\ +\sqrt{\frac{\sum_{r=1}^q (\sigma\sigma)_r(\bar{Y}) \cdot \sum_{r=1}^q (\zeta\zeta)_r(\bar{Y}) - (\sum_{r=1}^q (\sigma\zeta)_r(\bar{Y}))^2}{\sum_{r=1}^q (\sigma\sigma)_r(\bar{Y})}} d\bar{w}_2. \end{aligned} \tag{2.29}$$

We emphasize that the coefficients of (2.27)-(2.29) do not depend on $\bar{\Gamma}$. We also note that equation (2.27) and the systems (2.27), (2.28) and (2.27), (2.29) can be considered independently.

3. **Evaluation of the coefficients of the averaged system**

As it was mentioned, the averaged value $\bar{a}(\gamma, y)$ of $a(t, \gamma, y)$ (see (2.16)),

$$\bar{a}(\gamma, y) = \lim_{t \to \infty} \frac{1}{t} \int_0^t a(\tau, \gamma, y) d\tau,$$

depends on y only, i.e., $\bar{a}(\gamma, \gamma_1^2 + \gamma_2^2)$ is the function on y which was denoted as $\alpha(y)$. Analogous formulas are valid for the other averaged values mentioned in the previous section. In this section we prove these facts and derive formulas for $\alpha(y)$, $(\sigma\sigma)_r(y)$, $\beta(y; p)$, $(\sigma\zeta)_r(y)$, $(\zeta\zeta)_r(y)$, $\beta^0(y)$.

We begin with the averaging of $a(t, \gamma, y)$.

Lemma 3.1. *The following equalities hold:*

$$\sum_{k=1}^{2}(F^{-1}A_rF\gamma)_k\gamma_k=\sum_{k=1}^{2}(A_rF\gamma)_k(F\gamma)_k,\ r=0,1,...,q, \tag{3.1}$$

$$\sum_{k=1}^{2}(F^{-1}A_rF\gamma)_k^2=\sum_{k=1}^{2}(A_rF\gamma)_k^2,\ r=1,...,q, \tag{3.2}$$

Proof. The first two components of the vector $F^{-1}A_rF\gamma$ do not change if we take the matrix

$$\Phi=\begin{pmatrix}\cos\omega_1 t & -\sin\omega_1 t & 0 & 0\\ \sin\omega_1 t & \cos\omega_1 t & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

instead of F^{-1}. Therefore

$$\sum_{k=1}^{2}(F^{-1}A_rF\gamma)_k\gamma_k=\sum_{k=1}^{4}(\Phi A_rF\gamma)_k\gamma_k=(\Phi A_rF\gamma,\gamma)=(A_rF\gamma,\Phi^{\intercal}\gamma).$$

The first two components of the vector $\Phi^{\intercal}\gamma$ coincide with the first two components of the vector $F\gamma$ and the other two are equal to zero. Thus

$$(A_rF\gamma,\Phi^{\intercal}\gamma)=\sum_{k=1}^{2}(A_rF\gamma)_k(F\gamma)_k$$

and the equality (3.1) is proved.

Further we have

$$\sum_{k=1}^{2}(F^{-1}A_rF\gamma)_k^2=\sum_{k=1}^{4}(\Phi A_rF\gamma)_k^2=(\Phi A_rF\gamma,\Phi A_rF\gamma)$$

$$=(A_rF\gamma,\Phi^{\intercal}\Phi A_rF\gamma)=\sum_{k=1}^{2}(A_rF\gamma)_k^2.$$

The lemma is proved.

Thus, the dependence of $a(t,\gamma,y)$ on t and γ is realized through the superposition $F(t)\gamma$. It is clear from (2.16) and Lemma 3.1 that $a(t,\gamma,y)$ is a sum of a second order form and a fourth order form with respect to the coordinates $(F(t)\gamma)_k$, $k=1,2,3,4$.

Lemma 3.2. *Let* $\omega_1\neq\omega_2$, $\omega_1\neq 3\omega_2$, $3\omega_1\neq\omega_2$. *Then the following equalities hold:*

$$\overline{(F(t)\gamma)_k^2}=\frac{1}{2}y,\ k=1,2,\ \overline{(F(t)\gamma)_k(F(t)\gamma)_l}=0,\ k\neq l, \tag{3.3}$$
$$\overline{(F(t)\gamma)_k^2}=\frac{1}{2}(1-y),\ k=3,4,\quad y=\gamma_1^2+\gamma_2^2.$$

$$\overline{(F(t)\gamma)_k^4} = \frac{3}{8}y^2,\ k=1,2,\ \overline{(F(t)\gamma)_k^4} = \frac{3}{8}(1-y)^2,\ k=3,4, \tag{3.4}$$
$$\overline{(F(t)\gamma)_1^2(F(t)\gamma)_2^2} = \frac{1}{8}y^2,\ \overline{(F(t)\gamma)_3^2(F(t)\gamma)_4^2} = \frac{1}{8}(1-y)^2,$$
$$\overline{(F(t)\gamma)_k^2(F(t)\gamma)_l^2} = \frac{1}{4}y(1-y),\ k=1,2,\ l=3,4,\ y=\gamma_1^2+\gamma_2^2.$$

The averaged values of the other monomials of the fourth order with respect to $(F(t)\gamma)_k$, $k=1,2,3,4$, *are equal to zero:*

$$\overline{(F(t)\gamma)_k^3(F(t)\gamma)_l} = 0,\ k\neq l, \tag{3.5}$$
$$\overline{(F(t)\gamma)_k^2(F(t)\gamma)_l(F(t)\gamma)_m} = 0,\quad k\neq l,\ k\neq m,\ l\neq m,$$
$$\overline{(F(t)\gamma)_1^2(F(t)\gamma)_2(F(t)\gamma)_3(F(t)\gamma)_4} = 0.$$

Proof. For example, let us derive the formulae

$$\overline{(F(t)\gamma)_1^4} = \frac{3}{8}y^2$$

and

$$\overline{(F(t)\gamma)_1^3(F(t)\gamma)_3} = 0.$$

We have

$$(F(t)\gamma)_1^4 = (\gamma_1\cos\omega_1 t + \gamma_2\sin\omega_1 t)^4 = \gamma_1^4\cos^4\omega_1 t + 4\gamma_1^3\gamma_2\cos^3\omega_1 t\sin\omega_1 t$$
$$+ 6\gamma_1^2\gamma_2^2\cos^2\omega_1 t\sin^2\omega_1 t + 4\gamma_1\gamma_2^3\cos\omega_1 t\sin^3\omega_1 t + \gamma_2^4\sin^4\omega_1 t.$$

It is clear that

$$\overline{\cos^4\omega_1 t} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \cos^4\omega_1\tau d\tau = \frac{3}{8}$$

and

$$\overline{\sin^4\omega_1 t} = \frac{3}{8},\ \overline{\cos^2\omega_1 t\sin^2\omega_1 t} = \frac{1}{8},\ \overline{\cos^3\omega_1 t\sin\omega_1 t} = 0,\ \overline{\cos\omega_1 t\sin^3\omega_1 t} = 0.$$

So

$$\overline{(F(t)\gamma)_1^4} = \frac{3}{8}\gamma_1^4 + \frac{6}{8}\gamma_1^2\gamma_2^2 + \frac{3}{8}\gamma_2^4 = \frac{3}{8}(\gamma_1^2+\gamma_2^2)^2$$

and the first formula from above mentioned ones is proved.

We have also

$$(F(t)\gamma)_1^3(F(t)\gamma)_3 = (\gamma_1^3\cos^3\omega_1 t + 3\gamma_1^2\gamma_2\cos^2\omega_1 t\sin\omega_1 t$$
$$+3\gamma_1\gamma_2^2\cos\omega_1 t\sin^2\omega_1 t + \gamma_2^3\sin^3\omega_1 t)\cdot(\gamma_3\cos\omega_2 t + \gamma_4\sin\omega_2 t).$$

Further we get

$$\cos^3\omega_1 t\cos\omega_2 t = \frac{1+\cos 2\omega_1 t}{2}\cdot\frac{\cos(\omega_1+\omega_2)t + \cos(\omega_1-\omega_2)t}{2}$$
$$= \frac{1}{4}[\cos(\omega_1+\omega_2)t + \cos(\omega_1-\omega_2)t]$$
$$+\frac{1}{8}[\cos(3\omega_1+\omega_2)t + \cos(\omega_1-\omega_2)t + \cos(3\omega_1-\omega_2)t + \cos(\omega_1+\omega_2)t].$$

We obtain from here

$$\overline{(F(t)\gamma)_1^3(F(t)\gamma)_3} = \begin{cases} 0, & \text{if } \omega_1 \neq \omega_2,\ 3\omega_1 \neq \omega_2, \\ 3/8, & \text{if } \omega_1 = \omega_2, \\ 1/8, & \text{if } 3\omega_1 = \omega_2, \end{cases}$$

i.e., under the conditions of the lemma ($\omega_1 \neq \omega_2,\ 3\omega_1 \neq \omega_2$), the second formula is proved too. All the formulae of Lemma 3.2 can be proved analogously.

It follows from this lemma and the expressions for $a(t,\gamma,y)$, $\sigma_r(t,\gamma,y)$, $b(t,\gamma;p)$, $\zeta_r(t,\gamma)$, $b^0(t,\gamma)$ that all the functions $\bar{a}(\gamma,y)$, $\overline{(\sigma_r)^2}(\gamma,y)$, $\bar{b}(\gamma;p)$, $\overline{\sigma_r\zeta_r}(\gamma)$, $\overline{(\zeta_r)^2}(\gamma,y)$, $\overline{b^0}(\gamma)$ depend on y only. These functions were denoted correspondingly by $\alpha(y)$, $(\sigma\sigma)_r(y)$, $\beta(y;p)$, $(\sigma\zeta)_r(y)$, $(\zeta\zeta)_r(y)$, $\beta^0(y)$. Due to Lemma 3.2, they can be written explicitly (of course, after bulky but not difficult calculations). The exact formulas are given in the next lemma.

Lemma 3.3. *Let $\omega_1 \neq \omega_2$, $\omega_1 \neq 3\omega_2$, $3\omega_1 \neq \omega_2$. Then $\alpha(y)$ can be written as*

$$\begin{aligned}\alpha(y) &= (a_{11}+a_{22}-a_{33}-a_{44})_0 y(1-y) \\ &+\frac{1}{2}\sum_{r=1}^{q}((a_{13}(1-y)-a_{31}y)^2+(a_{14}(1-y)-a_{41}y)^2)_r(1-2y) \\ &+\frac{1}{2}\sum_{r=1}^{q}((a_{23}(1-y)-a_{32}y)^2+(a_{24}(1-y)-a_{42}y)^2)_r(1-2y) \\ &+\frac{1}{2}\sum_{r=1}^{q}(a_{11}^2+a_{21}^2+a_{12}^2+a_{22}^2-a_{33}^2-a_{43}^2-a_{34}^2-a_{44}^2)_r y(1-y) \\ &-\sum_{r=1}^{q}((a_{11}+a_{22})(a_{33}+a_{44}))_r y(1-y)(1-2y) \\ &-\frac{3}{2}\sum_{r=1}^{q}(a_{11}^2+a_{22}^2)_r y^2(1-y)-\frac{1}{2}\sum_{r=1}^{q}((a_{12}+a_{21})^2+2a_{11}a_{22})_r y^2(1-y) \\ &+\frac{3}{2}\sum_{r=1}^{q}(a_{33}^2+a_{44}^2)_r y(1-y)^2+\frac{1}{2}\sum_{r=1}^{q}((a_{34}+a_{43})^2+2a_{33}a_{44})_r y(1-y)^2.\end{aligned} \tag{3.6}$$

Here and below $(\cdot)_r$ means that the elements inside the parentheses are the elements of the matrix A_r.

The explicit expressions for $(\sigma\sigma)_r(y)$ and $\beta(y;p)$ are

$$\begin{aligned}(\sigma\sigma)_r(y) &= [(a_{11}+a_{22}-a_{33}-a_{44})^2+\frac{1}{2}(a_{11}-a_{22})^2 \\ &+\frac{1}{2}(a_{33}-a_{44})^2+\frac{1}{2}(a_{12}+a_{21})^2+\frac{1}{2}(a_{34}+a_{43})^2]_r y^2(1-y)^2 \\ &+((a_{13}(1-y)-a_{31}y)^2+(a_{14}(1-y)-a_{41}y)^2)_r y(1-y) \\ &+((a_{23}(1-y)-a_{32}y)^2+(a_{24}(1-y)-a_{42}y)^2)_r y(1-y),\end{aligned} \tag{3.7}$$

$$(3.8)\quad \beta(y;p) = \frac{p}{2}((a_{11}+a_{22})_0 y + (a_{33}+a_{44})_0(1-y))$$
$$+\frac{p}{4}\sum_{r=1}^{q}(|a^1|^2+|a^2|^2)_r y + (|a^3|^2+|a^4|^2)_r(1-y)$$
$$-\frac{3}{8}(p-\frac{1}{2}p^2)\sum_{r=1}^{q}(a_{11}^2+a_{22}^2)_r y^2 - \frac{1}{8}(p-\frac{1}{2}p^2)\sum_{r=1}^{q}((a_{12}+a_{21})^2+2a_{11}a_{22})_r y^2$$
$$-\frac{1}{2}(p-\frac{1}{2}p^2)\sum_{r=1}^{q}((a_{11}+a_{22})_r(a_{33}+a_{44})_r y(1-y)$$

$$-\frac{1}{4}(p-\frac{1}{2}p^2)\sum_{r=1}^{q}((a_{13}+a_{31})^2+(a_{14}+a_{41})^2+(a_{23}+a_{32})^2+(a_{24}+a_{42})^2)_r y(1-y)$$
$$-\frac{3}{8}(p-\frac{1}{2}p^2)\sum_{r=1}^{q}(a_{33}^2+a_{44}^2)_r(1-y)^2$$
$$-\frac{1}{8}(p-\frac{1}{2}p^2)\sum_{r=1}^{q}((a_{34}+a_{43})^2+2a_{33}a_{44})_r(1-y)^2,$$

where $(|a^k|^2)_r,\ k=1,2,3,4,$ *is the square of magnitude of the* k*-th column vector for the matrix* A_r*.*

The explicit expression for $(\sigma\zeta)_r(y)$ *is:*

$$(3.9)\quad (\sigma\zeta)_r(y) = \frac{3}{4}(a_{11}^2+a_{22}^2)_r y^2(1-y) + \frac{1}{4}((a_{12}+a_{21})^2+2a_{11}a_{22})_r y^2(1-y)$$
$$+\frac{1}{2}((a_{11}+a_{22})(a_{33}+a_{44}))_r y(1-y)$$
$$+\frac{1}{2}(a_{13}^2+a_{13}a_{31}+a_{14}^2+a_{14}a_{41}+a_{23}^2+a_{23}a_{32}+a_{24}^2+a_{24}a_{42})_r y(1-y)$$
$$-((a_{11}+a_{22})(a_{33}+a_{44}))_r y^2(1-y)$$
$$-\frac{1}{2}((a_{13}+a_{31})^2+(a_{14}+a_{41})^2+(a_{23}+a_{32})^2+(a_{24}+a_{42})^2)_r y^2(1-y)$$
$$-\frac{3}{4}(a_{33}^2+a_{44}^2)_r y(1-y)^2 - \frac{1}{4}((a_{34}+a_{43})^2+2a_{33}a_{44})_r y(1-y)^2.$$

The explicit expression for $(\zeta\zeta)_r(y)$ *is:*

$$(\zeta\zeta)_r(y) = \frac{3}{8}(a_{11}^2 + a_{22}^2)_r y^2 + \frac{1}{8}((a_{12}+a_{21})^2 + 2a_{11}a_{22})_r y^2$$
$$+\frac{3}{8}(a_{33}^2 + a_{44}^2)_r(1-y)^2 + \frac{1}{8}((a_{34}+a_{43})^2 + 2a_{33}a_{44})_r(1-y)^2$$
$$+\frac{1}{2}((a_{11}+a_{22})(a_{33}+a_{44}))_r y(1-y)$$
$$+\frac{1}{4}((a_{13}+a_{31})^2 + (a_{14}+a_{41})^2 + (a_{23}+a_{32})^2 + (a_{24}+a_{42})^2)_r y(1-y)\ . \tag{3.10}$$

The explicit expression for $\beta^0(y)$ *is:*

$$\beta^0(y) = \frac{1}{2}((a_{11}+a_{22})_0 y + (a_{33}+a_{44})_0(1-y))$$
$$+\frac{1}{4}\sum_{r=1}^{q}(|a^1|^2 + |a^2|^2)_r y + (|a^3|^2 + |a^4|^2)_r(1-y)$$
$$-\frac{3}{8}\sum_{r=1}^{q}(a_{11}^2 + a_{22}^2)_r y^2 - \frac{1}{8}\sum_{r=1}^{q}((a_{12}+a_{21})^2 + 2a_{11}a_{22})_r y^2$$
$$-\frac{1}{2}\sum_{r=1}^{q}((a_{11}+a_{22})_r(a_{33}+a_{44})_r y(1-y)$$
$$-\frac{1}{4}\sum_{r=1}^{q}((a_{13}+a_{31})^2 + (a_{14}+a_{41})^2 + (a_{23}+a_{32})^2 + (a_{24}+a_{42})^2)_r y(1-y)$$
$$-\frac{3}{8}\sum_{r=1}^{q}(a_{33}^2 + a_{44}^2)_r(1-y)^2 - \frac{1}{8}\sum_{r=1}^{q}((a_{34}+a_{43})^2 + 2a_{33}a_{44})_r(1-y)^2. \tag{3.11}$$

As a result, we obtain the following theorem.

Theorem 3.1. *Let* $\omega_1 \neq \omega_2,\ \omega_1 \neq 3\omega_2,\ 3\omega_1 \neq \omega_2$. *Then the averaged values* $\bar{Y},\ \bar{Z}_p,\ \bar{\rho}$ *(see* (2.27)-(2.29)*) satisfy the system*

$$d\bar{Y} = \alpha(\bar{Y})dt + \sqrt{\sigma(\bar{Y})}d\bar{w}_1, \tag{3.12}$$

$$d\bar{Z}_p = \beta(\bar{Y};p)\bar{Z}_p dt + p\frac{\delta(\bar{Y})}{\sqrt{\sigma(\bar{Y})}}\bar{Z}_p d\bar{w}_1 + p\frac{\sqrt{\sigma(\bar{Y})\zeta(\bar{Y}) - \delta^2(\bar{Y})}}{\sqrt{\sigma(\bar{Y})}}\bar{Z}_p d\bar{w}_2, \tag{3.13}$$

$$d\bar{\rho} = \beta^0(\bar{Y})dt + \frac{\delta(\bar{Y})}{\sqrt{\sigma(\bar{Y})}}d\bar{w}_1 + \frac{\sqrt{\sigma(\bar{Y})\zeta(\bar{Y}) - \delta^2(\bar{Y})}}{\sqrt{\sigma(\bar{Y})}}d\bar{w}_2, \tag{3.14}$$

where

$$\sigma(y) = \sum_{r=1}^{q}(\sigma\sigma)_r(y),\ \ \zeta(y) = \sum_{r=1}^{q}(\zeta\zeta)_r(y),\ \ \delta(y) = \sum_{r=1}^{q}(\sigma\zeta)_r(y). \tag{3.15}$$

4. **Averaged Lyapunov and moment Lyapunov exponents**

The averaged Lyapunov exponent is equal to

$$\bar{\lambda} = \lim_{t\to\infty} \frac{1}{t} E \ln \bar{Z}_1 = \lim_{t\to\infty} \frac{1}{t} E\bar{\rho} = \int_0^1 \beta^0(y)\mu(y)dy, \tag{4.1}$$

where $\mu(y)$ is an invariant density for the process $\bar{Y}$.

The following lemma can be proved straightforward.

Lemma 4.1. *Let $\omega_1 \neq \omega_2$, $\omega_1 \neq 3\omega_2$, $3\omega_1 \neq \omega_2$. The diffusion coefficient $\sigma(y)$ is positive on $(0,1)$ iff either*

$$\sum_{r=1}^{q}(a_{11}+a_{22}-a_{33}-a_{44})_r^2 \tag{4.2}$$

$$+\sum_{r=1}^{q}[(a_{11}-a_{22})^2+(a_{33}-a_{44})^2+(a_{12}+a_{21})^2+(a_{34}+a_{43})^2]_r > 0$$

or for any constant $0 < c < \infty$ there exists $r = 1,...,q$ such that

$$(a_{13}, a_{14}, a_{23}, a_{24})_r \neq c(a_{31}, a_{41}, a_{32}, a_{42})_r. \tag{4.3}$$

If

$$\sum_{r=1}^{q}(a_{13}^2+a_{14}^2+a_{23}^2+a_{24}^2)_r > 0,\ \sum_{r=1}^{q}(a_{31}^2+a_{41}^2+a_{32}^2+a_{42}^2)_r > 0, \tag{4.4}$$

then $\alpha(0) > 0$, $\alpha(1) < 0$. If (4.4) is fulfilled together with one of the inequalities (4.2), (4.3), then the function $\dfrac{\sigma(y)}{y(1-y)}$ is positive on the closed interval $[0,1]$.

We shall always assume that (4.4) is fulfilled together with one of the inequalities (4.2), (4.3). It is readily to see that in this case the process $\bar{Y}$ is ergodic. Let us give an explicit formula for the invariant density $\mu(y)$.

Lemma 4.2. *Let $\omega_1 \neq \omega_2$, $\omega_1 \neq 3\omega_2$, $3\omega_1 \neq \omega_2$, and conditions (4.4) together with one of the inequalities (4.2), (4.3) be fulfilled. Then the invariant density $\mu(y)$ of the process $\bar{Y}$ is equal to*

$$\mu(y) = C\exp(-\int_{1/2}^{y} q(y')dy'), \tag{4.5}$$

where

$$q(y) = \frac{\sigma'(y) - 2\alpha(y)}{\sigma(y)}$$

and C is the normalizing constant. The function $q(y)$ is continuous on the closed interval $[0,1]$.

Proof. The invariant density satisfies the Fokker-Planck equation

$$\frac{1}{2}\frac{d^2}{dy^2}[\sigma(y)\mu(y)] - \frac{d}{dy}[\alpha(y)\mu(y)] = 0. \tag{4.6}$$

Its general solution is

$$\mu = C\exp(-\int_{1/2}^{y} q(y')dy') \\ + C^*\exp(-\int_{1/2}^{y} q(y')dy')\cdot\int_{1/2}^{y}\frac{1}{\sigma(y')}\exp(-\int_{1/2}^{y'} q(y'')dy'')dy', \tag{4.7}$$

where C, C^* are arbitrary constants. It is not difficult to see that the polynomial $\sigma'(y) - 2\alpha(y)$ is equal to zero at $y = 0$ and $y = 1$, i.e., this polynomial has y and $(1-y)$ as factors. Therefore, due to the last assertion of Lemma 4.1, the function $q(y)$ is continuous on the closed interval $[0,1]$. If we suppose that $C^* > 0$ in (4.7), we obtain $\mu(y) \to -\infty$ as $y \downarrow 0$, If $C^* < 0$, we again obtain $\mu(y) \to -\infty$ as $y \uparrow 1$. It follows from here that $C^* = 0$ in (4.7) and the lemma is proved.

The averaged moment Lyapunov exponent is equal to

$$\bar{g}(p) = \lim_{t\to\infty}\frac{1}{t}\ln E\bar{Z}_p(t)\ . \tag{4.8}$$

Since

$$\beta(y,p) - \frac{1}{2}p^2\zeta^2(y) = p\beta^0(y),$$

we easily get

$$\bar{Z}_p(t) = \exp(p\bar{\rho}(t)) \tag{4.9}$$

and, consequently,

$$\bar{g}(p) = \lim_{t\to\infty}\frac{1}{t}\ln Ee^{p\bar{\rho}(t)}. \tag{4.10}$$

Therefore if we apply the Monte Carlo approach for evaluating $\bar{g}(p)$, we can use the same sample trajectories of (3.12), (3.14) for different p. Let us note that (4.9) is, as the final result, a consequence of (2.10). The formula (4.9) can be obtained by another way using less calculations, but we prefer the given direct derivation.

In [14], an effective deterministic method is proposed for evaluating moment Lyapunov exponents for second order stochastic systems. The method is based on the solution of a Sturm-Liouville problem. Most likely, the method of [14] can be carried over for evaluating $\bar{g}(p)$ in our case too. However, this requires additional investigations since the corresponding Sturm-Liouville problem in the considered case is, unlike [14], singular due to vanishing $\sigma(y)$ at $y = 0$ and $y = 1$.

5. **Lyapunov exponent for the original system**

In the next theorem we prove that the principal term of the Lyapunov exponent for the system (2.1) coincides with the averaged Lyapunov exponent $\bar{\lambda}$.

Theorem 5.1. *Let* $\omega_1 \neq \omega_2$, $\omega_1 \neq 3\omega_2$, $3\omega_1 \neq \omega_2$, *and conditions* (4.4) *together with one of the inequalities* (4.2), (4.3) *be fulfilled. Then the Lyapunov exponent* λ^ε *for the system* (2.1) *has the expansion*

$$\lambda^\varepsilon = \varepsilon\bar{\lambda} + o(\varepsilon), \tag{5.1}$$

where $\bar{\lambda}$ *is the Lyapunov exponent for the averaged system* (3.12), (3.14).

Proof. We have for any $\tilde{\Gamma}^\varepsilon(s)$ (see (2.20), (2.22), (2.25)):

$$\begin{aligned}\lambda^\varepsilon &= \lim_{T_1\to\infty}\frac{1}{T_1}E\rho^\varepsilon(T_1) = \lim_{T_1\to\infty}\frac{1}{T_1}\varepsilon E\int_0^{T_1} b^0(t,\Gamma^\varepsilon(t))dt \\ &= \varepsilon\lim_{T_1\to\infty}\frac{1}{\varepsilon T_1}E\int_0^{\varepsilon T_1} b^0(s/\varepsilon,\tilde{\Gamma}^\varepsilon(s))ds = \varepsilon\lim_{T\to\infty}\frac{1}{T}E\int_0^T b^0(s/\varepsilon,\tilde{\Gamma}^\varepsilon(s))ds.\end{aligned} \tag{5.2}$$

Thus, to get (5.1) we need to prove

$$\lim_{\varepsilon\to 0}\lim_{T\to\infty}\frac{1}{T}E\int_0^T b^0(s/\varepsilon,\tilde{\Gamma}^\varepsilon(s))ds = \bar{\lambda}. \tag{5.3}$$

Let $T = n\Delta$, $\Delta = \varepsilon^\alpha$, $\alpha > 0$ is a fixed number, $s_i = i\Delta$, $i = 0, ..., n$. Then

$$E\int_0^T b^0(s/\varepsilon,\tilde{\Gamma}^\varepsilon(s))ds = E\sum_{i=1}^n\int_{s_i}^{s_i+\Delta} b^0(s/\varepsilon,\tilde{\Gamma}^\varepsilon(s_i))ds + O(\Delta^{3/2}n). \tag{5.4}$$

Indeed

$$\begin{aligned}&E\sum_{i=1}^n\int_{s_i}^{s_i+\Delta}|b^0(s/\varepsilon,\tilde{\Gamma}^\varepsilon(s)) - b^0(s/\varepsilon,\tilde{\Gamma}^\varepsilon(s_i))|ds \\ &\le \sum_{i=1}^n\int_{s_i}^{s_i+\Delta}(E|b^0(s/\varepsilon,\tilde{\Gamma}^\varepsilon(s)) - b^0(s/\varepsilon,\tilde{\Gamma}^\varepsilon(s_i))|^2)^{1/2}ds.\end{aligned}$$

Further, due to the boundedness of b^0, $\partial b^0/\partial\gamma$, and ζ_r for $\gamma\in\mathbf{S}^3$, we get for $s_i \le s \le s_i + \Delta$:

$$E|b^0(s/\varepsilon,\tilde{\Gamma}^\varepsilon(s)) - b^0(s/\varepsilon,\tilde{\Gamma}^\varepsilon(s_i))|^2 = O(\Delta),$$

whence (5.4) follows.

Clearly, for any $t \ge 0$ and $\gamma\in\mathbf{S}^3$ we have

$$\frac{1}{S}E\int_t^{t+S} b^0(s,\gamma)ds = \bar{b}^0(\gamma) + O(1/S).$$

Thus

$$\int_{s_i}^{s_i+\Delta} b^0(s/\varepsilon, \tilde{\Gamma}^\varepsilon(s_i))ds = \Delta \frac{\varepsilon}{\Delta} \int_{s_i/\varepsilon}^{s_i/\varepsilon+\Delta/\varepsilon} b^0(t, \tilde{\Gamma}^\varepsilon(s_i))dt$$
$$= \Delta(\bar{b}^0(\tilde{\Gamma}^\varepsilon(s_i)) + O(\varepsilon/\Delta)) = \Delta \bar{b}^0(\tilde{\Gamma}^\varepsilon(s_i)) + O(\varepsilon).$$

Now we get from (5.4)

$$E\int_0^T b^0(s/\varepsilon, \tilde{\Gamma}^\varepsilon(s))ds = E\sum_{i=1}^n \Delta \cdot \bar{b}^0(\tilde{\Gamma}^\varepsilon(s_i)) + nO(\varepsilon) + O(\Delta^{3/2}n) \tag{5.5}$$
$$= E\sum_{i=1}^n \Delta \cdot \bar{b}^0(\tilde{\Gamma}^\varepsilon(s_i)) + \frac{T}{\varepsilon^\alpha}O(\varepsilon) + T \cdot O(\varepsilon^{\alpha/2}).$$

Analogously to (5.4)

$$E\int_0^T \bar{b}^0(\tilde{\Gamma}^\varepsilon(s))ds = E\sum_{i=1}^n \int_{s_i}^{s_i+\Delta} \bar{b}^0(\tilde{\Gamma}^\varepsilon(s_i))ds + O(\Delta^{3/2}n) \tag{5.6}$$
$$= E\sum_{i=1}^n \Delta \cdot \bar{b}^0(\tilde{\Gamma}^\varepsilon(s_i)) + T \cdot O(\varepsilon^{\alpha/2}).$$

Taking $\alpha = 2/3$, we obtain from (5.5) and (5.6):

$$\frac{1}{T}E\int_0^T b^0(s/\varepsilon, \tilde{\Gamma}^\varepsilon(s))ds = \frac{1}{T}E\int_0^T \bar{b}^0(\tilde{\Gamma}^\varepsilon(s))ds + O(\varepsilon^{1/3}), \tag{5.7}$$

where $|O(\varepsilon^{1/3})| \leq C\varepsilon^{1/3}$ with C independent of T.

We recall that $\bar{b}^0(\tilde{\Gamma}^\varepsilon(s)) = \beta^0(\tilde{Y}^\varepsilon(s))$, where $\tilde{Y}^\varepsilon = (\tilde{\Gamma}_1^\varepsilon)^2 + (\tilde{\Gamma}_2^\varepsilon)^2$, and that $(\tilde{\Gamma}_1^\varepsilon)^2 + (\tilde{\Gamma}_2^\varepsilon)^2$ weakly converges to $\bar{Y} = \bar{\Gamma}_1^2 + \bar{\Gamma}_2^2$ as $\varepsilon \to 0$, where $\bar{Y}$ is the corresponding solution of (3.12). In addition, let us recall that the process $\bar{Y}$ under conditions of the theorem is ergodic with invariant density $\mu(y)$. Let us denote by $\bar{Y}_\nu(t)$ a solution of (3.12) with an initial density $\nu(y)$. Then $\bar{Y}_\mu(t)$ is the stationary solution of (3.12). Due to ergodicity, for any $\delta > 0$ there exists $T_\delta > 0$ such that

$$|E\beta^0(\bar{Y}_\nu(t)) - E\beta^0(\bar{Y}_\mu(t))| \leq \delta \tag{5.8}$$

for any ν and $t \geq T_\delta$.

For any $\varepsilon > 0$, the system (2.2) has a stationary solution ${}_{st}\Lambda^\varepsilon$. Clearly, ${}_{st}Y^\varepsilon := ({}_{st}\Lambda_1^\varepsilon)^2 + ({}_{st}\Lambda_2^\varepsilon)^2$ (let us note that $(\Gamma_1^\varepsilon)^2 + (\Gamma_2^\varepsilon)^2 = (\Lambda_1^\varepsilon)^2 + (\Lambda_2^\varepsilon)^2$ due to (2.11)) has also a stationary distribution. The same is true for ${}_{st}\tilde{Y}^\varepsilon(s) ={}_{st} Y^\varepsilon(s/\varepsilon)$. Let $\tilde{\nu}^\varepsilon$ be the stationary distribution for ${}_{st}\tilde{Y}^\varepsilon(s)$. Let us fix some $T \geq T_\delta$. Then, due to the averaging principle, there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$

$$|E\beta^0(\tilde{Y}_\nu^\varepsilon(T)) - E\beta^0(\bar{Y}_\nu(T))| \leq \delta \tag{5.9}$$

for any initial distribution ν. According to (5.8) and (5.9), we get

$$|E\beta^0(\tilde{Y}_\nu^\varepsilon(T)) - E\beta^0(\bar{Y}_\mu(T))| \leq 2\delta. \tag{5.10}$$

If we take $\tilde{\nu}^\varepsilon$ instead of ν (i.e., $\tilde{Y}^\varepsilon_{\tilde{\nu}^\varepsilon}(s) =_{st} \tilde{Y}^\varepsilon(s)$) in (5.10), we obtain for $\varepsilon < \varepsilon_0$ and all $0 \le t < \infty$:

$$|E\beta^0({}_{st}\tilde{Y}^\varepsilon(t)) - E\beta^0(\bar{Y}_\mu(t))| \le 2\delta \tag{5.11}$$

since both the distribution of ${}_{st}\tilde{Y}^\varepsilon(t)$ and the distribution of $\bar{Y}_\mu(t)$ do not depend on t. Let us return to (5.7) to evaluate the integral

$$\frac{1}{T}E\int_0^T \bar{b}^0(\tilde{\Gamma}^\varepsilon(s))ds = \frac{1}{T}E\int_0^T \beta^0({}_{st}\tilde{Y}^\varepsilon(s))ds.$$

We take into account that for any $T > 0$

$$\frac{1}{T}E\int_0^T \beta^0(\bar{Y}_\mu(s))ds = E\beta^0(\bar{Y}_\mu(t)) = \int_0^1 \beta^0(y)\mu(y)dy = \bar{\lambda}.$$

Due to (5.11), we have for any $T > 0$

$$\bar{\lambda} - 2\delta \le \frac{1}{T}E\int_0^T \beta^0({}_{st}\tilde{Y}^\varepsilon(s))ds = \frac{1}{T}E\int_0^T \bar{b}^0(\tilde{\Gamma}^\varepsilon(s))ds \le \bar{\lambda} + 2\delta.$$

Letting $T \to \infty$ in (5.7), we obtain that for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$

$$\bar{\lambda} - 2\delta - |O(\varepsilon^{1/3})| \le \lim_{T\to\infty}\frac{1}{T}E\int_0^T b^0(s/\varepsilon, \tilde{\Gamma}^\varepsilon(s))ds = \frac{\lambda(\varepsilon)}{\varepsilon} \le \bar{\lambda} + 2\delta + |O(\varepsilon^{1/3})|,$$

i.e., (5.3) is fulfilled. The theorem is proved.

Remark 5.1. Most likely, the following assertion is true:

$$g^\varepsilon(p) = \varepsilon\bar{g}(p) + o(\varepsilon),$$

where $g^\varepsilon(p)$ is the moment Lyapunov exponent of the original system (2.1) (see (2.6)) and $\bar{g}(p)$ is the averaged moment Lyapunov exponent (see (4.8)). This fact can be easily verified for the single-degree-of-freedom oscillator systems considered in [13]. To this aim we can use the approach of averaging presented in Section 7 to get $\bar{g}(p)$ explicitly and then compare this $\bar{g}(p)$ with the expansion $g^\varepsilon(p)$ obtained in [13]. We underline that the approach of Section 7 is rigorous for the single-degree-of-freedom systems.

6. Gyroscopic systems with small noise and damping

We begin with transformation of the system (1.2).

1. The case $c_1 > 0,\ c_2 > 0$. By the change of variables

$$u_1 = \sqrt{c_1}y_1,\ u_2 = \sqrt{a_1}\dot{y}_1,\ u_3 = \sqrt{c_2}y_2,\ u_4 = \sqrt{a_2}\dot{y}_2 \tag{6.1}$$

in (1.2), we get

$$\dot{u} = Bu,\ B = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ -b_1 & 0 & 0 & g_1 \\ 0 & 0 & 0 & b_2 \\ 0 & -g_1 & -b_2 & 0 \end{pmatrix}, \tag{6.2}$$

where

$$b_1 = \sqrt{\frac{c_1}{a_1}},\ b_2 = \sqrt{\frac{c_2}{a_2}},\ g_1 = \frac{g}{\sqrt{a_1 a_2}}\ .$$

If $g = 0$, then the system (6.2) is of the form (1.5). If $g \neq 0$, then the change of variables

$$u = Qx,\ Q = \begin{pmatrix} b_1 & 0 & b_1 & 0 \\ 0 & \omega_1 & 0 & \omega_2 \\ 0 & \dfrac{\omega_1(b_2^2 - \omega_2^2)}{b_2 g_1} & 0 & \dfrac{\omega_2(b_2^2 - \omega_1^2)}{b_2 g_1} \\ \dfrac{b_1^2 - \omega_1^2}{g_1} & 0 & \dfrac{b_1^2 - \omega_2^2}{g_1} & 0 \end{pmatrix}, \tag{6.3}$$

in the system (6.2) gives

$$\dot{x} = Jx, \tag{6.4}$$

i.e., a system of the form (1.5). In (6.3) and (6.4), ω_1 and ω_2 are different positive numbers satisfying the equation

$$\omega^4 - (b_1^2 + b_2^2 + g_1^2)\omega^2 + b_1^2 b_2^2 = 0. \tag{6.5}$$

It is useful to write the inverse of Q :

$$Q^{-1} = \frac{1}{\omega_2^2 - \omega_1^2} \begin{pmatrix} \dfrac{\omega_2^2 - b_1^2}{b_1} & 0 & 0 & g_1 \\ 0 & \dfrac{b_2^2 - \omega_1^2}{\omega_1} & -\dfrac{b_2 g_1}{\omega_1} & 0 \\ \dfrac{b_1^2 - \omega_1^2}{b_1} & 0 & 0 & -g_1 \\ 0 & \dfrac{\omega_2^2 - b_2^2}{\omega_2} & \dfrac{b_2 g_1}{\omega_2} & 0 \end{pmatrix}. \tag{6.6}$$

2. The case $c_1 < 0,\ c_2 < 0,\ |g| > \sqrt{a_1|c_2|} + \sqrt{a_2|c_1|}$. Changing the variables

$$u_1 = \sqrt{|c_1|}y_1,\ u_2 = \sqrt{a_1}\dot{y}_1,\ u_3 = \sqrt{|c_2|}y_2,\ u_4 = \sqrt{a_2}\dot{y}_2 \tag{6.7}$$

in (1.2), we obtain

$$\dot{u} = Bu,\ B = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_1 & 0 & 0 & g_1 \\ 0 & 0 & 0 & b_2 \\ 0 & -g_1 & b_2 & 0 \end{pmatrix}, \tag{6.8}$$

where

$$b_1 = \sqrt{\frac{|c_1|}{a_1}},\ b_2 = \sqrt{\frac{|c_2|}{a_2}},\ g_1 = \frac{g}{\sqrt{a_1 a_2}}\ .$$

The further change of variables

$$u = Qx,\ Q = \begin{pmatrix} b_1 & 0 & b_1 & 0 \\ 0 & \omega_1 & 0 & \omega_2 \\ 0 & \dfrac{\omega_1(b_2^2+\omega_2^2)}{b_2 g_1} & 0 & \dfrac{\omega_2(b_2^2+\omega_1^2)}{b_2 g_1} \\ -\dfrac{b_1^2+\omega_1^2}{g_1} & 0 & -\dfrac{b_1^2+\omega_2^2}{g_1} & 0 \end{pmatrix}, \tag{6.9}$$

translates the system (6.8) to (6.4). Now ω_1 and ω_2 in (6.9) and (6.4) are different positive numbers satisfying the equation

$$\omega^4 + (b_1^2 + b_2^2 - g_1^2)\omega^2 + b_1^2 b_2^2 = 0. \tag{6.10}$$

The inverse of Q is equal to

$$Q^{-1} = \frac{1}{\omega_2^2 - \omega_1^2} \begin{pmatrix} \dfrac{b_1^2+\omega_2^2}{b_1} & 0 & 0 & g_1 \\ 0 & -\dfrac{b_2^2+\omega_1^2}{\omega_1} & \dfrac{b_2 g_1}{\omega_1} & 0 \\ -\dfrac{b_1^2+\omega_1^2}{b_1} & 0 & 0 & -g_1 \\ 0 & \dfrac{b_2^2+\omega_2^2}{\omega_2} & -\dfrac{b_2 g_1}{\omega_2} & 0 \end{pmatrix}. \tag{6.11}$$

3. Stabilization by noise. We consider the following special case of the SDEs (1.1):

$$\begin{aligned} &\ddot{y}_1 - g\dot{y}_2 + c_1 y_1 + \varepsilon k_1 \dot{y}_1 + \sqrt{\varepsilon}(\tilde{k}_1 \dot{y}_1 \circ \dot{w}_1 - \tilde{g}\dot{y}_2 \circ \dot{w}_3) = 0 \\ &\ddot{y}_2 + g\dot{y}_1 + c_2 y_2 - \varepsilon k_2 \dot{y}_2 + \sqrt{\varepsilon}(\tilde{k}_2 \dot{y}_2 \circ \dot{w}_2 + \tilde{g}\dot{y}_1 \circ \dot{w}_3) = 0\ , \end{aligned} \tag{6.12}$$

where $c_1 < 0,\ c_2 < 0,\ |g| > \sqrt{|c_2|} + \sqrt{|c_1|}$ (i.e., under $\varepsilon = 0$ the system is stable), $k_1 > 0,\ k_2 > 0$. If $\tilde{k}_1 = \tilde{k}_2 = \tilde{g} = 0$ (i.e., there is no noise), then the system is asymptotically stable iff $\varepsilon > 0$ and the following conditions are fulfilled (see (1.7)):

$$\begin{aligned} &\frac{c_2}{c_1}k_1 < k_2 < k_1, \\ &g^2 > |c_1| + |c_2| + \varepsilon^2 k_1 k_2 + \frac{(k_2 c_1 - k_1 c_2)^2 + c_1 c_2 (k_1 - k_2)^2}{(k_2 c_1 - k_1 c_2)(k_1 - k_2)}. \end{aligned} \tag{6.13}$$

Below we consider an example (Example 6.1) for unstable case when $k_2/k_1 \le c_2/c_1$ and succeed in stabilizing the system by noise.

The system (6.12) has three noises: the first noise is concerned to the positive damping, the second one – to the negative one, and the third noise – to the gyroscopic forces. Our goal is to investigate how a small positive and negative damping together with a small noise affect the stability of the gyroscopic system. Let us transform the system (6.12) to the standard form (2.1). Due to the change of variables (6.7)-(6.11), we obtain the

Stratonovich SDE

$$dX^\varepsilon = JX^\varepsilon dt + \varepsilon \tilde{A}_0 X^\varepsilon dt + \sqrt{\varepsilon}\sum_{r=1}^{3} A_r X^\varepsilon \circ dw_r(t), \tag{6.14}$$

where

$$\tilde{A}_0 = \frac{1}{\omega_2^2-\omega_1^2} \tag{6.15}$$
$$\times \begin{pmatrix} -k_2(|c_1|+\omega_1^2) & 0 & -k_2(|c_1|+\omega_2^2) & 0 \\ 0 & k_1(|c_2|+\omega_1^2) & 0 & \dfrac{k_1\omega_2(|c_2|+\omega_1^2)}{\omega_1} \\ k_2(|c_1|+\omega_1^2) & 0 & k_2(|c_1|+\omega_2^2) & 0 \\ 0 & -\dfrac{k_1\omega_1(|c_2|+\omega_2^2)}{\omega_2} & 0 & -k_1(|c_2|+\omega_2^2) \end{pmatrix},$$

$$A_1 = \frac{\tilde{k}_1}{\omega_2^2-\omega_1^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |c_2|+\omega_1^2 & 0 & \dfrac{\omega_2(|c_2|+\omega_1^2)}{\omega_1} \\ 0 & 0 & 0 & 0 \\ 0 & -\dfrac{\omega_1(|c_2|+\omega_2^2)}{\omega_2} & 0 & -(|c_2|+\omega_2^2) \end{pmatrix}, \tag{6.16}$$

$$A_2 = \frac{\tilde{k}_2}{\omega_2^2-\omega_1^2}\begin{pmatrix} |c_1|+\omega_1^2 & 0 & |c_1|+\omega_2^2 & 0 \\ 0 & 0 & 0 & 0 \\ -(|c_1|+\omega_1^2) & 0 & -(|c_1|+\omega_2^2) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.17}$$

(6.18)
$$A_3 = \frac{\tilde{g}}{\omega_2^2-\omega_1^2}\begin{pmatrix} 0 & -g\omega_1 & 0 & -g\omega_2 \\ \dfrac{(|c_1|+\omega_1^2)(|c_2|+\omega_1^2)}{g\omega_1} & 0 & \dfrac{(|c_1|+\omega_2^2)(|c_2|+\omega_1^2)}{g\omega_1} & 0 \\ 0 & g\omega_1 & 0 & g\omega_2 \\ -\dfrac{(|c_1|+\omega_1^2)(|c_2|+\omega_2^2)}{g\omega_2} & 0 & -\dfrac{(|c_1|+\omega_2^2)(|c_2|+\omega_2^2)}{g\omega_2} & 0 \end{pmatrix}.$$

The SDE (6.14) can be rewritten in the Ito form as

$$dX^\varepsilon = JX^\varepsilon dt + \varepsilon A_0 X^\varepsilon dt + \sqrt{\varepsilon}\sum_{r=1}^{3} A_r X^\varepsilon dw_r(t). \tag{6.19}$$

Here

$$A_0 = \tilde{A}_0 + \frac{1}{2}(A_1^2 + A_2^2 + A_3^2) \tag{6.20}$$

$$= \frac{1}{\omega_2^2-\omega_1^2}\begin{pmatrix} -\bar{k}_2(|c_1|+\omega_1^2) & 0 & -\bar{k}_2(|c_1|+\omega_2^2) & 0 \\ 0 & \bar{k}_1(|c_2|+\omega_1^2) & 0 & \frac{\bar{k}_1\omega_2(|c_2|+\omega_1^2)}{\omega_1} \\ \bar{k}_2(|c_1|+\omega_1^2) & 0 & \bar{k}_2(|c_1|+\omega_2^2) & 0 \\ 0 & -\frac{\bar{k}_1\omega_1(|c_2|+\omega_2^2)}{\omega_2} & 0 & -\bar{k}_1(|c_2|+\omega_2^2) \end{pmatrix}$$

with

(6.21) $$\bar{k}_1 = k_1 + \frac{\tilde{g}^2-\tilde{k}_1^2}{2},\ \bar{k}_2 = k_2 + \frac{\tilde{k}_2^2-\tilde{g}^2}{2}.$$

In particular, it follows from these expressions that $A_0 = \tilde{A}_0$ (Stratonovich and Ito SDEs coincide) for the case $\tilde{k}_1^2 = \tilde{k}_2^2 = \tilde{g}^2$.

Example 6.1. Let $c_1 = -2,\ c_2 = -1/2,\ g^2 = 5$ in (6.12). Then

(6.22) $$|c_1| = 2,\ |c_2| = \frac{1}{2},\ \omega_1^2 = \frac{1}{2},\ \omega_2^2 = 2$$

and

(6.23)

$$A_0 = \begin{pmatrix} -\frac{5}{3}\bar{k}_2 & 0 & -\frac{8}{3}\bar{k}_2 & 0 \\ 0 & \frac{2}{3}\bar{k}_1 & 0 & \frac{4}{3}\bar{k}_1 \\ \frac{5}{3}\bar{k}_2 & 0 & \frac{8}{3}\bar{k}_2 & 0 \\ 0 & -\frac{5}{6}\bar{k}_1 & 0 & -\frac{5}{3}\bar{k}_1 \end{pmatrix},\ A_1 = \tilde{k}_1\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{5}{6} & 0 & -\frac{5}{3} \end{pmatrix},$$

$$A_2 = \tilde{k}_2\begin{pmatrix} \frac{5}{3} & 0 & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{5}{3} & 0 & -\frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ A_3 = \tilde{g}\begin{pmatrix} 0 & -\frac{\sqrt{10}}{3} & 0 & -\frac{2\sqrt{10}}{3} \\ \frac{\sqrt{10}}{3} & 0 & \frac{8\sqrt{10}}{15} & 0 \\ 0 & \frac{\sqrt{10}}{3} & 0 & \frac{2\sqrt{10}}{3} \\ -\frac{5\sqrt{10}}{12} & 0 & -\frac{2\sqrt{10}}{3} & 0 \end{pmatrix}.$$

Further,

(6.24) $$\beta^0(y) = -\frac{5}{6}k_1 + \frac{4}{3}k_2 + (\frac{7}{6}k_1 - \frac{13}{6}k_2)y + \tilde{k}_1^2(\frac{37}{72} + \frac{41}{36}y - \frac{245}{144}y^2) + \tilde{k}_2^2(\frac{14}{9} + \frac{73}{18}y - \frac{409}{72}y^2) + \tilde{g}^2(\frac{533}{180} - \frac{533}{90}y + \frac{6929}{1440}y^2),$$

$$\sigma(y) = (\frac{127}{18}\tilde{k}_1^2 + \frac{427}{18}\tilde{k}_2^2)y^2(1-y)^2 + \tilde{k}_1^2(\frac{4}{3} - \frac{1}{2}y)^2 y(1-y) + \tilde{k}_2^2(\frac{8}{3} - y)^2 y(1-y) + \tilde{g}^2(\frac{328}{45} - \frac{1066}{45}y + \frac{6929}{360}y^2)y(1-y),$$

$$\alpha(y)=(\frac{7}{3}k_1-\frac{13}{3}k_2+\frac{53}{18}\tilde{k}_1^2+\frac{97}{9}\tilde{k}_2^2-y(\frac{127}{18}\tilde{k}_1^2+\frac{427}{18}\tilde{k}_2^2))y(1-y)$$
$$+\frac{\tilde{k}_1^2}{2}(\frac{4}{3}-\frac{1}{2}y)^2(1-2y)+\frac{\tilde{k}_2^2}{2}(\frac{8}{3}-y)^2(1-2y)+\frac{\tilde{g}^2}{2}(\frac{328}{45}-\frac{1066}{45}y+\frac{6929}{360}y^2)(1-2y).$$

We note that if $\tilde{k}_1^2=\tilde{k}_2^2=0$, then $\sigma(y)=0$ for $y=8/13$, i.e., this is the case for which the conditions of Lemma 5.1 are not fulfilled.

Due to (6.13), the system (6.12) in the absence of noise (i.e., $\tilde{k}_1=\tilde{k}_2=\tilde{g}=0$) is asymptotically stable for any $k_1>0,\ k_2>0$ such that $\frac{2}{5}<\frac{k_2}{k_1}<\frac{5}{8}$ if $\varepsilon>0$ is small enough. Let us try to stabilize the unstable case $\frac{k_2}{k_1}\le\frac{2}{5}$ by noise. Very often a noise acts as a negative friction and we may expect that $\tilde{k}_2$ acts as an increase of k_2 and $\tilde{k}_1$ as a decrease of k_1. Consequently, the presence of a damping noise ($\tilde{k}_1\neq 0,\ \tilde{k}_2\neq 0$) can act so as if the quotient k_2/k_1 increases and the system becomes stable. As for noise in the gyroscopic forces, we see from the expression for $\beta^0(y)$ that the quadratic form under $\tilde{g}^2$ is positive definite and, most likely, this noise does not lead to stability in this example. Let us put

(6.25) $$k_1=k,\ k_2=\nu k,\ \tilde{k}_1^2=\tilde{k}_2^2=\tilde{k}^2,\ \tilde{g}=0.$$

Then if $\nu=2/5$,

(6.26) $$\beta^0(y)=-\frac{3}{10}k(1-y)+\tilde{k}^2(\frac{149}{72}+\frac{187}{36}y-\frac{1063}{144}y^2).$$

We see that for any $k>0$ the quadratic form $\beta^0(y)<0$ for $0\le y\le 1$ if $\tilde{k}^2$ is sufficiently small. Therefore for the systems under consideration the stabilization by noise is possible. Let us underline that this is impossible for oscillating systems with one degree of freedom, see [13], [9], of course, we keep in mind the systems in the sense of Stratonovich. For the case (6.25), we have

(6.27) $$q(y)=\frac{\sigma'(y)-2\alpha(y)}{\sigma(y)}=\frac{(-42+78\nu)k+\tilde{k}^2(247-531.5y)}{\tilde{k}^2(80+217y-265.75y^2)}.$$

Further we use the formula (see (4.1), (4.5))

$$\bar{\lambda}=\int_0^1\beta^0(y)\mu(y)dy,\ \mu(y)=C\exp(-\int_{1/2}^{y}q(y')dy'),$$

where C is the normalizing constant.

In Table 6.1, the results of computing the Lyapunov exponent are given for $\nu=0.39,\ k_1=k=20,\ k_2=7.8$ (i.e. $\nu=0.39$), and different $\tilde{k}^2$.

Let us consider another case when

(6.28) $$k_1=k,\ k_2=\nu k,\ \tilde{k}_1^2=\tilde{k}_2^2=\tilde{g}^2.$$

We recall that in this case the systems in the sense of Stratonovich and Ito coincide. If $\nu=1/2$ then the considered system in the absence of noise is asymptotically stable. Our

TABLE 6.1. Results of computing the Lyapunov exponent in the case (6.25). The parameters are $\nu = 0.39$, $k = 20$.

$\tilde{k}^2$	0.05	0.1	0.3	0.7	1.0	1.2	1.5
$\bar{\lambda}$	0.1174	0.0681	−0.1274	−0.3029	−0.1378	0.0650	0.4524

aim is to determine the level of noise which destructs the stability. For $\nu = 1/2$,

$$\beta^0(y) = (-\frac{1}{6} + \frac{1}{12}y)k + \tilde{g}^2(\frac{1811}{360} - \frac{131}{180}y - \frac{3701}{1440}y^2),$$

$$q(y) = \frac{-3k + \tilde{g}^2(33.8 - 185.05y)}{\tilde{g}^2(145.6 + 3.8y - 92.525y^2)}.$$

For $k_1 = 20$, $k_2 = 10$ (i.e. $k = 20$), the calculations give $\bar{\lambda} = -1.564$ if $\tilde{g}^2 = 0.1$, $\bar{\lambda} = -0.3082$ if $\tilde{g}^2 = 0.6$, $\bar{\lambda} = 0.0245$ if $\tilde{g}^2 = 0.7$.

7. Another approach to averaging

Let us apply the following change of variables to the system (2.1) (for simplicity in writing we omit ε at X^ε):

$$\begin{aligned} X_1 &= r_1 \cos\varphi_1,\ X_2 = r_1 \sin\varphi_1, \\ X_3 &= r_2 \cos\varphi_2,\ X_4 = r_2 \sin\varphi_2. \end{aligned} \tag{7.1}$$

We get

$$\begin{aligned} dX_1 &= dr_1 \cos\varphi_1 - r_1 \sin\varphi_1 d\varphi_1 - \frac{1}{2} r_1 \cos\varphi_1 (d\varphi_1)^2 - \sin\varphi_1 dr_1 d\varphi_1 \\ dX_2 &= dr_1 \sin\varphi_1 + r_1 \cos\varphi_1 d\varphi_1 - \frac{1}{2} r_1 \sin\varphi_1 (d\varphi_1)^2 + \cos\varphi_1 dr_1 d\varphi_1. \end{aligned}$$

From here

$$\begin{aligned} dr_1 &= \frac{1}{2} r_1 (d\varphi_1)^2 + \cos\varphi_1 dX_1 + \sin\varphi_1 dX_2 \\ r_1 d\varphi_1 &= -\sin\varphi_1 dX_1 + \cos\varphi_1 dX_2 - dr_1 d\varphi_1. \end{aligned}$$

The second equation implies

$$(d\varphi_1)^2 = \frac{1}{r_1^2}(\cos\varphi_1 dX_2 - \sin\varphi_1 dX_1)^2$$

and consequently

$$
\begin{aligned}
(7.2)\qquad dr_1 &= \cos\varphi_1 dX_1 + \sin\varphi_1 dX_2 + \frac{1}{2r_1}(\cos\varphi_1 dX_2 - \sin\varphi_1 dX_1)^2 \\
&= \varepsilon\cos\varphi_1 (A_0X)_1 dt + \varepsilon \sin\varphi_1 (A_0X)_2 dt \\
&+\frac{\varepsilon}{2r_1}(\cos^2\varphi_1 \sum_{r=1}^{q}(A_rX)_2^2 + \sin^2\varphi_1 \sum_{r=1}^{q}(A_rX)_1^2 - \sin 2\varphi_1 \sum_{r=1}^{q}(A_rX)_1(A_rX)_2)dt \\
&+\sqrt{\varepsilon}\cos\varphi_1 \sum_{r=1}^{q}(A_rX)_1 dw_r + \sqrt{\varepsilon}\sin\varphi_1 \sum_{r=1}^{q}(A_rX)_2 dw_r.
\end{aligned}
$$

Analogously

$$
\begin{aligned}
(7.3)\qquad dr_2 &= \varepsilon\cos\varphi_2 (A_0X)_3 dt + \varepsilon \sin\varphi_2 (A_0X)_4 dt \\
&+\frac{\varepsilon}{2r_2}(\cos^2\varphi_2 \sum_{r=1}^{q}(A_rX)_4^2 + \sin^2\varphi_2 \sum_{r=1}^{q}(A_rX)_3^2 - \sin 2\varphi_2 \sum_{r=1}^{q}(A_rX)_3(A_rX)_4)dt \\
&+\sqrt{\varepsilon}\cos\varphi_2 \sum_{r=1}^{q}(A_rX)_3 dw_r + \sqrt{\varepsilon}\sin\varphi_2 \sum_{r=1}^{q}(A_rX)_4 dw_r.
\end{aligned}
$$

Now we substitute X_k, $k = 1,2,3,4$, according to (7.1) in (7.2)-(7.3). The obtained equations contain four stochastic variables: r_1, r_2, φ_1, φ_2. The variables r_1 and r_2 are slow and φ_1 and φ_2 are fast: $\varphi_1 \approx -\omega_1 t$, $\varphi_2 \approx -\omega_2 t$. Averaging gives us an autonomous stochastic system with respect to r_1 and r_2 which is homogeneous in r_1, r_2 of degree one (see [1]). Hence, the known procedures concerning Lyapunov exponents and moment Lyapunov exponents for linear systems can be employed to system (7.2)-(7.3).

We would like to emphasize that the derivation in this section (and in [1]) is not rigorous. In particular, difficulties in giving a rigorous proof are due to the fact that both r_1 and r_2 from the denominators of (7.2) and (7.3) can take zero values at some moments of time. At the same time a number of numerical experiments for evaluating principal terms of Lyapunov exponents show coincidence of the results obtained by two considered approaches.

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